

Ordinal sums of triangular norms on a bounded lattice

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Abstract

The ordinal sum construction provides a very effective way to generate a new triangular norm on the real unit interval from existing ones. One of the most prominent theorems concerning the ordinal sum of triangular norms on the real unit interval states that a triangular norm is continuous if and only if it is uniquely representable as an ordinal sum of continuous Archimedean triangular norms. However, the ordinal sum of triangular norms on subintervals of a bounded lattice is not always a triangular norm (even if only one summand is involved), if one just extends the ordinal sum construction to a bounded lattice in a naïve way. In the present paper, appropriately dealing with those elements that are incomparable with the endpoints of the given subintervals, we propose an alternative definition of ordinal sum of countably many (finite or countably infinite) triangular norms on subintervals of a complete lattice, where the endpoints of the subintervals constitute a chain. The completeness requirement for the lattice is not needed when considering finitely many triangular norms. The newly proposed ordinal sum is shown

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to be always a triangular norm. Several illustrative examples are given.

Keywords: Lattice; Triangular norm; Ordinal sum; Partially ordered monoid

1. Introduction

The ordinal sum construction provides a method to construct a new semigroup from existing ones [4]. Ling [14] and Schweizer and Sklar [18] applied this method to a special kind of semigroup, namely to triangular norms (t-norms, for short) on the real unit interval $[0, 1]$. One of the most prominent theorems concerning the ordinal sum of t-norms states that a t-norm is continuous if and only if it is uniquely representable as an ordinal sum of continuous Archimedean t-norms (see, e.g., [1, 13]).

T-norms on more general structures (e.g., posets [6, 19] and bounded lattices [5, 7]) have been proposed and extensively investigated. In 2006, Saminger [16] extended the ordinal sum of t-norms on the real unit interval $[0, 1]$ to the ordinal sum of t-norms on subintervals of a bounded lattice in a rather direct way without much consideration for the characteristics of a lattice, especially the existence of elements that are incomparable with the endpoints of the given subintervals. Unfortunately, Saminger's ordinal sum of t-norms on subintervals of a bounded lattice does not always yield a t-norm even in the case of a single summand. Some researchers [15, 17] characterized when Saminger's ordinal sum of t-norms always leads to a t-norm, while other researchers attempted to modify Saminger's ordinal sum or considered the ordinal sum problem for a particular class of lattices. For instance, Ertuğrul et al. [9] modified Saminger's ordinal sum for one special summand to make sure it results in a t-norm. El-Zekey [8] studied the ordinal sum of t-norms on bounded lattices that can be written as a lattice-

based sum of lattices. Up to now, the ordinal sum problem has not yet been solved completely.

Although Saminger's definition of ordinal sum of t-norms on a bounded lattice is a natural extension of the ordinal sum of t-norms on the real unit interval $[0, 1]$, it is not satisfactory since it does not always lead to a t-norm. This motivates the following question:

Does there exist a more appropriate definition of ordinal sum of t-norms on a bounded lattice?

We argue that any definition of ordinal sum of t-norms on a bounded lattice that reduces to the ordinal sum of t-norms on $[0, 1]$ could be a possible candidate for the answer to the above question. The key lies in whether it always leads to a t-norm. In this paper, appropriately dealing with those elements that are incomparable with the endpoints of the given subintervals, by synthesizing the techniques of [3] and [9], we propose an alternative definition of ordinal sum of countably many t-norms on subintervals of a complete lattice, where the endpoints of the subintervals constitute a chain. The completeness requirement for the lattice is not needed when considering finitely many t-norms. Our proposed ordinal sum is shown to be always a t-norm.

Admittedly, a t-norm on a bounded lattice is nothing else but a commutative and integral partially ordered monoid (pomonoid, for short) [11] and the ordinal sum of pomonoids has been investigated in the literature [10]. However, the ordinal sum of pomonoids is different from the above-mentioned ordinal sum of t-norms since the former is defined on the direct sum of the underlying posets, while the latter is defined on a bounded lattice that is not necessarily the direct sum of the underlying subintervals of the lattice. The poset product of pomonoids proposed in [12] generalizes both the or-

dinal sum of pomonoids and the direct product of pomonoids. Concretely speaking, the poset product of pomonoids reduces to the ordinal sum of pomonoids when the underlying index set is a chain, while it reduces to the direct product of pomonoids when the underlying index set is an antichain. Therefore, the poset product of pomonoids cannot cover our ordinal sum of t-norms due to the fact that in our ordinal sum of t-norms the underlying index set is a chain and the above-mentioned difference between the ordinal sum of pomonoids and the ordinal sum of t-norms.

The remainder of this paper is organized as follows. We recall some basic notions and results related to lattices and t-norms on a bounded lattice, and briefly review the progress in the study of ordinal sums of t-norms on a bounded lattice in Section 2. Section 3 is devoted to proposing an alternative definition of ordinal sum of t-norms on a bounded lattice and proving it to be a t-norm, while Section 4 shows some examples fitting in the newly proposed ordinal sum of t-norms on a bounded lattice. We end with some conclusions and future work in Section 5.

2. Preliminaries

In this section, we recall some basic notions and results related to lattices and t-norms on a bounded lattice, and briefly review the progress in the study of ordinal sums of t-norms on a bounded lattice.

2.1. *T-norms on a bounded lattice*

A *lattice* [2] is a nonempty set L equipped with a partial order \leq such that any two elements x and y have a greatest lower bound (called meet or infimum), denoted by $x \wedge y$, as well as a smallest upper bound (called join or supremum), denoted by $x \vee y$. For $a, b \in L$, the symbol $a < b$ means that

$a \leq b$ and $a \neq b$. If neither $a \leq b$ nor $b \leq a$, then we say that a and b are incomparable. The set of all elements of L that are incomparable with a is denoted by I_a . A lattice (L, \leq, \wedge, \vee) is called *bounded* if it has a top element and a bottom element, while it is said to be *complete* if for any $A \subset L$, the greatest lower bound $\bigwedge A$ and the smallest upper bound $\bigvee A$ of A exist. Obviously, any finite lattice is necessarily complete and any complete lattice is necessarily bounded.

Let (L, \leq, \wedge, \vee) be a lattice and $a, b \in L$ with $a \leq b$. The subinterval $[a, b]$ of L is defined as

$$[a, b] = \{x \in L \mid a \leq x \leq b\}.$$

Other subintervals such as $[a, b[$ and $]a, b[$ can be defined similarly. Obviously, $([a, b], \leq, \wedge, \vee)$ is a bounded lattice with top element b and bottom element a .

Definition 2.1. [5, 6] Let (L, \leq, \wedge, \vee) be a lattice and $[a, b]$ be a subinterval of L . A binary operation $T: [a, b] \times [a, b] \rightarrow [a, b]$ is said to be a t-norm on $[a, b]$ if, for any $x, y, z \in [a, b]$, the following conditions are fulfilled:

- (i) $T(x, y) = T(y, x)$ (commutativity);
- (ii) If $x \leq y$, then $T(x, z) \leq T(y, z)$ (increasingness);
- (iii) $T(T(x, y), z) = T(x, T(y, z))$ (associativity);
- (iv) $T(b, x) = x$ (neutrality).

Theorem 2.2. [5] Let (L, \leq, \wedge, \vee) be a lattice, $[a, b]$ be a subinterval of L and $c \in [a, b]$. The binary operation $T_c: [a, b] \times [a, b] \rightarrow [a, b]$ defined by

$$T_c(x, y) = \begin{cases} x \wedge y & \text{if } b \in \{x, y\} \\ x \wedge y \wedge c & \text{otherwise,} \end{cases}$$

is a t-norm on $[a, b]$.

If $c = b$ (resp. $c = a$), then we retrieve the strongest (resp. weakest) t-norm T_\wedge (resp. T_D) on $[a, b]$.

2.2. Progress in the study of ordinal sums of t-norms on a bounded lattice

The following result concerning ordinal sum of t-norms on the real unit interval $[0, 1]$ is well known.

Theorem 2.3. [13] *Let $\{[a_i, b_i]\}_{i \in I}$ be a family of (nonempty and) pairwise disjoint open subintervals of $[0, 1]$ and $\{T_i\}_{i \in I}$ be a family of t-norms on $[0, 1]$. Then the binary operation $T = \{\langle a_i, b_i, T_i \rangle\}_{i \in I}: [0, 1] \times [0, 1] \rightarrow [0, 1]$, called the ordinal sum of $\{T_i\}_{i \in I}$, defined by*

$$T(x, y) = \begin{cases} a_i + (b_i - a_i)T_i\left(\frac{x - a_i}{b_i - a_i}, \frac{y - a_i}{b_i - a_i}\right) & \text{if } (x, y) \in [a_i, b_i]^2 \\ \min\{x, y\} & \text{otherwise,} \end{cases}$$

is a t-norm on $[0, 1]$.

Remark 2.4. (i) It is well known that any set consisting of nonempty and pairwise disjoint open subintervals of the real unit interval $[0, 1]$ is countable.

(ii) For any $i \in I$, define $\tilde{T}_i: [a_i, b_i] \times [a_i, b_i] \rightarrow [a_i, b_i]$ as follows:

$$\tilde{T}_i(x, y) = a_i + (b_i - a_i)T_i\left(\frac{x - a_i}{b_i - a_i}, \frac{y - a_i}{b_i - a_i}\right).$$

It is easy to see that \tilde{T}_i is a t-norm on $[a_i, b_i]$. So, in the definition of ordinal sum of t-norms on the real unit interval $[0, 1]$, we can suppose that \tilde{T}_i is a t-norm on $[a_i, b_i]$ for any $i \in I$ and replace $a_i + (b_i - a_i)T_i\left(\frac{x - a_i}{b_i - a_i}, \frac{y - a_i}{b_i - a_i}\right)$ by $\tilde{T}_i(x, y)$. Based on this observation, one can naturally extend the notion of ordinal sum of t-norms from the real unit interval $[0, 1]$ to a bounded lattice, as Saminger [16] did in 2006.

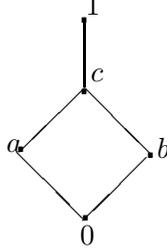


Figure 1: Hasse diagram of the lattice L in Example 2.6.

From here on, $(L, \leq, \wedge, \vee, 0, 1)$ denotes a bounded lattice with top element 1 and bottom element 0.

Definition 2.5. [16] Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice, $\{[a_i, b_i]\}_{i \in I}$ be a family of pairwise disjoint subintervals of L and $\{T_i\}_{i \in I}$ be a family of t-norms on these subintervals. The ordinal sum $T = \{\langle a_i, b_i, T_i \rangle\}_{i \in I} : L \times L \rightarrow L$ is given by

$$T(x, y) = \begin{cases} T_i(x, y) & \text{if } (x, y) \in [a_i, b_i]^2 \\ x \wedge y & \text{otherwise.} \end{cases}$$

According to Saminger [16], however, the above ordinal sum is not always a t-norm even if there is only one summand, as the following example shows.

Example 2.6. [16] Consider the complete lattice L with Hasse diagram shown in Figure 1. The ordinal sum $T = \{\langle a, 1, T_D \rangle\}$ given by Table 1 is not a t-norm on L , since

$$T(T(c, c), b) = T(a, b) = 0 \neq b = T(c, b) = T(c, T(c, b)).$$

Several researchers [15, 17] characterized when Saminger's ordinal sum of t-norms always leads to a t-norm, while other researchers attempted to modify Saminger's ordinal sum or considered the ordinal sum problem for a particular class of lattices. For instance, Ertuğrul et al. [9] modified Saminger's

Table 1: The ordinal sum $T = \{\langle a, 1, T_D \rangle\}$ in Example 2.6.

T	0	b	a	c	1
0	0	0	0	0	0
b	0	b	0	b	b
a	0	0	a	a	a
c	0	b	a	a	c
1	0	b	a	c	1

ordinal sum for one special summand in the following way to make sure it always results in a t-norm.

Theorem 2.7. [9] *Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice, $[a, 1]$ be a subinterval of L and T_1 be a t-norm on $[a, 1]$. Then the binary operation $T: L \times L \rightarrow L$ defined by*

$$T(x, y) = \begin{cases} T_1(x, y) & \text{if } (x, y) \in [a, 1]^2 \\ x \wedge y \wedge a & \text{if } (x, y) \in (I_a \times [0, 1[) \cup ([0, 1[\times I_a) \\ x \wedge y & \text{otherwise,} \end{cases} \quad (1)$$

is a t-norm on L .

Remark 2.8. Expression (1) looks different from the corresponding expression in Theorem 1 of [9], but they are essentially the same.

3. An alternative definition of ordinal sum of t-norms on a bounded lattice

In this section, appropriately dealing with those elements that are incomparable with the endpoints of subintervals, we propose an alternative

definition of ordinal sum of t-norms on a bounded lattice and prove it to always result in a t-norm.

We start by decomposing a bounded lattice with respect to a countable chain, which is crucial in our definition of ordinal sum of t-norms. Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice and $\{c_i\}_{i \in \mathbb{Z}} \subseteq L$ be given with $c_i \leq c_{i+1}$, where \mathbb{Z} is the set of all integers. Then $L = S_1 \cup S_2$ and $S_1 \cap S_2 = \emptyset$, where

$$S_1 = \{x \in L \mid (\exists i \in \mathbb{Z})(x \in I_{c_i})\} = \bigcup_{i \in \mathbb{Z}} I_{c_i}$$

and

$$S_2 = \{x \in L \mid (\forall i \in \mathbb{Z})(x \notin I_{c_i})\} = \bigcap_{i \in \mathbb{Z}} L \setminus I_{c_i} = L \setminus \bigcup_{i \in \mathbb{Z}} I_{c_i}.$$

Further, $S_1 = A_1 \cup A_2$ and $A_1 \cap A_2 = \emptyset$, where

$$A_1 = \{x \in S_1 \mid \inf\{i \in \mathbb{Z} \mid x \in I_{c_i}\} = -\infty\}$$

and

$$A_2 = \{x \in S_1 \mid \inf\{i \in \mathbb{Z} \mid x \in I_{c_i}\} \in \mathbb{Z}\}.$$

It is not difficult to prove that

$$A_2 = \bigcup_{i \in \mathbb{Z}} A_2^i,$$

where $A_2^i =]c_{i-1}, 1[\cap I_{c_i}$.

We furthermore divide S_2 into three subsets B_1 , B_2 and B_3 , i.e., $S_2 = B_1 \cup B_2 \cup B_3$, where

$$B_1 = \{x \in S_2 \mid (\forall i \in \mathbb{Z})(x \geq c_i)\} = \bigcap_{i \in \mathbb{Z}} [c_i, 1],$$

$$B_2 = \{x \in S_2 \mid (\forall i \in \mathbb{Z})(x \leq c_i)\} = \bigcap_{i \in \mathbb{Z}} [0, c_i]$$

and

$$B_3 = \{x \in S_2 \mid (\exists i \in \mathbb{Z})(x \in [c_{i-1}, c_i])\} = \bigcup_{i \in \mathbb{Z}} [c_{i-1}, c_i].$$

Let us further denote

$$\begin{aligned} \Delta_1 &= (A_1 \times [0, 1[) \cup ([0, 1[\times A_1) \\ \Delta_2^i &= (A_2^i \times [c_{i-1}, 1[) \cup ([c_{i-1}, 1[\times A_2^i). \end{aligned}$$

We give an example to illustrate the above decomposition.

Example 3.1. Let $L = [0, 1] \times [0, 1]$. Define the partial order \preceq on L componentwisely, i.e.,

$$x = (x^{(1)}, x^{(2)}) \preceq y = (y^{(1)}, y^{(2)}) \iff x^{(n)} \leq y^{(n)} \quad (n = 1, 2).$$

The meet \sqcap and the join \sqcup with respect to \preceq are given as follows:

$$\begin{aligned} (x^{(1)}, x^{(2)}) \sqcap (y^{(1)}, y^{(2)}) &= (x^{(1)} \wedge y^{(1)}, x^{(2)} \wedge y^{(2)}) \\ (x^{(1)}, x^{(2)}) \sqcup (y^{(1)}, y^{(2)}) &= (x^{(1)} \vee y^{(1)}, x^{(2)} \vee y^{(2)}). \end{aligned}$$

Obviously, $(L, \preceq, \sqcap, \sqcup, (0, 0), (1, 1))$ is a complete lattice.

Define $\{c_i\}_{i \in \mathbb{Z}} \subset L$ as follows:

$$c_i = (c_i^{(1)}, c_i^{(2)}) = \left(\frac{1}{3\pi} \arctan i + \frac{1}{2}, \frac{1}{3\pi} \arctan i + \frac{1}{2} \right).$$

Then $c_i \preceq c_{i+1}$ and $\bigwedge_{i \in \mathbb{Z}} c_i = \left(\frac{1}{3}, \frac{1}{3} \right)$. It is not difficult to see that

$$A_1 = \left([0, \frac{1}{3}] \times]\frac{1}{3}, 1] \right) \cup \left(]\frac{1}{3}, 1] \times [0, \frac{1}{3}] \right), \quad A_2 = \bigcup_{i \in \mathbb{Z}} A_2^i,$$

where

$$A_2^i = \left([c_{i-1}^{(1)}, c_i^{(1)}[\times]c_i^{(2)}, 1] \right) \cup \left(]c_i^{(1)}, 1] \times [c_{i-1}^{(2)}, c_i^{(2)}[\right),$$

$$B_1 = \left[\frac{2}{3}, 1\right] \times \left[\frac{2}{3}, 1\right], \quad B_2 = \left[0, \frac{1}{3}\right] \times \left[0, \frac{1}{3}\right], \quad \text{and } B_3 = \bigcup_{i \in \mathbb{Z}} [c_{i-1}, c_i],$$

where

$$[c_{i-1}, c_i] = \left\{ (x, y) \mid \frac{1}{3\pi} \arctan(i-1) + \frac{1}{2} \leq x, y \leq \frac{1}{3\pi} \arctan i + \frac{1}{2} \right\}.$$

So,

$$S_2 = B_1 \cup B_2 \cup B_3, \quad \text{and } S_1 = L \setminus S_2.$$

Moreover,

$$\Delta_1 = \left(A_1 \times [(0, 0), (1, 1)[\right) \cup \left([(0, 0), (1, 1)[\times A_1 \right)$$

and

$$\Delta_2^i = \left(A_2^i \times [c_{i-1}, (1, 1)[\right) \cup \left([c_{i-1}, (1, 1)[\times A_2^i \right).$$

We are now ready to propose our definition of ordinal sum of t-norms.

First, we consider the case of contiguous subintervals.

Definition 3.2. Let $(L, \leq, \wedge, \vee, 0, 1)$ be a complete lattice, $\{c_i\}_{i \in \mathbb{Z}} \subseteq L$ with $c_i \leq c_{i+1}$, $c = \bigwedge_{i \in \mathbb{Z}} c_i$ and $\{T_i\}_{i \in \mathbb{Z}}$ be a family of t-norms on the subintervals $\{[c_{i-1}, c_i]\}_{i \in \mathbb{Z}}$. The ordinal sum $T = \{\langle c_{i-1}, c_i, T_i \rangle\}_{i \in \mathbb{Z}}: L \times L \rightarrow L$ is given by

$$T(x, y) = \begin{cases} T_i(x, y) & \text{if } (x, y) \in [c_{i-1}, c_i]^2 \\ T_i(x \wedge c_i, y \wedge c_i) & \text{if } (x, y) \in \Delta_2^i \\ x \wedge y \wedge c & \text{if } (x, y) \in \Delta_1 \\ x \wedge y & \text{otherwise.} \end{cases} \quad (2)$$

Remark 3.3. (i) For any $i, j \in \mathbb{Z}$, it holds that

$$\Delta_2^i \cap \Delta_1 = [c_{i-1}, c_i]^2 \cap \Delta_1 = [c_{i-1}, c_i]^2 \cap \Delta_2^j = \emptyset.$$

In addition, for any $i, j \in \mathbb{Z}$ with $i \neq j$, it holds that

$$\Delta_2^i \cap \Delta_2^j = \emptyset.$$

Hence, the operation in (2) is well defined.

(ii) The completeness requirement for L is only used to ensure the existence of $\bigwedge_{i \in \mathbb{Z}} c_i$. We could just suppose that L is complete with respect to meet, but meet-completeness implies join-completeness since L has a top element. The completeness requirement is not needed when there exists $i \in \mathbb{Z}$ such that $c_j = c_i$ for any $j < i$, in particular when dealing with finitely many contiguous subintervals.

Second, we consider the case of not necessarily contiguous subintervals, whose endpoints form a chain. Let $(L, \leq, \wedge, \vee, 0, 1)$ be a complete lattice, $\{[a_i, b_i]\}_{i \in \mathbb{Z}}$ be a family of subintervals of L with $b_i \leq a_{i+1}$, $a = \bigwedge_{i \in \mathbb{Z}} a_i$ and $\{T_i\}_{i \in \mathbb{Z}}$ be a family of t-norms on these subintervals.

We intend to use (2) to define the ordinal sum $T = \{\langle a_i, b_i, T_i \rangle\}_{i \in \mathbb{Z}}$. The process is divided into three steps.

Step 1. Define $\{c_i\}_{i \in \mathbb{Z}} \subseteq L$ as follows:

$$c_{2i-1} = a_i \quad \text{and} \quad c_{2i} = b_i.$$

It holds that (1) $c_i \leq c_{i+1}$; (2) $\bigwedge_{i \in \mathbb{Z}} c_i = \bigwedge_{i \in \mathbb{Z}} a_i = a$; (3) $[c_{2i-1}, c_{2i}] = [a_i, b_i]$; (4) $[c_{2i}, c_{2i+1}] = [b_i, a_{i+1}]$.

Step 2. For any $i \in \mathbb{Z}$, endow $[c_{i-1}, c_i]$ with a t-norm \hat{T}_i as follows:

$$\hat{T}_{2i} = T_i \quad \text{and} \quad \hat{T}_{2i+1} = T_\wedge.$$

Step 3. Define the ordinal sum $T = \{\langle a_i, b_i, T_i \rangle\}_{i \in \mathbb{Z}}$ as $\{\langle c_{i-1}, c_i, \hat{T}_i \rangle\}_{i \in \mathbb{Z}}$,

i.e.,

$$T(x, y) = \begin{cases} \hat{T}_i(x, y) & \text{if } (x, y) \in [c_{i-1}, c_i]^2 \\ \hat{T}_i(x \wedge c_i, y \wedge c_i) & \text{if } (x, y) \in \Delta_2^i \\ x \wedge y \wedge a & \text{if } (x, y) \in \Delta_1 \\ x \wedge y & \text{otherwise.} \end{cases} \quad (3)$$

It is routine to check that (3) is the same as (4):

$$T(x, y) = \begin{cases} T_i(x, y) & \text{if } (x, y) \in [a_i, b_i]^2 \\ T_i(x \wedge b_i, y \wedge b_i) & \text{if } (x, y) \in \Lambda_3^i \\ x \wedge y \wedge a_i & \text{if } (x, y) \in \Lambda_2^i \\ x \wedge y \wedge a & \text{if } (x, y) \in \Lambda_1 \\ x \wedge y & \text{otherwise,} \end{cases} \quad (4)$$

where

$$\begin{aligned} \Lambda_3^i &= \Delta_2^{2i} = \left((]a_i, 1[\cap I_{b_i}) \times [a_i, 1[\right) \cup \left([a_i, 1[\times (]a_i, 1[\cap I_{b_i}) \right), \\ \Lambda_2^i &= \Delta_2^{2i-1} = \left((]b_{i-1}, 1[\cap I_{a_i}) \times [b_{i-1}, 1[\right) \cup \left([b_{i-1}, 1[\times (]b_{i-1}, 1[\cap I_{a_i}) \right) \end{aligned}$$

and

$$\Lambda_1 = \Delta_1 = \left(A'_1 \times [0, 1[\right) \cup \left([0, 1[\times A'_1 \right),$$

where

$$A'_1 = \{x \in L \mid (\exists i \in \mathbb{Z})(x \in I_{a_i}) \text{ and } \inf\{i \in \mathbb{Z} \mid x \in I_{a_i}\} = -\infty\}.$$

To summarize, for the case of not necessarily contiguous subintervals, we define the ordinal sum of t-norms as follows.

Definition 3.4. Let $(L, \leq, \wedge, \vee, 0, 1)$ be a complete lattice, $\{[a_i, b_i]\}_{i \in \mathbb{Z}}$ be a family of subintervals of L with $b_i \leq a_{i+1}$, $a = \bigwedge_{i \in \mathbb{Z}} a_i$ and $\{T_i\}_{i \in \mathbb{Z}}$ be a family of t-norms on these subintervals. The ordinal sum $T = \{ \langle a_i, b_i, T_i \rangle \}_{i \in \mathbb{Z}}$ is given by (4).

We are now going to prove our main theorems.

Theorem 3.5. *Let $(L, \leq, \wedge, \vee, 0, 1)$ be a complete lattice, $\{c_i\}_{i \in \mathbb{Z}} \subseteq L$ with $c_i \leq c_{i+1}$, $c = \bigwedge_{i \in \mathbb{Z}} c_i$ and $\{T_i\}_{i \in \mathbb{Z}}$ be a family of t -norms on the subintervals $\{[c_{i-1}, c_i]\}_{i \in \mathbb{Z}}$. Then the ordinal sum $T = \{\langle c_{i-1}, c_i, T_i \rangle\}_{i \in \mathbb{Z}}: L \times L \rightarrow L$ given by (2) is a t -norm on L .*

The following observations play a key role in simplifying the proof of Theorem 3.5.

Observation 1. The restriction $T|_{S_2}$ of T to $S_2 \times S_2$ is a t -norm on S_2 , where $T|_{S_2}: S_2 \times S_2 \rightarrow S_2$ is given by

$$T|_{S_2}(x, y) = \begin{cases} T_i(x, y) & \text{if } (x, y) \in [c_{i-1}, c_i]^2 \\ x \wedge y & \text{otherwise.} \end{cases}$$

In fact, $\{0, 1\} \subseteq S_2$ and $(S_2, \leq, \wedge, \vee, 0, 1)$ is a bounded lattice. In this lattice, $I_{c_i} = \emptyset$ for all $i \in \mathbb{Z}$. Therefore, it follows from Proposition 5.2 in [16] that $T|_{S_2}$ is a t -norm on S_2 .

Observation 2. $T(x, y) = T(x \wedge c, y)$ for any $x \in A_1$ and any $y \in [0, 1[$.

In fact, for any $x \in A_1$ and any $y \in [0, 1[$, it holds that $T(x, y) = x \wedge y \wedge c$. Note that $x \wedge c \leq c \leq c_i$ for any $i \in \mathbb{Z}$. If $y \in A_1$, then

$$T(x \wedge c, y) = (x \wedge c) \wedge y \wedge c = x \wedge y \wedge c = T(x, y).$$

Otherwise,

$$T(x \wedge c, y) = (x \wedge c) \wedge y = x \wedge y \wedge c = T(x, y).$$

Observation 3. $T(x, y) = T(x \wedge c_i, y)$ for any $x \in A_2^i$ and any $y \in [0, 1[$.

In fact, if $y \in A_1$, then

$$T(x, y) = x \wedge y \wedge c = (x \wedge c_i) \wedge y \wedge c = T(x \wedge c_i, y).$$

For $y \notin A_1$, we distinguish the following cases:

- If $y > c_i$, then

$$T(x, y) = T_i(x \wedge c_i, y \wedge c_i) = x \wedge c_i = x \wedge c_i \wedge y = T(x \wedge c_i, y).$$

- If $y \in [c_{i-1}, c_i]$ or $y \in A_2^i$, then

$$T(x, y) = T_i(x \wedge c_i, y \wedge c_i) = T(x \wedge c_i, y).$$

- If $y \in A_2^j$ for some $j < i$, then

$$T(x, y) = T_j(x \wedge c_j, y \wedge c_j) = T_j(x \wedge c_i \wedge c_j, y \wedge c_j) = T(x \wedge c_i, y).$$

- If $y \in [c_{j-1}, c_j]$ for some $j < i$ or $y \leq c$, then

$$T(x, y) = x \wedge y = x \wedge c_i \wedge y = T(x \wedge c_i, y).$$

Now we are ready to prove Theorem 3.5.

Proof of Theorem 3.5.

Obviously, T is commutative and 1 is the neutral element of T . We only need to show that T is increasing and associative.

Increasingness: Let $x, y, z \in L$ with $y \leq z$. We need to prove the following inequality

$$T(x, y) \leq T(x, z). \tag{5}$$

If $1 \in \{x, y, z\}$, then (5) trivially holds. In the following, we only consider $1 \notin \{x, y, z\}$.

By Observation 2, we can suppose that $x, y, z \notin A_1$.

In fact, if $x \in A_1$, then $x \wedge c \in [0, c]$ and (5) is equivalent to

$$T(x \wedge c, y) \leq T(x \wedge c, z).$$

If $y \in A_1$, then $y \wedge c \in [0, c]$, $y \wedge c \leq z$ and (5) is equivalent to

$$T(x, y \wedge c) \leq T(x, z).$$

If $z \in A_1$, then $z \wedge c \in [0, c]$. Note that $y \leq z$ implies either $y \in A_1$ or $y \leq c$.

In both cases, $T(x, y) = T(x, y \wedge c)$ and (5) is equivalent to

$$T(x, y \wedge c) \leq T(x, z \wedge c).$$

By Observation 3, we can also suppose that $x, y, z \notin A_2$.

In fact, if $x \in A_2$, i.e., there exists $i \in \mathbb{Z}$ such that $x \in A_2^i$, then $x \wedge c_i \in [c_{i-1}, c_i]$ and (5) is equivalent to

$$T(x \wedge c_i, y) \leq T(x \wedge c_i, z).$$

If $y \in A_2$, i.e., there exists $j \in \mathbb{Z}$ such that $y \in A_2^j$, then $y \wedge c_j \in [c_{j-1}, c_j]$, $y \wedge c_j \leq z$ and (5) is equivalent to

$$T(x, y \wedge c_j) \leq T(x, z).$$

If $z \in A_2$, i.e., there exists $k \in \mathbb{Z}$ such that $z \in A_2^k$, then $z \wedge c_k \in [c_{k-1}, c_k]$.

Note that $y \leq z$, we distinguish the following cases:

- If $y \in A_2^j$ for some $j \leq k$, then $y \wedge c_j \leq z \wedge c_k$ and (5) is equivalent to

$$T(x, y \wedge c_j) \leq T(x, z \wedge c_k).$$

- If $y \in [c_{j-1}, c_j]$ for some $j \leq k$ or $y \in [0, c]$, then $y = y \wedge c_k \leq z \wedge c_k$ and (5) is equivalent to

$$T(x, y) \leq T(x, z \wedge c_k).$$

- If $y \in A_1$, then $y \wedge c \leq z \wedge c_k$ and (5) is equivalent to

$$T(x, y \wedge c) \leq T(x, z \wedge c_k).$$

Based on the discussion above, it suffices to verify that (5) holds for $x, y, z \in S_2 = B_1 \cup B_2 \cup B_3$. However, in that case the proof follows from Observation 1.

Associativity: Let $x, y, z \in L$. We need to prove the following equality

$$T(T(x, y), z) = T(x, T(y, z)). \quad (6)$$

If $1 \in \{x, y, z\}$, then (6) trivially holds. We only consider $1 \notin \{x, y, z\}$.

In a similar way as in the case of the increasingness property, we can prove that it suffices to consider $x, y, z \in S_2$, in which case the proof follows from Observation 1.

To conclude, we have proved that T is commutative, increasing, associative and has neutral element 1, i.e., T is a t-norm on L . \square

Theorem 3.6. *Let $(L, \leq, \wedge, \vee, 0, 1)$ be a complete lattice, $\{[a_i, b_i]\}_{i \in \mathbb{Z}}$ be a family of subintervals of L with $b_i \leq a_{i+1}$, $a = \bigwedge_{i \in \mathbb{Z}} a_i$ and $\{T_i\}_{i \in \mathbb{Z}}$ be a family of t-norms on these subintervals. Then the ordinal sum $T = \{ \langle a_i, b_i, T_i \rangle \}_{i \in \mathbb{Z}}$ given by (4) is a t-norm on L .*

Proof. It follows from Theorem 3.5 and the fact that (4) is actually deduced from (2). \square

Theorem 3.6 also applies to a finite sequence of subintervals $\{[a_i, b_i]\}_{i=1}^n$ on a bounded lattice L (in this case L need not be complete). To this end, it suffices to let $a_i = b_i = a_1$ for any $i \in \mathbb{Z}$ with $i < 1$ and $b_i = a_i = b_n$ for any $i \in \mathbb{Z}$ with $i > n$.

In the finite case, it holds that

$$\Lambda_1 = (I_{a_1} \times [0, 1[) \cup ([0, 1[\times I_{a_1}).$$

Theorem 3.7. *Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice, $\{[a_i, b_i]\}_{i=1}^n$ be a finite sequence of subintervals on L with $b_i \leq a_{i+1}$ and $\{T_i\}_{i=1}^n$ be a finite sequence of t -norms on these subintervals. Then the ordinal sum $T = \{\langle a_i, b_i, T_i \rangle\}_{i=1}^n: L \times L \rightarrow L$ defined by*

$$T(x, y) = \begin{cases} T_i(x, y) & \text{if } (x, y) \in [a_i, b_i]^2 \\ T_i(x \wedge b_i, y \wedge b_i) & \text{if } (x, y) \in \Lambda_3^i \\ x \wedge y \wedge a_i & \text{if } (x, y) \in \Lambda_2^i \\ x \wedge y & \text{otherwise,} \end{cases} \quad (7)$$

is a t -norm on L , where $\Lambda_2^1 \triangleq \Lambda_1$.

Setting $n = 1$, we get the ordinal sum with one summand.

Theorem 3.8. *Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice, $[a, b]$ be a subinterval of L and T_1 be a t -norm on $[a, b]$. Then the ordinal sum $T = \{\langle a, b, T_1 \rangle\}: L \times L \rightarrow L$ defined by*

$$T(x, y) = \begin{cases} T_1(x, y) & \text{if } (x, y) \in [a, b]^2 \\ T_1(x \wedge b, y \wedge b) & \text{if } (x, y) \in \Lambda \\ x \wedge y \wedge a & \text{if } (x, y) \in (I_a \times [0, 1[) \cup ([0, 1[\times I_a) \\ x \wedge y & \text{otherwise,} \end{cases} \quad (8)$$

is a t -norm on L , where

$$\Lambda = \left((]a, 1[\cap I_b) \times [a, 1[\right) \cup \left([a, 1[\times (]a, 1[\cap I_b) \right).$$

Setting $b = 1$, (8) reduces to (1).

To conclude this section, we give an example to show that, in our definition of ordinal sum, the condition that the endpoints of the subintervals constitute a chain is indispensable since it assures the well-definedness of our ordinal sum.

Example 3.9. Consider the complete lattice L with Hasse diagram shown in Figure 2. Let T_1 be a t-norm on $[a, b]$ and T_2 be a t-norm on $[c, d]$

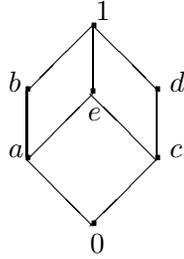


Figure 2: Hasse diagram of the lattice L in Example 3.9.

(note that these t-norms are unique and coincide with \wedge). Note that both $e \in]a, 1[\cap I_b$ and $e \in]c, 1[\cap I_d$. For the ordinal sum $T = \{\langle a, b, T_1 \rangle, \langle c, d, T_2 \rangle\}$ defined by (7), we have both $T(e, e) = T_1(e \wedge b, e \wedge b) = a$ and $T(e, e) = T_2(e \wedge d, e \wedge d) = c$. Therefore, T is not well defined.

4. Examples

In this section, we present two examples that fit in our proposed ordinal sum of t-norms on a bounded lattice.

Example 4.1. Consider the bounded lattice L with Hasse diagram shown in Figure 3. Consider the ordinal sum $T = \{\langle a, d, T_c \rangle, \langle f, h, T_D \rangle\}$ defined by (7). It is routine to check that T (shown in Table 2) is a t-norm on L .

Example 4.2. Consider the complete lattice $(L, \preceq, \sqcap, \sqcup, (0, 0), (1, 1))$ and the chain $\{c_i\}_{i \in \mathbb{Z}}$ introduced in Example 3.1. For any $i \in \mathbb{Z}$, consider the t-norm $T_i = \hat{T}_i \times \hat{T}_i$ (see [5]) on $[c_{i-1}, c_i]$ given by

$$T_i((x^{(1)}, x^{(2)}), (y^{(1)}, y^{(2)})) = \left(\hat{T}_i(x^{(1)}, y^{(1)}), \hat{T}_i(x^{(2)}, y^{(2)}) \right),$$

Table 2: The ordinal sum $\{\langle a, d, T_c \rangle, \langle f, h, T_D \rangle\}$ in Example 4.1.

T	0	a	b	c	d	e	f	g	h	i	j	k	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0
a	0	a	0	a									
b	0	a	a	a	b	a	b	b	b	b	b	0	b
c	0	a	a	c	0	c							
d	0	a	b	c	d	c	d	d	d	d	d	0	d
e	0	a	a	c	c	c	e	e	e	e	e	0	e
f	0	a	b	c	d	e	f	f	f	f	f	0	f
g	0	a	b	c	d	e	f	f	g	f	f	0	g
h	0	a	b	c	d	e	f	g	h	f	g	0	h
i	0	a	b	c	d	e	f	f	f	f	f	0	i
j	0	a	b	c	d	e	f	f	g	f	f	0	j
k	0	0	0	0	0	0	0	0	0	0	0	0	k
1	0	a	b	c	d	e	f	g	h	i	j	k	1

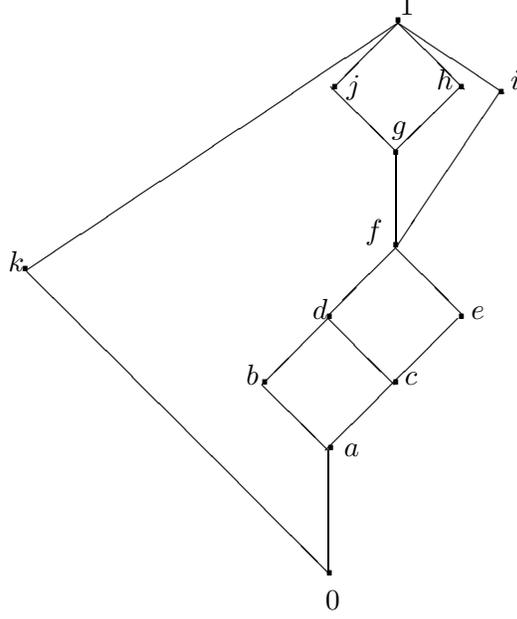


Figure 3: Hasse diagram of the lattice L in Example 4.1.

where \hat{T}_i is a t-norm on $[\frac{1}{3\pi} \arctan(i-1) + \frac{1}{2}, \frac{1}{3\pi} \arctan i + \frac{1}{2}]$.

The ordinal sum $T = \{\langle c_{i-1}, c_i, T_i \rangle\}_{i \in \mathbb{Z}}: L \times L \rightarrow L$ defined by (2) is given by $T((x^{(1)}, x^{(2)}), (y^{(1)}, y^{(2)})) = (T^{(1)}(x^{(1)}, y^{(1)}), T^{(2)}(x^{(2)}, y^{(2)}))$, where

$$T^{(1)}(x^{(1)}, y^{(1)}) = \begin{cases} \hat{T}_i(x^{(1)}, y^{(1)}) & \text{if } (x, y) \in [c_{i-1}, c_i]^2 \\ \hat{T}_i(x^{(1)} \wedge c_i^{(1)}, y^{(1)} \wedge c_i^{(1)}) & \text{if } (x, y) \in \Delta_2^i \\ x^{(1)} \wedge y^{(1)} \wedge \frac{1}{3} & \text{if } (x, y) \in \Delta_1 \\ x^{(1)} \wedge y^{(1)} & \text{otherwise} \end{cases}$$

and

$$T^{(2)}(x^{(2)}, y^{(2)}) = \begin{cases} \hat{T}_i(x^{(2)}, y^{(2)}) & \text{if } (x, y) \in [c_{i-1}, c_i]^2 \\ \hat{T}_i(x^{(2)} \wedge c_i^{(2)}, y^{(2)} \wedge c_i^{(2)}) & \text{if } (x, y) \in \Delta_2^i \\ x^{(2)} \wedge y^{(2)} \wedge \frac{1}{3} & \text{if } (x, y) \in \Delta_1 \\ x^{(2)} \wedge y^{(2)} & \text{otherwise.} \end{cases}$$

5. Conclusion

In this paper, we have proposed an alternative definition of ordinal sum of countably many t-norms on subintervals of a complete lattice, where the endpoints of the subintervals constitute a chain. The completeness requirement for the lattice is not needed when considering finitely many t-norms. The newly proposed ordinal sum is shown to be always a t-norm. Obviously, our approach can be applied to define the ordinal sum of triangular conorms on a bounded lattice in a dual way.

Note that we have only partially solved the ordinal sum problem. For future work, it is interesting to consider how to define the ordinal sum of t-norms on subintervals of a bounded lattice in case the endpoints of the subintervals do not constitute a chain (see Example 3.9).

Acknowledgement

This work has been supported in part by National Natural Science Foundation of China (No. 11571106 and No. 11571006), Natural Science Foundation of Zhejiang Province (No. LY20A010006) and NUPTSF (No. NY220029).

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