UNIVERSITÀ POLITECNICA DELLE MARCHE

DIPARTIMENTO DI SCIENZE ECONOMICHE E SOCIALI

BONFERRONI AND DE VERGOTTINI ARE BACK: NEW SUBGROUP DECOMPOSITIONS AND BIPOLARIZATION MEASURES

MARIATERESA CIOMMI, CHIARA GIGLIARANO,
GIOVANNI MARIA GIORGI

QUADERNO DI RICERCA n. 439

Ottobre 2019

Comitato scientifico:

Giulia Bettin
Marco Gallegati
Stefano Staffolani
Alessandro Sterlacchini

Collana curata da:
Massimo Tamberi

Abstract

Aim of this paper is to propose new bipolarization indices based on he Bonferroni and De Vergottini indices of inequality. The new bipolarization indices follow the approach of Foster and Wolfson and are based on new subgroup decomposition of Bonferroni and De Vergottini indices of inequality. We also provide the conditions under which the new polarization indices satisfy the Increased Spread and Increased Bipolarity axioms. Finally, a simulation study has been carried out to compare the different sensitivity of the new bipolarization indices to progressive transfers. Also, an empirical application based on EU-SILC data for Italy ober the period 2007-2011 show the appeal of our proposal.

JEL Class.: D31; D63; C43; I32

Keywords: Bonferroni index; De Vergottini index; Gini index; Inequality; Subgroup decomposition; Polarization measurement.

Indirizzo: *Ciommi Mariateresa*: Universita' Politecnica delle Marche. E-mail: `m.ciommi@univpm.it`;
Chiara Gigliarano: Universita' degli Studi dell'Insubria. E-mail: `chiara.gigliarano@uninsubria.it`;
Giovanni Maria Giorgi: Sapienza Universita' di Roma. E-mail: `giovanni.giorgi@uniroma1.it`

Bonferroni and De Vergottini are back: new subgroup decompositions and bipolarization measures*

Mariateresa Ciommi, Chiara Gigliarano,

Giovanni Maria Giorgi

1 Introduction

In recent years, the concept of income polarization has received an increasing attention from scholars of different fields (economists, statisticians, econometricians, sociologists, etc.). The main reasons for the interest in income polarization can be identified in the link between homogeneous groups and social tensions (Esteban and Ray [24]).

Monitoring the degree of polarization in a given income distribution means measuring not only how poorer are getting the poor but also how richer are getting the rich and hence how distant these two groups are one from the other. The further the two groups are one from the other and at the same time the more cohesive inside they are, the harder it will be communicating and interacting one to the other.

Two strands are distinguished in the income polarization literature: the first one, going back to Wolfson [60] and [61] and Foster and Wolfson [26] and [27], is focused to measure the shrinking middle class, monitoring how the income distribution spreads out from its center. The second strand, originating from Esteban and Ray [24], focuses on the rise of separated income groups: polarization increases if the population groups are getting more homogeneous inside and more separate one to the other. These pioneering contributions have been followed by many others, such

Authors thank Gaia Rocchetti for her valuable collaboration provided in the design phase of the paper. It is understood that the responsibility of the work is exclusively of the authors.

as Wang and Tsui [59], Gradin [38], Chakravarty and Majumder [15], D'Ambrosio [19], Zhang and Kanbur [64], Duclos *et al.* [22], Anderson [4], Esteban *et al.* [25], Massari *et al.* [44], Chakravarty and D'Ambrosio [14], Lasso de la Vega *et al.* [42], Yitzhaki [63], Pittau *et al.* [51], Permanyer [47], Silber *et al.* [55], Lasso de la Vega and Urrutia [41], Chakravarty and Majumder [16].

In this paper we will follow the second strand of income polarization literature, based on the Foster and Wolfson approach. Starting from the generalization of Foster and Wolfson index proposed by Rodriguez and Salas [53], we introduce two new polarization indices that are based on Bonferroni [11] and De Vergottini [20] inequality indices, respectively. Our aim is therefore to investigate whether both new indices provide additional or more detailed information than traditional measures of polarization.

In a period in which Gini's scientific ideas heavy influenced the research activity of the Italian statisticians (see, e.g., Giorgi [31]; Giorgi and Gubbiotti [35]), Carlo Emilio Bonferroni [11], in his book entitled *Elements of General Statistics*, proposed the Bonferroni inequality index. His purpose was simply to highlight the possibility of constructing indices as simple as the Gini concentration index with similar properties. In fact, the Bonferroni index is more sensitive than the Gini index to the lower levels of income distribution in the sense that it gives more weights to transfers among poor (Nygard and Sandstrom [46]). This makes the Bonferroni index particularly suitable to investigate poverty (Giorgi and Crescenzi [33]; Giordani and Giorgi [29]). For about fifty years, the Bonferroni index remained almost forgotten as it was opposed by Corrado Gini and his followers, who tried to prevent that any measures of inequality could overshadow the concentration ratio.

A fundamental contribution to the renewed interest in the Bonferroni and other inequality indices (including De Vergottini) is due to Walter Piesch [49], who discussed, among other things, a series of links between inequality measures in his book *Statistische Konzentrationsmasse.* Subsequently, the work of Nygard and Sandstrom [46] enlarged and deepened such studies, reaching a larger audience of scholars.

Recently, the Bonferroni index has been revalued since its features and new interesting applications in social and economic contexts have been studied (Barcena-

Martin and Silber [8], [9], [10]; Chakravarty and Muliere [18]; Chakravarty [13]). Various other important properties have been analyzed by, among others, Aaberge [2], Aaberge *et al.* [3], Barcena and Imedio [7] and Imedio-Olmedo *et al.* [39]. Some inferential results of the Bonferroni index have also been investigated (e.g., Giorgi and Mondani [36] and [37]; Giorgi and Crescenzi [32]).

Some years later, in 1950, Mario De Vergottini [20] proposed another index of inequality, the De Vergottini index, which, compared to the Gini index, is more sensitive to the right tail of the income distribution, i.e. it is more sensitive to income transfers among the rich. He also obtained a general formula from which various indices of inequality can be derived (including Gini, Bonferroni and De Vergottini), highlighting, as previously Bonferroni did, that the Gini index is only one of the many suggested indices having similar features and properties.

Motivated by these very different types of sensitivity, in this paper we will consider and compare the three indices (Gini, Bonferroni and De Vergottini) from the point of view of polarization.

The present work is organized as follows: in Section 2 we review the Gini, the Bonferroni and the De Vergottini inequality indices, while in Section 3 we provide a brief review of the the main contributions proposed in the polarization measurement literature. In Section 4 we propose a new subgroup decomposition for Bonferroni and De Vergottini inequality indices, which will be used in Section 5 to propose new polarization indices based on those indices of inequality. Section 6 shows some of their properties through a simulation study, while Section 7 illustrates a simple application to EU-SILC data referred to Italy, and in Section 8 we conclude. Appendix A contains the proofs of all propositions.

2 Gini, Bonferroni and De Vergottini inequality indices

2.1 Notation

Let us assume that, for a given population, the income distribution can be represented by a continuous non-negative random variable X, with positive support on $[x_1, x_n]$, $x_n \geq x_1 \geq 0$. Let $F(x)$ and μ denote the cumulative distribution function and the mean income, respectively.

Similarly, let $\boldsymbol{x} = (x_1, x_1, ..., x_n)$ indicate a positive non-decreasingly ordered income distribution, which corresponds to an empirical distribution F that attaches equal weights to each of the n points $x_1, x_2, \ldots, x_n$ and let $\mu(\boldsymbol{x})$ be the corresponding mean. By $\mu_i(\boldsymbol{x})$ we denote the average income of the individuals at the left of individual i, that is: $\mu_i(\boldsymbol{x}) = \frac{1}{i}\sum_{j=1}^{i} x_j$. Under the assumption that the incomes are non-decreasingly ordered, we denote by $\overline{n} = \frac{n+1}{2}$ the position of the median individual, thus x_- represents the sub-vector of the income distribution such that $x_i : i < \overline{n}$ and $x+$ represents the sub-vector of the income distribution such that $x_i : i > \overline{n}$. Consequently, if n is a even number, $\boldsymbol{x} = (x-, x+)$, whereas, if n is odd, $\boldsymbol{x} = (x-, m(\boldsymbol{x}), x+)$ where $m(\boldsymbol{x})$ denotes the median income. In addition, for any $\boldsymbol{x}$ and $\boldsymbol{y}$, $\boldsymbol{y} > \boldsymbol{x}$ ($\boldsymbol{y} < \boldsymbol{x}$ and $\boldsymbol{y} = \boldsymbol{x}$) means that $y_i > x_i$ ($y_i < x_i$ and $y_i = x_i$, respectively) for all $i = 1, \cdots, n$. Finally, n_i denotes the position of an individual whose income is x_i, $n_i \leq n$.

Definition 1 *A polarization index $P^n(\boldsymbol{x})$ is a continuous function $P^n : X_n \to \mathbb{R}^+$, where X_n is the set of all possible income distributions for a population of n individuals.*

As stressed by Permanyer [48], income polarization indices can be classified in two subgroups: *bipolarization index* and *multipolar index*. Here, we focus on the first group, that is a polarization index that measures the extent to which an income distribution is clustered around two antipodal groups: the *poor* and the *rich*.

In what follows, we recall two axioms that are the cornerstones upon which income bipolarization measures are based. The first one, the so-called *Increased*

Spread Axiom, introduced by Chakravarty and Majumder [15] is a monotonicity principle, it requires that polarization increases if the distance between the two groups below and above the median increases. The second important axiom is the *Increased Bipolarity Axiom*, which states that polarization should increase if a progressive transfer between individuals belonging to the same income group takes place. Formally:

Axiom 1 (Increased Spread (IS)) *Let* $\mathbf{x}, \mathbf{y}$ *be two income distributions with the same median income* $m(x) = m(y)$*. If* $y- \leq x-$ *and* $y+ \geq x+$*, then* $P^n(\mathbf{y}) > P^n(\mathbf{x})$.

Axiom 2 (Increased Bipolarity (IB)) *Let* $\mathbf{x}, \mathbf{y}$ *be two income distributions with the same median income* $m(\mathbf{x}) = m(\mathbf{y})$*. Consider the following scenarios:*

(i) $x+ = y+, \quad y-PDx-$

(ii) $x- = y-, \; y+PDx+$

(iii) $y-PDx-, \; y+PDx+,$

where PD *denotes that the* Pigou-Dalton transfers principle *is satisfied. If one condition among* (i)*,* (ii) *or* (iii) *holds, then* $P^n(\mathbf{y}) > P^n(\mathbf{x})$.

Using the previous notation, we recall now the formulation of the Lorenz curve in the continuous case (see [43]), given by:

$$L(p) = \frac{1}{\mu} \int_0^p F^{-1}(t)dt, \;\; with \, p \in (0,1] \, ,$$

where $F^{-1}(t)$ is the left-continuous version of the inverse of F, defined as $F^{-1}(t) = inf\{x : F(x) \geq t\}$; see Pietra [50] and Gastwirth [28].

The corresponding discrete Lorenz curve is obtained by linearly interpolating the following n points:

$$\left(\frac{i}{n}; \frac{1}{n\mu(\mathbf{x})} \sum_{j=1}^{i} x_j \right), \; i = 1, \ldots, n.$$

The Gini concentration index, the Bonferroni concentration index and the De Vergottini concentration index can all be written as function of the Lorenz curve (see, among others, Amato [5], Tarsitano [57] and Barcena and Imedio [7]). We will present both their continuous and their discrete formulation.

The population (or continuous) Gini index is defined as twice the area between the equidistribution line (p) and the Lorenz curve $L(p)$:

$$G = 2\int_0^1 [p - L(p)]\, dp = 1 - 2\int_0^1 L(p)dq.$$

The Gini index G ranges in $[0, 1]$, where the lower extreme value ($G = 0$) is achieved when the income is equally distributed among individuals, while the upper extreme ($G = 1$) is reached when one person owns the overall income and all the others have zero income.

One of the discrete formulation for the Gini index[1] in the discrete case is:

$$I_G(\mathbf{x}) = 1 - \frac{2}{n(n+1)\mu(\mathbf{x})}\sum_{i=1}^{n}\sum_{j=1}^{i} x_j.$$

The index $I_G(\mathbf{x})$ ranges in $\left[0, \frac{n-1}{n+1}\right]$.

The Bonferroni index corresponds to the area between the line of perfect equality (horizontal line at height 1) and the Bonferroni curve $B(p) = L(p)/p$ (see Giorgi and Crescenzi[34]):

$$B = 1 - \int_0^1 \frac{L(p)}{p} dp = \int_0^1 \frac{[p - L(p)]}{p} dp \tag{1}$$

Note that B displays the same range as G.

The discrete Bonferroni index I_B can be written as[2]:

$$I_B(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\mu(\mathbf{x}) - \mu_i(\mathbf{x})}{\mu(\mathbf{x})}\right) = 1 - \frac{1}{n\mu(\mathbf{x})}\sum_{i=1}^{n}\mu_i(\mathbf{x}) = 1 - \frac{1}{n\mu(\mathbf{x})}\sum_{i=1}^{n}\frac{1}{i}\sum_{j=1}^{i} x_j,$$

The index $I_B(\mathbf{x})$ ranges in $\left[0, \frac{n-1}{n}\right]$.

The De Vergottini index [20] corresponds to the area between the De Vergottini

[1] See Giorgi [30] and Yitzhaki [62].

[2] See Nygard and Sandstrom [46] and Barcena and Imedio [7].

curve $V(p) = (1 - L(p))/(1 - p)$ and the line of perfect equality:

$$V = \int_0^1 \frac{1 - L(p)}{1 - p} dp - 1 = \int_0^1 \frac{[p - L(p)]}{1 - p} dp. \tag{2}$$

Index V has a lower bound equal to 0, when income is equally distributed over the population, and an upper bound equal to $\frac{x_n}{\mu} - 1$, in case of maximum concentration. This implies that its maximum depends on the income of the richer individual (Barcena and Imedio [7]).

The discrete formulation of the De Vergottini index is:

$$I_V(\mathbf{x}) = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{M_i(\mathbf{x}) - \mu(\mathbf{x})}{\mu(\mathbf{x})} \right) = \frac{1}{n\mu(\mathbf{x})} \sum_{i=1}^{n} M_i(\mathbf{x}) - 1,$$

where $M_i(\mathbf{x}) = \frac{1}{(n-i+1)} \sum_{j=i}^{n} x_j$. The De Vergottini index can be interpreted as a weighted average of the relative differences between the mean of the population and the partial means of the i-th richest group (see Tarsitano [58]). The index $I_V(\mathbf{x})$ ranges in $\left[0, \left(\sum_{j=1}^{n} \frac{1}{n-j+1}\right) - 1\right]$. [3]

The inequality indices G, B and V are pure numbers (relative indices) and satisfy the following properties: *i*) the index ranges in $[0, 1]$ with higher values denoting greater concentration (see Bonferroni [11])[4]; *ii*) transfer sensitivity: the index increases as a result of a progressive income transfer from a richer to a poorer individual (Pigou-Dalton principle). The Bonferroni index satisfies a stronger version of the Pigou-Dalton principle, namely the *positional transfer sensitivity*, which ensures that the reduction in inequality due to a progressive income transfer is higher if the incomes involved are smaller than the average (see Mehran [45] and Zoli [65]). Also the De Vergottini index satisfies the *positional transfer sensitivity*, with an opposite interpretation: the reduction in inequality due to a progressive income transfer is higher if the the individuals involved in the transfer have income higher than the

[3] Note that the De Vergottini index does not have a unit upper bound. The maximum inequality corresponds to the income profile in which only one individual holds the total income, i.e. $x_i = n\mu(x)$ and $x_j = 0$ for $j = 1, \ldots, n$, $j \neq i$. The upper bound of I_V can be written as: $V^{MAX} = \sum_{j=2}^{n} \frac{1}{j}$. In this way it is easy to see that V^{MAX} only depends on the population size.

[4] The value of V^{MAX} can be used to normalize V in order to ensure $V \in [0, 1]$, see Aristondo et al. [6].

average income.

2.2 A broader class of inequality indices

The Gini, Bonferroni and De Vergottini indices belong to the class of linear measures introduced by Mehran [45] and defined as:

$$I_{\theta(p)} = \int_0^1 \theta(p) \cdot (p - L(p))\, dp.$$

Indeed, assuming particular formulations for the parameter $\theta(p)$, we trace back to the three inequality indices as follows: for $\theta(p) = 2$ we get the Gini index; for $\theta(p) = 1/p$ with $0 < p \leq 1$ we obtain the Bonferroni index and for $\theta(p) = 1/(1-p)$ with $0 \leq p < 1$ the De Vergottini index.

As already stressed, these three indices show a different sensitivity to progressive income transfers occurring at different segments of the distribution. Indeed, this is due to the different weights attached to the differences $(p - L(p))$ in equation (3). In particular, for the Gini index, these differences are multiplied by a constant, $\theta(p) = 2$, revealing that the index attaches the same importance to the differences $(p - L(p))$, regardless of their position in the income distribution. In the Bonferroni index, instead, the differences $(p - L(p))$ are multiplied by the decreasing convex function $\theta(p) = 1/p$. This implies that the income transfers involving the poorer individuals are attached a greater impact on the index variation. In other word, the Bonferroni inequality index is more sensitive to income transfers occurring among poor people. Finally, in the De Vergottini index the differences $(p - L(p))$ are multiplied by the increasing convex function $\theta(p) = 1/(1-p)$. Consequently, the index is more sensitive to income transfers occurring among richer people.

The different types of sensitivity of the indices are even more evident considering their sample formulations. In particular, the sample version of all the three indices can be written as a weighted mean, and with different weighting systems.

According to Tarsitano [57], the discrete Bonferroni index can be written as a linear combination of units with weights depending on the individual ranks. That is:

$$I_B(\mathbf{x}) = 1 - \frac{1}{n\mu(\mathbf{x})}\sum_{i=1}^{n}\frac{1}{i}\sum_{j=1}^{i} x_j = \frac{1}{n\mu(\mathbf{x})}\sum_{i=1}^{n} w_i x_i$$

where the weights are such that

$$w_i = 1 - \sum_{j=i}^{n}\frac{1}{j}, \;\; w_{i+1} = w_i + \frac{1}{i}, \quad \sum_{i=1}^{n} w_i = 0. \tag{3}$$

Barcena-Martin and Imedio [7] propose a similar expression for the Gini index:

$$I_G(\mathbf{x}) = \frac{1}{n\mu(\mathbf{x})}\sum_{i=1}^{n}\gamma_i x_i,$$

where

$$\gamma_i = \left(\frac{2i-1}{n}\right) - 1, \quad \gamma_{i+1} = \gamma_i + \frac{2}{n}, \quad \sum_{i=1}^{n}\gamma_i = 0 \tag{4}$$

and for the De Vergottini index:

$$I_V(\mathbf{x}) = \frac{1}{n\mu(\mathbf{x})}\sum_{i=1}^{n}\xi_i x_i,$$

where

$$\xi_i = \sum_{j=1}^{i}\frac{1}{n-j+1} - 1 \quad \xi_{i+1} = \xi_i + \frac{1}{n-i} \quad \sum_{i=1}^{n}\xi_i = 0. \tag{5}$$

In the three weighting systems (3), (4) and (5), the weight associated to the individual's income depends on his position in the income distribution and increases with the individual's rank in the distribution. Thus, we have three weight systems, w_i, γ_i, ξ_i that are all increasing with the individual ranks $i = 1, ..., n$ but at different rates: for the Gini index, the weight sequence increases constantly, with an absolute increment of $2/n$, whereas both Bonferroni's and De Vergottini's weights grow at a decreasing rate (the absolute increment is equal to $1/i$ and $1/(n-i)$, respectively). For the Gini index, therefore, the weighing system is such that the variation in inequality recorded as a result of an income transfer depends only on the distance between the individuals involved, regardless of their position in the distribution. On the contrary, for Bonferroni and De Vergottini's indexes, the effect of a transfer also depends on the position of individuals, making Bonferroni index more sensitive to transfers that occur at the lower end of the income distribution and De Vergottini index more sensitive to variations among the richest.

3 Brief review of polarization measures

In this section we briefly review the most common univariate polarization measures and their connection to well-known inequality measures. Since in this paper we follow the second strand of income polarization literature, based on the Foster and Wolfson approach ([26] and [27]), here we will briefly review the main contributions based on this approach.

In the Foster and Wolfson's approach the middle class constitutes a crucial element. By middle class the authors mean a group of people who are close enough in their economic status to be able to cooperate and form a common political will. A strong middle class has a beneficial influence on the society, as it provides a buffer between the extreme tendencies of the lower and upper social classes; see Pressman [52]. Easterly [23] for example shows that a higher share of income for the middle class is associated with higher growth, more education, better health status and less political instability in the society. In this context, the decline of the middle class in a developed country signifies a threat for economic growth and socio-political stability.

Foster and Wolfson [26] and the authors who have followed their approach (in particular, Wang and Tsui [59], Chakravarty and Majumder [15], Rodriguez and Salas [53], Chakravarty *et al.* [17] and Chakravarty and D'Ambrosio [14]) define the middle class using the median income as a reference point, considering the middle class as the group of individuals whose income is exactly equal to the median income. The closer the incomes are to the median the less polarized is the distribution, while the presence of two well separated poles at the right and at the left of the median income identifies a highly polarized income distribution.

Foster and Wolfson [26] define the bipolarization measure to be

$$P^{FW} = \frac{2\mu(\mathbf{x})}{m(\mathbf{x})} \left(1 - 2L(0.5) - G\right) , \tag{6}$$

where $m(\mathbf{x})$ is the median of the incomes and $L(z)$ be the value of the Lorenz curve at the z-quantile of $\boldsymbol{x}$.

It is easy to prove that the index P^{FW} is consistent with the *Increased Bipolarity* axiom, that is, it measures an increase in polarization in case of a progressive transfer occurring either within **x-** or within **x+** and with the Increased Spread axiom, measuring an increase in polarization in case of a regressive transfer between **x-** and **x+**.

An alternative expression of the Foster and Wolfson measure is given by

$$P^{FW} = \frac{2\mu(\mathbf{x})}{m(\mathbf{x})} (G^B - G^W) . \tag{7}$$

Therefore, the Foster and Wolfson polarization measure is a normalized function of the difference between the Gini index between groups G^B and the Gini index within groups G^W. As Rodriguez and Salas [53] pointed out, this formulation clearly shows that there is a difference between adding up the two components, as it is done in inequality measurement, and taking the differences, as it is in polarization measurement.

Several other polarization measures are explicitly constructed as functions of inequality measures. In particular, Rodriguez and Salas [53] propose an extension of the Foster and Wolfson's polarization measure that includes an additional sensitivity parameter v. Their measure is based on a subgroup decomposition of the extended Gini coefficient introduced by Donaldson and Weymark [21] in the case of population divided by the median, and is defined as

$$P^{RS}(\boldsymbol{x}, v) = G^B(\boldsymbol{x}, v) - G^W(\boldsymbol{x}, v) , \quad v \in [2, 3] .$$

The idea of measuring polarization in terms of the difference of a between-group inequality component and a within-group inequality component (measured through the Gini index) is also present in the polarization index of Silber *et al.* [55], which is defined as

$$P^{SDH} = \frac{G^B - G^W}{G} \,,$$

where it is assumed that the population is divided by the median income.

Rather than using the difference between the two inequality components, Zhang and Kanbur [64] use the ratio of the two components and define their polarization measure to be the ratio of the between-component over the within-group component of Theil's measure of inquality. Since this measure is not defined if the within-group inequality is zero, Silber *et al.* [55] propose a slight modification for the Zhang and Kanbur measure.

4 Subgroup decomposition of Bonferroni and De Vergottini inequality indices

The decomposition of income inequality indices by groups is of great interest to researchers, since it allows to detect possible drivers of inequality, thus constituting a valid tool for policy-makers. Decomposition by groups, indeed, aims at explaining the contribution to total income inequality of some characteristics that affect income, such as age, gender, education, and geographical area.

Moreover, as already discussed in Section 3, the subgroup decomposition of the Gini index has been used to propose bipolarisation indices; see Wolfson [60] and [61], Foster and Wolfson [26] and [27]. In particular, Wolfson's index and its generalization proposed by Rodriguez and Salas [53], based on the extended Gini index, are defined as the difference between the *between-group* inequality and the *within-group* inequality, in case of two groups divided by the median income.

In this context, since different inequality indices put different emphasis to changes on different distribution segments and, consequently, they give different weight to the *between* and *within* components (Shorrocks [56]), the use of a given inequality index to measure polarization allows to characterise polarization indices according to their sensitivity to transfers.

In particular, we will provide a subgroup decomposition of the Bonferroni and the de Vergottini concentration indices following the approach used by Lambert and Aronson [40] for the Gini index. We will first discuss the case of a generic number of groups that can also overlap. Then, we will show the decompositions for the special case of two non-overlapping groups split by the median (which is the case of interest for bipolarization analysis). In the next section we will employ these decompositions to propose new polarization indexes based on the Bonferroni and on the De Vergottini concentration indices.

4.1 The case of k generic groups

Let us suppose that a society with n individuals can be partitioned in $K \geq 2$ groups, each with n_h individuals, for $h = 1, 2, \ldots K$, and with group mean $\mu_h(\mathbf{x})$. Let us also assume that the groups may overlap.

We now propose the decomposition for the Bonferroni and De Vergottini indices.

4.1.1 The Bonferroni decomposition

For the Bonferroni index we adopt the decomposition proposed by Barcena-Martin and Silber [9][5] that is based on the decomposition for the Gini index discussed in Lambert and Aronson [40].

The inequality *between group* can be obtained assuming that each individual in the subgroup has income equal to the subgroup mean. We denote with $B_B(q)$ the Bonferroni curve for this distribution.

$C_{Bonf}(q)$ represents the Bonferroni curve computed after a two-stage reordering. First, the individuals are divided into subgroups and those subgroups are oredered according to the mean income of the group. Then, in each group, individuals are reordered in a non-decreasing order with respect to their income. Formally

$$C_{Bonf}(q) = \frac{1}{\mu} \frac{\sum_{i<h-1} n_i \mu_i + n_h \mu_h L_h(p_h)}{nq}$$

[5]Note that these authors consider a slightly different definition of the Bonferroni curve, and therefore their decomposition slightly differs from ours.

Thus, $C_{Bonf}(q)$ is a concentration curve, and, for a given percentile q, the ordinate of the curve corresponds to the ratio between the mean of the first nq observations and the mean of the whole distribution.

Proposition 1 *For the Bonferroni index defined in 1, let the areas A_B, A_W and A_O be defined as follows:*

$$A_B = \int_0^1 [1 - B_B(q)]\, dq$$

$$A_W = \int_0^1 [B_B(q) - C_{Bonf}(q))]\, dq$$

$$A_O = \int_0^1 [C_{Bonf}(q) - B(q)]\, dq.$$

Then

$$B = A_B + A_W + A_O$$

with

$$A_B = B_{BET_{means}},\ A_W = B_{WITH} + B_{residualA_W},\ A_O = OV$$

where $B_B(q)$ is the ratio between the mean of the first nq individuals, under the assumption that each individual has the mean income of the subgroup, and the overall mean, namely: $B_B(q) = \frac{1}{\mu}\frac{\sum_{i\leq h-1} n_i\mu_i + n_h\mu_h p_h}{nq}$; $B_{BET_{means}}$ is the value of the Bonferroni index calculated in the hypothesis that each individual has the average income of the subgroup to which he belongs, formally: $B_{BET_{means}} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{1/n}\left(\frac{i}{n} - L_B\left(\frac{i}{n}\right)\right)$; the within component, *$B_{WITH} = \sum_{h=1}^{k}\nu_h w_h I_B^h$ is obtained as the sum of the Bonferroni indices calculated in each subgroup, weighted by a coefficient including the share of income and population of the subgroup and $B_{residualA_W}$[6] takes into account the role of the rank of observations in calculating the Bonferroni index. Finally, OV measures the degree of overlap between income distributions in subgroups.*

Proof: See appendix A.1.

In the discrete case, the area A_W can be written as a sum of the Bonferroni indices computed in each subgroup and weighted by a coefficient that includes the

[6]The $B_{residualA_W}$ component is a consequence of the fact that the Bonferroni index does not obey Dalton's principle of population (see Barcena-Martin and Silber [9]).

income shares and those of the population of the subgroup. Formally:

$$A_W = \sum_{p=1}^{K} \nu_p w_p I_B^p + \sum_{p=1}^{K} \nu_p w_p I_{B-reranking}^p$$

where for $i = 1, \cdots, K$, I_B^i represents the Bonferroni index computed in each subgroup, $\nu_i = \frac{n_i}{n}$ is the population share in the subgroup, $w_i = \frac{n_i \mu_i}{n\mu}$ the corresponding income share and $I_{B-reranking}^i$ is the Bonferroni index computed for each subgroup rescaling the Bonferroni curve and the area that defines the index so that, if we divide the population in subgroups, the units are ranked differently compared with their position in the original distribution. Thus, for a given subgroup h, p_h denotes the position of an individual j, that is, $p_h = j/n_h$ we have:

$$I_{B-reranking}^p = \frac{1}{n_h} \sum_{i=1}^{n_h} (\frac{p_h}{q} - 1) - \frac{1}{n_h} \sum_{i=1}^{n_h} (\frac{p_h}{q} - 1) \frac{\mu_{[i]}}{\mu_h} \quad where \quad \frac{p_h}{q} = \frac{\frac{i}{n_h}}{\frac{\sum_{j \le h-1} n_j + i}{n}}$$

and $\mu_{[i]}$ is the mean until individual i, assuming that, within eanch subgroup, the individuals are orfered by increasing individual incomes.

The presence of the residual depends on the *positional transfer sensitivity* property, fulfilled by the Bonferroni's index (see Aaberge [1]) and by the fact that the index is not *replication invariant* (Giorgi [31], Chakravarty [12] and Barcena-Martin and Imedio [7]).

Thus, the residual component $B_{residualA_W}$ represents the effect of the rank of observations on the calculation of the Bonferroni index. We recall that the index can be expressed as a linear combination of units with decreasing weights to increase the rank (Tarsitano [57]):

$$I_B = \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} x_i} \quad \text{with} \quad w_i = 1 - \sum_{j=i}^{n} \frac{1}{j} \quad w_{i+1} = w_i + \frac{1}{i} \quad \sum_{i=1}^{n} w_i = 1$$

The weights depend on the position of units in the income distribution of income.

To better understand the role of the residual it is useful to compare the decomposition of the Bonferroni index with that of the Gini index (see Lambert and Aronson [40] and Silber [54]).

For the Gini index, the decomposition consists of:

$$I_G = G_{BET_{means}} + \sum_h \nu_h w_h G_h + OV$$

where $G_{BET_{means}}$ is the Gini index computed supposing that each individual has he average income of the subgroup in which he belongs; G_h is the Gini index for the $h-$th groups, ν_i and w_i are the population share in the subgroup and the corresponding income share, respectively and OV measures the degree of overlapping among subgroup income distributions.

In this case, the weighted sum of the indices computed for subgroups returns the A_W area. In fact, the weighted sum takes into account the idea that the calculation of subindices, the relative weights associated to the observations change. Thus, rather than being divided by n/μ, they are reported to n_h/μ_h.

However, in the case of the Bonferroni index, we need to take into account the fact that ranges vary, in addition to the different numbers and averages in subgroups. Therefore, the area A_W is obtained as the sum of the *within* component and a residual that corrects the weight of the modified individuals in the subgroup.

4.1.2 The De Vergottini decomposition

We now move to the De Vergottini index. Let $V_B(q)$ denote the average (income) of the last $n(1-q)$ individuals under the assumption that each individual has the subgroup average income over the overall income:

$$V_B(q) = \frac{\sum_{i>h} n_i \mu_i + n_h \mu_h (1 - p_h)}{n(1-q)}$$

and $C_{DeVe}(q)$ be the concentration curve for De Vergottini index, defined as the ratio between the average of the income of the last $n(1-q)$ individuals and the overall average income:

$$C_{DeVe}(q) = \frac{\sum_{i>h} n_i \mu_i + n_h \mu_h (1 - L(p_h))}{n(1-q)}$$

where $L(\cdot)$ is the Lorenz curve computed in a given percentile of the distribution.

Proposition 2 *Thus, for the De Vergottini index defined in (2) we have*

$$
\begin{aligned}
A_B &= \int_0^1 \left[V_B(q) - 1\right] dq \\
A_W &= \int_0^1 \left[C_{DeVe}(q) - V_B(q)\right] dq \\
A_O &= \int_0^1 \left[B(q) - C_{DeVe}(q)\right] dq.
\end{aligned}
$$

Then

$$V = A_B + A_W + A_O$$

with

$$A_B = V_{BET_{means}},\ A_W = V_{WITH} + V_{residualA_W},\ A_O = OV$$

where $V_{BET_{means}}$ *represents the value of De Vergottini's index computed assuming that each individual has income equal to his group's average income. The* within *component* $V_{WITH} = \sum_{h=1}^{K} \nu_h w_h I_V^h$ *is the weighted average of the subgroup Bonferroni indices, namely* $\nu_h = \frac{n_h}{n}$ *and* $w_h = \frac{n_h \mu_h}{n\mu}$*. The quantity* $V_{residualA_W}$ *takes into account the role of the rank of observations in De Vergottini's index calculation. Finally,* OV *measures the degree of overlap between income subgroup distributions.*

Proof: See appendix $A.2$.

Note that in the discrete case, the area A_W can be written as sum of the De Vergottinin inequality indices calculated in each subgroup, weighted by a coefficient that includes the income shares and those of the population of the subgroup:

$$A_W = \sum_{p=1}^{K} \nu_p w_p I_V^p + \sum_{p=1}^{K} \nu_p w_p I_{V-reranking}^p,$$

where I_V^p denotes the I_V index defined in **??** computed in the $p-th$ subgroup, ν_i and w_i are the population share in the subgroup and the corresponding income share, respectively and, for each group, the *reranking* component, $I_{V-reranking}^p$ are calculated by *re-scaling* the curve $V(p)$, and the area defining the index, so that, in the subdivision of the population into subgroups, they take into account that the units have a different rank than they had in the original distribution. Thus, we have

$$I^p_{V-reranking} = \frac{1}{n_h}\sum_{i=1}^{n_h}(\frac{1-p_h}{1-q}-1)\frac{M_i}{\mu_h} - \frac{1}{n_h}\sum_{i=1}^{n_h}(\frac{1-p_h}{1-q}-1)$$

with

$$\frac{1-p_h}{1-q} = \frac{1-i/n_h}{1-\frac{\left(\sum_{j\leq h-1} n_j+i\right)}{n}}.$$

Again, the residual represents the effect of the rank of observations on De Vergottini's index calculation. Compared with the decomposition of the Gini index, the same considerations made for the Bonferroni index hold: in this case, the weighting of the indices calculated in the subgroups also takes into account both the variation of the relative weights of the observations which, rather than being divided by $n\mu$, they are related to $n_h\mu_h$, and the fact that the ranks vary. Therefore, the A_W area is obtained as the sum of the *within* component and a residue that corrects the weight that was changed when subgroup are created.

4.1.3 Synthesis framework for the decomposition

The following Table 1 summarizes the decomposition's components for the Bonferroni and De Vergottini indices, baaed on the Lambert and Aronson's approach. For comparison we have also reported the Gini decomposition.

Table 2 reports the decomposition for the special case of k non-overlapping groups. If groups do not overlap we have $C_{Lorenz}(q) = L(q)$, $C_{Bonf}(q) = B(q)$ and $C_{DeVe}(q) = V(q)$. Therefore, the *overlap* term A_O is null and the *within* component A_W is calculated accordingly.

4.2 The special case of two non-overlapping groups

We now illustrate how the subgroup decompositions discussed in the previous section simplify when we are in the special case of two groups divided by the median income $m(\mathbf{x})$. This is the case of interest in the construction of bipolarization indices.
Recall that denote with **x-** the vector of incomes that are below the median and with **x+** the vector of incomes that are above that threshold. Obviously, the two groups

Table 1: Decompositions of Gini, Bonferroni and De Vergottini inequality indices with K overlapping groups.

	Between groups (A_B)	Within groups (A_W)	Overlap (A_O)
Gini	$2\int_0^1 [q - L_B(q)]\,dq$	$2\int_0^1 [L_B(q) - C_{Lorenz}(q)]\,dq$	$2\int_0^1 [C_{Lorenz}(q) - L(q)]\,dq$
Bonferroni	$\int_0^1 [1 - B_B(q)]\,dq =$ $= \int_0^1 \left[1 - \frac{L_B(q)}{q}\right] dq =$ $= \int_0^1 \left[\frac{1}{q}(q - L_B(q))\right] dq$	$\int_0^1 [B_B(q) - C_{Bonf}(q)]\,dq =$ $= \int_0^1 \left[\frac{L_B(q)}{q} - \frac{C_{Lorenz}(q)}{q}\right] dq =$ $= \int_0^1 \left[\frac{1}{q}(L_B(q) - C_{Lorenz}(q))\right] dq$	$\int_0^1 [C_{Bonf}(q) - B(q)]\,dq =$ $\int_0^1 \left[\frac{C_{Lorenz}(q)}{q} - \frac{L(q)}{q}\right] dq =$ $= \int_0^1 \left[\frac{1}{q}(C_{Lorenz}(q) - L(q))\right] dq$
De Vergottini	$\int_0^1 [V_B(q) - 1]\,dq =$ $= \int_0^1 \left[\frac{1-L_B(q)}{1-q} - 1\right] dq =$ $= \int_0^1 \left[\frac{1}{1-q}(q - L_B(q))\right] dq$	$\int_0^1 [C_{DeVe}(q) - V_B(q)]\,dq =$ $= \int_0^1 \left[\frac{1-C_{Lorenz}(q)}{1-q} - \frac{1-L_B(q)}{1-q}\right] dq =$ $= \int_0^1 \left[\frac{1}{1-q}(L_B(q) - C_{Lorenz}(q))\right] dq$	$\int_0^1 [V(q) - C_{DeVe}(q)]\,dq =$ $\int_0^1 \left[\frac{1-L(q)}{1-q} - \frac{1-C_{DeVe}(q)}{1-q}\right] dq =$ $= \int_0^1 \left[\frac{1}{1-q}(C_{DeVe}(q) - L(q))\right] dq$

Table 2: Decompositions of Gini, Bonferroni and De Vergottini inequality indices with K non-overlapping groups.

	Between groups (A_B)	Within groups (A_W)
Gini	$2\int_0^1 [q - L_B(q)]\,dq$	$2\int_0^1 [L_B(q) - L(q)]\,dq$
Bonferroni	$\int_0^1 [1 - B_B(q)]\,dq = \int_0^1 \left[1 - \frac{L_B(q)}{q}\right] dq =$ $= \int_0^1 \left[\frac{1}{q}(q - L_B(q))\right] dq$	$\int_0^1 [B_B(q) - B(q)]\,dq = \int_0^1 \left[\frac{L_B(q)}{q} - \frac{L(q)}{q}\right] dq =$ $= \int_0^1 \left[\frac{1}{q}(L_B(q) - L(q))\right] dq$
De Vergottini	$\int_0^1 [V_B(q) - 1]\,dq = \int_0^1 \left[\frac{1-L_B(q)}{1-q} - 1\right] dq =$ $= \int_0^1 \left[\frac{1}{1-q}(q - L_B(q))\right] dq$	$\int_0^1 [V(q) - V_B(q)]\,dq = \int_0^1 \left[\frac{1-L(q)}{1-q} - \frac{1-L_B(q)}{1-q}\right] dq =$ $= \int_0^1 \left[\frac{1}{1-q}(L_B(q) - L(q))\right] dq$

do not overlap, and, moreover, the order of the individuals remain the same as in the overall distribution. In this special case, the computation of the *within* inequality component reproduces the overall order of the individuals and, consequently, we have $C_{Lorenz}(q) = B(q)$ for the Gini index and $C_{Bonf}(q) = L(q)$ for Bonferroni index. Similarly, for the De Vergottini index we have: $C_{DeVe}(q) = V(q)$

4.2.1 Subgroup decomposition of the Gini index

The decomposition of the Gini index in case of two groups divided by the median reduces to:

$$G = G_{BETmeans} + G_{WITH},$$

where, if n is an even number,

$$G_{BETmeans} = \frac{1}{4\mu(x)}\left(\mu(\mathbf{x+}) - \mu(\mathbf{x-})\right)$$

and

$$\begin{aligned} G_{WITH} = \frac{1}{2\mu(\mathbf{x})}\left[\frac{1}{2}\mu(\mathbf{x-}) - \frac{1}{n/2(n/2+1)}\sum_{i=1}^{n/2}\sum_{j=1}^{i} x_j\right] + \\ \frac{1}{2\mu(\mathbf{x})}\left[\frac{1}{2}\mu(\mathbf{x+}) - \frac{1}{n/2(n/2+1)}\sum_{i=1}^{n/2}\sum_{j=1}^{i} x_j\right]. \end{aligned}$$

4.2.2 Subgroup decomposition of the Bonferroni index

The decomposition of the Bonferroni index in case of two groups divided by the median is given by:

$$B = B_{BETmeans} + B_{WITH},$$

where, if n is an even number,

$$B_{BETmeans} = \tfrac{1}{2\mu(\mathbf{X})}\left(\mu(\mathbf{x+}) - \mu(\mathbf{x-})\right)\left(\textstyle\sum_{j=1}^{n/2}\frac{1}{n/2+j}\right). \tag{8}$$

Recall that for the Bonferroni index, differently from what happens for the Gini index, the *within* component includes the residual. Thus, considering the two income groups **x-** and **x+**, from A_W we calculate the *within* component.

Thus, if n is an even number[7], we have:

$$B_{WITH} = \frac{1}{2n\mu(x)}\left[\frac{n}{2}\mu(x-) - \sum_{i=1}^{n/2}\mu_i(x-)\right] + \frac{1}{2n\mu(x)}\left[\frac{n}{2}\mu(x+) - \sum_{i=1}^{n/2}\mu_i(x+)\right]. \tag{9}$$

For the residual term, since $\mu(x-) \leq \mu(x+)$ and the subgroups are disjointed, the lexicographic order coincides with the original one and, consequently, the *overlapping* is null and the concentration curve $C_{Bonf}(q)$ coincides with the Bonferroni curve $B(q)$. Therefore, the residual term reduces to:

$$\begin{aligned} B_{residualA_W} &= \frac{1}{2n\mu(x)}\left[\frac{n}{2}\mu(x-) - \sum_{i=1}^{n/2}\mu_i(x-)\right] + \\ &+ \frac{1}{2n\mu(x)}\left[\mu(x+)\sum_{i=1}^{n/2}\left(\frac{(i-n)/2}{(i+n)/2}\right) - \left(\frac{(i-n)/2}{(i+n)/2}\right)\mu_i(x+)\right]. \end{aligned}$$

The effect of reranking, which corresponds to the correction made by the residual term, becomes as stronger as the observation gets closer to the median.

4.2.3 Subgroup decomposition of the De Vergottini index

In case of two groups divided by the median, the De Vergottini inequality index can be decomposed as follows:

$$V = V_{BET_{means}} + V_{WITH}.$$

[7] If n is an odd number:

$$\begin{aligned} B_{WITH} &= \frac{((n-1)/2)^2}{n^2}\frac{\mu(x-)}{\mu(x)}I_B^{(n-1)/2}(x-) + \frac{((n-1)/2)^2}{n^2}\frac{\mu(x+)}{\mu(x)}I_B^{(n-1)/2}(x+) = \\ &= (\tfrac{n-1}{2n})^2\frac{\mu(x-)}{\mu(x)}I_B^{(n-1)/2}(x-) + (\tfrac{n-1}{2n})^2\frac{\mu(x+)}{\mu(x)}I_B^{(n-1)/2}(x+) \end{aligned}$$

with $I_B^{(n-1)/2}(x-) = 1 - \frac{2}{n-1}\sum_{i=1}^{(n-1)/2}\sum_{j=1}^{i}\frac{x_j}{i\mu(x-)}$ and $I_B^{(n-1)/2}(x+) = 1 - \frac{2}{n-1}\sum_{i=1}^{(n-1)/2}\sum_{j=1}^{i}\frac{x_j}{i\mu(x+)}$. We observe that for n large enough, $((n-1)/(2n))^2 \rightarrow 1/4$ and, consequently, the two expression coincides.

If n is an even number, then the between component reduces to

$$
\begin{aligned}
V_{BET_{means}} &= \frac{1}{n\mu(x)} \sum_{i=1}^{n/2} \frac{1}{(n-i+1)} \sum_{j=i, j \geq n/2} [\mu(x+) - \mu(x-)] \\
&= \frac{1}{n\mu(x)} \frac{n}{2} (\mu(x+) - \mu(x-)) \left(\sum_{i=1}^{n/2} \frac{1}{n-i+1} \right) \\
&= \frac{1}{2\mu(x)} (\mu(x+) - \mu(x-)) \left(\sum_{i=1}^{n/2} \frac{1}{n-i+1} \right).
\end{aligned}
$$

Let us remember that for the De Vergottini index, the *within* term includes a residual term. Thus, in case of two non overlapping groups and assuming n even, the within term reduces to:

$$
V_{WITH} = \frac{1}{2n\mu(x)} \left[\sum_{i=1}^{n/2} M_i(x-) - \frac{n}{2}\mu(x-) \right] + \frac{1}{2n\mu(x)} \left[\sum_{i=1}^{n/2} M_i(x+) - \frac{n}{2}\mu(x+) \right]. \tag{10}
$$

Moving to the residual term, since $\mu(x-) \leq \mu(x+)$ and since the groups are disjoint, the lexicographic order coincides with the original one and, consequently, the *overlap* is zero and the $C_{DeVe}(q)$ concentration curve coincides with the curve $V(q)$. Therefore, the residual is equal to:

$$
V_{residualAW} = \frac{1}{2n\mu(x)} \left[\sum_{i=1}^{n/2} \left(\frac{i}{i-n} \right) M_i(x-) - \sum_{i=1}^{n/2} \left(\frac{i}{i-n} \right) \mu(x-) \right] + \tag{11}
$$

$$
+ \frac{1}{2n\mu(x)} \left[\sum_{i=1}^{n/2} M_i(x+) - \frac{n}{2}\mu(x+) \right]. \tag{12}
$$

5 Bonferroni and De Vergottini based bipolarization indices

Following the approach described in Wolfson [60] and [61] and Foster and Wolfson [26] and [27] and generalized in Rodriguez and Salas [53] , we will now propose new bipolarization indices based, respectively, on the Bonferroni concentration index and on the De Vergottini concentration index.

Both Foster and Wolfson's index and Rodriguez and Salas' index of bipolarization are function of the difference between the inequality *between groups* and the inequality *within groups*. Here, we follow the same approach using the decompositions of the Bonferroni and the De Vergottini indices proposed in the previous section, for the case of two non-overlapping groups, to obtain new bipolarization measures.

In the following, we introduce two new bipolarization measures, denoted by P^B and P^V, discussing the conditions under which they are consistent with the *Increased bipolarity axiom (IB)*[8] and the *Increased spread axiom (IS)* defined in Section 2.1[9].

5.1 A new bipolarization index based on the Bonferroni index

Since the *within groups* inequality component contains a residual part representing the role played by the individuals' rank in the calculation of the Bonferroni index, we propose the following *Bonferroni- based bipolarization index*:

$$\begin{aligned} P^B &= \tfrac{2\mu(x)}{m(x)}\left[B_{BETmeans} - (B_{WITH} + B_{residualA_W})\right] = \tfrac{2\mu(x)}{m(x)}\left[A_B - A_W\right] = \\ &= \tfrac{2\mu(x)}{m(x)}\left[\int_0^1 \left(\tfrac{1}{q}\right)(q - L_B(q))\,dq - \int_0^1 \left(\tfrac{1}{q}\right)(L_B(q) - L(q))\,dq\right]. \end{aligned}$$

Proposition 3 *Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be the income distribution of a population of n individuals, s be the position of the individual belonging to group $x-$ and transferring part of his income, with $s \in [1, n/2]$, and k be the position of the individual belonging to group $x+$ and receiving the transfer, with $k \in [1, n/2]$ and $s \geq k$, then the bipolarization index P^B satisfies IB and IS axioms if $s \geq \min(k, n/6)$.*

Proof: See appendix $A.3$.

[8] IB axiom requires that polarization increases in presence of a progressive transfer occurring either below or above the median.

[9] According to IS axiom, polarization increases in presence of a regressive transfer occurring between one income smaller than the median and another income greater than the median.

Proposition 3 gives a condition in order to ensure that the bipolarization index P^B is consistent with the so-called *second polarization curve*[10] (Wolfson [60] and [61], Foster and Wolfson [26] [27]).

5.2 A new bipolarization index based on the De Vergottini index

Also for the De Vergottini index, the *within groups* inequality component contains a residual part representing the role of observation rank in the calculation of I_V. Therefore, the *De Vergottini-based bipolarization index* that we proposed is the following:

$$\begin{aligned} P^V &= \tfrac{2\mu(x)}{m(x)} \left[V_{BETmeans} - (V_{WITH} + V_{residualA_W})\right] = \tfrac{2\mu(x)}{m(x)} \left[A_B - A_W\right] = \\ &= \tfrac{2\mu(x)}{m(x)} \int_0^1 \left(\tfrac{1}{1-q}\right)(q - L_B(q))\, dq - \tfrac{2\mu(x)}{m(x)} \int_0^1 \left(\tfrac{1}{1-q}\right)(L_B(q) - L(q))\, dq \end{aligned}$$

Proposition 4 *Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be the income distribution of a population of n individuals, s be the position of the individual belonging to group $x-$ and transferring part of his income, with $s \in [1, n/2]$, and k the position of the individual belonging to group $x+$ and receiving the transfer, with $k \in [1, n/2]$ and $s \geq k$. The index P^V is consistent with the IB and IS axioms if $k \leq \min\left(s, \frac{n}{3} + 1\right)$.*

Proof: See appendix $A.4$.

As for proposition 3, proposition 4 gives a range of admissible position values (k) for an individual that receives the transfer in order to guarantee to P^V to satisfies the two fundamental axioms for bipolarization measures.

6 Simulation study

We now illustrate through a simulation study the sensitivity of the new bipolarization indices proposed, P^B and P^V, with respect to the regressive transfers through the median (IS axiom) and Pigou-Dalton transfers above or below the median (IB

[10] A bipolarization index is consistent with the second polarization curve if a progressive median-preserving transfer within (between) polar subgroups never reduces (increases) polarization

axiom). We will study the sensitivity of the new bipolarization indices to different types of transfer, which vary in terms of the amount of income transferred, namely α, with $\alpha > 0$.

To analyze the sensitivity of the indices to the IS and IB axioms, we generate an income distribution by extracting a sample of $n = 1000$ observations from a mixture of two normal distributions. For the sake of simplicity, we will assume that the population of n (even) observations is composed of two sub-populations with same number of individuals $(n/2)$: the income distribution of the first sub-population is a realization from the random variable $X_1 \sim N(\mu_1,\ \sigma_1)$, whereas the distribution of the second group is a realization of $X_2 \sim N(\mu_2,\ \sigma_2)$, with $(\mu_1,\ \sigma_1) = (375, 100)$ and $(\mu_2,\ \sigma_2) = (625, 100)$. It should be noted that here the simulated distribution is not intended to reproduce the features of an effective income distribution, but rather to provide an initial distribution, starting from which we will gradually introduce increasing degrees of bi-polarisation.

To ensure robustness to our analysis, $10,000$ independent extractions are carried out for s and k, that are the positions of the individuals involved into the transfer and the index variations are calculated for each draw. The average percentage variations of the indices and the respective confidence intervals are then computed.

Let $s*$ and $k*$ be the empirical threshold values obtained by simulating an IS transfer. More in detail, $s*$ and $k*$ are the threshold values that guarantee the Bonferroni and De Vergottini indices, respectively, to satisfy the IS transfer. That is, for a given k, $s*$ is the minimum value of s such that $\Delta P_B(s, k, n) > 0$ for $s > s*$ and $\Delta P_B(s, k, n) < 0$ for $s \leq s*$. Similarly, for a given s, $k*$ is the maximum value of k such that $\Delta P_V(s, k, n) < 0$ for $k \geq k*$ and $\Delta P_V(s, k, n) > 0$ for $k < k*$.

In this section, we will focus on transfers between individuals with position s and k, respectively, in the *benchmark* scenario[11] to evaluate the effect of IS and Pigou-Dalton transfers (IB above and below the median) both on inequality and bipolarization indices. For the simulation illustrated in Table 3 we choose $s^* = 50$ and $k^* = 946$.

[11]In what follows, we call *benchmark* scenario a situation in which we select two individuals in the positions $s > s*$ and $k < k*$ such that P_B and P_V indices are consistent with an IS transfer, and these indices can be considered polarization indices

Table 3 compares the different sensitivity degrees of the inequality indices with respect to a transfer α, as α varies. For the IS transfer we note that, on average, the Gini index has a greater variation than both the Bonferroni and De Vergottini indices. This difference increases with the increase of the transferred income.[12]

By definition, the IB transfers, occurring either below or above the median value, determine a decrease in inequality. Given the positional transfer sensitivity that characterizes both the Bonferroni and the De Vergottini index, we observe that the Bonferroni index, which attaches more weight to the poor people, is the most sensitive to the IB transfers below the median value. Conversely, De Vergottini index, which gives more weight to the income variation among the richer individuals in the distribution, is more sensitive to IB transfers above the median. For both type of transfers, the variation of the Gini index is in between the variations of the other two indices. Moreover, in case of IB transfers, the difference between index variations increases with α. Concluding, the analysis of the overall effect of IS and IB transfers reveals that the Gini index has a greater sensitivity than the other two indices for all α values. Consequently, if we take into account the impact of transfer on the inequality, the impact of the IS transfer seems prevailing on those of the IB one.

We now move to analyze the transfer sensitivity of the three bipolarization indexes, Foster and Wolfson (P^{FW}), Bonferroni-based (P^B) and De Vergottini-based (P^V), as α varies (see Table 4). First, we note that, for all the three indices, a regressive IS transfer induces an increase of both inequality and polarization; these results are in line with the literature. On the contrary, progressive transfers that increase concentration on one, or both, income distribution segments above and below the median, cause a decrease in inequality and an increase in polarization.

For each value of α, the variation of P^B and P^V indices with respect to the IS transfer is always higher than that of the Foster and Wolfson index, since the index based on Bonferroni is more responsive to a regressive transfer. Also P^V index shows a greater sensitivity to IS transfers than the Foster and Wolfson index. Moreover,

[12]Note that this characteristic of index variation, depicted by the simulation, also occurs if the positions are chosen without imposing constraints on the variation of s and k. However, the difference among the variation of the indices is less accentuated.

Table 3: Variation (mean and variance) of the inequality indices in case of IS and an IB transfer for different values of the amount of transferred income (α). 10,000 Bootstrap replications

	Average variation			**Variance of the variation**		
	Gini	**Bonf**	**DeVe**	**Gini**	**Bonf**	**DeVe**
IS transfer						
$\alpha = 10$	0.99%	0.78%	0.79%	0.00%	0.00%	0.00%
$\alpha = 20$	2.05%	1.64%	1.66%	0.01%	0.01%	0.01%
$\alpha = 30$	3.19%	2.57%	2.61%	0.01%	0.02%	0.02%
$\alpha = 40$	4.40%	3.58%	3.64%	0.02%	0.04%	0.04%
$\alpha = 50$	5.68%	4.68%	4.76%	0.03%	0.06%	0.06%
IB transfer below the median						
$\alpha = 10$	-0.32%	-0.61%	-0.14%	0.00%	0.00%	0.00%
$\alpha = 20$	-0.49%	-0.68%	-0.22%	0.00%	0.00%	0.00%
$\alpha = 30$	-0.62%	-0.86%	-0.28%	0.00%	0.01%	0.00%
$\alpha = 40$	-0.68%	-0.94%	-0.30%	0.01%	0.02%	0.00%
$\alpha = 50$	-0.67%	-0.92%	-0.30%	0.01%	0.03%	0.00%
IB transfer above the median						
$\alpha = 10$	-0.32%	-0.14%	-0.62%	0.00%	0.00%	0.00%
$\alpha = 20$	-0.47%	-0.21%	-0.65%	0.00%	0.00%	0.00%
$\alpha = 30$	-0.58%	-0.26%	-0.81%	0.00%	0.00%	0.01%
$\alpha = 40$	-0.61%	-0.27%	-0.85%	0.01%	0.00%	0.02%
$\alpha = 50$	-0.56%	-0.24%	-0.78%	0.01%	0.00%	0.03%
IS and IB transfers						
$\alpha = 10$	0.35%	0.02%	0.03%	0.00%	0.01%	0.01%
$\alpha = 20$	1.08%	0.74%	0.77%	0.01%	0.01%	0.01%
$\alpha = 30$	1.97%	1.44%	1.51%	0.02%	0.03%	0.03%
$\alpha = 40$	3.08%	2.36%	2.47%	0.04%	0.05%	0.05%
$\alpha = 50$	4.42%	3.49%	3.66%	0.06%	0.09%	0.09%

the difference between the variations in the indices is the higher the bigger is α. For what concerns the IB transfer, its effect on the polarization indexes is similar to what is observed for the inequality indices on which these are built: P^B has the greatest sensitivity to IB transfers below the median, while the P^V index is the most sensitive to IB transfers above the median. Finally, considering the overall effect, we observe that the P^B and P^V indices are more sensitive to transfers that bi-polarize the income distribution, with P^B showing the greatest reactivity. This characteristic becomes more evident as the transferred income (α) increases. The results of the simulation show that P^B and P^V indices not only better report the phenomenon of the emptying of the middle class, but their sensitivity is the more accentuated the stronger is the change in the income distribution.

Table 4: Variation (mean and variance) of the polarization indices in case of IS and an IB transfer for different values of the amount of transferred income (α). 10,000 Bootstrap replications

	Average variation			**Variance of the variation**		
	$\mathbf{P^{FW}}$	$\mathbf{P^{B}}$	$\mathbf{P^{V}}$	$\mathbf{P^{FW}}$	$\mathbf{P^{B}}$	$\mathbf{P^{V}}$
IS transfer						
$\alpha = 10$	2.44%	3.76%	3.51%	0.01%	0.02%	0.02%
$\alpha = 20$	4.73%	7.26%	6.78%	0.03%	0.09%	0.08%
$\alpha = 30$	6.87%	10.50%	9.80%	0.06%	0.22%	0.18%
$\alpha = 40$	8.84%	13.48%	12.55%	0.10%	0.42%	0.34%
$\alpha = 50$	10.68%	16.17%	15.03%	0.16%	0.69%	0.56%
IB transfer below the median						
$\alpha = 10$	0.69%	2.12%	0.44%	0.00%	0.05%	0.00%
$\alpha = 20$	1.01%	2.29%	0.65%	0.02%	0.07%	0.01%
$\alpha = 30$	1.16%	2.81%	0.71%	0.06%	0.16%	0.04%
$\alpha = 40$	1.16%	3.00%	0.68%	0.10%	0.27%	0.07%
$\alpha = 50$	0.98%	2.84%	0.54%	0.15%	0.41%	0.10%
IB transfer above the median						
$\alpha = 10$	0.70%	0.48%	1.95%	0.01%	0.01%	0.05%
$\alpha = 20$	1.13%	0.84%	2.15%	0.03%	0.03%	0.06%
$\alpha = 30$	1.64%	1.30%	2.92%	0.08%	0.09%	0.15%
$\alpha = 40$	2.11%	1.80%	3.49%	0.17%	0.20%	0.30%
$\alpha = 50$	2.54%	2.34%	3.90%	0.29%	0.36%	0.50%
IS anf IB transfers						
$\alpha = 10$	3.82%	6.34%	5.89%	0.02%	0.08%	0.07%
$\alpha = 20$	6.85%	10.36%	9.55%	0.08%	0.19%	0.15%
$\alpha = 30$	9.64%	14.56%	13.38%	0.20%	0.47%	0.37%
$\alpha = 40$	12.08%	18.21%	16.67%	0.37%	0.89%	0.69%
$\alpha = 50$	14.16%	21.28%	19.40%	0.60%	1.47%	1.15%

7 Empirical application to EU-SILC data

In this section, we illustrate the results of an empirical exercise in which we have applied the new polarization indexes proposed to real data, with the aim of estimating the degree of income bipolarization in Italy over the period between 2007 and 2011. We used data from the European Union Statistics on Income and Living Conditions (EU-SILC), referred to Italy (sample size of about 20,000 each year).

As income variable we considered the household disposable income, defined as the sum of the personal income components of all household members plus the family income components, net of income tax and social contributions.[13] Negative incomes have been excluded from the analysis.

For the estimation of the inequality and polarization we will use simple weighted estimators, constructed from the discrete formulation of the indices above presented.

More in details, we estimate the Gini index of inequality with

$$\widehat{I_G}(\mathbf{x}) = 1 - \frac{2}{n(n-1)\mu(x)} \sum_{i=1}^{n-1} \sum_{j=1}^{i} w_j x_j, \tag{13}$$

the Bonferrroni index of inequality with

$$\widehat{I_B}(\mathbf{x}) = 1 - \frac{1}{(n-1)\mu(x)} \sum_{i=1}^{n-1} \frac{1}{i} \sum_{j=1}^{i} w_j x_j, \tag{14}$$

and the De Vergottini inequality index with

$$\widehat{I_V}(\mathbf{x}) = \frac{1}{\left(1 + n\sum_{s=2}^{n-1} \frac{1}{s}\right)\mu(x)} \sum_{i=1}^{n-1} \frac{1}{i} \sum_{j=n-i+1}^{n} w_j x_j - 1 \tag{15}$$

where the sums are up to $n-1$ to ensure the accuracy of the indices and w_j are the sample weights[14]. To estimate the absolute sampling error and build confidence intervals we applied a bootstrap resampling technique.[15]

[13] For more details, see variable HY020.

[14] Variable $DB090$. These weights are obtained starting from the inverse of the inclusion probability of the family, corrected for the overall non-response rate.

[15] Since estimators (13), (14) and (15) have a complex formulation, the standard methodology for sampling variance cannot be applied.

Table 5 reports the results of the bootstrap procedure for the estimators (13), (14) and (15). We note that, in all the years considered, the proposed estimators have a negligible bias. Moreover, such bias is not systematic, since it is negative in some years and positive in others. Moreover, the estimated standard error and confidence intervals reveal a good accuracy of the proposed estimates. However, we observe that the standard error $\sigma(\hat{\theta}(x))$, is higher for P^B and P^V than for P^{FW}.

Table 5: Bootstrap estimates for the polarization indices (1,000 replications) and their 95% confidence intervals

	year	$\hat{\theta}(x)$	θ^*	$\theta^* - \hat{\theta}(x)$	$\sigma(\hat{\theta}(x)))$	$\theta^* - z_{1-\frac{\alpha}{2}}\sigma(\hat{\theta}(x))$	$\theta^* + z_{1-\frac{\alpha}{2}}\sigma(\hat{\theta}(x))$
P^{FW}	**2007**	0.3300	0.3299	-0.000117	0.002558	0.325	0.335
	2008	0.3320	0.3317	-0.000220	0.002686	0.327	0.337
	2009	0.3245	0.3244	-0.000140	0.00267	0.320	0.330
	2010	0.3269	0.3270	0.000029	0.002785	0.321	0.333
	2011	0.3294	0.3292	-0.000185	0.002842	0.324	0.335
P^B	**2007**	0.4847	0.4845	-0.000168	0.005256	0.474	0.496
	2008	0.4942	0.4940	-0.000202	0.005694	0.483	0.506
	2009	0.4765	0.4767	0.0001746	0.005857	0.465	0.487
	2010	0.4887	0.4884	-0.000326	0.007269	0.475	0.503
	2011	0.4927	0.4931	0.0004162	0.007073	0.478	0.505
P^V	**2007**	0.7240	0.7238	-0.000191	0.005453	0.713	0.735
	2008	0.7334	0.7333	-0.000108	0.005904	0.722	0.745
	2009	0.7139	0.7137	-0.000177	0.005554	0.703	0.725
	2010	0.7222	0.7224	0.0002136	0.005666	0.711	0.733
	2011	0.7300	0.7300	$-7.26E-06$	0.006162	0.718	0.742

Where:

$\hat{\theta}(x)$ is the estimator of the parameter $\theta(x)$

$\theta^* = \frac{1}{B}\sum_{b=1}^{B}\hat{\theta}(x_b^*)$ and x_b^* for $b = 1, \cdots, B$ are sample of equal size drawn from the population X.

$\theta^* - \hat{\theta}(x)$ represents the bias of the estimator.)

$\sigma(\hat{\theta}(x))$ is the sample variance

$\theta^* \pm z_{1-\frac{\alpha}{2}}\sigma(\hat{\theta}(x))$ are the lower and upper bound of the confidence interval.

Table 6 shows the annual percent variations of the polarization indices, as estimated in the column $\hat{\theta}(x)$ of Table 5.

Let P^t be the polarization index at time t. Then, the annual percent variation is:

$$\Delta P^{t-1/t} = \frac{P^t - P^{t-1}}{P^{t-1}} \cdot 100.$$

Looking at Table 6, we observe a growing income polarization over the years 2007 and 2011, with the only exception of the year 2009.

Over the period of interest, the three indexes provide variations having the same sign. The variations are higher for P^B and P^V.

More in details, we note that between the years 2007 and 2008 and between 2010 and 2009 the increase in polarization was recorded with greater intensity by P^B index, which has increased by 2% and 2.6%, respectively, compared to 0.6% and 0.7% for the Foster and Wolfson index and 1.1% and 1.3% for P^V. The differences in the variation of the indexes between the years 2008 and 2009, when the polarization decreased, are also notable: there is -2.2% for the Foster and Wolfson index, -3.6% for P^B and -2.7% for P^V.

Table 6: Annual percent variation of polarization indices (estimated on EU-SILC 2008-2012 (household disposable income for 2007-2011)

	P^FW	$\mathbf{P}^B$	$\mathbf{P}^V$
$\boldsymbol{\Delta}\mathbf{P}^{\mathbf{2008/2007}}$	0.6%	2.0%	1.3%
$\boldsymbol{\Delta}\mathbf{P}^{\mathbf{2009/2008}}$	−2.2%	−3.6%	−2.7%
$\boldsymbol{\Delta}\mathbf{P}^{\mathbf{2010/2009}}$	0.7%	2.6%	1.2%
$\boldsymbol{\Delta}\mathbf{P}^{\mathbf{2011/2010}}$	0.8%	0.8%	1.1%

8 Conclusions

The paper provides a theoretical contribution to the study of income bipolarization measurement.

Following Wolfson [60] and Foster and Wolfson ([26] and [27]), we have proposed two new polarization indices, based on the Bonferroni index [11] and the index of De Vergottini [20], respectively. The proposed indices show different degrees of sensitivity to progressive transfers that accentuate the bipolarization in the distribution of income.

A simulation study and an empirical illustration, based on EU-SILC for the years 2007 - 2011, have been also performed in order to compare the proposed indices to the existing ones. By means of the simulation study, the analysis of the overall effect of a regressive transfer through the median (IS) and a Pigou-Dalton transfer above or below the median(IB) reveals that the polarization index based on Bonferroni has the greatest sensitivity to IB transfers below the median, while the the De

Vergottini-based polarization index is the most sensitive to IB transfers above the median.

Finally, with the empirical analysis, we estimate the income polarization levels in Italy over the period between 2007 and 2011, showing the percentage changes on an annual basis of the three polarization indices.

References

[1] Aaberge R., Characterization of Lorenz curves and income distributions, *Social Choice and Welfare*, (2000) 17, 639-653.

[2] Aaberge R., Gini's Nuclear Family, *Journal of Economic Inequality*, (2007) 5(3), 305-322.

[3] Aaberge R., Bjerve S. and K. Doksum, Decomposing of Rank-dependent Measures of Inequality by Subgroups, *Metron*, (2005) 63(3), 493-503.

[4] Anderson G., Towards an empirical analysis of polarization, *Journal of Econometrics*. (2004) 12, 1-26.

[5] Amato V., Contributo della Scuola Statistica Italiana alla costruzione di classi di indici di concentrazione, in Zenga M. (editor) *La distribuzione personale del reddito: problemi di formazione, di ripartizione e di misurazione Vita e Pensiero, Milano* (1987), 3-25.

[6] Aristondo, O., J. L. Garcia-Lapresta, C. Lasso de la Vega, R.A. Marques Pereira, Classical inequality indices, welfare and illfare functions, and the dual decomposition, *Fuzzy Sets and Systems*, (2013), 228, 114-136.

[7] Barcena-Martin E. and L.J. Imedio, The Bonferroni, Gini and De Vergottini indices. Inequality, welfare and deprivation in the European Union in 2000, *Research on Economic Inequality*, (2008) 16, 231-257.

[8] Barcena-Martin E. and J. Silber, On the Concepts of Bonferroni Segregation Index and Curve, *Rivista Italiana di Economia, Demografia e Statistica*, (2011) 65(2), 57-74.

[9] Barcena-Martin E. and J. Silber, On the generalization and decomposition of the Bonferroni index, *Social Choice and Welfare*, (2013) 41, 763-787.

[10] Barcena-Martin E. and J. Silber, The Bonferroni index and the measurement of distributional change, *Metron*, (2017) 75(1), 1-16.

[11] Bonferroni C.E., Elementi di statistica generale, 1930, Seber, Firenze.

[12] Chakravarty S.R., The Bonferroni indices of Inequality, *International conference in memory of Gini C. and Lorenz M.O.*, May 23-26, 2015, Siena (IT), (2005).

[13] Chakravarty S.R., A Deprivation-based Axiomatic Characterization of the Absolute Bonferroni Index of Inequality, *Journal of Economic Theory*, (2007) 5(3), 339-351.

[14] Chakravarty, S.R. and C. D'Ambrosio, Polarization orderings of income distributions, *Review of Income and Wealth*, (2010) 56, 47-64.

[15] Chakravarty S.R. and A. Majumder, Inequality, polarization and welfare: Theory and applications, *Australian Economic Papers*, (2001) 40, 1-13.

[16] Chakravarty S.R. and A. Majumder, Measuring ethnic polarization, *Social Choice and Welfare*, (2011) 37, 431-452.

[17] Chakravarty S.R., Majumder A. and S. Roy, A treatment of absolute indices of polarization, *Japanese Economic Review*, (2007) 58, 273-293.

[18] Chakravarty S.R. and P. Muliere, Welfare Indicators: A Review and New Perspectives. 2. Measurement of Poverty, *Metron*, (2004) 62(2), 247-281.

[19] D'Ambrosio C., Household characteristics and the distribution of income in Italy: An application of a social distance measures, *Review of Income and Wealth*, (2001) 47, 43-64.

[20] De Vergottini M., Methods of measuring the concentration of wealth, *Statistica*, (1950) X(4), 445-454.

[21] Donaldson D. and J. Weymark, A single-parameter generalization of the Gini indices of inequality, *Journal of Economic Theory*, (1980) 22, 67-86.

[22] Duclos J., Esteban J. and D. Ray, Polarization: concepts, measurements, estimation, *Econometrica*, (2004) 22(6), 1737-1772.

[23] Easterly W., The middle class consensus and economic development, *Journal of Economic Growth*, (2001) 6, 317-335.

[24] Esteban J. and D. Ray, On the measurement of polarization, *Econometrica*, (1994) 62(4), 819-851.

[25] Esteban J., Gradin C. and Ray D., An extension of a measure of polarization, with an application to the income distribution of five OECD countries, *Journal of Economic Inequality*, (2007) 5, 1-19.

[26] Foster J.E. and M.C. Wolfson, Polarization and the decline of the Middle Class: Canada and the US, *OPHI Working Paper*, (1992) No.31, University of Oxford.

[27] Foster J.E. and M.C. Wolfson, Polarization and the decline of the Middle Class: Canada and the US, *Journal of Economic Inequality*, (2010) 8, 247-273.

[28] Gastwirth J.L., A General Definition of the Lorenz Curve, *Econometrica*, (1971) 31(6), 1037-1039.

[29] Giordani P. and G.M. Giorgi,Fuzzy Logic Approach to Poverty Analysis Based on the Gini and Bonferroni Inequality Indices, *Statistical Methods and Applications*, (2010) 19(4), 587-607.

[30] Giorgi G. M., Il rapporto di concentrazione di Gini: genesi, evoluzione ed una bibliografia commentata, Siena (IT): Libreria Editrice Ticci, (1992).

[31] Giorgi G. M., Concentration index, Bonferroni, in Kotz S. Read CB, Banks DS (eds), *Encyclopedia of Statistical Sciences*, (1998) Vol.2, Wiley, NY, 141-146.

[32] Giorgi G. M. and M. Crescenzi, Bayesian Estimation of the Bonferroni Index in a Pareto-Type I Population, *Statistical Methods and Applications*, (2001) 10(1-3), 41-48.

[33] Giorgi G. M. and M. Crescenzi, A Proposal of Poverty Measures Based on the Bonferroni Inequality Index, *Metron*, (2001) 59(3-4), 3-15.

[34] Giorgi G. M. and M. Crescenzi, A look at the Bonferroni inequality measure in a reliability framework, *Statistica*, (2001) 61(4), 571 -583.

[35] Giorgi G. M. and S. Gubbiotti, Celebrating the memory of Corrado Gini: a personality out of the ordinary, *International Statistical Review*, (2017) 85(2), 325-339.

[36] Giorgi G. M. and R. Mondani, The exact sampling distribution of the Bonferroni concentration index, *Metron*, (1994) 52(1-2), 5-41.

[37] Giorgi G. M. and R. Mondani, Sampling distribution of the Bonferroni inequality index from exponential population, *Sankhya*, series B, (1995) 57, 10-18.

[38] Gradín C., Polarization by sub-populations in spain, 1973-91, *Review of Income and Wealth*, (2000) 46, 457-474.

[39] Imesio-Olmedo L.J.- Barcena-Martin E. and E.M. Parrado-Gallardo, A Class of Bonferroni Inequality index, *Journal of Public Economic Theory*, (2011) 13(1), 97-124.

[40] Lambert P. J. and R. J. Aronson, Inequality Decomposition Analysis and the Gini Coefficient Revisited, *The Economic Journal*, (1993) 103(420), 1221-1227.

[41] Lasso de la Vega C. and A. Urrutia, An alternative formulation of the Esteban-Gradin-Ray extended measure of bipolarization, *Journal of Income Distribution*, (2006) 15, 42-54.

[42] Lasso de la Vega C., Urrutia A. and H. Diez, Unit consistency and bipolarization of income distributions, *Review of Income and Wealth*, (2010) 56(1), 65-83.

[43] Lorenz M. O., Methods of measuring the concentration of wealth, *Quarterly Pubblications of the American Statistical Association*, (1905) 9, 209-219.

[44] Massari R., Pittau M.G. and R. Zelli, A dwindling middle class? Italian evidence in the 2000s, *The Journal of Economic Inequality* (2009) 7, 333-350.

[45] Mehran F., Linear Measures of Income Inequality, *Econometrica*, (1976) 44, 805-809.

[46] Nygard F. and A. Sandstrom, Measuring Income Inequality, (1981) Acta Universitatis Stockholmiensis, Almqvist & Wiksell International Stockholm.

[47] Permanyer, I., The conceptualization and measurement of social polarization, *The Journal of Economic Inequality* (2012) 10, 45-74.

[48] Permanyer, I., Income and social polarization: theoretical approaches. (Ch.19), in *Handbook of Research on Economic and Social Well-Being*, Edited by Conchita D'Ambrosio, (2018) 434-459, Edward Elgar Publishing.

[49] Piesch W., Statistische Konzentrationsmasse, Tubingen: J.B.C. Mohr (Paul Siebeck) (1975).

[50] Pietra G., Delle Relazioni tra gli Indici di Variabilità, I, II., *Atti del Reale Istituto Veneto di Scienze, Lettere ed Arti* (1915), LXXIV(II): 775-804. English translation: On the relationships between variability indices (Note I), *Metron* (2014) 72(1), 5-16.

[51] Pittau M.G., Zelli R. and P.A. Johnson, Mixture models, convergence clubs, and polarization, *Review of Income and Wealth*, (2010) 56, 102-122.

[52] Pressman S., The Decline of the Middle Class: An International Perspective, *Journal of Economic Issues*, (2007) XLI, 181-200.

[53] Rodriguez J. G. e R. Salas, Extended bi-polarization and inequality measures, *Research on Economic Inequality*, (2003) 9, 69-3.

[54] Silber J., Factor components, population subgroups and the computation of the Gini index of inequality, *Review of Economics and Statistics*, (1989) 71, 107-115.

[55] Silber J., Deutsch J. and Hanoka M., On the link between the concepts of kurtosis and bipolarization, *Economics Bulletin* (2007) 4, 1-6.

[56] Shorrocks A. S., The Class of Additively Decomposable Inequality Measures, *Econometrica*, (1980) 48(3), 613-625.

[57] Tarsitano A., Methods of measuring the concentration of wealth, in Dagum C. and Zenga M. (eds.) *Income and wealth distribution, inequality and poverty*, Berlin: Springer-Verlag (1990), 228-242.

[58] Tarsitano A., A new class of inequality measures based on a ratio of L-statistics, *Metron*, (2004), 62(1-2), 137-160.

[59] Wang Y. and k. Tsui, Polarization orderings and new classes of polarization indices, *Journal of Public Economic Theory* (2000) 2, 349-363.

[60] Wolfson M. C., When Inequality Diverge, *The American Economic Review*, (1994) 84(2), 353-358.

[61] Wolfson M. C., Divergent Inequalities: Theory and Empirical Results, *Review of Income and Wealth*, (1997) 43(4), 401-421.

[62] Yitzhaki S., More than a dozen alternative ways of spelling Gini, *Research on Economic Inequality*, Greenwich (CT): JAI Press (1998), 8, 13-30.

[63] Yitzhaki S., Is there room for polarization?, *Review of Income and Wealth*, (2010) 56, 7-22.

[64] Zhang X. and R. Kanbur, What differences do polarization measures make? An application to China, *Journal of Development Studies*, (2001) 37(3), 85-98.

[65] Zoli C., Intersecting generalized Lorenz curves and the Gini index, *Social Choice and Welfare*, (1999) 16, 183-196.

Appendix A

A.1 Proof of Proposition 1

We want to prove that $I_B = A_B + A_W + A_O$ where $A_B = B_{BET\_means}$, $A_W = B_{WITH} + B_{residualA_W}$ and $A_O = OV$.

We will proceed by steps.

Firstly we compute the three components.

(1) The *between groups* inequality component can be obtained by dividing the observations into subgroups, and assuming that each observation has the average income of the subgroup in which it belongs. Let $B_B(q)$ denote corresponding the Bonferroni curve and q be the rank of individuals in the income distribution, defined according to a *lexicographic* order, that is, first, the observations are sorted according to their group average and then, in each group, the observations are ordered in a non-increasing order relative to their income. The ordinates of the curve at the q-th percentile correspond to the ratio between the cumulatives of the average of the group until q, and the the average μ of the distribution.

Thus, the A_B area is obtained by subtracting the area under the curve $B_B(q)$ to the equi-distribution area, that is the area under the curve $B(q) = 1$[16], for any q

$$A_B = \int_0^1 [1 - B_B(q)]\, dq.$$

(2) The *within* inequality component can be obtained by assigning to each observation its income, maintaining the particular order imposed in step (1). We obtain the concentration curve $C_{Bonf}(q)$, that is, the Bonferroni curve computed on the ordered observations respect to the *lexicographic* order. The ordinates of this curve at the q-th percentile correspond to the ratio between the average of the first nq observations and the average of the distribution.

[16]In the case of equi-distribution $L(p) = p$ from which $B(p) = L(p)/p$, and $B(p) = 1$.

The A_W area is obtained by subtracting the area below the $C_{Bonf}(q)$ curve to the equidistribution area, that is the area under $B_B(q)$ for each q.

$$A_W = \int_0^1 [B_B(q) - C_{Bonf}(q))]\, dq.$$

(3) The final step consists in arranging the observations as in the original distribution, thus isolating the effect of *overlapping* between groups. We get the Bonferroni curve ($B(q)$), whose ordinates are equal to the ratio between the partial average up to q and the average of the distribution, for each q.

Thus, the A_O area is obtained as the difference between the $C_{Bonf}(q)$ curve and the Bonferroni curve $B(q)$:

$$A_O = \int_0^1 [C_{Bonf}(q) - B(q)]\, dq.$$

We will prove that

$$I_B = A_B + A_W + A_O$$

where $A_B = B_{BET\_means}$, $A_W = B_{WITH} + B_{residualA_W}$, $A_O = OV$.

For construction $A_B = B_{BET\_means}$.

We compute the A_W area by isolating the two terms, B_{WITH} and $B_{residualA_W}$.

Let now considere the $h-$th subgroup in which the Bonferroni curve is defined: $B_h(p_h) = \frac{L_h(p_h)}{p_h}$ where p_h dentotes the rank of units in the $h-$th group that, for the j-th unit, is equal to $p_h = j/n_h$.

The unit having rank p_h in the $h-$th subgroup, in the original distribution has rank equal to q being:

$$q = \frac{\sum_{i \leq h-1} n_i + p_h n_h}{n}.$$

The concetration curve $C_{Bonf}(q)$ is defined as the ratio between the average of the first nq units and the overall average. It follows that

$$C_{Bonf}(q) = \left[\frac{1}{\mu}\frac{\sum_{i \leq h-1} n_i \mu_i + n_h \mu_h L_h(p_h)}{nq}\right]$$

where $L_h(p_h)$ represents the Lorenz curve at the p_h percentile, that is, the income portion of the $h-$th group that has individuals with income less than or equal to the income of the individual having rank equal to p_h.

By contrast, the $B_B(q)$ curve is defined as the ratio between the average of the first nq units if each unit has the average income of the subgroup and the overall average

$$B_B(q) = \left[\frac{1}{\mu} \frac{\sum_{i \leq h-1} n_i \mu_i + n_h \mu_h p_h}{nq} \right]$$

Multiplying both curves by imu and subtracting the second expression from the first one we get:

$$n\mu \left[B_B(q) - C(q) \right] = n_h \mu_h \frac{(p_h - L_h(p_h))}{q}$$

From the expression of q we get: $dq = (n_h/n)dp_h$ and by multiplying both sides by dq we have:

$$n\mu \left[B_B(q) - C(q) \right] dq = \frac{n_h^2 \mu_h}{n} \frac{(p_h - L_h(p_h))}{q} dp_h.$$

Now, integrating in subgroups over p_h between 0 and 1 and scrolling through the group index (h)[17], we get

$$\begin{aligned} n\mu A_W &= \sum_h (n_h^2 \mu_h / n) \int_0^1 \tfrac{(p_h - L_h(p_h))}{q} dp_h \\ &= \sum_h (n_h^2 \mu_h / n) \int_0^1 \tfrac{(p_h - L_h(p_h))}{q} \tfrac{p_h}{p_h} dp_h \\ &= \sum_h (n_h^2 \mu_h / n) \int_0^1 \tfrac{(p_h - L_h(p_h))}{p_h} \tfrac{p_h}{q} dp_h \pm \sum_h (n_h^2 \mu_h / n) \int_0^1 \tfrac{(p_h - L_h(p_h))}{p_h} dp_h \end{aligned}$$

Assuming $v_h = \frac{n_h}{n}$ and $w_h = \frac{n_h \mu_h}{n\mu}$ we get:

$$n\mu A_W = \sum_h v_h w_h n\mu \int_0^1 \frac{(p_h - L_h(p_h))}{p_h} dp_h + \sum_h v_h w_h n\mu \int_0^1 \frac{(p_h - L_h(p_h))}{p_h} (\frac{p_h}{q} - 1) dp_h$$

and by dividing both sides by $n\mu$, we have:

$$\begin{aligned} A_W &= \sum_h v_h w_h \int_0^1 \tfrac{(p_h - L_h(p_h))}{p_h} dp_h + \sum_h v_h w_h \int_0^1 \tfrac{(p_h - L_h(p_h))}{p_h} (\tfrac{p_h}{q} - 1) dp_h \\ &= B_{WITH} + B_{residual A_W}. \end{aligned}$$

■

[17] This is equivalent to integrating for q between 0 and 1

A2. Proof of Proposition 2

We know that $M_i \geq \mu$, for $i = 1, \ldots, n$. This implies that the equi-distribution line is below the $V(p)$, $V_B(p)$ and $C_{DeVe}(q)$ curves.

Similarly to the Bonferroni index, we prove that:

$$I_V = V_{WITH} + V_{BET\_means} + V_{residualA_W} + OV.$$

By construction $A_B = V_{BET\_means}$. Let us compute the AA_W area by isolating the V_{WITH} and $V_{residualA_W}$ terms. Let us consider the subgroup h in which the curve of De Vergottini is defined:

$$V_h(p_h) = \frac{1 - L_h(p_h)}{1 - p_h}$$

where p_h denotes the rank of units in the h group that, for the j-th unit, is $p_h = j/n_h$. The rank unit p_h in the h-th subgroup has rank equals to q in the original distribution, since q is equal to:

$$q = \frac{\sum_{i \leq h-1} n_i + p_h n_h}{n}.$$

and, $dq = (n_h/n)dp_h$.

The concentration curve $C_{DeVe}(q)$ is defined as the ratio between the average of the last $n(1-q)$ units, that is, the units that have more than the income of the unit having rank equal to 1, and the overall average. That is.

$$C_{DeVe}(q) = \left[\frac{1}{\mu} \frac{\sum_{i>h} n_i \mu_i + n_h \mu_h (1 - L_h(p_h))}{n(1-q)}\right]$$

We observe that $L_h(p_h)$ represents the Lorenz curve at p_h percentile that is the the income share of the $h-$th group possessed by individulas with income lower or equal to the income of the individual having rank p_h, whereas $1 - L_h(p_h)$ represents the share of individuals who have a higher income than the p_h individual.

The $V_B(q)$ curve is defined as the ration between the average of the last $n(1-q)$ unit if each unit has the average income of the subgroup and the average of the distribution

$$V_B(q) = \left[\frac{1}{\mu} \frac{\sum_{i>h} n_i\mu_i + n_h\mu_h(1-p_h)}{n(1-q)} \right].$$

Thus we have:

$$n\mu\left[C(q) - V_B(q)\right] = n_h\mu_h \frac{(p_h - L_h(p_h))}{1-q}.$$

Then, multiplying for dq and integrating in the subgroups between 0 and 1 over p_h through the h-th group index, we get:

$$\begin{aligned} n\mu A_W = & \quad \textstyle\sum_h \frac{n_h^2\mu_h}{n} \int_0^1 \frac{(p_h - L_h(p_h))}{1-q} dp_h = \\ = & \quad \textstyle\sum_h \frac{n_h^2\mu_h}{n} \int_0^1 \frac{(p_h - L_h(p_h))(1-p_h)}{(1-p_h)(1-q)} \pm \sum_h \frac{n_h^2\mu_h}{n} \int_0^1 \frac{(p_h - L_h(p_h))}{(1-p_h)} dp_h \end{aligned}$$

Let be $v_h = \frac{n_h}{n}$ and $w_h = \frac{n_h\mu_h}{n\mu}$, we have:

$$\begin{aligned} n\mu A_W = & \sum_h v_h w_h n\mu \int_0^1 \frac{(p_h - L_h(p_h))}{(1-p_h)} dp_h + \\ & + \sum_h v_h w_h n\mu \int_0^1 \frac{(p_h - L_h(p_h))}{(1-p_h)} \left(\frac{1-p_h}{1-q} - 1 \right) dp_h \end{aligned}$$

And, dividing for $n\mu$ we get:

$$\begin{aligned} A_W = & \sum_h v_h w_h \int_0^1 \frac{p_h - L_h(p_h}{1-p_h} dp_h + \sum_h v_h w_h \int_0^1 \frac{(p_h - L_h(p_h))}{(1-p_h)} \left(\frac{1-p_h}{1-q} - 1 \right) dp_h \\ = & V_{WITH} + V_{residual A_W}. \end{aligned}$$

■

A3. Proof of Proposition 3

Let's verify if the polarization increases as a result of the IB transfer.

To this end we study:

$$\Delta P_B = P_B^{t_1} - P_B^{t_0} = \frac{2\mu(x)}{m(x)} \left[\Delta B_{BET\_means} - (\Delta B_{WITH} + \Delta B_{residualA_W})\right]$$

where

$$\begin{aligned}&\Delta B_{BET\_means} = \\ &= \int_0^1 \frac{1}{q}\left(q - L_B^{t_1}(q)\right) dq - \int_0^1 \frac{1}{q}\left(q - L_B^{t_0}(q)\right) dq = \int_0^1 \frac{1}{q}\left(L_B^{t_0}(q) - L_B^{t_1}(q)\right) dq\end{aligned}$$

and t_1 denotes the period after the transfer of income and t_0 the *status quo,* and

$$\begin{aligned}\Delta\left(B_{WITH} + B_{residualA_W}\right) &= \\ = \int_0^1 \frac{1}{q}\left(L_B^{t_1}(q) - L^{t_1}(q)\right) dq - \int_0^1 \frac{1}{q}\left(L_B^{t_0}(q) - L^{t_0}(q)\right) dq &= \\ = \int_0^1 \frac{1}{q}\left(L_B^{t_1}(q) - L_B^{t_0}(q)\right) dq - \int_0^1 \frac{1}{q}\left(L^{t_1}(q) - L^{t_0}(q)\right) dq& \\ = -\Delta B_{BET\_means} - \int_0^1 \frac{1}{q}\left(L^{t_1}(q) - L^{t_0}(q)\right) dq.&\end{aligned}$$

Thus we get:

$$\Delta P_B = \frac{2\mu(x)}{m(x)}\left[2\Delta B_{BET\_means} + \int_0^1 \frac{1}{q}\left(L^{t_1}(q) - L^{t_0}(q)\right) dq\right].$$

Under IB, a α progressive transfer on one side of the median (or on both segments but without crossing the median) has the effect of $\Delta B_{BET\_means} = 0$.

Transfers that do not cross the median are such that both average incomes on each segment and the general average $\mu(x)$ remain unchanged. Thus, $L_B^{t_1}(q) = L_B^{t_0}(q)\ \forall q \in [0,1]$.

As for the *within groups* inequality component, including the residual, the progressive income transfers on one side of the median have the effect of $\Delta(B_{WITH} + B_{residualA_W})\ le0$. In fact, remembering that $\Delta(B_{WITH}+B_{residualA_W}) = -\Delta B_{BET\_means} - \int_0^1 \frac{1}{q}\left(L^{t_1}(q) - L^{t_0}(q)\right) dq$, a progressive transfer to the $x-$ segment of the income distribution has the effect of reducing inequality and, consequently:

$$\int_0^{1/2} \frac{1}{q} \left(L^{t_1}(q) - L^{t_0}(q)\right) dq > 0.$$

Similarly, a progressive transfer on $x+$ is such that $\int_{1/2}^{1} \frac{1}{q} \left(L^{t_1}(q) - L^{t_0}(q)\right) dq > 0$.

Consequently, under IB, we have $\Delta P_B > 0$.

Now, we compute the effect of an IS transfer considering a regressive income transfer of amount equal to α from the individual having rank s in the $x-$ population subgroup to the individual rankedd at k in the $x+$ subgroup.

For each pair (s,k) $s,k \in [1, n/2]$, as a result of the transfer, the average income below the median decreases of $\alpha/(n/2)$, whereas the average income above the median increases by the same amount.

Thus, we have:

$$\begin{aligned}
\Delta B_{BET_means} &= B^{t_1}_{BET_means} - B^{t_0}_{BET_means} = \\
&= \frac{1}{2\mu(x)} \sum_{j=1}^{n/2} \left(\frac{1}{n/2+j}\right) \left(\mu^{t_1}(x+) - \mu^{t_1}(x-)\right) + \\
&\quad - \frac{1}{2\mu(x)} \sum_{j=1}^{n/2} \left(\frac{1}{n/2+j}\right) \left(\mu^{t_0}(x+) - \mu^{t_0}(x-)\right) = \\
&= \frac{1}{2\mu(x)} \sum_{j=1}^{n/2} \left(\frac{1}{n/2+j}\right) \left(\mu^{t_1}(x+) - \mu^{t_1}(x-) - \mu^{t_0}(x+) + \mu^{t_0}(x-)\right) = \\
&= \frac{1}{2\mu(x)} \sum_{j=1}^{n/2} \left(\frac{1}{n/2+j}\right) \left[\left(\mu^{t_1}(x+) - \mu^{t_0}(x+)\right) - \left(\mu^{t_1}(x-) - \mu^{t_0}(x-)\right)\right] = \\
&= \frac{1}{2\mu(x)} \sum_{j=1}^{n/2} \left(\frac{1}{n/2+j}\right) \left(\frac{\alpha}{n/2} + \frac{\alpha}{n/2}\right) = \\
&= \frac{1}{2\mu(x)} \sum_{j=1}^{n/2} \frac{2\alpha}{n/2} \left(\frac{1}{n/2+j}\right) = \\
&= \frac{2\alpha}{n\mu(x)} \sum_{j=1}^{n/2} \left(\frac{1}{n/2+j}\right) > 0 \quad \forall s,k \in [1, n/2].
\end{aligned} \tag{16}$$

For the *within groups* component, we have:

$$\Delta B_{WITH} = \frac{1}{2\mu(x)}\frac{\alpha}{n}\left[\sum_{j=s}^{n/2}\frac{1}{j} - \sum_{j=k}^{n/2}\frac{1}{j}\right] \leq 0, \quad for\ s \geq k \tag{17}$$

where s and k denote the position of the $s-$th and $k-$th individual on $x-$ and $x+$, respectively.

Indeed

$$\Delta B_{WITH} = B_{WITH}^{t_1} - B_{WITH}^{t_0} =$$

$$= \left[\frac{1}{4}\frac{\mu^{t_1}(x-)}{\mu(x)} I_{B,t_1}^{n/2}(x-) - \frac{1}{4}\frac{\mu^{t_0}(x-)}{\mu(x)} I_{B,t_0}^{n/2}(x-)\right] +$$

$$\left[\frac{1}{4}\frac{\mu^{t_1}(x+)}{\mu(x)} I_{B,t_1}^{n/2}(x+) - \frac{1}{4}\frac{\mu^{t_0}(x+)}{\mu(x)} I_{B,t_0}^{n/2}(x+)\right]$$

$$= \frac{1}{4\mu(x)}\left[\mu^{t_1}(x-)\left(1 - \frac{1}{n/2\mu^{t_1}(x-)}\sum_{j=1}^{n/2}\mu_j^{t_1}(x-)\right) + \right.$$

$$\left. -\mu^{t_0}(x-)\left(1 - \frac{1}{n/2\mu^{t_0}(x-)}\sum_{j=1}^{n/2}\mu_j^{t_0}(x-)\right)\right] +$$

$$+\frac{1}{4\mu(x)}\left[\mu^{t_1}(x+)\left(1 - \frac{1}{n/2\mu^{t_1}(x+)}\sum_{j=1}^{n/2}\mu_j^{t_1}(x+)\right) + \right.$$

$$\left. -\mu^{t_0}(x+)\left(1 - \frac{1}{n/2\mu^{t_0}(x+)}\sum_{j=1}^{n/2}\mu_j^{t_0}(x+)\right)\right] =$$

$$= \frac{1}{2n\mu(x)}\left[\frac{n}{2}\left(\mu^{t_1}(x-) - \mu^{t_0}(x-)\right) - \sum_{j=1}^{n/2}\left(\mu_j^{t_1}(x-) - \mu_j^{t_0}(x-)\right)\right] +$$

$$+\frac{1}{2n\mu(x)}\left[\frac{n}{2}\left(\mu^{t_1}(x+) - \mu^{t_0}(x+)\right) - \sum_{j=1}^{n/2}\left(\mu_j^{t_1}(x+) - \mu_j^{t_0}(x+)\right)\right] =$$

$$= \frac{1}{2n\mu(x)}\left[\frac{n}{2}\left(-\frac{\alpha}{n/2}\right) + \alpha\sum_{j=s}^{n/2}\frac{1}{j}\right] + \frac{1}{2n\mu(x)}\left[\frac{n}{2}\left(\frac{\alpha}{n/2}\right) - \alpha\sum_{j=k}^{n/2}\frac{1}{j}\right] =$$

$$= \frac{\alpha}{2n\mu(x)} \left[\sum_{j=s}^{n/2} \frac{1}{j} - \sum_{j=k}^{n/2} \frac{1}{j} \right].$$

Equation (17) shows that the sign of ΔB_{WITH} depends on the relative positions of the income earners. In particular, the variation of the *within* component is not positive, that is it has the desired effect on the polarization index, only if $s \geq k$ with $s \geq 1$ and "$k \leq n/2$.

Note 1 *Note that, as shown in the previous paragraph, the variation of the* within *component depends on s and k also in the case of the Gini index. However, by construction, the Foster and Wolfson's index is consistent with the second polarization curve. Therefore, necessarily, the* between *component compensates for that* within *for each pair s, k.*

Now we prove that this is true also for the P_B index, by finding the condition on s and k for which ΔB_{BET_means} is greater than $\Delta B_{WITH} + \Delta B_{residualAw}$[18].

Since

$$B_{residualA_W} = \frac{1}{2n\mu(x)} \left[\frac{n}{2}\mu(x-) - \sum_{i=1}^{n/2} \mu_i(x-) \right] +$$

$$\frac{1}{2n\mu(x)} \left[\mu(x+) \sum_{i=1}^{n/2} \left(\frac{i - n/2}{i + n/2} \right) - \sum_{i=1}^{n/2} \left(\frac{i - n/}{i + n/2} \right) \mu_i(x+) \right]$$

we have:

$$\Delta B_{residualAw} = \left(B^{t_1}_{residualAw} - B^{t_0}_{residualAw} \right) =$$

[18]The residual variation could introduce a sort of correction. Recall, in fact, that in the residue the weights for which we multiply the observations above the median are negative and increasing and, consequently, the correction made by the residue is all the stronger as the observation is close to the median.

$$= \frac{1}{2n\mu(x)}\left[\frac{n}{2}\mu^{t_1}(x-) - \sum_{i=1}^{n/2}\mu_i^{t_1}(x-) - \frac{n}{2}\mu^{t_0}(x-) + \sum_{i=1}^{n/2}\mu_i^{t_0}(x-)\right] +$$

$$+ \frac{1}{2n\mu(x)}\left[\mu^{t_1}(x+)\sum_{i=1}^{n/2}\left(\frac{i-n/2}{i+n/2}\right) - \sum_{i=1}^{n/2}\left(\frac{i-n/2}{i+n/2}\right)\mu_i^{t_1}(x+) - \mu^{t_0}(x+)\sum_{i=1}^{n/2}\left(\frac{i-n/2}{i+n/2}\right) + \right.$$

$$\left. + \sum_{i=1}^{n/2}\left(\frac{i-n/2}{i+n/2}\right)\mu_i^{t_0}(x+)\right] =$$

$$= \frac{1}{2n\mu(x)}\left[\frac{n}{2}\left(\mu^{t_1}(x-) - \mu^{t_0}(x-)\right) - \sum_{i=1}^{n/2}\left(\mu_i^{t_1}(x-) - \mu_i^{t_0}(x-)\right)\right] +$$

$$+ \frac{1}{2n\mu(x)}\left[\left(\mu^{t_1}(x+) - \mu^{t_0}(x+)\right)\sum_{i=1}^{n/2}\left(\frac{i-n/2}{i+n/2}\right) - \sum_{i=1}^{n/2}\left(\frac{i-n/2}{i+n/2}\right)\left(\mu_i^{t_1}(x+) - \mu_i^{t_0}(x+)\right)\right] =$$

$$= \frac{1}{2n\mu(x)}\left[\frac{n}{2}\left(\frac{-\alpha}{n/2}\right) + \alpha\sum_{j=s}^{n/2}\frac{1}{j}\right] + \frac{1}{2n\mu(x)}\left[\frac{\alpha}{n/2}\sum_{j=1}^{n/2}\left(\frac{j-n/2}{j+n/2}\right) - \alpha\sum_{j=k}^{n/2}\frac{1}{j}\left(\frac{j-n/2}{j+n/2}\right)\right] =$$

$$= \frac{1}{2n\mu(x)}\left[\frac{\alpha}{n/2}\sum_{j=1}^{n/2}\left(\frac{j-n/2}{j+n/2} - 1\right) + \alpha\sum_{j=s}^{n/2}\frac{1}{j} - \alpha\sum_{j=k}^{n/2}\frac{1}{j}\left(\frac{j-n/2}{j+n/2}\right)\right] =$$

$$= \frac{\alpha}{2n\mu(x)}\left[\sum_{j=s}^{n/2}\frac{1}{j} - \sum_{j=k}^{n/2}\frac{1}{j}\left(\frac{j-n/2}{j+n/2}\right) + \frac{1}{n/2}\sum_{j=1}^{n/2}\left(\frac{j-n/2}{j+n/2} - 1\right)\right] =$$

$$= \frac{\alpha}{2n\mu(x)}\left[\sum_{j=s}^{n/2}\frac{1}{j} + \sum_{j=k}^{n/2}\frac{1}{j}\left(\frac{n/2-j}{n/2+j}\right) - 2\sum_{j=1}^{n/2}\left(\frac{1}{n/2+j}\right)\right]$$

Now we sum the B_{WITH} and $B_{residualA_W}$ components:

$$\begin{aligned}
\Delta B_{WITH} + \Delta B_{residualAw} &= \tfrac{\alpha}{2n\mu(x)}\left[\textstyle\sum_{j=s}^{n/2}\tfrac{1}{j} - \sum_{j=k}^{n/2}\tfrac{1}{j}\right] + \tfrac{\alpha}{2n\mu(x)}\left[\textstyle\sum_{j=s}^{n/2}\tfrac{1}{j} + \right. \\
&\quad \left. + \textstyle\sum_{j=k}^{n/2}\tfrac{1}{j}\left(\tfrac{n/2-j}{n/2+j}\right) - 2\sum_{j=1}^{n/2}\left(\tfrac{1}{n/2+j}\right)\right] = \\
&= \tfrac{\alpha}{2n\mu(x)}\left[2\textstyle\sum_{j=s}^{n/2}\tfrac{1}{j} - \sum_{j=k}^{n/2}\tfrac{1}{j}\left(1 - \left(\tfrac{n/2-j}{n/2+j}\right)\right) - 2\sum_{j=1}^{n/2}\left(\tfrac{1}{n/2+j}\right)\right] = \\
&= \tfrac{\alpha}{2n\mu(x)}\left[2\textstyle\sum_{j=s}^{n/2}\tfrac{1}{j} - 2\sum_{j=k}^{n/2}\left(\tfrac{1}{n/2+j}\right) - 2\sum_{j=1}^{n/2}\left(\tfrac{1}{n/2+j}\right)\right] = \\
&= \tfrac{\alpha}{n\mu(x)}\left[\textstyle\sum_{j=s}^{n/2}\tfrac{1}{j} - \sum_{j=k}^{n/2}\left(\tfrac{1}{n/2+j}\right) - \sum_{j=1}^{n/2}\left(\tfrac{1}{n/2+j}\right)\right].
\end{aligned}$$

Consequently, we have:

$$\begin{aligned}
\Delta P_B &= \tfrac{2\mu(x)}{m(x)}\left[\Delta B_{BET_means} - (\Delta B_{WITH} + \Delta B_{residualAw})\right] = \\
&= \tfrac{2\mu(x)}{m(x)}\tfrac{2\alpha}{n\mu(x)}\textstyle\sum_{j=1}^{n/2}\left(\tfrac{1}{n/2+j}\right) - \tfrac{2\mu(x)}{m(x)}\tfrac{\alpha}{n\mu(x)}\left[\sum_{j=s}^{n/2}\tfrac{1}{j} - \sum_{j=1}^{n/2}\left(\tfrac{1}{n/2+j}\right) - \sum_{j=k}^{n/2}\left(\tfrac{1}{n/2+j}\right)\right] = \\
&= \tfrac{2\alpha}{nm(x)}\left[\textstyle\sum_{j=1}^{n/2}\left(\tfrac{3}{n/2+j}\right)\right] + \tfrac{2\alpha}{nm(x)}\left[\sum_{j=k}^{n/2}\left(\tfrac{1}{n/2+j}\right) - \sum_{j=s}^{n/2}\tfrac{1}{j}\right] = \\
&= \tfrac{2\alpha}{nm(x)}\left[\textstyle\sum_{j=1}^{n/2}\left(\tfrac{3}{n/2+j}\right) - \sum_{j=s}^{n/2}\tfrac{1}{j} + \sum_{j=k}^{n/2}\left(\tfrac{1}{n/2+j}\right)\right].
\end{aligned}$$

We note there exists a value for s such that:

$$\sum_{j=s*}^{n/2}\frac{1}{j} > \sum_{j=1}^{n/2}\left(\frac{3}{n/2+j}\right) + \sum_{j=k}^{n/2}\left(\frac{1}{n/2+j}\right) \quad and \quad \Delta P_B < 0$$

However, the analytic detection of this threshold is quite complicated, since it is the solution of the equation to the differences $\Delta P_B[s,k,n] > 0$, as (s,k,n) vary. By restricting to the case where $s \geq k$ we can obtain a sufficient condition on s for $\Delta P_B > 0$ under IS.

Thus we have:

$$\begin{aligned}
\Delta P_B &= \tfrac{2\alpha}{nm(x)}\left[\textstyle\sum_{j=1}^{n/2}\left(\tfrac{3}{n/2+j}\right) - \sum_{j=s}^{n/2}\tfrac{1}{j} + \sum_{j=k}^{n/2}\left(\tfrac{1}{n/2+j}\right)\right] \geq \\
&\geq \tfrac{2\alpha}{nm(x)}\left[\textstyle\sum_{j=k}^{n/2}\left(\tfrac{3}{n/2+j}\right) - \sum_{j=s}^{n/2}\tfrac{1}{j} + \sum_{j=k}^{n/2}\left(\tfrac{1}{n/2+j}\right)\right] = \\
&= \tfrac{2\alpha}{nm(x)}\left[\textstyle\sum_{j=k}^{n/2}\left(\tfrac{4}{n/2+j}\right) - \sum_{j=s}^{n/2}\tfrac{1}{j}\right] = \\
&= \tfrac{2\alpha}{nm(x)}\left[\textstyle\sum_{j=k}^{s-1}\left(\tfrac{4}{n/2+j}\right) + \sum_{j=s}^{n/2}\left(\tfrac{4}{n/2+j}\right) - \sum_{j=s}^{n/2}\tfrac{1}{j}\right] = \\
&= \tfrac{2\alpha}{nm(x)}\left[\textstyle\sum_{j=k}^{s-1}\left(\tfrac{4}{n/2+j}\right) + \sum_{j=s}^{n/2}\left(\tfrac{4}{n/2+j} - \tfrac{1}{j}\right)\right] = \\
&= \tfrac{2\alpha}{nm(x)}\left[\textstyle\sum_{j=k}^{s-1}\left(\tfrac{4}{n/2+j}\right) + \sum_{j=s}^{n/2}\left(\tfrac{3j-n/2}{j(n/2+j)}\right)\right]
\end{aligned}$$

That is non-negative for $j \in [s, n/2]$ such that $3j - n/2 \geq 0$. That is, for $3s - n/2 \geq 0$. Thus we get $s \geq \min(k, n/6)$.

■

A4. Proof of Proposition 4

Let's verify if P_V increases after the IB transfer.

We have

$$\Delta P_V = P_V^{t_1} - P_V^{t_0} = \frac{2\mu(x)}{m(x)} \left[\Delta V_{BET\_means} - (\Delta V_{WITH} + \Delta V_{residualA_W})\right].$$

Now, the expression of the ΔV_{BET_means} is:

$$\begin{aligned} \Delta V_{BET\_means} &= \int_0^1 \tfrac{1}{1-q}\left(q - L_B^{t_1}(q)\right) dq - \int_0^1 \tfrac{1}{1-q}\left(q - L_B^{t_0}(q)\right) dq = \\ &= \int_0^1 \tfrac{1}{1-q}\left(L_B^{t_0}(q) - L_B^{t_1}(q)\right) dq \end{aligned}$$

where t_1 denotes the period after the transfer of income and t_0 the *status quo,* and

$$\Delta\left(V_{WITH} + V_{residualA_W}\right) =$$

$$\begin{aligned} &= \int_0^1 \tfrac{1}{1-q}\left(L_B^{t_1}(q) - L^{t_1}(q)\right) dq - \int_0^1 \tfrac{1}{1-q}\left(L_B^{t_0}(q) - L^{t_0}(q)\right) dq = \\ &= \int_0^1 \tfrac{1}{1-q}\left(L_B^{t_1}(q) - L_B^{t_0}(q)\right) dq - \int_0^1 \tfrac{1}{1-q}\left(L^{t_1}(q) - L^{t_0}(q)\right) dq = \\ &= \Delta V_{BET\_means} - \int_0^1 \tfrac{1}{1-q}\left(L^{t_1}(q) - L^{t_0}(q)\right) dq. \end{aligned}$$

Thus we have:

$$\Delta P_V = \frac{2\mu(x)}{m(x)} \left[2\Delta V_{BET\_means} + \int_0^1 \frac{1}{1-q}\left(L^{t_1}(q) - L^{t_0}(q)\right) dq\right].$$

Now, let us consider a progressive income transfer[19] α that does not cross the median, fom the $k-$th individual to the $s-$th one, with $s < k$.

As consequence of an IB transfer, $\Delta V_{BET\_means} = 0$ and $\Delta(V_{WITH} + V_{residualA_W}) \leq 0$. Consequently, $\Delta P_V > 0$.

To analyze the effect of an IS transfer, let us consider a regressive income transfer α from the $s - th$ individual to the $k-$th one, with $s < k$).

In the case of the transfer, the average income below the median decreases by $alpha/(n/2)$, while the average income above the median increases of the same amount.

[19]The transfer can take place in only one of the subgroups $x-$ and $x+$, or both.

Consequently:

$$\Delta V_{BET\_means} = V^{t_1}_{BET\_means} - V^{t_0}_{BET\_means}$$

$$\begin{aligned}
&= \tfrac{1}{2\mu(x)} \textstyle\sum_{j=1}^{n/2} \left(\tfrac{1}{n-j+1}\right) \left(\mu^{t_1}(x+) - \mu^{t_1}(x-)\right) - \tfrac{1}{2\mu(x)} \textstyle\sum_{j=1}^{n/2} \left(\tfrac{1}{n-j+1}\right) \left(\mu^{t_0}(x+) - \mu^{t_0}(x-)\right) \\
&= \tfrac{1}{2\mu(x)} \textstyle\sum_{j=1}^{n/2} \left(\tfrac{1}{n-j+1}\right) \left(\mu^{t_1}(x+) - \mu^{t_1}(x-) - \mu^{t_0}(x+) + \mu^{t_0}(x-)\right) \\
&= \tfrac{1}{2\mu(x)} \textstyle\sum_{j=1}^{n/2} \left(\tfrac{1}{n-j+1}\right) \left[\left(\mu^{t_1}(x+) - \mu^{t_0}(x+)\right) - \left(\mu^{t_1}(x-) - \mu^{t_0}(x-)\right)\right] \\
&= \tfrac{1}{2\mu(x)} \textstyle\sum_{j=1}^{n/2} \left(\tfrac{1}{n-j+1}\right) \left(\tfrac{\alpha}{n/2} + \tfrac{\alpha}{n/2}\right) \\
&= \tfrac{1}{2\mu(x)} \textstyle\sum_{j=1}^{n/2} \tfrac{2\alpha}{n/2} \left(\tfrac{1}{n-j+1}\right) \\
&= \tfrac{2\alpha}{n\mu(x)} \textstyle\sum_{j=1}^{n/2} \left(\tfrac{1}{n-j+1}\right) > 0.
\end{aligned}$$

Note 2 *We observe that, for a given n,*

$$\Delta B_{BET\_means} = \Delta V_{BET\_means}$$

since

$$\sum_{j=1}^{n/2} \left(\frac{2}{n+2j}\right) = \sum_{j=1}^{n/2} \left(\frac{1}{n-j+1}\right).$$

In addition, since the two additions are less than $1/2$, we have

$$\Delta B_{BET\_means} < \Delta G_{BET\_means} \quad \textit{and} \quad \Delta V_{BET\_means} < \Delta G_{BET\_means}.$$

Recalling equation (10), the variation of the *within* term is equal to

$$\Delta V_{WITH} = V^{t_1}_{WITH} - V^{t_0}_{WITH} =$$

$$\begin{aligned}
&= \tfrac{1}{2n\mu(x)} \left[\textstyle\sum_{j=1}^{n/2} \left(M_j^{t_1}(x-) - M_j^{t_0}(x-)\right) - \tfrac{n}{2}\left(\mu^{t_1}(x-) - \mu^{t_0}(x-)\right)\right] + \\
&\quad + \tfrac{1}{2n\mu(x)} \left[\textstyle\sum_{j=1}^{n/2} \left(M_j^{t_1}(x+) - M_j^{t_0}(x+)\right) - \tfrac{n}{2}\left(\mu^{t_1}(x+) - \mu^{t_0}(x+)\right)\right] \\
&= \tfrac{1}{2n\mu(x)} \left[-\alpha \textstyle\sum_{j=1}^{s} \left(\tfrac{1}{n/2-j+1}\right) - \tfrac{n}{2}\left(-\tfrac{\alpha}{n/2}\right)\right] + \tfrac{1}{2n\mu(x)} \left[\alpha \textstyle\sum_{j=1}^{k} \left(\tfrac{1}{n/2-j+1}\right) - \tfrac{n}{2}\left(\tfrac{\alpha}{n/2}\right)\right] = \\
&= \tfrac{\alpha}{2n\mu(x)} \left[\textstyle\sum_{j=1}^{k} \left(\tfrac{1}{n/2-j+1}\right) - \textstyle\sum_{j=1}^{s} \left(\tfrac{1}{n/2-j+1}\right)\right].
\end{aligned}$$

Consequently,

$$\Delta V_{WITH} = \frac{\alpha}{2n\mu(x)}\left[\sum_{j=1}^{k}\left(\frac{1}{n/2-j+1}\right) - \sum_{j=1}^{s}\left(\frac{1}{n/2-j+1}\right)\right] \leq 0, \quad for\, s \geq k \tag{18}$$

where s denotes the position of the $s-$th individual in the $x-$ distribution and k the position of the $k-$th individual in the $x+$ distribution.

Equation ((18)) shows that, as already noted for the polarization indexes based on the concentration indices of Gini and Bonferroni, the sign of ΔV_{WITH} depends on the relative positions of the income earners. In particular, the variation of the *within* component is non-positive, that is it has the desired effect on the polarization index, only in case $s \geq k$, with $s \geq 1$ and $k \leq n/2$.

Similarly to the Bonferroni concentration index case, we study the condition on s and k for which variation ΔV_{BET_means} is greater of $\Delta V_{WITH} + \Delta V_{residualAw}$.

We remember that

$$V_{residualAw} = \frac{1}{2n\mu(x)}\left[\sum_{i=1}^{n/2}\left(\frac{i}{i-n}\right)M_i(x-) - \sum_{i=1}^{n/2}\left(\frac{i}{i-n}\right)\mu(x-)\right] +$$

$$+\frac{1}{2n\mu(x)}\left[\sum_{i=1}^{n/2}M_i(x+) - \frac{n}{2}\mu(x+)\right]$$

end, consequently:

$$\Delta V_{residualAw} = V_{residualAw}^{t_1} - V_{residualAw}^{t_0} =$$

$$
\begin{aligned}
&= \tfrac{1}{2n\mu(x)}\left[\sum_{i=1}^{n/2}\left(\tfrac{i}{i-n}\right)M_i^{t_1}(x-) - \sum_{i=1}^{n/2}\left(\tfrac{i}{i-n}\right)\mu^{t_1}(x-)\right] + \\
&\quad + \tfrac{1}{2n\mu(x)}\left[\sum_{i=1}^{n/2} M_i^{t_1}(x+) - \tfrac{n}{2}\mu^{t_1}(x+)\right] - \tfrac{1}{2n\mu(x)}\left[\sum_{i=1}^{n/2}\left(\tfrac{i}{i-n}\right)M_i^{t_0}(x-) + \right. \\
&\quad \left. - \sum_{i=1}^{n/2}\left(\tfrac{i}{i-n}\right)\mu^{t_0}(x-)\right] - \tfrac{1}{2n\mu(x)}\left[\sum_{i=1}^{n/2} M_i^{t_0}(x+) - \tfrac{n}{2}\mu^{t_0}(x+)\right] = \\
&= \tfrac{1}{2n\mu(x)}\left[\sum_{i=1}^{n/2}\left(\tfrac{i}{i-n}\right)\left(M_i^{t_1}(x-) - M_i^{t_0}(x-)\right) - \sum_{i=1}^{n/2}\left(\tfrac{i}{i-n}\right)\left(\mu^{t_1}(x-) - \mu^{t_0}(x-)\right)\right] + \\
&\quad + \tfrac{1}{2n\mu(x)}\left[\sum_{i=1}^{n/2}(M_i^{t_1}(x+) - M_i^{t_0}(x+)) - \sum_{i=1}^{n/2}(\mu^{t_1}(x+) - \mu^{t_0}(x+))\right] = \\
&= \tfrac{1}{2n\mu(x)}\left[-\alpha\sum_{j=1}^{s}\left(\tfrac{j}{j-n}\right)\left(\tfrac{1}{n/2-j+1}\right) + \tfrac{\alpha}{n/2}\sum_{j=1}^{n/2}\left(\tfrac{j}{j-n}\right)\right] + \\
&\quad + \tfrac{1}{2n\mu(x)}\left[\alpha\sum_{j=1}^{k}\left(\tfrac{1}{n/2-j+1}\right) - \alpha\right] = \\
&= \tfrac{\alpha}{2n\mu(x)}\left[-\sum_{j=1}^{s}\left(\tfrac{j}{j-n}\right)\left(\tfrac{1}{n/2-j+1}\right) + \sum_{j=1}^{k}\left(\tfrac{1}{n/2-j+1}\right) + \tfrac{1}{n/2}\sum_{j=1}^{n/2}\left(\tfrac{j}{j-n}\right) - 1\right] = \\
&= \tfrac{\alpha}{2n\mu(x)}\left[\sum_{j=1}^{s}\left(\tfrac{j}{n-j}\right)\left(\tfrac{1}{n/2-j+1}\right) + \sum_{j=1}^{k}\left(\tfrac{1}{n/2-j+1}\right) - 2\sum_{j=1}^{n/2}\left(\tfrac{1}{n-j}\right)\right].
\end{aligned}
$$

Thus we have:

$$\Delta V_{WITH} + \Delta V_{residualAw}$$

$$
\begin{aligned}
&= \tfrac{\alpha}{2n\mu(x)}\left[2\sum_{j=1}^{k}\left(\tfrac{1}{n/2-j+1}\right) - \sum_{j=1}^{s}\left(\tfrac{1}{n/2-j+1}\right) + \right. \\
&\quad \left. + \sum_{j=1}^{s}\left(\tfrac{j}{n-j}\right)\left(\tfrac{1}{n/2-j+1}\right) - 2\sum_{j=1}^{n/2}\left(\tfrac{1}{n-j}\right)\right] = \\
&= \tfrac{\alpha}{2n\mu(x)}\left[2\sum_{j=1}^{k}\left(\tfrac{1}{n/2-j+1}\right) + \sum_{j=1}^{s}\left(\tfrac{j}{n-j} - 1\right)\left(\tfrac{1}{n/2-j+1}\right) - 2\sum_{j=1}^{n/2}\left(\tfrac{1}{n-j}\right)\right] = \\
&= \tfrac{\alpha}{2n\mu(x)}\left[\sum_{j=1}^{k}\left(\tfrac{2}{n/2-j+1}\right) - \sum_{j=1}^{s}\left(\tfrac{2j-n}{j-n}\right)\left(\tfrac{1}{n/2-j+1}\right) - 2\sum_{j=1}^{n/2}\left(\tfrac{1}{n-j}\right)\right] = \\
&= \tfrac{\alpha}{2n\mu(x)}\left[\sum_{j=1}^{k}\left(\tfrac{2}{n/2-j+1}\right) - \sum_{j=1}^{s}\left(\tfrac{2j-n}{j-n}\right)\left(\tfrac{1}{n/2-j+1}\right) + 2\sum_{j=1}^{n/2}\left(\tfrac{1}{j-n}\right)\right]
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\Delta V_{BET\_means} - (\Delta V_{WITH} + \Delta V_{residualAw}) \\
&= \tfrac{2\alpha}{n\mu(x)}\sum_{j=1}^{n/2}\left(\tfrac{1}{n-j+1}\right) - \tfrac{\alpha}{2n\mu(x)}\left[\sum_{j=1}^{n/2}\left(\tfrac{2}{j-n}\right) + \sum_{j=1}^{k}\left(\tfrac{2}{n/2-j+1}\right) - \sum_{j=1}^{s}\left(\tfrac{2j-n}{j-n}\right)\left(\tfrac{1}{n/2-j+1}\right)\right] \\
&= \tfrac{2\alpha}{2n\mu(x)}\sum_{j=1}^{n/2}\left(\tfrac{2}{n-j+1}\right) - \tfrac{2\alpha}{2n\mu(x)}\left[\sum_{j=1}^{n/2}\left(\tfrac{1}{j-n}\right) + \sum_{j=1}^{k}\left(\tfrac{1}{n/2-j+1}\right) - \sum_{j=1}^{s}\left(\tfrac{j-n/2}{j-n}\right)\left(\tfrac{1}{n/2-j+1}\right)\right] \\
&= \tfrac{\alpha}{n\mu(x)}\left[\sum_{j=1}^{n/2}\left(\tfrac{2}{n-j+1}\right) - \sum_{j=1}^{n/2}\left(\tfrac{1}{j-n}\right) - \sum_{j=1}^{k}\left(\tfrac{1}{n/2-j+1}\right) + \sum_{j=1}^{s}\left(\tfrac{j-n/2}{j-n}\right)\left(\tfrac{1}{n/2-j+1}\right)\right]
\end{aligned}
$$

and

$$\Delta P_V =$$

$$
\begin{aligned}
&= \tfrac{2\mu(x)}{m(x)}\left[\Delta V_{BET\_means} - (\Delta V_{WITH} + \Delta V_{residualAw})\right] = \\
&= \tfrac{2\alpha}{nm(x)}\left[\sum_{j=1}^{n/2}\left(\tfrac{2}{n-j+1} + \tfrac{1}{n-j}\right) - \sum_{j=1}^{k}\left(\tfrac{1}{n/2-j+1}\right) + \sum_{j=1}^{s}\left(\tfrac{j-n/2}{j-n}\right)\left(\tfrac{1}{n/2-j+1}\right)\right].
\end{aligned}
$$

Similarly to ΔP_B, we note that the ΔP_V variation is made up of the difference between the sum of the constant term and the sum that depends on s and the sum

k.. The latter one assumes a decreasing value as k increases and therefore, there will be a value of k such that

$$\sum_{j=1}^{k}\left(\frac{1}{n/2-j+1}\right) > \sum_{j=1}^{s}\left(\frac{j-n/2}{j-n}\right)\left(\frac{1}{n/2-j+1}\right) + \sum_{j=1}^{n/2}\left(\frac{2}{n-j+1}+\frac{1}{n-j}\right)$$

and

$$\Delta P_V > 0.$$

Again, the analytic detection of the threshold, that is the solution of the equation to the differences $\Delta P_V[s,k,n] > 0$ as (s,k,n) vary, is rather complicated. However, if $s \geq k$, we can get a sufficient condition on k such that $\Delta P_V > 0$ under IS.

Thus, we have:

$$\begin{aligned}
\Delta P_V &= \tfrac{2\alpha}{nm(x)}\left[\sum_{j=1}^{n/2}\left(\tfrac{2}{n-j+1}+\tfrac{1}{n-j}\right)-\sum_{j=1}^{k}\left(\tfrac{1}{n/2-j+1}\right)+\sum_{j=1}^{s}\left(\tfrac{j-n/2}{j-n}\right)\left(\tfrac{1}{n/2-j+1}\right)\right] \geq \\
&\geq \tfrac{2\alpha}{nm(x)}\left[\sum_{j=1}^{k}\left(\tfrac{2}{n-j+1}+\tfrac{1}{n-j}\right)-\sum_{j=1}^{k}\left(\tfrac{1}{n/2-j+1}\right)+\sum_{j=1}^{k}\left(\tfrac{j-n/2}{j-n}\right)\left(\tfrac{1}{n/2-j+1}\right)\right] = \\
&= \tfrac{2\alpha}{nm(x)}\left[\sum_{j=1}^{k}\left(\tfrac{2}{n-j+1}+\tfrac{1}{n-j}+\tfrac{1}{(j-n)}\tfrac{(j-n/2)}{(n/2-j+1)}-\tfrac{1}{n/2-j+1}\right)\right] = (*)
\end{aligned}$$

where $k \leq s \leq n/2$. In addition, since

$$\frac{1}{(j-n)}\frac{(j-n/2)}{(n/2-j+1)} = \frac{1}{(n-j)}\frac{(j-n/2)}{(j-n/2-1)} > \frac{1}{(n-j)}$$

$$\begin{aligned}
(*) &\geq \tfrac{2\alpha}{nm(x)}\left[\sum_{j=1}^{k}\left(\tfrac{2}{(n-j+1)}+\tfrac{1}{(n-j)}-\tfrac{1}{(n/2-j+1)}+\tfrac{1}{(n-j)}\right)\right] = \\
&=< \tfrac{2\alpha}{nm(x)}\left[\sum_{j=1}^{k}\left(\tfrac{2}{(n-j+1)}+\tfrac{2}{(n-j)}-\tfrac{1}{(n/2-j+1)}\right)\right] \geq \\
&\geq \tfrac{2\alpha}{n\cdot m(x)}\left[\sum_{j=1}^{k}\left(\tfrac{2}{(n-j+1)}+\tfrac{2}{(n-j+1)}-\tfrac{1}{(n/2-j+1)}\right)\right] = \\
&= \tfrac{2\alpha}{n\cdot m(x)}\left[\sum_{j=1}^{k}\left(\tfrac{4}{(n-j+1)}-\tfrac{1}{(n/2-j+1)}\right)\right].
\end{aligned}$$

The quantity $\left[\sum_{j=1}^{k}\left(\frac{4}{(n-j+1)}-\frac{1}{(n/2-j+1)}\right)\right]$ is positive for j such that

$$\frac{4}{(n-j+1)}-\frac{1}{(n/2-j+1)} > 0,$$

that is for $j \leq \frac{n}{3}+1$.

Since $j = 1, \ldots, k$, the sufficient condition that ensures $\Delta P_V \geq 0$ is

$$k \leq \min\left(s, \frac{n}{3} + 1\right).$$

■