On the uniqueness of L-fuzzy sets in the representation of families of sets*

Peng He[†], Xue-ping Wang[‡]
College of Mathematics and Software Science, Sichuan Normal University, Chengdu, Sichuan 610066, People's Republic of China

Abstract

This paper deals with the uniqueness of L-fuzzy sets in the representation of a given family of subsets of a nonempty set. It first shows a formula of the number of L-fuzzy sets whose collection of cuts coincides with a given family of subsets of a nonempty set, and then provides a necessary and sufficient condition under which such L-fuzzy set is unique.

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1 Introduction

In the classical paper [19], Zadeh introduced a notion of a fuzzy set of a set X as a function from X into [0,1]. In 1967, Goguen [3] gave a generalized version of the notion which is called an L-fuzzy set. Since then, L-fuzzy sets and structures have been widely studied. It is well-known that L-fuzzy mathematics attracts more and more interest in many branches, for instance, algebraic theories including order-theoretic structures (see e.g., [4, 6, 10, 18]), automata and tree series (see e.g., [1]) and theoretical computer science (see e.g., [2]). Among all the topics on L-fuzzy mathematics, the representation of a poset by an L-fuzzy set is very interesting, which in the case of a fuzzy set had been studied by Šešelja and Tepavčević [14, 15, 17], and by Jaballah and Saidi [8] who investigated the characterization of all fuzzy sets of X that can be identified with a given arbitrary family C of subsets of X together with a given arbitrary subset of [0,1], in particular, they

*Supported by National Natural Science Foundation of China (No.61573240)
[†]*E-mail address*: 443966297@qq.com
[‡]Corresponding author. xpwang1@hotmail.com; fax: +86-28-84761393

gave necessary and sufficient conditions for both existence and uniqueness of such fuzzy sets. Later, Saidi and Jaballah investigated the problem of uniqueness under different considerations (see e.g., [11–13]). Also, Gorjanac-Ranitović and Tepavčević [5] formulated a necessary and sufficient condition, under which for a given family of subsets $\mathcal{F}$ of a set X and a fixed complete lattice L there is an L-fuzzy set μ such that the collection of cuts of μ coincides with $\mathcal{F}$. Further, Jiménez, Montes, Šešelja and Tepavčević [9] showed a necessary and sufficient condition, under which a collection of crisp up-sets (down-sets) of a poset X consists of cuts of a lattice valued up-set (down-set). Therefore, a natural problem is: What is the condition under which the uniqueness of L-fuzzy set whose collection of cuts coincides with a given family of subsets of a nonempty set is guaranteed? Unfortunately, there are no results about the problem till now. In this paper, we shall discuss the condition under which such L-fuzzy set is unique.

The paper is organized as follows. For the sake of convenience, some notions and previous results are given in Section 2. Section 3 first shows a formula of the number of L-fuzzy sets whose collection of cuts is equal to a given family of subsets of a nonempty set, and then provides a necessary and sufficient condition under which such L-fuzzy set is unique. Conclusions are drawn in Section 4.

2 Preliminaries

We list some necessary notions and relevant properties from the classical order theory in the sequel. For more comprehensive presentation, see e.g., book [7].

A poset is a structure $(P, \leq)$ where P is a nonempty set and $\leq$ an ordering (reflexive, antisymmetric and transitive) relation on P. A poset $(P, \leq_d)$ is a dual poset to the poset $(P, \leq)$, where $\leq_d$ is a dual ordering relation, defined by $x \leq_d y$ if and only if $y \leq x$. A sub-poset of a poset $(P, \leq)$ is a poset $(Q, \leq)$ where Q is a nonempty subset of P and $\leq$ on Q is restricted from P. A complete lattice is a poset $(L, \leq)$ in which every subset M has the greatest lower bound, infimum, meet, denoted by $\bigwedge M$, and the least upper bound, supremum, join, denoted by $\bigvee M$. A complete lattice L possesses the top element 1_L and the bottom element 0_L.

In the following, we present some notations from the theory of L-fuzzy sets. More details about the relevant properties can be found e.g., in [5, 9, 16, 17].

An L-fuzzy set, Lattice-valued or L-valued set here is a mapping $\mu : X \to L$ from a nonempty set X (domain) into a complete lattice $(L, \wedge, \vee, 0_L, 1_L)$ (co-domain) (see [3]).

If $\mu : X \to L$ is an L-fuzzy set on X then, for $p \in L$, the set

$$\mu_p = \{x \in X \mid \mu(x) \geq p\}$$

is called the p-cut, a cut set or simply a cut of μ. Let L^μ be defined by

$$L^\mu = \{p \in L \mid p = \bigwedge B \text{ with } B \subseteq \mu(X)\}, \tag{1}$$

where $\mu(X) = \{\mu(x) \mid x \in X\}$. Then L^μ is a complete lattice (see [9]).

We say that a mapping $f\colon L_1 \to L_2$ from a complete lattice L_1 to a complete lattice L_2 preserves all infima if $f(\bigwedge_{s\in S} s) = \bigwedge_{s\in S} f(s)$ for all $S \subseteq L_1$, and it preserves the top element if $f(1_{L_1}) = 1_{L_2}$. Then the following two statements present some characterizations of the collection of cuts of an L-fuzzy set.

Proposition 2.1 ([5]) *Let $(L, \vee_L, \wedge_L)$ and $(L_1, \vee_{L_1}, \wedge_{L_1})$ be complete lattices and let $\phi : L \to L_1$ be the injection from L to L_1 which maps the top element of L to the top element of L_1, such that for all $x, y \in L, \phi(x \wedge_L y) = \phi(x) \wedge_{L_1} \phi(y)$. Let $\mu : X \to L$ be an L-fuzzy set. Let L-fuzzy set $\nu : X \to L_1$ be defined by $\nu(x) = \phi(\mu(x))$. Then the two L-fuzzy sets μ and ν have the same families of cuts and $\mu_p = \nu_{\phi(p)}$ for all $p \in L$.*

Theorem 2.1 ([5]) *Let L be a fixed complete lattice. Necessary and sufficient conditions under which $\mathcal{F} \subseteq \mathcal{P}(X)$ is the collection of cut sets of an L-fuzzy set with domain X are:*
(1) *$\mathcal{F}$ is closed under arbitrary intersections and contains X.*
(2) *The dual poset of $\mathcal{F}$ under inclusion can be embedded into L, such that all infima and the top element are preserved under the embedding.*

3 Uniqueness of L-fuzzy sets

In this section we shall investigate the condition for the uniqueness of L-fuzzy sets in the representation of families of sets.

Let $(M, \leq)$ be a sub-poset of $(N, \leq)$. Define a mapping $\iota_{(M,N)} : M \to N$ by

$$\iota_{(M,N)}(x) = x \tag{2}$$

for all $x \in M$.

Definition 3.1 Let $(L_1, \leq)$ be a sub-poset of a complete lattice $(L, \leq)$. The mapping $\iota_{(L_1,L)}$ is called an $\iota_{(L_1,L)}$-embedding if all infima and the top element are preserved under $\iota_{(L_1,L)}$.

Let $\mu_L = \{\mu_p \mid p \in L\}$ if $\mu : X \to L$ is an L-fuzzy set. Then we have the following theorem.

Theorem 3.1 *Let L be a complete lattice and let $\mu : X \to L$ be an L-fuzzy set. Then:*
(a) *$(\mu_L, \supseteq) \cong (L^\mu, \leq)$;*
(b) *L^μ can be embedded into L by $\iota_{(L^\mu,L)}$-embedding;*
(c) *$\mu = \iota_{(L^\mu,L)} \circ \nu$ where the mapping $\nu : X \to L^\mu$ is defined by $\nu(x) = \mu(x)$ for all $x \in X$, $\nu_{L^\mu} = \mu_L$ and $(L^\mu)^\nu = L^\mu$.*

Proof. (a) Let $\varphi : \mu_L \to L^\mu$ be defined by

$$\varphi(\mu_p) = \bigwedge_{x\in\mu_p} \mu(x) \tag{3}$$

for all $\mu_p \in \mu_L$. It is easy to see that for all $p \in L$

$$\varphi(\mu_p) \geq p. \tag{4}$$

In what follows, we further prove that

$$\varphi(\mu_p) = p \tag{5}$$

for all $p \in L^\mu$.

Indeed, let $p \in L^\mu$. Then $p = \bigwedge_{B_p \subseteq \mu(X)} B_p$ by formula (1). If $B_p = \emptyset$ then $p = 1_L$. Clearly, $\varphi(\mu_p) = \varphi(\mu_{1_L}) = 1_L = p$. Now, suppose $B_p \neq \emptyset$. Then from $B_p \subseteq \mu(X) = \{\mu(x) \mid x \in X\}$, we have $B_p \subseteq \{\mu(x) \in \mu(X) \mid \mu(x) \geq p\}$ since $b \in B_p$ implies $p \leq b$. Thus by formulas (3) and (4),

$$p = \bigwedge_{B_p \subseteq \mu(X)} B_p \geq \bigwedge_{\mu(x) \geq p} \mu(x) = \bigwedge_{x \in \mu_p} \mu(x) = \varphi(\mu_p) \geq p,$$

i.e., $\varphi(\mu_p) = p$. Therefore, $\varphi(\mu_p) = p$ for all $p \in L^\mu$.

Formula (5) means that φ is surjective. Moreover, one can check that φ is injective by formulas (3) and (4). Consequently, the mapping φ is a bijection.

In what follows, we shall prove that both φ and φ^{-1} preserve the orders $\supseteq$ and $\leq$, respectively. In fact, if $\mu_p \subseteq \mu_q$ then obviously $\varphi(\mu_p) = \bigwedge_{x \in \mu_p} \mu(x) \geq \bigwedge_{x \in \mu_q} \mu(x) = \varphi(\mu_q)$, i.e., $\varphi(\mu_p) \geq \varphi(\mu_q)$. At the same time, if $r, e \in L^\mu$ and $r \leq e$ then by the definition of a cut set, we have $\mu_r \supseteq \mu_e$. Thus, by formula (5), $\varphi^{-1}(r) = \mu_r \supseteq \mu_e = \varphi^{-1}(e)$, i.e., $r \leq e$ implies $\varphi^{-1}(r) \supseteq \varphi^{-1}(e)$.

Therefore, φ is an isomorphism from $(\mu_L, \supseteq)$ to $(L^\mu, \leq)$.

(b) From formula (1), $1_L = \bigwedge \emptyset \in L^\mu$ since $\emptyset \subseteq \mu(X)$. Thus by Definition 3.1, due to formulas (1) and (2), L^μ can be embedded into L by $\iota_{(L^\mu, L)}$-embedding.

(c) It is easy to see $\mu = \iota_{(L^\mu, L)} \circ \nu$. By (b), all the conditions of Proposition 2.1 are fulfilled. Thus $\nu_{L^\mu} = \mu_L$. Moreover, by applying (a) to L^μ-fuzzy set ν, we know that $(\nu_{L^\mu}, \supseteq) \cong ((L^\mu)^\nu, \leq)$. Then $(\mu_L, \supseteq) \cong ((L^\mu)^\nu, \leq)$, which together with $(\mu_L, \supseteq) \cong (L^\mu, \leq)$ yields that $(L^\mu, \leq) \cong ((L^\mu)^\nu, \leq)$. Therefore, from (b), $(L^\mu)^\nu = L^\mu$ since $(L^\mu)^\nu$ is embedded into L^μ by $\iota_{((L^\mu)^\nu, L^\mu)}$-embedding. □

Lemma 3.1 *Let $\mathcal{F}$ be a family of some subsets of a nonempty set X which is closed under intersections and contains X, and let L be a complete lattice. If $(\mathcal{F}, \supseteq) \cong (L_0, \leq)$ and L_0 can be embedded into L by $\iota_{(L_0, L)}$-embedding, then:*
(i) *there exists an L_0-fuzzy set $\nu : X \to L_0$ such that $\nu_{L_0} = \mathcal{F}$ and $L_0^\nu = L_0$;*
(ii) *$\mu = \iota_{(L_0, L)} \circ \nu$ satisfies both $\mu_L = \mathcal{F}$ and $L^\mu = L_0$.*

Proof. By the hypotheses, $(\mathcal{F}, \supseteq)$ can be embedded into L_0, such that all infima and the top element are preserved under the embedding. Then by Theorem 2.1, there exists an L_0-fuzzy set $\nu : X \to L_0$ such that $\nu_{L_0} = \mathcal{F}$. Thus by applying (a) of Theorem 3.1 to ν, we have $(L_0^\nu, \leq) \cong (\nu_{L_0}, \supseteq) = (\mathcal{F}, \supseteq) \cong (L_0, \leq)$, i.e., $(L_0^\nu, \leq) \cong (L_0, \leq)$. Therefore, $L_0 = L_0^\nu$ since $L_0 \supseteq L_0^\nu$ by formula (1).

Now, let $\mu = \iota_{(L_0,L)} \circ \nu$. Since L_0 can be embedded into L by $\iota_{(L_0,L)}$-embedding, we know that all the conditions of Proposition 2.1 are fulfilled. Then $\mu_L = \mathcal{F}$ since $\nu_{L_0} = \mathcal{F}$. Moreover, by formula (1) we have $L^\mu = L_0^\nu$ since $\mu(x) = \nu(x)$ for all $x \in X$ and L_0 can be embedded into L by $\iota_{(L_0,L)}$-embedding. Therefore, $L^\mu = L_0$ since $L_0^\nu = L_0$. $\square$

Let $\mathcal{F}$ be a family of some subsets of a nonempty set X which is closed under intersections and contains X. Given a complete lattice L, we let $OI(L)$ denote the set of lattice isomorphisms from L onto itself. Also, denote

$$H_{(L_0,L_0,\mathcal{F})} = \{\mu \mid \mu : X \to L_0, L_0^\mu = L_0 \text{ and } \mu_{L_0} = \mathcal{F}\} \text{ and}$$
$$H_{(L,L_0,\mathcal{F})} = \{\mu \mid \mu : X \to L, L^\mu = L_0 \text{ and } \mu_L = \mathcal{F}\}.$$

Let

$$\mathcal{N}(L,\mathcal{F}) = \{f \mid f : X \to L, f_L = \mathcal{F}\}$$

and

$$\mathcal{S}(L,\mathcal{F}) = \{P \mid P \text{ can be embedded into } L \text{ by } \iota_{(P,L)}\text{-embedding and } (P,\leq) \cong (\mathcal{F},\supseteq)\}.$$

Then, we have the following lemma.

Lemma 3.2 *Let $\mathcal{F}$ be a family of some subsets of a nonempty set X which is closed under intersections and contains X, and let L be a complete lattice. Then*

$$\mathcal{N}(L,\mathcal{F}) = \bigcup_{L_0 \in \mathcal{S}(L,\mathcal{F})} H_{(L,L_0,\mathcal{F})}.$$

Proof. Obviously, $\mathcal{N}(L,\mathcal{F}) \supseteq \bigcup_{L_0\in\mathcal{S}(L,\mathcal{F})} H_{(L,L_0,\mathcal{F})}$. On the other hand, let $\mu \in \mathcal{N}(L,\mathcal{F})$. Then $\mathcal{F} = \mu_L$. Thus, by Theorem 3.1, $(\mathcal{F},\supseteq) = (\mu_L,\supseteq) \cong (L^\mu,\leq)$ and L^μ can be embedded into L by $\iota_{(L^\mu,L)}$-embedding. Therefore $L^\mu \in \mathcal{S}(L,\mathcal{F})$ and $\mu \in H_{(L,L^\mu,\mathcal{F})}$, and we conclude $\mathcal{N}(L,\mathcal{F}) \subseteq \bigcup_{L_0\in\mathcal{S}(L,\mathcal{F})} H_{(L,L_0,\mathcal{F})}$. $\square$

Theorem 3.2 *Under the assumptions of Lemma 3.1, we have:*
(I) $H_{(L,L_0,\mathcal{F})} = \{\beta \mid \beta = \iota_{(L_0,L)} \circ \mu, \mu \in H_{(L_0,L_0,\mathcal{F})}\}$;
(II) *Let $g \in H_{(L_0,L_0,\mathcal{F})}$. Then $f \in H_{(L_0,L_0,\mathcal{F})}$ if and only if $f = \eta \circ g$ for some $\eta \in OI(L_0)$.*

Proof. (I) We first note that $H_{(L_0,L_0,\mathcal{F})} \neq \emptyset$ by Lemma 3.1. Then, by using Theorem 3.1 and Lemma 3.1, we have $H_{(L,L_0,\mathcal{F})} = \{\beta \mid \beta = \iota_{(L_0,L)} \circ \mu, \mu \in H_{(L_0,L_0,\mathcal{F})}\}$.

(II) Let $f \in H_{(L_0,L_0,\mathcal{F})}$. Define $\eta : L_0 \to L_0$ by

$$\eta(\bigwedge_{x\in B\subseteq X} g(x)) = \bigwedge_{x\in B\subseteq X} f(x)$$

and $\eta_1 : L_0 \to L_0$ by

$$\eta_1(\bigwedge_{x\in B\subseteq X} f(x)) = \bigwedge_{x\in B\subseteq X} g(x).$$

The following proof is made in three parts:

Part (1). Both η and η_1 are mappings from L_0 to L_0.

Note that, from $f, g \in H_{(L_0,L_0,\mathcal{F})}$, we have $L_0^f = L_0$ and $L_0^g = L_0$, which together with (3) and (5) imply that

$$t = \bigwedge_{x \in g_t} g(x) = \bigwedge_{x \in f_t} f(x) \tag{6}$$

for any $t \in L_0$. Then

$$\eta(t) = \eta(\bigwedge_{x \in g_t} g(x)) = \bigwedge_{x \in g_t} f(x) \in L_0 \text{ and } \eta_1(t) = \eta_1(\bigwedge_{x \in f_t} f(x)) = \bigwedge_{x \in f_t} g(x) \in L_0$$

for all $t \in L_0$.

Therefore, in order to prove that η is a mapping from L_0 to L_0, we just need to prove

$$\eta(\bigwedge_{x \in B_1 \subseteq X} g(x)) = \eta(\bigwedge_{x \in B_2 \subseteq X} g(x)) \tag{7}$$

while $\bigwedge_{x \in B_1 \subseteq X} g(x) = \bigwedge_{x \in B_2 \subseteq X} g(x)$.

Now let $p = \bigwedge_{x \in B \subseteq X} g(x)$. Also, we can let $p = \bigwedge_{x \in g_p} g(x)$ by (6). We shall show that

$$\eta(\bigwedge_{x \in B \subseteq X} g(x)) = \eta(\bigwedge_{x \in g_p} g(x)) \tag{8}$$

because formula (8) implies (7).

First, it is easy to see that $B \subseteq g_p$. Then from $f, g \in H_{(L_0,L_0,\mathcal{F})}$, $f_{L_0} = g_{L_0}$. Thus $g_p \in g_{L_0} = f_{L_0}$, and again by formula (5), there exists a unique element $w \in L_0^f$, i.e., $w \in L_0$ such that

$$f_w = g_p \tag{9}$$

since $L_0 = L_0^f$. So that, by (6),

$$\eta(\bigwedge_{x \in g_p} g(x)) = \bigwedge_{x \in g_p} f(x) = \bigwedge_{x \in f_w} f(x) = w. \tag{10}$$

On the other hand, by the definition of η and $L_0 = L_0^f$, we have $\eta(\bigwedge_{x \in B \subseteq X} g(x)) = \bigwedge_{x \in B \subseteq X} f(x) \in L_0$. Let $\bigwedge_{x \in B \subseteq X} f(x) = r$, i.e.,

$$\eta(\bigwedge_{x \in B \subseteq X} g(x)) = r. \tag{11}$$

Then

$$B \subseteq f_r, \tag{12}$$

and $r \geq w$ since $B \subseteq g_p$. Thus

$$f_r \subseteq f_w. \tag{13}$$

Using (6), we have $\bigwedge_{x \in B \subseteq X} f(x) = r = \bigwedge_{x \in f_r} f(x)$. Clearly $f_r \in f_{L_0} = g_{L_0}$. Again, by formula (5), there exists a unique element $q \in L_0^g$, i.e., $q \in L_0$ such that

$$f_r = g_q \tag{14}$$

since $L_0 = L_0^g$. Thus, by formulas (12), (13)and (9), $B \subseteq g_q \subseteq g_p$. Hence, by (6) we know that

$$p = \bigwedge_{x \in B \subseteq X} g(x) \geq \bigwedge_{x \in g_q} g(x) = q \geq \bigwedge_{x \in g_p} g(x) = p$$

i.e., $p = q$. Then $g_p = g_q$, and which together with (9) and (14) means that $f_r = f_w$. Thus, by (5), we conclude that $r = w$, i.e.,

$$\eta(\bigwedge_{x \in B \subseteq X} g(x)) = \eta(\bigwedge_{x \in g_p} g(x))$$

by (10) and (11). This completes the proof of formula (8).

Consequently η is a mapping from L_0 to itself.

Similarly, η_1 is also a mapping from L_0 to itself.

Part (2). Both η and η_1 preserve the order.

Assume that $t_1, t_2 \in L_0$ and $t_1 \leq t_2$. Then $g_{t_1} \supseteq g_{t_2}$ by the definition of a cut set. Thus by (6), $\eta(t_1) = \bigwedge_{x \in g_{t_1}} f(x) \leq \bigwedge_{x \in g_{t_2}} f(x) = \eta(t_2)$, and which means that η preserves the order. Also, we can similarly check that η_1 preserves the order.

Part (3). The mapping η is a bijection and $\eta_1 = \eta^{-1}$.

Let $k \in L_0$. Then from (6),

$$\eta_1 \circ \eta(k) = \eta_1 \circ \eta(\bigwedge_{x \in g_k} g(x)) = \eta_1(\bigwedge_{x \in g_k} f(x)) = \bigwedge_{x \in g_k} g(x) = k.$$

Thus $\eta_1 \circ \eta$ is an identity mapping on L_0. Similarly, we can check that $\eta \circ \eta_1$ is also an identity mapping on L_0. Therefore, the mapping η is a bijection and $\eta_1 = \eta^{-1}$.

From Parts (1), (2) and (3), we conclude that $\eta \in OI(L_0)$ and $f = \eta \circ g$.

Conversely, let $\eta \in OI(L_0)$ and $\eta \circ g = f$. We shall show that $f \in H_{(L_0, L_0, \mathcal{F})}$.

Since f is an L_0-fuzzy set on X, we need to prove that $f_{L_0} = \mathcal{F}$ and $L_0^f = L_0$.

First, we prove $f_{L_0} = \mathcal{F}$, i.e., prove that $f_{L_0} = g_{L_0}$ since $g \in H_{(L_0, L_0, \mathcal{F})}$.

From $\eta \in OI(L_0)$,

$$\begin{aligned} f_{\eta(s)} &= \{x \in X \mid \eta \circ g(x) \geq \eta(s)\} \\ &= \{x \in X \mid g(x) \geq \eta^{-1}(\eta(s)) = s\} \\ &= g_s \end{aligned}$$

for all $s \in L_0$. Therefore $g_{L_0} \subseteq f_{L_0}$. Similarly, we can prove that $g_{L_0} \supseteq f_{L_0}$ since $g = \eta^{-1} \circ f$ and $\eta^{-1} \in OI(L_0)$. Hence,

$$f_{L_0} = g_{L_0}. \tag{15}$$

Secondly, we shall prove $L_0^f = L_0$.

By Theorem 3.1, we know that $(f_{L_0}, \supseteq) \cong (L_0^f, \leq)$ and $(g_{L_0}, \supseteq) \cong (L_0^g, \leq)$. Thus, by (15), $(L_0^f, \leq) \cong (L_0^g, \leq)$. Note that $g \in H_{(L_0, L_0, \mathcal{F})}$ implies that $(L_0^g, \leq) = (L_0, \leq)$. Therefore $(L_0^f, \leq) \cong (L_0, \leq)$. This follows that $L_0^f = L_0$ since $L_0^f \subseteq L_0$.

Finally, $f \in H_{(L_0, L_0, \mathcal{F})}$. □

From Theorem 3.2, we easily deduce the following consequence.

Corollary 3.1 *Under the assumptions of Lemma 3.1. Let $g \in H_{(L_0, L_0, \mathcal{F})}$. Then*

$$H_{(L_0, L_0, \mathcal{F})} = \{\eta \circ g \mid \eta \in OI(L_0)\} \text{ and}$$
$$H_{(L, L_0, \mathcal{F})} = \{\iota_{(L_0, L)} \circ \eta \circ g \mid \eta \in OI(L_0)\}.$$

Given any set A, we denote its cardinality by $|A|$. Then from Corollary 3.1 and Lemma 3.2, we have:

Theorem 3.3 *Let $\mathcal{F}$ be a family of some subsets of a nonempty set X which is closed under intersection and contains X, and let L be a complete lattice. Then*

$$|\mathcal{N}(L, \mathcal{F})| = |\mathcal{S}(L, \mathcal{F})||OI(\mathcal{F})|.$$

Proof. From Lemma 3.2, this result obviously holds if $\mathcal{S}(L, \mathcal{F}) = \emptyset$. Let $L_0 \in \mathcal{S}(L, \mathcal{F})$. Then $(L_0, \leq) \cong (\mathcal{F}, \supseteq)$, which means that

$$|OI(\mathcal{F})| = |OI(L_0)|. \tag{16}$$

Now, by Corollary 3.1, we have

$$|H_{(L, L_0, \mathcal{F})}| = |H_{(L_0, L_0, \mathcal{F})}|. \tag{17}$$

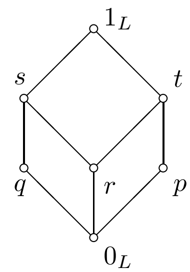
Let $g \in H_{(L_0, L_0, \mathcal{F})}$. Suppose $f_1, f_2 \in H_{(L_0, L_0, \mathcal{F})}$. Then by Corollary 3.1, there exist two mapping $\eta_1, \eta_2 \in OI(L_0)$ such that $f_1 = \eta_1 \circ g$ and $f_2 = \eta_2 \circ g$. Thus $f_1 \neq f_2$ if and only if $\eta_1 \neq \eta_2$, which results in

$$|H_{(L_0, L_0, \mathcal{F})}| = |OI(L_0)|. \tag{18}$$

Let $L_1 \in \mathcal{S}(L, \mathcal{F})$ with $L_1 \neq L_0$. Then it is clear that $H_{(L, L_1, \mathcal{F})} \bigcap H_{(L, L_0, \mathcal{F})} = \emptyset$, and $|H_{(L, L_1, \mathcal{F})}| = |H_{(L, L_0, \mathcal{F})}|$ since $(L_1, \leq) \cong (L_0, \leq)$. Therefore, from Lemma 3.2 and formulas (16), (17) and (18), we have $|\mathcal{N}(L, \mathcal{F})| = |\mathcal{S}(L, \mathcal{F})||OI(\mathcal{F})|$. □

The following example illustrates Theorem 3.3.

Example 3.1 Let us consider a set $X = \{a, b, c\}$ and the complete lattice $(L, \leq)$ represented in Fig.1. Let $\mathcal{F} = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$ be a family of subsets of X.



L

Fig.1 Hasse diagram of L

It is clear that $\mathcal{S}(L, \mathcal{F}) = \{(\{0_L, p, r, t, 1_L\}, \leq), (\{0_L, q, r, s, 1_L\}, \leq)\}$ and $|OI(\mathcal{F})| = 2$. Therefore, from Theorem 3.3, $|\mathcal{N}(L, \mathcal{F})| = 4$. On the other hand, all the L-fuzzy sets of $\mathcal{N}(L, \mathcal{F})$ are

X	a	b	c
δ	r	t	p

X	a	b	c
γ	p	t	r

X	a	b	c
β	q	s	r

X	a	b	c
α	r	s	q

where we consider a tabular representation for the L-fuzzy set on X, respectively.

The following theorem follows immediately from Theorem 3.3.

Theorem 3.4 *Let $\mathcal{F}$ be a family of some subsets of a nonempty set X which is closed under intersection and contains X, and let L be a complete lattice. There exists a unique L-fuzzy μ on X such that $\mathcal{F} = \mu_L$ if and only if $|\mathcal{S}(L, \mathcal{F})| = |OI(\mathcal{F})| = 1$.*

4 Conclusions

This contribution gave a necessary and sufficient condition under which L-fuzzy sets whose collection of cuts equals a given family of subsets of a nonempty set are unique. Using Theorem 12 in [9], one can verify that all our results made for L-fuzzy sets could be applied to L-fuzzy up-sets (L-fuzzy down-sets) since an L-fuzzy up-set (L-fuzzy down-set) is just a particularity of the concept of an L-fuzzy set.

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