Antonyms of predicates on n-tuplas of fuzzy sets. A characterization of involutions on $[0,1]^n$ ⋆

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Abstract. Obtaining the complement of a fuzzy set is usually done through a negation function. On the other hand, the antonym of a predicate, which in classical logic could be considered to be the complement, is radically different from it in uncertain environments. To model it in fuzzy logic or in its extensions, it is common to use involutions, functions on the universe that satisfy some boundary, monotony and involution conditions. In particular, to obtain the antonym of a fuzzy predicate determined by n arguments, we will need an involution on $[0,1]^n$. As involutions on $[0,1]^2$ were characterized in a previous work, in the present paper we firstly focus on involutions on $[0,1]^3$, suggesting how involutions on $[0,1]^n$ could be. We then obtain the main result, the characterization of involutions in this set $[0,1]^n$.

Keywords: Fuzzy sets · Complement · Negation · Antonym · Involution.

1 Introduction

Since Zadeh introduced fuzzy sets in 1965 ([21]), a lot of work has been done in this area. In particular, some extensions of fuzzy sets have been developed to face the uncertainty or the vagueness in a more suitable way. For example, the interval-valued fuzzy sets, Atanassov's fuzzy sets, rough fuzzy sets, or type-2 fuzzy sets, have been defined and their properties have been deeply studied (some examples of these studies are [1–3, 9–11, 14, 15]).

An essential problem in this framework is the design of the complement of a given fuzzy set. In classical theory, this task is immediate: Given a subset A

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of a universe X, with its membership function (characteristic function) $\varphi_A : X \to \{0,1\}$, the complement is the set given by $\varphi_{A^c} : X \to \{0,1\}$, where $\varphi_{A^c}(x) = 1 - \varphi_A(x)$. But this is a bit more complicated when we are working with fuzzy logic.

In general, in the extensions of classical logic, each set A in a universe X is given by its membership function:

$$\chi_A : X \to B$$

where B is a bounded partially ordered set (poset) $(B, \leq, 1_B, 0_B)$, with maximum element 1_B and minimum element 0_B. Usually, the most appropriate structure for B is the lattice, although there are cases where B is only a bounded poset.

Then, the usual way to obtain the complement of A in B (denoted by A^c), is throughout a function $C : B \to B$, satisfying the boundary and decreasing conditions:

1. $C(1_B) = 0_B$, $C(0_B) = 1_B$.
2. If $x \leq y$, then $C(y) \leq C(x)$.

Considering such a function, the membership function of the complement A^c will be

$$\chi_{A^c} : X \to B$$

where $\chi_{A^c}(x) = C(\chi_A(x))$.

The function C is called a fuzzy complementation or fuzzy negation. If the fuzzy negation is a continuous function and is strictly decreasing, it is called a strict fuzzy negation. Moreover, strict fuzzy negations are bijective functions. In fact the concepts of strict fuzzy negation and bijective fuzzy negation are equivalent. It is worth noting that fuzzy negations have been widely used, not only for modeling complementarity, but also in many applications in the field of fuzzy logic (see, for example, [13] to construct necessity and possibility operators, and [16] to obtain measures of entropy). Studies and characterizations of negations in fuzzy sets and their extensions have been obtained in [1–6, 9–11, 17, 19].

Another concept related to the idea of complement is that of antonym. In fact, in classical logic, these two terms have the same meaning. However, this is not necessarily the case in fuzzy logic and its extensions, when working in environments with ambiguity or uncertainty ([18]). For example, with the predicate 'big' and its antonym 'small', it is not equivalent saying 'the house is not big', and 'the house is small'. Consequently we need to develop a new method to obtain the antonym of a predicate or a set.

The most common way to define the antonym of a subset A in a bounded partially ordered set $(X, \leq, 1_X, 0_X)$, when A is characterized by a monotonic membership function with $\chi_A : X \to B$, involves the use of a function $\alpha : X \to X$ satisfying three conditions:

1. $\alpha(1_X) = 0_X$, $\alpha(0_X) = 1_X$.
2. If $x \leq y$, then $\alpha(y) \leq \alpha(x)$.
3. $\alpha(\alpha(x)) = x$, $\forall x \in X$.

In this case the antonym of A, within the universe X, will be aA (with $\chi_{aA} : X \to B$) where:

$$\chi_{aA}(x) = \chi_A(\alpha(x)).$$

The function α will be said an involution in X.

In [20] the authors extend the process to obtain antonyms to the general case of sets with non-monotonic membership functions. In this case, a different function α will be determined to each subset of X in which the function is monotonic.

Remark 1. It is important to notice that from a functional point of view, the two conditions imposed to fuzzy negations are identical to the first two conditions imposed to involutions. Moreover, when we add the involutive condition to negations, the result is what we usually call *strong fuzzy negations*. Consequently, in this paper strong negation functions (negations with the involution property) and involution functions are considered functionally similar but conceptually different. The difference we establish relies on the way they are applied: while involutions work on the universe of the variable (from X to X), negations act on membership values (from B to B).

Let us now illustrate the application of the concepts of complement and antonym with an example.

Example 1. To express the level of a student in a subject, three terms could be considered: 'high level', its negation 'not high level', and its antonym 'low level'. In order to determine the level of a student, one, two or n exam tests can be taken into account, each one with values from 0 to 1. In this case, the degree in which the student has a 'high level' should be modelled with a membership function

$$\aleph_{highlevel} : [0,1]^n \to [0,1]$$

Note that $\aleph_{highlevel}$ must be a monotonically increasing function, and therefore, the antonym can be obtained through a single involution. Furthermore, it will be $\aleph_{highlevel}(0,\ldots,0) = 0$, $\aleph_{highlevel}(1,\ldots,1) = 1$, and monotonically increasing respect to the usual order in $[0,1]^n$.

Now, to obtain the predicate 'not high level' we should use a negation function $N : [0,1] \to [0,1]$ $(B = [0,1])$, obtaining:

$$\aleph_{nothighlevel} : [0,1]^n \to [0,1],$$

$$\aleph_{nothighlevel}(x_1,\ldots,x_n) = N(\aleph_{highlevel}(x_1,\ldots,x_n)),\ \forall (x_1,\ldots,x_n) \in [0,1]^n.$$

Similarly if the values of membership functions are in other sets different to the set $[0,1]$, as it is the case of the extensions of the fuzzy logic.

But if we want to determine the predicate 'low level', an involution $\alpha : [0,1]^n \to [0,1]^n$ should be considered $(X = [0,1]^n)$. So

$$\aleph_{lowlevel} : [0,1]^n \to [0,1],$$

$$\aleph_{lowlevel}(x_1, \ldots, x_n) = \aleph_{highlevel}(\alpha(x_1, \ldots, x_n)).$$

Note that a 'low level' is not exactly the same than a 'not high level'. In fact, 'low level' implies 'not high level', but not the opposite. This relation will be further investigated in Section 4.

In order to obtain the complement or the antonym of different sets, many researchers have faced the problem of finding characterizations of the strong negations or involutions. These characterizations are very different depending on the base set. In this paper, and considering the antonyms in the Example 1, we will focus on obtaining the characterization of involutions in the set $[0,1]^n$.

Taking this into account the rest of the paper is structured as follows. Section 2 focuses on antonyms of predicates applied to n-tuples of fuzzy sets, and shows a characterization of the involutions on $[0,1]^3$. Why considering $[0,1]^3$ and not directly studying the general case (from which this will be a particular case)? This characterization will provide us with some clues to define how the involutions in $[0,1]^n$ could be, finding a characterization that could not be obtained from the previous cases directly. This characterization is showed in Section 3, being this the main result of this paper. However, this approach provides us with a functional view of antonyms, that should be completed with a conceptual review. To do so, the intuitive condition of maintaining the antonym below the negation is considered in Section 4, where an example helps us to illustrate how some attempts to create antonyms according to what is obtained in Section 3 could fail in satifying this conceptual requirement. Finally, some conclusions are presented in Section 5.

2 Antonyms of predicates on n-tuples of fuzzy sets. Involutions on $[0,1]^3$

In [8] antonyms of some fuzzy predicates were studied. In particular, predicates depending on two fuzzy sets as arguments. Such predicates determine, for example, how much two fuzzy sets are supplementary or contradictory ([7]). Note that in this case, a predicate is given by its membership function: $M : [0,1]^X \times [0,1]^X \to [0,1]$. So, in order to search an antonym, we need involutions on $[0,1]^X \times [0,1]^X \approx ([0,1] \times [0,1])^X$, and then, the aim is to find out and, if possible, to characterize the involutions on that set. But this is equivalent to find out involutions on $I^2 = [0,1]^2 = [0,1] \times [0,1]$. In fact, if $\alpha : [0,1]^2 \to [0,1]^2$ is an involution respect to the usual order on I^2 ($(a,b) \leq (a',b') \Leftrightarrow a \leq a', b \leq b'$), then the function:

$$\Lambda : ([0,1] \times [0,1])^X \to ([0,1] \times [0,1])^X,$$

defined as

$$(\Lambda(f))(x) = \alpha(f(x)),$$

is an involution on $([0,1] \times [0,1])^X$, respect to the usual order on functions ($f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in X$). Moreover, $(I^2, \leq)$ is a complete lattice with the smallest element $(0,0)$, and the greatest element $(1,1)$.

In this sense, we already obtained the following result.

Theorem 1. Involutions on $I^2 = [0,1]^2$. Representation Theorem *([8])* *Let α be an involution on $(I^2, \leq)$. Then only two cases are possible:*

1. *There exist two strong fuzzy negations N_1 and N_2 in $[0,1]$ such that $\alpha(a_1, a_2) = (N_1(a_1), N_2(a_2))$, for any $(a_1, a_2) \in I^2$, or*
2. *There exists a strict fuzzy negation N (not necessarily strong) such that $\alpha(a_1, a_2) = (N(a_2), N^{-1}(a_1))$ for all $(a_1, a_2) \in I^2$.*

But in many applications we may need to work with predicates with three or more arguments, as we have already seen in Example 1.

That is why it is worth studying the involutions on $[0,1]^n$ and, if possible, to obtain a characterization of them. In this sense, a partial result is presented in this Section when $n = 3$, with a characterization theorem.

Remark 2. We want to note that the characterizations for $n = 2$ and $n = 3$ were decisive to understand and to be able to find the characterization of the involutions in $[0,1]^n$. This is the reason why in this paper we include the case $n = 3$ and do not directly give the proof of the general case. However, we will only present a few steps of the proof.

In the following we consider the usual partial order in I^3, $(a_1, a_2, a_3) \leq (a'_1, a'_2, a'_3) \Leftrightarrow a_1 \leq a'_1, a_2 \leq a'_2, a_3 \leq a'_3$. Moreover, $(I^3, \leq)$ is a complete lattice with the smallest element $(0,0,0)$, and the greatest element $(1,1,1)$.

Theorem 2. Representation Theorem of involutions on $I^3 = [0,1]^3$.
Let α be an involution in $(I^3, \leq)$. where $(a,b,c) \leq (a',b',c') \Leftrightarrow a \leq a', b \leq b'$ and $c \leq c'$. Then, only four cases are possible:

1. *There exist three strong fuzzy negations N_1, N_2 and N_3 in $[0,1]$ such that $\alpha(a_1, a_2, a_3) = (N_1(a_1), N_2(a_2), N_3(a_3))$ for all $(a_1, a_2, a_3) \in I^3$.*
2. *There exist a strict fuzzy negation N and a strong fuzzy negation N_1, such that $\alpha(a_1, a_2, a_3) = (N(a_2), N^{-1}(a_1), N_1(a_3))$ for all $(a_1, a_2, a_3) \in I^3$.*
3. *There exist a strict fuzzy negation N and a strong fuzzy negation N_1, such that $\alpha(a_1, a_2, a_3) = (N(a_3), N_1(a_2), N^{-1}(a_1))$ for all $(a_1, a_2, a_3) \in I^3$.*
4. *There exist a strict fuzzy negation N and a strong fuzzy negation N_1, such that $\alpha(a_1, a_2, a_3) = (N_1(a_1), N(a_3), N^{-1}(a_2))$ for all $(a_1, a_2, a_3) \in I^3$.*

The proof will be developed in several steps through a lemma and some propositions.

Remark 3. First, let us observe that if two elements in $[0,1]^3$ are equal except for the value of one of its coordinates, then both elements are comparable. That is, given $(a,b,c), (a',b',c') \in [0,1]^3$, if $a = a'$ and $b = b'$, or $a = a'$ and $c = c'$, or $b = b'$ and $c = c'$, then either $(a,b,c) \leq (a',b',c')$ or $(a',b',c') \leq (a,b,c)$.

Obviously this is also true when those coordinates being equal take value 0 or 1. Therefore, the order on each edge of the cube $[0,1]^3$, is a total (linear) order.

In the following lemma we will show that the image by an involution α, of a vertex of the cube having two coordinates 0, must be a vertex with two coordinates 1, and vice-versa.

Lemma 1. *If α is an involution in $(I^3, \leq)$, then:*

$$\alpha(0,0,1), \alpha(0,1,0), \alpha(1,0,0) \in \{(0,1,1),(1,0,1),(1,1,0)\}, \textit{ and}$$

$$\alpha(0,1,1), \alpha(1,0,1), \alpha(1,1,0) \in \{(0,0,1),(0,1,0),(1,0,0)\}.$$

Proof. Firstly, let us show that the image by the involution α, of the vertices with two coordinates 0, must have two coordinates 1 and vice-versa. That is, we are going to prove the following inequalities:

- $\alpha(0,0,1) \geq (0,1,1)$, or $\alpha(0,0,1) \geq (1,0,1)$, or $\alpha(0,0,1) \geq (1,1,0)$. That is, if $\alpha(0,0,1) = (a,b,c)$, at least two of the values a, b, c should be 1.
  Similarly,
- $\alpha(0,1,0) \geq (0,1,1)$, or $\alpha(0,1,0) \geq (1,0,1)$, or $\alpha(0,1,0) \geq (1,1,0)$.
- $\alpha(1,0,0) \geq (0,1,1)$, or $\alpha(1,0,0) \geq (1,0,1)$, or $\alpha(1,0,0) \geq (1,1,0)$.
- $\alpha(0,1,1) \leq (0,0,1)$, or $\alpha(0,1,1) \leq (0,1,0)$, or $\alpha(0,1,1) \leq (1,0,0)$.
- $\alpha(1,0,1) \leq (0,0,1)$, or $\alpha(1,0,1) \leq (0,1,0)$, or $\alpha(1,0,1) \leq (1,0,0)$.
- $\alpha(1,1,0) \leq (0,0,1)$, or $\alpha(1,1,0) \leq (0,1,0)$, or $\alpha(1,1,0) \leq (1,0,0)$.

These inequalities are ilustrated in Figure 1.

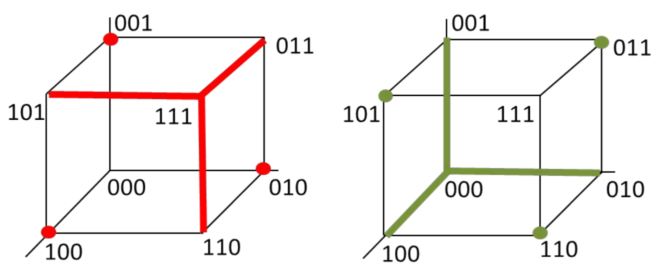

Fig. 1. The images of 001, 010 and 100 are greater than 011, or 101, or 110, and the images of 011, 101,110 are smaller than 001, or 010, or 100

Let $\alpha(0,0,1) = (a,b,c)$. Let us suppose $a, b < 1$, and let $a_1 > a$, $b_1 > b$. Then $(a,b,c) \leq (a_1,b,c)$ and $(a,b,c) \leq (a,b_1,c)$. More, (a_1,b,c) and (a,b_1,c) are not comparable.

Nevertheless $\alpha(a_1,b,c) \leq \alpha(a,b,c) = (0,0,1)$ and $\alpha(a,b_1,c) \leq \alpha(a,b,c) = (0,0,1)$. Then, taking into account the previous remark, $\alpha(a_1,b,c)$ and $\alpha(a,b_1,c)$ should be comparable, and so should (a_1,b,c) and (a,b_1,c), attaining a contradiction. Similarly if $a, c < 1$ or $b, c < 1$.

The rest of inequalities are proved in a similar way.

Let us see now that these inequalities are, in fact, equalities.

If $\alpha(0,0,1) \geq (0,1,1)$, it is $(0,0,1) \leq \alpha(0,1,1)$, and as $\alpha(0,1,1)$ should be smaller than $(0,0,1)$ or $(0,1,0)$ or $(1,0,0)$, therefore:

- $(0,0,1) \leq \alpha(0,1,1) \leq (0,0,1)$, or
- $(0,0,1) \leq \alpha(0,1,1) \leq (0,1,0)$, or
- $(0,0,1) \leq \alpha(0,1,1) \leq (1,0,0)$.

Nevertheless, the last two cases are not possible, and the only possibility is: $\alpha(0,1,1) = (0,0,1)$, and as α is an involution, $\alpha(0,0,1) = (0,1,1)$.

Similarly we have:

If $\alpha(0,0,1) \geq (1,0,1)$ then $\alpha(0,0,1) = (1,0,1)$ and $\alpha(1,0,1) = (0,0,1)$.
If $\alpha(0,0,1) \geq (1,1,0)$ then $\alpha(0,0,1) = (1,1,0)$ and $\alpha(1,1,0) = (0,0,1)$.

Therefore, $\alpha(0,0,1) \in \{(0,1,1),(1,0,1),(1,1,0)\}$.

The rest of assertions are proved in a similar way. □

Proposition 1. *If α is an involution in $(I^3, \leq)$ such that $\alpha(0,0,1) = (0,1,1)$, then there exists a strict fuzzy negation N and a strong fuzzy negation N_1, such that $\alpha(a,b,c) = (N(c), N_1(b), N^{-1}(a))$ for all $(a,b,c) \in I^3$.*

Proof. 1. As $\alpha(0,0,1) = (0,1,1)$ it could be either $\alpha(0,1,0) = (1,0,1)$ and $\alpha(1,0,0) = (1,1,0)$ (see Figure 2, Case b), or $\alpha(0,1,0) = (1,1,0)$ and $\alpha(1,0,0) = (1,0,1)$ (see Figure 2, Case a); nevertheless, in the later case, as $\alpha(0,1,1) = (0,0,1) \leq (1,0,1) = \alpha(1,0,0)$ it would be $\alpha(1,0,1) = (1,0,0) \leq \alpha(0,0,1) = (0,1,1)$,
attaining a contradiction. Then it should be :
$\alpha(0,0,1) = (0,1,1)$, $\alpha(0,1,0) = (1,0,1)$, and $\alpha(1,0,0) = (1,1,0)$.

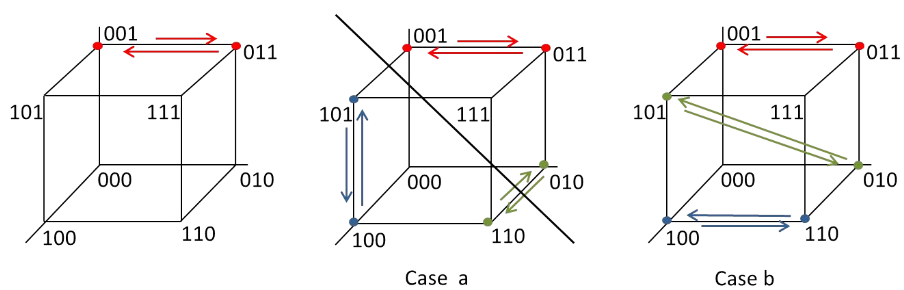

Fig. 2. Cases when $\alpha(0,0,1) = (0,1,1)$. Case **a** is not possible

2. On the other hand, since $\alpha(0,0,1) = (0,1,1)$ and $(0,0,0) \leq (0,0,a) \leq (0,0,1)$, then $(0,1,1) = \alpha(0,0,1) \leq \alpha(0,0,a) \leq \alpha(0,0,0) = (1,1,1)$, and the image of the edge $\{(0,0,a); a \in [0,1]\}$ (elements between $(0,0,0)$ and

$(0,0,1))$ will be $\{(b,1,1); b \in [0,1]\}$ (elements between $(0,1,1)$ and $(1,1,1)$), and vice-versa (see Figure 3a). Therefore, we can define two functions $N, N^* : [0,1] \to [0,1]$, such that $N(a) = b$, if $\alpha(0,0,a) = (b,1,1)$ (and so $\alpha(0,0,a) = (N(a),1,1)$), and $N^*(a) = b$ if $\alpha(a,1,1) = (0,0,b)$. So $\alpha(a,1,1) = (0,0,N^*(a))$. These functions are well defined and it is easy to show that they are strict fuzzy negations not necessarily strong. Furthermore, as α is involutive

$$(0,0,a) = \alpha(\alpha(0,0,a)) = \alpha(N(a),1,1) = (0,0,N^*(N(a))),$$

getting $N^*(N(a)) = a$, $\forall a \in [0,1]$; and

$$(a,1,1) = \alpha(\alpha(a,1,1)) = \alpha(0,0,N^*(a)) = (N(N^*(a)),1,1),$$

getting $N(N^*(a)) = a$, $\forall a \in [0,1]$.
In summary, $N^* = N^{-1}$.

3. As $\alpha(0,1,0) = (1,0,1)$ and $(0,0,0) \leq (0,a,0) \leq (0,1,0)$, the edge $\{(0,a,0); a \in [0,1]\}$ is transformed into the edge $\{(1,b,1); b \in [0,1]\}$, and vice-versa (see Figure 3b). We can define two fuzzy negations $N_1, N_1^* : [0,1] \to [0,1]$, satisfying $\alpha(1,a,1) = (0,N_1(a),0)$ and $\alpha(0,a,0) = (1,N_1^*(a),1)$. These fuzzy negations are well defined and, as in the previous case, $N_1^* = N_1^{-1}$.

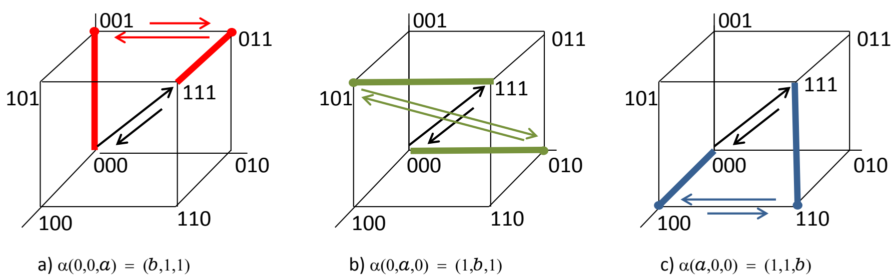

Fig. 3. Images of the different edges

4. In a similar way, since $\alpha(1,0,0) = (1,1,0)$ and $(0,0,0) \leq (a,0,0) \leq (1,0,0)$, the image of the edge $\{(a,0,0); a \in [0,1]\}$ by the involution is $\{(1,1,b); b \in [0,1]\}$, and vice-versa (see Figure 3c), allowing to define $N_2, N_2^* : [0,1] \to [0,1]$, such that $N_2(a) = b$, if $\alpha(a,0,0) = (1,1,b)$ (and then $\alpha(a,0,0) = (1,1,N_2(a))$), and $N_2^*(a) = b$ if $\alpha(1,1,a) = (b,0,0)$ (and then, $\alpha(1,1,a) = (N_2^*(a),0,0)$).
These functions are well defined and are fuzzy negations, but they are not necessarily strong. Again, $N_2^* = (N_2)^{-1}$.
5. For any (a,b,c), it is:
   $(0,0,c) \leq (a,b,c)$, and then, $\alpha(a,b,c) \leq (N(c),1,1)$.
   $(0,b,0) \leq (a,b,c)$, and $\alpha(a,b,c) \leq (1,N_1^*(b),1)$.
   $(a,0,0) \leq (a,b,c)$, and $\alpha(a,b,c) \leq (1,1,N_2(a))$.

Then $\alpha(a,b,c) \leq (N(c), N_1^*(b), N_2(a))$.
Furthermore,
$(a,b,c) \leq (1,1,c)$, and then, $(N_2^*(c),0,0) \leq \alpha(a,b,c)$.
$(a,b,c) \leq (a,1,1)$, and $(0,0,N^*(a)) \leq \alpha(a,b,c)$.
$(a,b,c) \leq (1,b,1)$, and $(0,N_1(b),0) \leq \alpha(a,b,c)$.
Then $(N_2^*(c), N_1(b), N^*(a)) \leq \alpha(a,b,c)$, and, as α is involutive, $N^* = N^{-1}$, $N_1^* = (N_1)^{-1}$ and $N_2^* = (N_2)^{-1}$, we get

$$\alpha(\alpha(a,b,c)) = (a,b,c) \leq \alpha(N_2^*(c), N_1(b), N^*(a)) \leq$$
$$\leq (N(N^*(a)), N_1^*(N_1(b)), N_2(N_2^*(c))) = (a,b,c),$$

and consequently

$$\alpha(a,b,c) = (N_2^*(c), N_1(b), N^*(a)).$$

6. Finally, as α is involutive, for all (a,b,c)

$$\alpha(\alpha(a,b,c)) = \alpha(N_2^*(c), N_1(b), N^*(a)) =$$
$$= (N_2^*(N^*(a)), N_1(N_1(b)), N^*(N_2^*(c))) = (a,b,c).$$

Then we obtain that N_1 is a strong fuzzy negation, and N_2^* and N^* are inverse. So

$$\alpha(a,b,c) = ((N^*)^{-1}(c), N_1(b), N^*(a)) = (N(c), N_1(b), N^{-1}(a)).$$

□

The proofs of Propositions 2, 3 and 4, follow similar steps as Proposition 1. Then, they will be omitted since our main object is the general case with any n.

Proposition 2. *If α is an involution in $(I^3, \leq)$ such that $\alpha(0,0,1) = (1,0,1)$, then there exists a strict fuzzy negation N and a strong fuzzy negation N_1, satisfying*

$$\alpha(a,b,c) = (N_1(a), N(c), N^{-1}(b)) \text{ for all } (a,b,c) \in I^3 \text{ (see figures 4 and 5).}$$

Proposition 3. *If α is an involution in $(I^3, \leq)$ such that $\alpha(0,0,1) = (1,1,0)$, $\alpha(1,0,0) = (1,0,1)$, and $\alpha(0,1,0) = (0,1,1)$, then there exists a strict fuzzy negation N and a strong fuzzy negation N_1, such that*

$$\alpha(a,b,c) = (N(b), N^{-1}(a), N_1(c)) \text{ for all } (a,b,c) \in I^3 \text{ (see Figure 6).}$$

Proposition 4. *If α is an involution in $(I^3, \leq)$ such that $\alpha(0,0,1) = (1,1,0)$, $\alpha(1,0,0) = (0,1,1)$ and $\alpha(0,1,0) = (1,0,1)$, then there exist three strong fuzzy negations N_1, N_2, N_3, satisfying*

$$\alpha(a,b,c) = (N_1(a), N_2(b), N_3(c)) \text{ for all } (a,b,c) \in I^3 \text{ (see Figure 7).}$$

Then, with these propositions, we have proved Theorem 2, obtaining the only four options for involutions in $(I^3, \leq)$.

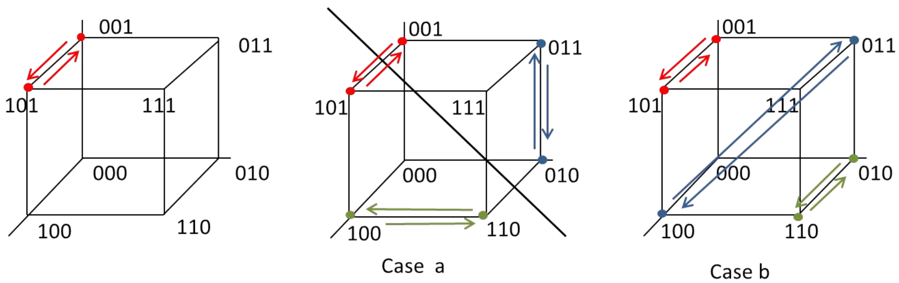

Fig. 4. Cases when $\alpha(0,0,1) = (1,0,1)$. Case **a** is not possible

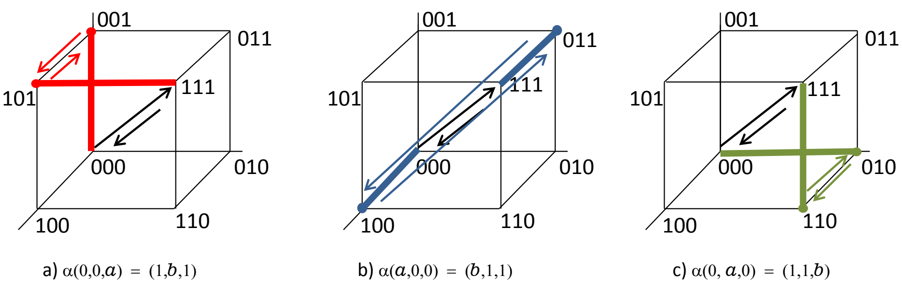

Fig. 5. Images of the different edges, when $\alpha(0,0,1) = (1,0,1)$

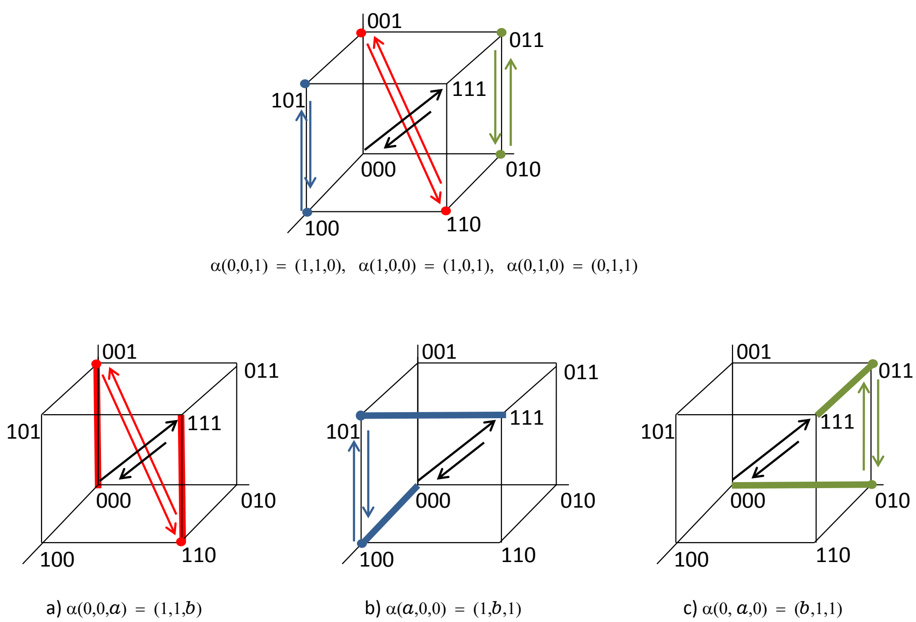

Fig. 6. Images in the case of Proposition 3

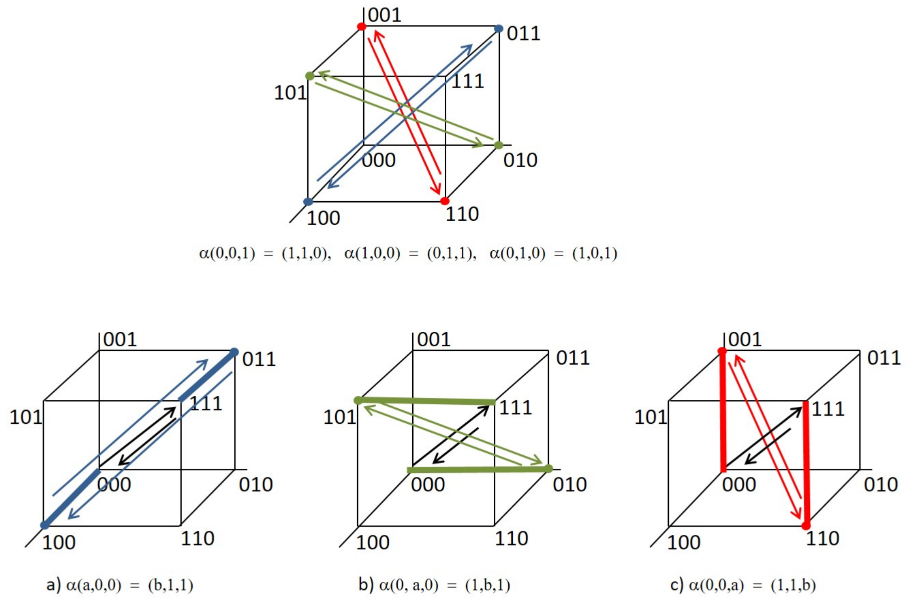

Fig. 7. Images in the case of Proposition 4

3 Representation Theorem of Involutions on $[0,1]^n$

This Section is devoted to reach a characterization theorem of involutions in $[0,1]^n$ through bijective negations in $[0,1]$. Starting from the cases of $n = 2$, and particullarly from that of $n = 3$, let's proceed now to extend to any value of n.

From $n = 2$ we learnt that in addition to applying strong negations on each component of the predicate, we can also obtain an involution by exchanging the two components and then applying two strict negations, being one the inverse of the other (see Theorem 1).

From $n = 3$ we know, according to Theorem 2, that it is either possible to use three strong negations (for the three components), or one strong negation (in one of the components) and two inverse strict negations on the other two components (after exchanging them). But what is more important, we have shown that we can't use any permutation exchanging the position of the three components, no matter if we then apply strong or strict negations.

We will try now to extend these ideas to the general case. As in the case of $n = 3$, we consider the usual partial order in I^n, $(a_1, \ldots, a_n) \leq (a'_1, \ldots, a'_n) \Leftrightarrow a_1 \leq a'_1, \ldots, a_n \leq a'_n$. Then, $(I^n, \leq)$ is a complete lattice with smallest element $(0, \ldots, 0)$, and greatest element $(1, \ldots, 1)$.

As in the case of $n = 3$, if two elements of the n-cube $[0,1]^n$ are equal except for the value of one of their coordinates, then both elements are comparable. Consequently, the order on each edge of the n-cube $[0,1]^n$, is a total order.

In addition, we can prove with a similar reasoning as in Lemma 1, that the image by an involution α of a vertex of the n-cube having $n-1$ coordinates 0, must be a vertex with $n-1$ coordinates 1, and vice-versa. Then in the following lemma we will obtain a permutation associated with those transformations.

Lemma 2. *If α is an involution in $(I^n, \leq)$, then there exists a permutation (a bijection) $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that*

$$\alpha(0, \ldots, 0, \overset{i}{1}, 0, \ldots, 0) = (1, \ldots, 1, \overset{\sigma(i)=j}{0}, 1, \ldots, 1), \text{ and}$$

$$\alpha(1, \ldots, 1, \overset{j=\sigma(i)}{0}, 1, \ldots, 1) = (0, \ldots, 0, \overset{\sigma^{-1}(j)=i}{1}, 0, \ldots, 0).$$

Proof. Taking into account the previous comment, we have that for each $i \in \{1, \ldots, n\}$, $\exists\, j \in \{1, \ldots, n\}$ such that

$$\alpha(0, \ldots, 0, \overset{i}{1}, 0, \ldots, 0) = (1, \ldots, 1, \overset{j}{0}, 1, \ldots, 1),$$

and

$$\alpha(1, \ldots, 1, \overset{j}{0}, 1, \ldots, 1) = (0, \ldots, 0, \overset{i}{1}, 0, \ldots, 0).$$

Now, we define $\sigma : \{0, \ldots, n\} \to \{1, \ldots, n\}$ such that $\sigma(i) = j$ if

$$\alpha(0, \ldots, 0, \overset{i}{1}, 0, \ldots, 0) = (1, \ldots, 1, \overset{j}{0}, 1, \ldots, 1).$$

Let us see that σ is a bijective function (permutation):

- σ is well defined since α is a map.
- σ is bijective: if $i \neq i'$ then, $(0, \ldots, 0, \overset{i}{1}, 0, \ldots, 0) \neq (0, \ldots, 0, \overset{i'}{1}, 0, \ldots, 0)$ and it is $\alpha(0, \ldots, 0, \overset{i}{1}, 0, \ldots, 0) \neq \alpha(0, \ldots, 0, \overset{i'}{1}, 0, \ldots, 0)$ since α is injective. Therefore, $(1, \ldots, 1, \overset{\sigma(i)}{0}, 1, \ldots, 1) \neq (1, \ldots, 1, \overset{\sigma(i')}{0}, 1, \ldots, 1)$ and so $\sigma(i) \neq \sigma(i')$. Thus, σ is injective and consequently is surjective.

Moreover, $\alpha(1, \ldots, 1, \overset{\sigma(i)=j}{0}, 1, \ldots, 1) = (0, \ldots, 0, \overset{\sigma^{-1}(j)=i}{1}, 0, \ldots, 0)$. □

Lemma 3. *Given α an involution in $(I^n, \leq)$, then the permutation σ associated with α, obtained in Lemma 2, cannot have cycles of length greater than or equal to three.*

Proof. Let us suppose that σ has a cycle of length greater than or equal to three, that is, $\exists\, i_1, i_2, i_3, \ldots, i_k \in \{1, \ldots, n\}$, all of them different from each other such that $\sigma(i_1) = i_2$, $\sigma(i_2) = i_3, \ldots, \sigma(i_k) = i_1$, with $k \geq 3$:

$$(0, \ldots, 0, \overset{i_1}{1}, 0, \ldots, 0) \xrightarrow{\alpha} (1, \ldots, 1, \overset{i_2=\sigma(i_1)}{0}, 1, \ldots, 1)$$

$$(0,\dots,0,\overset{i_2}{1},0,\dots,0) \xrightarrow{\alpha} (1,\dots,1,\overset{i_3=\sigma(i_2)}{0},1,\dots,1)$$

Then, $\alpha(1,\dots,1,\overset{i_2}{0},1,\dots,1) = (0,\dots,0,\overset{i_1}{1},0,\dots,0) \overset{\overset{i_1\neq i_3}{\downarrow}}{\leq} (1,\dots,1,\overset{i_3}{0},1,\dots,1) = \alpha(0,\dots,0,\overset{i_2}{1},0,\dots,0)$, therefore $(0,\dots,0,\overset{i_2}{1},0,\dots,0) \leq (1,\dots,1,\overset{i_2}{0},1,\dots,1)$, attaining a contradiction. □

Remark 4. From Lemma 3 it is obtained that given α an involution in $(I^n,\leq)$ and σ the permutation associated to α, then σ can only have cycles of length two and elements that remain fixed, in other words, for each $i\in\{1,\dots,n\}$ there are only two possibilities:

$$\sigma(i)=i \text{ or } \sigma(i)\neq i \text{ with } \sigma(\sigma(i))=i.$$

In both cases $\sigma^{-1}=\sigma$, and consequently, for each $i\in\{1,\dots,n\}$ it is

$$\alpha(0,\dots,0,\overset{i}{1},0,\dots,0) = (1,\dots,1,\overset{\sigma(i)}{0},1,\dots,1), \text{ and}$$

$$\alpha(1,\dots,1,\overset{i}{0},1,\dots,1) = (0,\dots,0,\overset{\sigma^{-1}(i)=\sigma(i)}{1},0,\dots,0)$$

Lemma 4. *Let α be an involution in $(I^n,\leq)$ and σ the permutation associated to α, then for each $a\in(0,1)$ there are $b,b'\in(0,1)$ such that*

$$\alpha(0,\dots,0,\overset{i}{a},0,\dots,0) = (1,\dots,1,\overset{\sigma(i)}{b},1,\dots,1), \textit{ and}$$

$$\alpha(1,\dots,1,\overset{i}{a},1,\dots,1) = (0,\dots,0,\overset{\sigma(i)}{b'},0,\dots,0).$$

Proof. Let $a\in(0,1)$, it is $(0,\dots,0) \lneqq (0,\dots,0,\overset{i}{a},0,\dots,0) \lneqq (0,\dots,0,\overset{i}{1},0,\dots,0)$ then, $(1,\dots,1) \gneqq \alpha(0,\dots,0,\overset{i}{a},0,\dots,0) \gneqq (1,\dots,1,\overset{\sigma(i)}{0},1,\dots,1)$. Therefore,

$$\alpha(0,\dots,0,\overset{i}{a},0,\dots,0) = (1,\dots,1,\overset{\sigma(i)}{b},1,\dots,1) \text{ with } b\in(0,1).$$

Since $(1,\dots,1,\overset{i}{0},1,\dots,1) \lneqq (1,\dots,1,\overset{i}{a},1,\dots,1) \lneqq (1,\dots,1)$, it is

$$(0,\dots,0,\overset{\sigma(i)}{1},0,\dots,0) \gneqq \alpha(1,\dots,1,\overset{i}{a},1,\dots,1) \gneqq (0,\dots,0).$$

Therefore, $\alpha(1,\dots,1,\overset{i}{a},1,\dots,1) = (0,\dots,0,\overset{\sigma(i)}{b'},0,\dots,0)$ with $b'\in(0,1)$. □

As a conclusion of Lemmas 2, 3 and 4 we have that for each $i\in\{1,\dots,n\}$,

$$\alpha(0,\dots,0,\overset{i}{1},0,\dots,0) = (1,\dots,1,\overset{\sigma(i)}{0},1,\dots,1), \text{ and}$$

$$\alpha(1,\ldots,1,\overset{i}{0},1,\ldots,1)=(0,\ldots,0,\overset{\sigma(i)}{1},0,\ldots,0)$$

and moreover, for each $a \in (0,1)$, $\exists\, b, b' \in (0,1)$ such that

$$\alpha(0,\ldots,0,\overset{i}{a},0,\ldots,0)=(1,\ldots,1,\overset{\sigma(i)}{b},1,\ldots,1),\text{and}$$

$$\alpha(1,\ldots,1,\overset{i}{a},1,\ldots,1)=(0,\ldots,0,\overset{\sigma(i)}{b'},0,\ldots,0).$$

From these transformations we can define the following applications in $[0,1]$ that are going to be bijective negations, and some of them inverse of others.

Definition 1. *Let α be an involution in $(I^n,\leq)$, and σ the permutation associated to α. For each $i \in \{1,\ldots,n\}$, let us define $N_{i\sigma(i)} : [0,1] \to [0,1]$ such that*

$$N_{i\sigma(i)}(a)=b \textit{ if and only if } \alpha(0,\ldots,0,\overset{i}{a},0,\ldots,0)=(1,\ldots,1,\overset{\sigma(i)}{b},1,\ldots,1),$$

for all $a \in [0,1]$. That is,

$$\alpha(0,\ldots,0,\overset{i}{a},0,\ldots,0)=(1,\ldots,1,\overset{\sigma(i)}{N_{i\sigma(i)}(a)},1,\ldots,1).$$

Definition 2. *Let α be an involution in $(I^n,\leq)$, and σ the permutation associated to α. For each $i \in \{1,\ldots,n\}$, let us define $N^*_{i\sigma(i)} : [0,1] \to [0,1]$ such that*

$$N^*_{i\sigma(i)}(a)=b' \textit{ if and only if } \alpha(1,\ldots,1,\overset{i}{a},1,\ldots,1)=(0,\ldots,0,\overset{\sigma(i)}{b'},0,\ldots,0),$$

for all $a \in [0,1]$. That is,

$$\alpha(1,\ldots,1,\overset{i}{a},1,\ldots,1)=(0,\ldots,0,\overset{\sigma(i)}{N^*_{i\sigma(i)}(a)},0,\ldots,0).$$

Proposition 5. *Let α be an involution in $(I^n,\leq)$, σ the permutation associated to α, and let $N_{i\sigma(i)}$ and $N^*_{i\sigma(i)}$ be the functions given by Definitions 1 and 2, then $N_{i\sigma(i)}$ and $N^*_{i\sigma(i)}$ are bijective negations on $[0,1]$, $\forall i \in \{1,\ldots,n\}$. Moreover, $N^*_{i\sigma(i)} = (N_{\sigma(i)i})^{-1}$.*

Proof. 1. Firstly, we prove that the functions $N_{i\sigma(i)}$ and $N^*_{i\sigma(i)}$ are negations on $[0,1]$.

- $N_{i\sigma(i)}(0)=1$ since $\alpha(0,\ldots,0,\overset{i}{0},0,\ldots,0)=(1,\ldots,1,\overset{\sigma(i)}{1},1,\ldots,1)$.
- $N_{i\sigma(i)}(1)=0$ since $\alpha(0,\ldots,0,\overset{i}{1},0,\ldots,0)=(1,\ldots,1,\overset{\sigma(i)}{0},1,\ldots,1)$.

- $N_{i\sigma(i)}$ is decreasing: let $a, b \in [0,1]$ such that $a \leq b$, then

$$(0,\ldots,0,\overset{i}{a},0,\ldots,0) \leq (0,\ldots,0,\overset{i}{b},0,\ldots,0),$$

therefore

$$(1,\ldots,1,\overset{\sigma(i)}{N_{i\sigma(i)}(b)},1,\ldots,1) = \alpha(0,\ldots,0,\overset{i}{b},0,\ldots,0) \leq$$

$$\leq \alpha(0,\ldots,0,\overset{i}{a},0,\ldots,0) = (1,\ldots,1,\overset{\sigma(i)}{N_{i\sigma(i)}(a)},1,\ldots,1),$$

and so $N_{i\sigma(i)}(b) \leq N_{i\sigma(i)}(a)$.

Thus, $N_{i\sigma(i)}$ is a negation on $[0,1]$.
In a similar way, we can prove that $N^*_{i\sigma(i)}$ is a negation.

2. Now, let us see that $N_{i\sigma(i)}$ and $N^*_{i\sigma(i)}$ are bijective functions.
   - $N_{i\sigma(i)}$ is injective: let $a, b \in [0,1]$ such that $a \neq b$, then $(0,\ldots,0,\overset{i}{a},0,\ldots,0) \neq (0,\ldots,0,\overset{i}{b},0,\ldots,0)$, therefore $\alpha(0,\ldots,0,\overset{i}{a},0,\ldots,0) \neq \alpha(0,\ldots,0,\overset{i}{b},0,\ldots,0)$ since α is a bijective function, and so $(1,\ldots,1,\overset{\sigma(i)}{N_{i\sigma(i)}(a)},1,\ldots,1) \neq (1,\ldots,1,\overset{\sigma(i)}{N_{i\sigma(i)}(b)},1,\ldots,1)$. Thus, $N_{i\sigma(i)}(a) \neq N_{i\sigma(i)}(b)$.
   - $N_{i\sigma(i)}$ is surjective: Let $b \in [0,1]$. We have $(1,\ldots,\overset{\sigma(i)}{b},\ldots,1) \in [0,1]^n$. Then $\exists\ (a_1,\ldots,a_n) \in [0,1]^n$ such that $\alpha(a_1,\ldots,a_n) = (1,\ldots,\overset{\sigma(i)}{b},\ldots,1)$, since α is surjective. Taking into account that $\alpha(\alpha(a_1,\ldots,a_n)) = \alpha(1,...,\overset{\sigma(i)}{b},\ldots,1) = (0,\ldots,\overset{\sigma(\sigma(i))}{a},\ldots,0) = (0,\ldots,\overset{i}{a},\ldots,0)$, it is $\alpha(0,\ldots,\overset{i}{a},\ldots,0) = (1,\ldots,\overset{\sigma(i)}{b},\ldots,1)$. So, $N_{i\sigma(i)}(a) = b$.
   - $(N_{\sigma(i)i})^{-1} = N^*_{i\sigma(i)}\ \forall i = 1,\ldots,n$, or $(N_{i\sigma(i)})^{-1} = N^*_{\sigma(i)i}$.
     $(1,\ldots,\overset{i}{a},\ldots,1) = \alpha(\alpha(1,\ldots,\overset{i}{a},\ldots,1)) = \alpha(0,\ldots,\overset{\sigma(i)}{N^*_{i\sigma(i)}(a)},\ldots,0) = (1,\ldots,N_{\sigma(i)\sigma(\sigma(i))}\left(N^*_{i\sigma(i)}(a)\right),\ldots 1)$.
     Then $N_{\sigma(i)i}\left(N^*_{i\sigma(i)}(a)\right) = a\ \forall a \in [0,1]$, and $N^*_{i\sigma(i)} = (N_{\sigma(i)i})^{-1}$.

□

Finally, we get the desired result, a characterization of involutions in $[0,1]^n$.

Theorem 3. (Characterization Theorem) $\alpha : [0,1]^n \to [0,1]^n$ *is an involution if and only if there exists a permutation* $\sigma : \{1,\ldots,n\} \to \{1,\ldots,n\}$, *with* $\sigma \circ \sigma = Id$, *and* n *negations in* $[0,1]$, $N_{\sigma(1)},\ldots,N_{\sigma(i)},\ldots,N_{\sigma(n)}$ *such that, for all* $(a_1,\ldots,a_n)$ *it is* $\alpha(a_1,\ldots,a_n) = (N_{\sigma(1)}(a_{\sigma(1)}),\ldots,N_{\sigma(n)}(a_{\sigma(n)}))$ *satisfying that if* $\sigma(i) = i$, *then* $N_{\sigma(i)}$ *is a strong negation, and if* $\sigma(i) \neq i$, *then* $N_{\sigma(i)} = (N_i)^{-1}$ *with* $N_{\sigma(i)}$ *and* N_i *negations not necessarily strong.*

Proof. ($\Rightarrow$) Let $\alpha : [0,1]^n \to [0,1]^n$ be an involution. Considering the associated permutation σ and the negations $N_{\sigma(i)i}$ and $N^*_{i\sigma(i)}$, we have that $\forall i = 1, 2, \ldots, n$,

$$(0, \ldots, a_i, \ldots, 0) \leq (a_1, \ldots, a_i, \ldots, a_n) \Rightarrow$$

$$\alpha(a_1, \ldots, a_i, \ldots, a_n) \leq (1, \ldots, \overset{\sigma(i)}{N_{i\sigma(i)}(a_i)}, \ldots, 1) = (1, \ldots, \overset{j=\sigma(i)}{N_{\sigma(j)j}(a_{\sigma(j)})}, \ldots, 1),$$

and consequently

$$\alpha(a_1, \ldots, a_j, \ldots, a_n) \leq (N_{\sigma(1)1}(a_{\sigma(1)}), \ldots, N_{\sigma(j)j}(a_{\sigma(j)}), \ldots, N_{\sigma(n)n}(a_{\sigma(n)})). \tag{1}$$

Moreover, for any $i \in \{1, \ldots, n\}$,

$$(a_1, \ldots, a_i, \ldots, a_n) \leq (1, \ldots, a_i, \ldots 1) \Rightarrow$$

$$(0, \ldots, \overset{\sigma(i)}{N^*_{i\sigma(i)}(a_i)}, \ldots, 0) = (0, \ldots, \overset{j=\sigma(i)}{N^*_{\sigma(j)j}(a_{\sigma(j)})}, \ldots, 0) \leq \alpha(a_1, \ldots, a_j, \ldots, a_n),$$

and consequently

$$(N^*_{\sigma(1)1}(a_{\sigma(1)}), \ldots, N^*_{\sigma(j)j}(a_{\sigma(j)}), \ldots, N^*_{\sigma(n)}(a_{\sigma(n)})) \leq \alpha(a_1, \ldots, a_n). \tag{2}$$

Then, for all $(a_1, \ldots, a_n)$ we have that:

$$(a_1, \ldots, a_n) = \alpha(\alpha(a_1, \ldots, a_n)) \leq \text{ (according to inequality 2)}$$

$$\alpha(N^*_{\sigma(1)1}(a_{\sigma(1)}), \ldots, N^*_{\sigma(j)j}(a_{\sigma(j)}), \ldots, N^*_{\sigma(n)}(a_{\sigma(n)})) \leq \text{ (by inequality 1)}$$

$$\left(N_{\sigma(1)1}(N^*_{1\sigma(1)}(a_1)), \ldots, N_{\sigma(i)i}(N^*_{i\sigma(i)}(a_i)), \ldots, N_{\sigma(n)n}(N^*_{n\sigma(n)}(a_n))\right) = (a_1, \ldots, a_n)$$

(considering now that $N^*_{i\sigma(i)} = (N_{\sigma(i)i})^{-1}$).
Consequently, $(a_1, \ldots, a_n) = \alpha(N^*_{\sigma(1)1}(a_{\sigma(1)}), \ldots, N^*_{\sigma(i)i}(a_{\sigma(i)}), \ldots, N^*_{\sigma(n)n}(a_{\sigma(n)}))$, and $\alpha(a_1, \ldots, a_n) = (N^*_{\sigma(1)1}(a_{\sigma(1)}), \ldots, N^*_{\sigma(i)i}(a_{\sigma(i)}), \ldots, N^*_{\sigma(n)n}(a_{\sigma(n)}))$.

More, as α is involutive, we have:
$\alpha(\alpha(a_1, \ldots, a_n)) = (a_1, \ldots, a_n) =$
$\alpha\left(N^*_{\sigma(1)1}(a_{\sigma(1)}), \ldots, N^*_{\sigma(i)i}(a_{\sigma(i)}), \ldots, N^*_{\sigma(n)n}(a_{\sigma(n)})\right) =$
$\left(N^*_{\sigma(1)1}(N^*_{1\sigma(1)}(a_1)), \ldots, N^*_{\sigma(i)i}(N^*_{i\sigma(i)}(a_i)), \ldots, N^*_{\sigma(n)n}(N^*_{n\sigma(n)}(a_n))\right) = (a_1, \ldots, a_n)$.
This implies that $\forall i = 1, \ldots, n$, $N^*_{\sigma(i)i}(N^*_{i\sigma(i)}(a_i)) = a_i \Rightarrow N^*_{\sigma(i)i} = (N^*_{i\sigma(i)})^{-1}$.
More, if $\sigma(i) = i$, then N^*_{ii} is a strong negation.
In addition, taking $N_{\sigma(i)} = N^*_{\sigma(i)i}$, we obtain the proof of the necessity condition.

($\Leftarrow$) Let us suppose that there exists a permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$, with $\sigma \circ \sigma = Id$, and n negations in $[0,1]$, $N_{\sigma(1)}, \ldots, N_{\sigma(i)}, \ldots, N_{\sigma(n)}$ such that, for all $(a_1, \ldots, a_n)$ it is $\alpha(a_1, \ldots, a_n) = (N_{\sigma(1)}(a_{\sigma(1)}), \ldots, N_{\sigma(n)}(a_{\sigma(n)}))$ satisfying that if $\sigma(i) = i$, then $N_{\sigma(i)}$ is a strong negation, and if $\sigma(i) \neq i$, then $N_{\sigma(i)} = (N_i)^{-1}$, with $N_{\sigma(i)}$ and N_i negations not necessarily strong.

Let us prove that α is an involution.

- $\alpha(0,\ldots,0) = (N_{\sigma(1)}(0),\ldots,N_{\sigma(n)}(0)) = (1,\ldots,1)$.
- $\alpha(1,\ldots,1) = (N_{\sigma(1)}(1),\ldots,N_{\sigma(n)}(1)) = (0,\ldots,0)$.
- If $(a_1,\ldots,a_n) \le (b_1,\ldots,b_n)$, $N_{\sigma(i)}(a_{\sigma(i)}) \ge N_{\sigma(i)}(b_{\sigma(i)})$ for all $i \in \{1,\ldots,n\}$, $(N_{\sigma(1)}(a_{\sigma(1)}),\ldots,N_{\sigma(n)}(a_{\sigma(n)})) \ge (N_{\sigma(1)}(b_{\sigma(1)}),\ldots,N_{\sigma(n)}(b_{\sigma(n)}))$, and $\alpha(a_1,\ldots,a_n) \ge \alpha(b_1,\ldots,b_n)$.
- Finally, $\alpha(\alpha(a_1,\ldots,a_n)) = \alpha(N_{\sigma(1)}(a_{\sigma(1)}),\ldots,N_{\sigma(n)}(a_{\sigma(n)}))$ $= \big(N_{\sigma(1)}(N_{\sigma(\sigma(1))}(a_{\sigma(\sigma(1))})),\ldots,N_{\sigma(n)}(N_{\sigma(\sigma(n))}(a_{\sigma(\sigma(n))}))\big)$ $= \big(N_{\sigma(1)}(N_1(a_1)),\ldots,N_{\sigma(n)}(N_n(a_n)\big) = (a_1,\ldots,a_n)$.

□

4 Maintaining the antonym below the complement

It is important to remind that in Example 1 we stated that 'low level' must imply 'not high level', but not the other way around. Furthermore, in their definition of N-antonym, De Soto et al. in [12] and Trillas et al. in [20], forced the antonym to remain below the complement.

Does any involution defined according to Theorem 3 generate an antonym satisfying this condition?

To consider this question let us continue defining Example 1.

Let us consider that in order to qualify a student, three exam tests are considered in such a way that: the first one corresponds to a part of the subject (weighted as β), the second one to another part (weighted as γ), and the third one to the remaining contents (consequently weighted as $1-\beta-\gamma$), with $0 < \beta,\gamma,\beta+\gamma < 1$. In that framework we can establish the degree in which the student has a 'high level' through the function

$$\aleph_{highlevel}(x_1,x_2,x_3) = \beta x_1 + \gamma x_2 + (1-\beta-\gamma)x_3 \in [0,1]$$

where x_1, x_2 and x_3 are the scores obtained in each of the three tests.

From this point, to obtain the degree in which a student has a 'low level', we need an involution α on the set $[0,1]^3$, and to obtain the degree in which it has a 'non-high level' we need a fuzzy negation N in $[0,1]$. Then we obtain the following membership functions.

$$\aleph_{lowlevel}(x_1,x_2,x_3) = \aleph_{highlevel}(\alpha(x_1,x_2,x_3)),$$

$$\aleph_{non-highlevel}(x_1,x_2,x_3) = N(\aleph_{highlevel}(x_1,x_2,x_3)).$$

Now we can ask how the involution α, and the fuzzy negation N, should be defined to ensure the previously mentioned condition being satisfied in our model. That is, to attain $\aleph_{lowlevel}(x_1,x_2,x_3) \le \aleph_{non-highlevel}(x_1,x_2,x_3)$.

We have to consider that the model must be valid for any distribution of the test, that is for any value of β and γ such that $0 < \beta,\gamma,\beta+\gamma < 1$. Without trying to cover all possible options, in what follows we present the study of some cases.

Among the different options to define the complement, the common choice is considering a strong negation. So we will define N as a strong negation. On the other hand, Theorem 2 offers four possibilities to choose the involution (α) defining the antonym. Those four options can be grouped into two: using three strong negations (the first option in Theorem 2) and using a strict negation plus a strong negation (the three other options).

We will first consider involutions built upon three strong negations.

1. The simplest case is the one considering the standard fuzzy negation (being a strong negation) both for N and for the strong negations in α. If we take $N_1 = N_2 = N_3 = N = 1 - Id$, then we have

$$\begin{aligned}\alpha(x_1, x_2, x_3) &= (N_1(x_1), N_2(x_2), N_3(x_3)) \\ &= (1 - x_1, 1 - x_2, 1 - x_3),\end{aligned}$$

   and consequently

$$\begin{aligned}\aleph_{lowlevel}(x_1, x_2, x_3) &= \aleph_{highlevel}(\alpha(x_1, x_2, x_3)) \\ &= \beta(1 - x_1) + \gamma(1 - x_2) + (1 - \beta - \gamma)(1 - x_3) \\ &= 1 - \beta x_1 - \gamma x_2 - (1 - \beta - \gamma)x_3,\end{aligned}$$

   On the other hand

$$\begin{aligned}\aleph_{non-highlevel}(x_1, x_2, x_3) &= N(\aleph_{highlevel}(x_1, x_2, x_3)) \\ &= 1 - (\beta x_1 + \gamma x_2 + (1 - \beta - \gamma)x_3)\,.\end{aligned}$$

   Therefore, in this case, for all (x_1, x_2, x_3)

$$\aleph_{lowlevel}(x_1, x_2, x_3) = \aleph_{non-highlevel}(x_1, x_2, x_3).$$

   This is the boundary of those selections attaining the target inequality. From this point, if we want to achieve a strict inequality there are two options: moving the antonym down or the complement up.

   (a) To move the antonym down it would be enough to take N_1, N_2, N_3 strong fuzzy negations smaller than the standard fuzzy negation so that $\aleph_{lowlevel} \leq \aleph_{non-highlevel}$. For example, Sugeno's negation,

$$N_\lambda(x) = \frac{1 - x}{1 + \lambda x}, \text{ with } \lambda \geq 0.$$

   (b) To move the complement up, it would be enough to take a fuzzy negation N greater than the standard fuzzy negation. For example, the strong fuzzy negation

$$N_p(x) = (1 - x^p)^{1/p}\,, \text{ with } p \geq 2,$$

   or Sugeno's negation,

$$N_\lambda(x) = \frac{1 - x}{1 + \lambda x}, \text{ with } -1 < \lambda < 0.$$

Both options ensure the antonym being below the complement.

2. Let us consider again strong fuzzy negations for N_1, N_2, N_3, and N. In this case we choose N being any concave fuzzy negation[1], with $N_1 \not\leq N$. In this situation we have

$$\alpha(x_1, x_2, x_3) = (N_1(x_1), N_2(x_2), N_3(x_3)).$$

Let $a \in (0, 1)$ such that $N_1(a) > N(a)$, and we take $x_2 = x_3 = 1$,

$$\aleph_{lowlevel}(a, 1, 1) = \aleph_{highlevel}(N_1(a), 0, 0) = \beta N_1(a),$$

and by the properties of concave functions it is

$$\begin{aligned} \aleph_{non-highlevel}(a, 1, 1) &= N\left(\beta a + \gamma + (1 - \beta - \gamma)\right) \\ &= N(\beta a + (1 - \beta)1) \end{aligned}$$

$$\leq \beta N(a) + (1 - \beta)N(1) = \beta N(a) < \beta n_1(a).$$

Therefore, $\aleph_{lowlevel}(a, 1, 1) > \aleph_{non-highlevel}(a, 1, 1)$, and the desired relation among antonym and negation fails:

$$\aleph_{lowlevel} \not\leq \aleph_{non-highlevel}.$$

3. Another option, again with N_1, N_2, N_3, and N strong negations, is considering $N_1, N_2, N_3 \leq N$ and N a convex fuzzy negation[2], then by the properties of convex functions it is

$$\begin{aligned} &\beta N(x_1) + \gamma N(x_2) + (1 - \beta - \gamma)N(x_3) \\ &\leq N(\beta x_1 + \gamma x_2 + (1 - \beta - \gamma)x_3). \end{aligned}$$

Therefore,

$$\begin{aligned} &\aleph_{lowlevel}(x_1, x_2, x_3) \\ &= \beta N_1(x_1) + \gamma N_2(x_2) + (1 - \beta - \gamma)N_3(x_3) \\ &\leq \beta N(x_1) + \gamma N(x_2) + (1 - \beta - \gamma)N(x_3) \\ &\leq \aleph_{non-highlevel}(x_1, x_2, x_3). \end{aligned}$$

In this case the desired relation among antonym and complement is achieved.

Once that involutions built upon three strong fuzzy negations have been considered, let us present now a case using a strict fuzzy negation and a strong fuzzy negation.

[1] In the fuzzy framework, f is a concave function on $[0, 1]$ if for all $x, y, \lambda \in [0, 1]$, $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$.

[2] f is a convex function on $[0, 1]$ if for all $x, y, \lambda \in [0, 1]$, $f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y)$.

4. Let us consider now an antonym based on a strict fuzzy negation ($N_1 = (N_2)^{-1}$), and a strong fuzzy negation (N_3), plus a strong fuzzy negation N (any strict fuzzy negation could also be used in this case). This is a situation linked to the second option in Theorem 2 but could be adapted to the third and fourth options as well.
   As the relation among antonym and complement should be maintained for any weighting of the three tests, that is, for any value of β and γ ($0 < \beta, \gamma, \beta + \gamma < 1$), let us consider that β and γ are such that $N(\beta) = \beta$ and $\gamma = \frac{1-\beta}{2}$ (then, $1 - \beta - \gamma = \gamma$). In this situation,
$$\alpha(x_1, x_2, x_3) = (N_1(x_2), N_2(x_1), N_3(x_3)),$$
$$\aleph_{lowlevel}(x_1, x_2, x_3) = \beta N_1(x_2) + \gamma N_2(x_1) + \gamma N_3(x_3),$$
$$\aleph_{non-highlevel}(x_1, x_2, x_3) = N(\beta x_1 + \gamma x_2 + \gamma x_3).$$
   Assume now that the scores are $x_2 = 0$, $x_1 = x_3 = 1$, it is
$$\aleph_{lowlevel}(1, 0, 1) = \beta, \text{ and}$$
$$\aleph_{non-highlevel}(1, 0, 1) = N(\beta + \gamma) < N(\beta) = \beta.$$

Therefore,
$\aleph_{lowlevel}(1, 0, 1) > \aleph_{non-highlevel}(1, 0, 1)$, and so $\aleph_{lowlevel} \not\leq \aleph_{non-highlevel}$.

We will finally consider one of the previous cases with some specific choices and values.

Example 2. Let us take three strong fuzzy negations of Sugeno $N_1, N_2, N_3 : [0, 1] \to [0, 1]$ such that,
$$N_1(x) = \frac{1-x}{1+x}, \quad N_2(x) = \frac{1-x}{1+2x}, \quad N_3(x) = 1 - x,$$
let us consider the involution $\alpha : [0, 1]^3 \to [0, 1]^3$, given by
$$\begin{aligned}\alpha(x_1, x_2, x_3) &= (N_1(x_1), N_2(x_2), N_3(x_3)) \\ &= \left(\tfrac{1-x_1}{1+x_1}, \tfrac{1-x_2}{1+2x_2}, 1 - x_3\right),\end{aligned}$$
and N the strong fuzzy negation
$$N(x) = \sqrt{1 - x^2}$$
Let us suppose an assessment in such a way that the first exam test covers (weights) $\frac{1}{3}$ of the subject, the second one $\frac{1}{2}$, and the third one $\frac{1}{6}$. Then we can establish the degree in which the student has a 'high level' through the function
$$\aleph_{highlevel}(x_1, x_2, x_3) = \frac{1}{3}x_1 + \frac{1}{2}x_2 + \frac{1}{6}x_3$$

Therefore, we obtain the membership functions

$$\begin{aligned}\aleph_{lowlevel}(x_1,x_2,x_3) &= \aleph_{highlevel}(\alpha(x_1,x_2,x_3))\\ &= \aleph_{highlevel}(N_1(x_1),N_2(x_2),N_3(x_3))\\ &= \aleph_{highlevel}\left(\tfrac{1-x_1}{1+x_1},\tfrac{1-x_2}{1+2x_2},1-x_3\right)\\ &= \tfrac{1}{3}\left(\tfrac{1-x_1}{1+x_1}\right)+\tfrac{1}{2}\left(\tfrac{1-x_2}{1+2x_2}\right)+\tfrac{1}{6}(1-x_3),\end{aligned}$$

and

$$\begin{aligned}\aleph_{non-highlevel}(x_1,x_2,x_3) &= N(\aleph_{highlevel}(x_1,x_2,x_3))\\ &= \sqrt{1-\left(\tfrac{1}{3}x_1+\tfrac{1}{2}x_2+\tfrac{1}{6}x_3\right)^2}.\end{aligned}$$

For example, if the three test scores of a student are $(0.84, 0.80, 0.90)$, we have that

$$\begin{aligned}\aleph_{highlevel}(0.84,0.80,0.90) &= \tfrac{1}{3}\cdot 0.84+\tfrac{1}{2}\cdot 0.80+\tfrac{1}{6}\cdot 0.90\\ &= 0.28+0.40+0.15=0.83,\end{aligned}$$

and

$$\begin{aligned}&\aleph_{lowlevel}(0.84,0.80,0.90)\\ &= \tfrac{1}{3}\cdot 0.087+\tfrac{1}{2}\cdot 0.077+\tfrac{1}{6}\cdot 0.10=0.084.\end{aligned}$$

Moreover, the degree in which the student has 'non-high level' is

$$\begin{aligned}&\aleph_{non-highlevel}(0.84,0.80,0.90)\\ &= N(\aleph_{highlevel}(0.84,0.80,0.90))\\ &= N(0.83)=\sqrt{1-0.83^2}=0.5577.\end{aligned}$$

Let us observe that, according to our required condition, the degree in which the student has a *lowlevel* is less than the degree in which he/she has a $non-highlevel$.

In summary, we can say that given an involution defined according to Theorem 3, and a negation, the antonym generated by the involution and the complement defined by the negation will not necesarilly satisfy the condition of maintaining the antonym below the complement. In fact, in the case of choosing a strict fuzzy negation and its inverse, exchanging the order of the variables, it is always possible to find out an evaluation for which the requirement is not satisfied.

5 Conclusions

In this paper we have reconsidered the modeling of antonyms in fuzzy logic and its extensions. In this sense, in many papers throughout the literature, the determination of an antonym involves involutions in the universe. In particular, the antonyms of predicates defined on n-tuples of fuzzy sets need involutions on the set $[0,1]^n$. Therefore, it is of interest to characterize them, which is not an easy task. As far as we know, this has only been done in a previous paper for $n = 2$.

In this work, an example has been first presented showing the difference between the complement and the antonym of a predicate and their modelization.

Then, a characterization of involutions on the set $[0,1]^3$ has been attained. This is considered as a preliminary step to be able to reach the more general case $[0,1]^n$. In fact, the way in which an involution in $[0,1]^2$ or in $[0,1]^3$ can be expressed by means of strict or strong fuzzy negations, has been the guiding steps for a generalization to the case of $[0,1]^n$, being the main result of this paper.

Finally, the relative and intuitive ordering among antonym and complement (antonym should be below complement), has been considered. In fact, not all antonyms obtained through involutions satisfy this condition. For this reason, we have carried out an initial study with some specific cases in $[0,1]^3$, that is, in predicates with three variables.

A last comment is that one could have considered studying the involutions in the ordered set of the real numbers R. In this case, as R is not bounded, only the conditions of monotony and involution would be required. This study is beyond the scope of this article, but in a future research we can expect that the results would be similar.

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