

Properties of several metric spaces of fuzzy sets ^{*}

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Abstract

This paper discusses the properties the spaces of fuzzy sets in a metric space equipped with the endograph metric and the sendograph metric, respectively. We first give some relations among the endograph metric, the sendograph metric and the Γ -convergence, and then investigate the level characterizations of the endograph metric and the Γ -convergence. By using the above results, we give some relations among the endograph metric, the sendograph metric, the supremum metric and the d_p^* metric, $p \geq 1$. On the basis of the above results, we present the characterizations of total boundedness, relative compactness and compactness in the space of fuzzy sets whose α -cuts are compact when $\alpha > 0$ equipped with the endograph metric, and in the space of compact support fuzzy sets equipped with the sendograph metric, respectively. Furthermore, we give completions of these metric spaces, respectively.

Keywords: Endograph metric; Sendograph metric; Hausdorff metric; Total boundedness; Relative compactness; Compactness; Completion

1. Introduction

Fuzzy set is a fundamental tool to investigate fuzzy phenomenon [3, 4, 7, 10, 11, 26–28]. A fuzzy set can be identified with its endograph. Also, a fuzzy set can be identified with its sendograph. So convergence structures on fuzzy sets can be defined on their endographs or sendographs. The the endograph metric H_{end} convergence, the sendograph metric H_{send} convergence and the Γ -convergence are this kind of convergence structures. These three convergence structures are related to each other [14, 23].

^{*}Project supported by Natural Science Foundation of Fujian Province of China(No. 2020J01706)

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The endograph metric on fuzzy sets is shown to have significant advantages [21, 22]. The sendograph metric has attracted deserving attentions [5, 8]. Compactness is one of the central concepts in topology and analysis and is useful in applications (see [18, 26]). There is a lot of work devoted to characterizations of compactness in various fuzzy set spaces endowed with different topologies [5, 8, 9, 13, 14, 24, 25, 29]. It is natural to consider what the completion of a metric space is. The recent results on completions of fuzzy set spaces include [13, 14].

In [14], we presented the relations and level characterizations of the endograph metric H_{end} and the Γ -convergence. Based on this, we have given the characterizations of total boundedness, relative compactness and compactness of fuzzy set spaces equipped with the endograph metric H_{end} . We also gave the completions of fuzzy set spaces according to the endograph metric H_{end} .

The common fuzzy sets used in theoretical research and practical applications are fuzzy sets in a metric space whose α -cuts are nonempty compact sets when $\alpha > 0$. Common compact fuzzy sets are common fuzzy sets whose support sets are compact. Throughout this paper, we suppose that X is a nonempty set and d is the metric on X . For simplicity, we also use X to denote the metric space (X, d) . The symbols $F_{USCG}^1(X)$ and $F_{USCB}^1(X)$ are used to denote the family of common fuzzy sets in X and the family of common compact fuzzy sets in X , respectively. We use $F_{USC}^1(X)$ to denote the family of normal and upper semi-continuous fuzzy sets in X . $F_{USCB}^1(X)$ is a subset of $F_{USCG}^1(X)$. $F_{USCG}^1(X)$ is a subset of $F_{USC}^1(X)$.

The results in [14] are obtained on the realm of fuzzy sets in the m -dimensional Euclidean space \mathbb{R}^m (\mathbb{R}^1 is also written as \mathbb{R}). \mathbb{R}^m is a special type of metric space. Of course, it is worth to study the fuzzy sets in a metric space [8, 9, 17]. In this paper, the results are obtained on the realm of fuzzy sets in a general metric space X . We mainly discuss H_{end} metric and H_{send} metric on $F_{USC}^1(X)$ including the relations among H_{end} metric, H_{send} metric and other convergence structures, and properties of H_{end} metric and H_{send} metric.

The first part of this paper is devoted to the relations among the H_{end} metric, the H_{send} metric and the Γ -convergence, the level characterizations of the endograph metric H_{end} and the Γ -convergence, and the relations among the supremum metric d_∞ , the H_{end} metric, the H_{send} metric and the d_p^* metric. The d_p^* metric is an expansion of the L_p -type d_p metric on $F_{USC}^1(X)$.

To aid discussion, we introduce the sets $P_{USC}^1(X)$ and $P_{USCB}^1(X)$. An

element of $P_{USC}^1(X)$ (respectively, $P_{USCB}^1(X)$) is a subset of $X \times [0, 1]$, which is equal to or in a specific way slightly larger than the sendograph of a certain fuzzy set in $F_{USC}^1(X)$ (respectively, $F_{USCB}^1(X)$). $F_{USC}^1(X)$ and $F_{USCB}^1(X)$ can be viewed as the subsets of $P_{USC}^1(X)$ and $P_{USCB}^1(X)$, respectively. $P_{USCB}^1(X)$ is a subset of $P_{USC}^1(X)$.

We define the H_{send} distance and the H_{end} distance on $P_{USC}^1(X)$, and give the relations among the H_{send} distance, the H_{end} distance and the Kuratowski convergence on $P_{USC}^1(X)$. Then, as corollaries, we obtain the relations among the H_{send} metric, the H_{end} metric and the Γ -convergence on $F_{USC}^1(X)$.

We discuss the level characterizations of the Γ -convergence and the endograph metric H_{end} on fuzzy sets in $F_{USC}^1(X)$. It is shown that under some conditions, the Γ -convergence of fuzzy sets can be decomposed to the Kuratowski convergence of certain α -cuts, and the H_{end} metric convergence of fuzzy sets can be decomposed to the Hausdorff metric convergence of certain α -cuts.

The understanding of the relations among the H_{end} metric, the H_{send} metric and the Γ -convergence is beneficial for the understanding of themselves. The level characterizations help to study these three convergence structures on fuzzy sets by using the properties of the corresponding α -cuts.

A H_{send} metric convergent sequence must be a H_{end} metric convergent sequence. A H_{end} metric convergent sequence must be a Γ -convergent sequence. So the knowledge of the Γ -convergent sequences can help us to analyse the properties of the H_{end} convergent sequences and the H_{send} convergent sequences. For this reason, we give the level characterizations of a Γ -convergent sequence in this paper.

Based on the results in the first part, we give the other results of this paper. The second part of this paper is devoted to the characterizations of total boundedness, relative compactness and compactness in $(F_{USCG}^1(X), H_{\text{end}})$ and $(F_{USCB}^1(X), H_{\text{end}})$, respectively. Here we mention that the characterization of relative compactness in $(F_{USCB}^1(X), H_{\text{send}})$ has already been given by Greco [8].

The total boundedness is the key property of compactness in metric space. We show that a set U in $(F_{USCG}^1(X), H_{\text{end}})$ is totally bounded (respectively, relatively compact) if and only if for each $\alpha \in (0, 1]$, the union of all the α -cuts of $u \in U$ is totally bounded (respectively, relatively compact) in (X, d) . We also show that a set U in $(F_{USCB}^1(X), H_{\text{send}})$ is totally bounded if and only if the union of all the 0-cuts of $u \in U$ is totally bounded in (X, d) (see Section 7). These results indicate that for a set U in $(F_{USCG}^1(X), H_{\text{end}})$ or

$(F_{USCB}^1(X), H_{\text{send}})$, the total boundedness, relative compactness and compactness of U are closely related to the total boundedness, relative compactness and compactness of the union of all the α -cuts of $u \in U$ in (X, d) , respectively.

We point out that some part of the proof of the characterizations in this paper is similar to the corresponding part in [14]. But in general, since a set in X need not have the properties of the set in \mathbb{R}^m , the proof of the conclusions in this paper requires deep understandings of the problem.

The third part is devoted to the completions of several common fuzzy set spaces under the H_{end} metric and the H_{send} metric, respectively.

Let \tilde{X} denote the completion of X . We show that the space $(P_{USCB}^1(\tilde{X}), H_{\text{send}})$ is a completion of the fuzzy set space $(F_{USCB}^1(X), H_{\text{send}})$. Then we show that $(F_{USCG}^1(\tilde{X}), H_{\text{end}})$ is a completion of $(F_{USCB}^1(X), H_{\text{end}})$. So, of course, $(F_{USCG}^1(\tilde{X}), H_{\text{end}})$ is also a completion of $(F_{USCG}^1(X), H_{\text{end}})$.

These conclusions indicate that in the case of the H_{end} metric, a completion of the space of common compact fuzzy set in X is the space of common fuzzy set in \tilde{X} ; in the case of the H_{send} metric, a completion of the space of common compact fuzzy set in X is a metric space in which the space of common compact fuzzy set in \tilde{X} can be isometrically embedded, and each element of which is a nonempty compact set in $\tilde{X} \times [0, 1]$.

The conclusions for the completions of the spaces of fuzzy set in X given in this paper apply to not only the cases that X is a complete metric space but also the cases that X is an incomplete metric space.

The remainder of this paper is organized as follows. In Section 2, we recall and give some basic notions and fundamental results related to fuzzy sets and convergence structures on them. In Section 3, we discuss the properties and relations of H_{send} , H_{end} and Kuratowski convergence on $P_{USC}^1(X)$. Based on this, we give some relations among H_{end} , H_{send} and Γ -convergence on $F_{USC}^1(X)$. In Sections 4 and 5, we investigate the level characterizations of the Γ -convergence and the H_{end} convergence, respectively. By using the above results, Section 6 discusses some relations among the d_∞ metric, the d_p^* metric, the H_{end} metric and the H_{send} metric. In Section 7, on the basis of the conclusions in previous sections, we give characterizations of total boundedness, relative compactness and compactness in $(F_{USCG}^1(X), H_{\text{end}})$ and $(F_{USCB}^1(X), H_{\text{send}})$, respectively. In Section 8, we give completions of $(F_{USCG}^1(X), H_{\text{end}})$ and $(F_{USCB}^1(X), H_{\text{send}})$, respectively. At last, we draw the conclusions in Section 9.

2. Fuzzy sets and convergence structures on them

In this section, we recall and give some basic notions and fundamental results related to fuzzy sets and convergence structures on them. Readers can refer to [4, 19, 28] for related contents.

A *fuzzy set* u in X can be seen as a function $u : X \rightarrow [0, 1]$. A subset S of X can be seen as a fuzzy set in X . If there is no confusion, the fuzzy set corresponding to S is often denoted by χ_S ; that is,

$$\chi_S(x) = \begin{cases} 1, & x \in S, \\ 0, & x \in X \setminus S. \end{cases}$$

For simplicity, for $x \in X$, we will use \hat{x} to denote the fuzzy set $\chi_{\{x\}}$ in X . In this paper, if we want to emphasize a specific metric space X , we will write the fuzzy set corresponding to S in X as $S_{F(X)}$, and the fuzzy set corresponding to $\{x\}$ in X as $\hat{x}_{F(X)}$.

The symbol $F(X)$ is used to denote the set of all fuzzy sets in X . For $u \in F(X)$ and $\alpha \in [0, 1]$, let $\{u > \alpha\}$ denote the set $\{x \in X : u(x) > \alpha\}$, and let $[u]_\alpha$ denote the α -cut of u , i.e.

$$[u]_\alpha = \begin{cases} \{x \in X : u(x) \geq \alpha\}, & \alpha \in (0, 1], \\ \text{supp } u = \overline{\{u > 0\}}, & \alpha = 0, \end{cases}$$

where \overline{S} denotes the topological closure of S in (X, d) .

For $u \in F(X)$, define

$$\begin{aligned} \text{end } u &:= \{(x, t) \in X \times [0, 1] : u(x) \geq t\}, \\ \text{send } u &:= \{(x, t) \in X \times [0, 1] : u(x) \geq t\} \cap ([u]_0 \times [0, 1]). \end{aligned}$$

$\text{end } u$ and $\text{send } u$ are called the *endograph* and the *sendograph* of u , respectively.

The symbol $K(X)$ and $C(X)$ are used to denote the set of all nonempty compact subsets of X and the set of all nonempty closed subsets of X , respectively. The symbol \mathbb{N} is used to denote the set of all positive integers.

Let $F_{USC}^1(X)$ denote the set of all normal and upper semi-continuous fuzzy sets $u : X \rightarrow [0, 1]$, i.e.,

$$F_{USC}^1(X) := \{u \in F(X) : [u]_\alpha \in C(X) \text{ for all } \alpha \in [0, 1]\}.$$

We introduce some subclasses of $F_{USC}^1(X)$, which will be discussed in this paper. Define

$$\begin{aligned} F_{USCB}^1(X) &:= \{u \in F_{USC}^1(X) : [u]_0 \in K(X)\}, \\ F_{USCG}^1(X) &:= \{u \in F_{USC}^1(X) : [u]_\alpha \in K(X) \text{ for all } \alpha \in (0, 1]\}. \end{aligned}$$

Clearly,

$$F_{USCB}^1(X) \subseteq F_{USCG}^1(X) \subseteq F_{USC}^1(X).$$

The following representation theorems for $F_{USCB}^1(X)$, $F_{USCG}^1(X)$, $F_{USC}^1(X)$ are similar to Theorem 3.3 in [14].

Proposition 2.1. *Let $u \in F_{USCB}^1(X)$ (respectively, $u \in F_{USCG}^1(X)$, $u \in F_{USC}^1(X)$). Then (i) $[u]_\alpha \in K(X)$ for all $\alpha \in [0, 1]$ (respectively, $[u]_\alpha \in K(X)$ for all $\alpha \in (0, 1]$, $[u]_\alpha \in C(X)$ for all $\alpha \in [0, 1]$); (ii) $[u]_\alpha = \bigcap_{\beta < \alpha} [u]_\beta$ for all $\alpha \in (0, 1]$; (iii) $[u]_0 = \overline{\bigcup_{\alpha > 0} [u]_\alpha}$.*

Moreover, if the family of sets $\{v_\alpha : \alpha \in [0, 1]\}$ satisfies (i) through (iii), then there exists a unique $u \in F_{USCB}^1(X)$ (respectively, $u \in F_{USCG}^1(X)$, $u \in F_{USC}^1(X)$) such that $[u]_\alpha = v_\alpha$ for each $\alpha \in [0, 1]$.

Using Proposition 2.1, we can define a certain type fuzzy set by giving the family of its α -cuts. For brevity, in the sequel, we often directly point out that what we defined is a certain type fuzzy set without mentioning the use of Proposition 2.1 since it is easy to see.

Let (X, d) be a metric space. We use \mathbf{H} to denote the **Hausdorff distance** on $C(X)$ induced by d , i.e.,

$$\mathbf{H}(U, V) = \max\{H^*(U, V), H^*(V, U)\}$$

for arbitrary $U, V \in C(X)$, where

$$H^*(U, V) = \sup_{u \in U} d(u, V) = \sup_{u \in U} \inf_{v \in V} d(u, v).$$

The metric \bar{d} on $X \times [0, 1]$ is defined as follows: for $(x, \alpha), (y, \beta) \in X \times [0, 1]$,

$$\bar{d}((x, \alpha), (y, \beta)) = d(x, y) + |\alpha - \beta|.$$

Throughout this paper, we suppose that *the metric on $X \times [0, 1]$ is \bar{d}* . For simplicity, we also use $X \times [0, 1]$ to denote the metric space $(X \times [0, 1], \bar{d})$.

If there is no confusion, we also use H to denote the Hausdorff distance on $C(X \times [0, 1])$ induced by \bar{d} .

Remark 2.2. ρ is said to be a *metric* on Y if ρ is a function from $Y \times Y$ into \mathbb{R} satisfying positivity, symmetry and triangle inequality. At this time, (Y, ρ) is said to be a metric space.

ρ is said to be an *extended metric* on Y if ρ is a function from $Y \times Y$ into $\mathbb{R} \cup \{+\infty\}$ satisfying positivity, symmetry and triangle inequality. At this time, (Y, ρ) is said to be an extended metric space.

We can see that for arbitrary metric space (X, d) , the Hausdorff distance H on $K(X)$ induced by d is a metric. So the Hausdorff distance H on $K(X \times [0, 1])$ induced by \bar{d} on $X \times [0, 1]$ is a metric.

The Hausdorff distance H on $C(X)$ induced by d on X is an extended metric, but may not be a metric, because $H(A, B)$ could be equal to $+\infty$ for certain metric space X and $A, B \in C(X)$. Clearly, if H on $C(X)$ induced by d is not a metric, then H on $C(X \times [0, 1])$ induced by \bar{d} is also not a metric. So the Hausdorff distance H on $C(X \times [0, 1])$ induced by \bar{d} on $X \times [0, 1]$ is an extended metric but may not be a metric. We can see that H on $C(\mathbb{R}^m)$ is an extended metric but not a metric, and then the same is H on $C(\mathbb{R}^m \times [0, 1])$.

We call the Hausdorff distance H the Hausdorff metric (respectively, the Hausdorff extended metric) if H is a metric (respectively, an extended metric). In this paper, for simplicity, we refer to both the Hausdorff extended metric and the Hausdorff metric as the Hausdorff metric.

The Hausdorff metric has the following important properties.

Theorem 2.3. [19, 24] *Let (X, d) be a metric space and let H be the Hausdorff metric induced by d . Then*

- (i) (X, d) is complete $\iff (K(X), H)$ is complete;
- (ii) (X, d) is separable $\iff (K(X), H)$ is separable;
- (iii) (X, d) is compact $\iff (K(X), H)$ is compact.

Rojas-Medar and Román-Flores [23] introduced the Γ -convergence of a sequence of fuzzy sets in a metric space based on the Kuratowski convergence of a sequence of sets in a certain metric space.

Let (X, d) be a metric space. Let C be a set in X and $\{C_n\}$ a sequence of sets in X . $\{C_n\}$ is said to **Kuratowski converge** to C , if

$$C = \liminf_{n \rightarrow \infty} C_n = \limsup_{n \rightarrow \infty} C_n,$$

where

$$\liminf_{n \rightarrow \infty} C_n = \{x \in X : x = \lim_{n \rightarrow \infty} x_n, x_n \in C_n\},$$

$$\limsup_{n \rightarrow \infty} C_n = \{x \in X : x = \lim_{j \rightarrow \infty} x_{n_j}, x_{n_j} \in C_{n_j}\} = \bigcap_{n=1}^{\infty} \overline{\bigcup_{m \geq n} C_m}.$$

In this case, we will write $\mathbf{C} = \lim_{n \rightarrow \infty}^{(K)} C_n$.

Remark 2.4. Definition 3.1.4 in [19] gives the definitions of $\liminf C_n$, $\limsup C_n$ and $\lim C_n$ for a net of subsets $\{C_n, n \in D\}$ in a topological space. When $\{C_n, n = 1, 2, \dots\}$ is a sequence of subsets of a metric space, $\liminf C_n$, $\limsup C_n$ and $\lim C_n$ according to Definition 3.1.4 in [19] are $\liminf_{n \rightarrow \infty} C_n$, $\limsup_{n \rightarrow \infty} C_n$ and $\lim_{n \rightarrow \infty}^{(K)} C_n$ according to the above definitions, respectively. So Corollary 3.2.13 in [19] implies Theorem 4.2.

Let $u, u_n, n = 1, 2, \dots$, be fuzzy sets in $F(X)$. $\{u_n\}$ is said to **Γ -converge** to u , denoted by $\mathbf{u} = \lim_{n \rightarrow \infty}^{(\Gamma)} \mathbf{u}_n$, if $\text{end } u = \lim_{n \rightarrow \infty}^{(K)} \text{end } u_n$. Here $\lim_{n \rightarrow \infty}^{(K)} \text{end } u_n$ is considered with respect to $(X \times [0, 1], \bar{d})$.

Let (X, d) be a metric space and let $u \in F(X)$. Then from basic analysis, the following three properties are equivalent: (i) u is upper semi-continuous; (ii) $\text{end } u$ is closed in $(X \times [0, 1], \bar{d})$; (iii) $\text{send } u$ is closed in $(X \times [0, 1], \bar{d})$.

Kloeden [20] introduced the endograph metric H_{end} . The endograph metric H_{end} and the sendograph metric H_{send} on $F_{USC}^1(X)$ are defined as follows. For $u, v \in F_{USC}^1(X)$,

$$\begin{aligned} H_{\text{end}}(\mathbf{u}, \mathbf{v}) &:= H(\text{end } u, \text{end } v), \\ H_{\text{send}}(\mathbf{u}, \mathbf{v}) &:= H(\text{send } u, \text{send } v), \end{aligned}$$

where H is the Hausdorff metric on $C(X \times [0, 1])$ induced by \bar{d} on $X \times [0, 1]$.

The d_{∞} metric on $F_{USC}^1(X)$ is defined as follows: for $u, v \in F_{USC}^1(X)$,

$$d_{\infty}(\mathbf{u}, \mathbf{v}) := \sup\{H([u]_{\alpha}, [v]_{\alpha}) : \alpha \in [0, 1]\}.$$

Below property (1) may be known. We gave this kind of conclusion in [12].

$$\text{For each } u, v \in F_{USC}^1(X), \quad d_{\infty}(u, v) \geq H_{\text{send}}(u, v) \geq H_{\text{end}}(u, v). \quad (1)$$

Remark 2.5. We can see that H_{end} is a metric on $F_{USC}^1(X)$ with $H_{\text{end}}(u, v) \leq 1$ for all $u, v \in F_{USC}^1(X)$. Both d_{∞} and H_{send} are metrics on $F_{USCB}^1(X)$. However, each one of d_{∞} and H_{send} on $F_{USC}^1(X)$ is an extended metric but may not be a metric. See also Remark 3.3 in [15]. We can see that both d_{∞} and H_{send} on $F_{USCG}^1(\mathbb{R}^m)$ are not metrics, they are extended metrics.

For simplicity, in this paper, we call H_{send} on $F_{USC}^1(X)$ the H_{send} metric or the sendograph metric H_{send} . We call d_{∞} on $F_{USC}^1(X)$ the d_{∞} metric or the supremum metric d_{∞} .

3. H_{end} , H_{send} and Kuratowski convergence on $P_{USC}^1(X)$

In this section, we introduce $P_{USC}^1(X)$ and its subset $P_{USCB}^1(X)$, and define the H_{send} distance and the H_{end} distance on $P_{USC}^1(X)$. We discuss the properties and relations of H_{send} , H_{end} and Kuratowski convergence on $P_{USC}^1(X)$. Based on this, we give some relations among H_{end} , H_{send} and Γ -convergence on $F_{USC}^1(X)$.

For $u \subseteq X \times [0, 1]$ and $\alpha \in [0, 1]$, define $\langle u \rangle_\alpha := \{x : (x, \alpha) \in u\}$. $P_{USC}^1(X)$ and $P_{USCB}^1(X)$ are subsets of the power set of $X \times [0, 1]$ defined by

$$\begin{aligned} P_{USC}^1(X) &:= \{u \subseteq X \times [0, 1] : \langle u \rangle_\alpha = \bigcap_{\beta < \alpha} \langle u \rangle_\beta \text{ for all } \alpha \in (0, 1]; \\ &\quad \langle u \rangle_\alpha \in C(X) \text{ for all } \alpha \in [0, 1]\}, \\ P_{USCB}^1(X) &:= \{u \in P_{USC}^1(X) : \langle u \rangle_\alpha \in K(X) \text{ for all } \alpha \in [0, 1]\}. \end{aligned}$$

Clearly $P_{USCB}^1(X) \subseteq P_{USC}^1(X)$.

By Proposition 2.1, we can see $F_{USC}^1(X)$ (respectively, $F_{USCB}^1(X)$) as a subset of $P_{USC}^1(X)$ (respectively, $P_{USCB}^1(X)$) by identifying a fuzzy set with its sendograph. So the conclusions on $P_{USC}^1(X)$ and $P_{USCB}^1(X)$ are useful for the discussions of fuzzy sets in this paper.

From the basic analysis, we can obtain that

- (i) if $u \in P_{USC}^1(X)$, then $u \in C(X \times [0, 1])$;
- (ii) if $u \in C(X \times [0, 1])$, then $\langle u \rangle_\alpha \in C(X) \cup \{\emptyset\}$ for all $\alpha \in [0, 1]$;
- (iii) if $u \in P_{USCB}^1(X)$, then $u \in K(X \times [0, 1])$;
- (iv) if $u \in K(X \times [0, 1])$, then $\langle u \rangle_\alpha \in K(X) \cup \{\emptyset\}$ for all $\alpha \in [0, 1]$.

So we have

$$\begin{aligned} P_{USC}^1(X) &= \{u \in C(X \times [0, 1]) : \langle u \rangle_1 \neq \emptyset, \langle u \rangle_\alpha = \bigcap_{\beta < \alpha} \langle u \rangle_\beta \text{ for all } \alpha \in (0, 1]\}, \\ P_{USCB}^1(X) &= \{u \in P_{USC}^1(X) : u \in K(X \times [0, 1])\}. \end{aligned}$$

We define the H_{send} distance and the H_{end} distance on $P_{USC}^1(X)$: for each $u, v \in P_{USC}^1(X)$,

$$H_{\text{send}}(u, v) := H(u, v), \quad H_{\text{end}}(u, v) := H(\underline{u}, \underline{v}),$$

where H is the Hausdorff metric on $C(X \times [0, 1])$ induced by \bar{d} on $X \times [0, 1]$, and $\underline{u} := u \cup (X \times \{0\})$.

As H_{send} on $P_{USC}^1(X)$ is the restriction to $P_{USC}^1(X) \times P_{USC}^1(X)$ of the Hausdorff extended metric H on $C(X \times [0, 1])$, we have that H_{send} is an extended metric on $P_{USC}^1(X)$.

Obviously $H_{\text{end}} \leq 1$ on $P_{USC}^1(X)$. H_{end} is a pseudometric on $P_{USC}^1(X)$; that is, for all $u, v \in P_{USC}^1(X)$, $H_{\text{end}}(u, v) \in \mathbb{R}$ and $H_{\text{end}}(u, u) = 0$, and H_{end} satisfies symmetry and triangle inequality. H_{end} is a metric on $P_{USC}^1(X)$ if and only if X is a singleton. The “if” part is obvious. To show the “only if” part, assume that X contains more than one point. Let $x, y \in X$ with $x \neq y$. Define $u \in P_{USCB}^1(X)$ by putting $\langle u \rangle_\alpha = \{x\}$ for all $\alpha \in [0, 1]$, and $v \in P_{USCB}^1(X)$ by putting $\langle v \rangle_\alpha = \{x\}$ for $\alpha \in (0, 1]$ and $\langle v \rangle_0 = \{x, y\}$. Then $u \neq v$ and $H_{\text{end}}(u, v) = 0$. So H_{end} does not satisfy the positivity and hence is not a metric on $P_{USC}^1(X)$.

Define a function $f : F_{USC}^1(X) \rightarrow P_{USC}^1(X)$ by $f(u) = \text{send } u$. Then f is an isometric embedding of $(F_{USC}^1(X), H_{\text{send}})$ in $(P_{USC}^1(X), H_{\text{send}})$. So $(F_{USC}^1(X), H_{\text{send}})$ can be embedded isometrically in $(P_{USC}^1(X), H_{\text{send}})$. Note that $f(F_{USCB}^1(X)) \subseteq P_{USCB}^1(X)$. Thus $(F_{USCB}^1(X), H_{\text{send}})$ can be embedded isometrically in $(P_{USCB}^1(X), H_{\text{send}})$.

For $u \in F_{USC}^1(X)$, we define $\vec{u} := f(u) = \text{send } u$. Then $\vec{u} \in P_{USC}^1(X)$.

Let $v \in P_{USC}^1(X)$. Define $v' \in f(F_{USC}^1(X)) \subseteq P_{USC}^1(X)$ by putting

$$\langle v' \rangle_\alpha = \begin{cases} \langle v \rangle_\alpha, & \alpha \in (0, 1], \\ \bigcup_{\alpha > 0} \langle v \rangle_\alpha, & \alpha = 0. \end{cases}$$

Define $\overleftarrow{v} \in F_{USC}^1(X)$ by putting $[\overleftarrow{v}]_\alpha = \langle v' \rangle_\alpha$ for $\alpha \in [0, 1]$. Then $\overleftarrow{\overleftarrow{v}} = f(\overleftarrow{v}) = v'$. $\langle v \rangle_0 \supseteq \langle v' \rangle_0$ as $\langle v \rangle_0 \in C(X)$ and $\langle v \rangle_0 \supseteq \bigcup_{\alpha > 0} \langle v \rangle_\alpha$. Note that $\langle v \rangle_0 = \langle v' \rangle_0 \cup \overline{\langle v \rangle_0 \setminus \langle v' \rangle_0}$. Thus we have (a) $v = v' \cup (\overline{\langle v \rangle_0 \setminus \langle v' \rangle_0} \times \{0\})$. So $P_{USC}^1(X) = \{\vec{u} \cup (A \times \{0\}) : u \in F_{USC}^1(X), A \in C(X) \cup \{\emptyset\}\}$ (“ \subseteq ” follows from (a), and “ \supseteq ” is obvious). Clearly $\langle v \rangle_0 \in K(X)$ if and only if $\langle v' \rangle_0 \in K(X)$ and $\overline{\langle v \rangle_0 \setminus \langle v' \rangle_0} \in K(X) \cup \{\emptyset\}$. So $P_{USCB}^1(X) = \{\vec{u} \cup (A \times \{0\}) : u \in F_{USCB}^1(X), A \in K(X) \cup \{\emptyset\}\}$ (“ \supseteq ” is obvious). We can see that the above two expressions of $P_{USC}^1(X)$ and $P_{USCB}^1(X)$ remain true if “ $\cup \{\emptyset\}$ ” is deleted in them.

For a subset U of $F_{USC}^1(X)$, we use \vec{U} to denote the set $\{\vec{u} : u \in U\}$. For a subset U of $P_{USC}^1(X)$, we use \overleftarrow{U} to denote the set $\{\overleftarrow{u} : u \in U\}$. Clearly $\overleftarrow{P_{USC}^1(X)} = F_{USC}^1(X)$ and $\overleftarrow{P_{USCB}^1(X)} = F_{USCB}^1(X)$.

Proposition 3.1. *Let $v \in P_{USC}^1(X)$. Suppose the conditions: (i) $v \in F_{USC}^1(X)$; (ii) $\langle v \rangle_0 = \bigcup_{\delta > 0} \langle v \rangle_\delta$; (iii) $v = v'$; (iv) $\langle v \rangle_0 = [\overleftarrow{v}]_0$; (v) $v \in$*

- $\overrightarrow{F_{USCB}^1(X)}$; $(vi) \lim_{\delta \rightarrow 0+} H(\langle v \rangle_\delta, \langle v \rangle_0) = 0$.
 (a) $(v) \Rightarrow (vi) \Rightarrow (i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv)$.
 (b) If $v \in P_{USCB}^1(X)$, then $(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv) \Leftrightarrow (v) \Leftrightarrow (vi)$.

Proof. Clearly $(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv)$. By Lemma 5.4(v), $(v) \Rightarrow (vi)$. From Remark 3.6(i), $(vi) \Rightarrow (ii)$. So (a) is proved.

If $v \in P_{USCB}^1(X)$, then $\langle v \rangle_0 \in K(X)$. Assume that (i) is true; that is, there is a $u \in F_{USC}^1(X)$ with $v = \overline{u}$. Then $[u]_0 = \langle v \rangle_0$, and thus $u \in F_{USCB}^1(X)$. Hence $(i) \Rightarrow (v)$. So from (a), we have that (b) is true. \square

Theorem 3.2. Let (X, d) be a metric space. For $u, v \in P_{USC}^1(X)$:

- (i) $H_{\text{end}}(u, v) \leq H_{\text{send}}(u, v)$;
 (ii) $H(\langle u \rangle_0, \langle v \rangle_0) \leq H_{\text{send}}(u, v)$;
 (iii) If $H_{\text{end}}(u, v) < 1$, then $H_{\text{send}}(u, v) \leq H_{\text{end}}(u, v) + H(\langle u \rangle_0, \langle v \rangle_0)$.
 For a sequence $\{u_n\}$ in $P_{USC}^1(X)$ and u in $P_{USC}^1(X)$:
 (iv) $H_{\text{send}}(u_n, u) \rightarrow 0$ if and only if $H_{\text{end}}(u_n, u) \rightarrow 0$ and $H(\langle u_n \rangle_0, \langle u \rangle_0) \rightarrow 0$;
 (v) $\lim_{n \rightarrow \infty}^{(K)} u_n = u$ if and only if $\lim_{n \rightarrow \infty}^{(K)} \underline{u_n} = \underline{u}$ and $\lim_{n \rightarrow \infty}^{(K)} \langle u_n \rangle_0 = \langle u \rangle_0$.

Proof. Clearly (i) and (ii) are true. To show (iii), let $(x, \alpha) \in u$. It suffices to verify the following property (a) if $H_{\text{end}}(u, v) < 1$, then

$$\overline{d}((x, \alpha), v) \leq H_{\text{end}}(u, v) + H(\langle u \rangle_0, \langle v \rangle_0). \quad (2)$$

We claim that (b) $\overline{d}((x, \alpha), v) \leq \alpha + d(x, \langle v \rangle_0) \leq \alpha + H(\langle u \rangle_0, \langle v \rangle_0)$, and (c) if $\overline{d}((x, \alpha), \underline{v}) < \alpha$, then $\overline{d}((x, \alpha), v) = \overline{d}((x, \alpha), \underline{v}) \leq H_{\text{end}}(u, v)$.

Notice that $\overline{d}((x, \alpha), v) \leq \overline{d}((x, \alpha), \langle v \rangle_0 \times \{0\}) \leq \overline{d}((x, \alpha), (x, 0)) + \overline{d}((x, 0), \langle v \rangle_0 \times \{0\}) = \alpha + d(x, \langle v \rangle_0)$. Thus (b) is true. Note that $\overline{d}((x, \alpha), \underline{v}) = \min\{\overline{d}((x, \alpha), v), \overline{d}((x, \alpha), X \times \{0\})\} = \min\{\overline{d}((x, \alpha), v), \alpha\}$. So if $\overline{d}((x, \alpha), \underline{v}) < \alpha$, then $\overline{d}((x, \alpha), v) = \overline{d}((x, \alpha), \underline{v})$. Thus (c) is true.

If $\alpha \in [0, H_{\text{end}}(u, v)]$, then by (b), (2) holds. If $H_{\text{end}}(u, v) < 1$ and $\alpha \in (H_{\text{end}}(u, v), 1]$, then $\overline{d}((x, \alpha), \underline{v}) < \alpha$. Hence by (c), $\overline{d}((x, \alpha), v) \leq H_{\text{end}}(u, v)$, and thus (2) holds. So (a) is true and then (iii) is proved.

Let $u, v \in P_{USC}^1(X)$. Suppose the following conditions (d-1) $H_{\text{end}}(u, v) = 0$, (d-2) $H(\langle u \rangle_0, \langle v \rangle_0) = 0$, and (d-3) $H(\langle u \rangle_0, \langle v \rangle_0) = +\infty$. If (d-1) holds, then by (ii) and (iii), $H_{\text{send}}(u, v) = H(\langle u \rangle_0, \langle v \rangle_0)$. If (d-2) holds, then $\langle u \rangle_0 = \langle v \rangle_0$ and so $H_{\text{send}}(u, v) = H_{\text{end}}(u, v)$. If (d-3) holds, then by (ii), $H_{\text{send}}(u, v) = +\infty$. So if one of the conditions (d-1), (d-2), and (d-3) holds, then $H_{\text{send}}(u, v) = H_{\text{end}}(u, v) + H(\langle u \rangle_0, \langle v \rangle_0)$. However, the converse is false.

(iv) follows immediately from (i), (ii) and (iii). Below we verify (v).

Suppose that $\lim_{n \rightarrow \infty}^{(K)} u_n = u$. To show $\lim_{n \rightarrow \infty}^{(K)} \underline{u}_n = \underline{u}$ and $\lim_{n \rightarrow \infty}^{(K)} \langle u_n \rangle_0 = \langle u \rangle_0$, we only need to show that

$$\begin{aligned} \underline{u} &\subseteq \liminf_{n \rightarrow \infty} \underline{u}_n, \quad \limsup_{n \rightarrow \infty} \underline{u}_n \subseteq \underline{u}, \\ \langle u \rangle_0 &\subseteq \liminf_{n \rightarrow \infty} \langle u_n \rangle_0, \quad \limsup_{n \rightarrow \infty} \langle u_n \rangle_0 \subseteq \langle u \rangle_0. \end{aligned}$$

Let $(x, \alpha) \in \underline{u}$. If $\alpha = 0$, then clearly $(x, \alpha) \in \liminf_{n \rightarrow \infty} \underline{u}_n$. If $\alpha > 0$, then $(x, \alpha) \in u$, and thus $(x, \alpha) \in \liminf_{n \rightarrow \infty} u_n \subseteq \liminf_{n \rightarrow \infty} \underline{u}_n$. So $\underline{u} \subseteq \liminf_{n \rightarrow \infty} \underline{u}_n$.

Let $(x, \alpha) \in \limsup_{n \rightarrow \infty} \underline{u}_n$. If $\alpha = 0$, then clearly $(x, \alpha) \in \underline{u}$. If $\alpha > 0$, then $(x, \alpha) \in \limsup_{n \rightarrow \infty} u_n = u \subseteq \underline{u}$. So $\limsup_{n \rightarrow \infty} \underline{u}_n \subseteq \underline{u}$.

Let $x \in \langle u \rangle_0$. Then $(x, 0) \in u = \liminf_{n \rightarrow \infty} u_n$. Thus there is a sequence $\{(x_n, \alpha_n)\}$ such that $(x_n, \alpha_n) \in u_n$, $n = 1, 2, \dots$ and $(x, 0) = \lim_{n \rightarrow \infty} (x_n, \alpha_n)$. Hence $x_n \in \langle u_n \rangle_0$ and $x = \lim_{n \rightarrow \infty} x_n$. So $\langle u \rangle_0 \subseteq \liminf_{n \rightarrow \infty} \langle u_n \rangle_0$.

Let $x \in \limsup_{n \rightarrow \infty} \langle u_n \rangle_0$. Then there is a sequence $\{x_{n_i}\}$ such that $x_{n_i} \in \langle u_{n_i} \rangle_0$, $i = 1, 2, \dots$ and $x = \lim_{i \rightarrow \infty} x_{n_i}$. Thus $(x, 0) = \lim_{i \rightarrow \infty} (x_{n_i}, 0) \in \limsup_{n \rightarrow \infty} u_n = u$. Hence $x \in \langle u \rangle_0$. So $\limsup_{n \rightarrow \infty} \langle u_n \rangle_0 \subseteq \langle u \rangle_0$.

Suppose that $\lim_{n \rightarrow \infty}^{(K)} \underline{u}_n = \underline{u}$ and $\lim_{n \rightarrow \infty}^{(K)} \langle u_n \rangle_0 = \langle u \rangle_0$. To show $\lim_{n \rightarrow \infty}^{(K)} u_n = u$, we only need to show that

$$u \subseteq \liminf_{n \rightarrow \infty} u_n, \quad \limsup_{n \rightarrow \infty} u_n \subseteq u.$$

Let $(x, \alpha) \in u$. If $\alpha = 0$, then $x \in \langle u \rangle_0 = \lim_{n \rightarrow \infty}^{(K)} \langle u_n \rangle_0$. Thus there is a sequence $\{x_n\}$ such that $x_n \in \langle u_n \rangle_0$, $n = 1, 2, \dots$ and $x = \lim_{n \rightarrow \infty} x_n$. Hence $(x, \alpha) = (x, 0) = \lim_{n \rightarrow \infty} (x_n, 0) \in \liminf_{n \rightarrow \infty} u_n$.

If $\alpha > 0$, then from $(x, \alpha) \in \underline{u} = \lim_{n \rightarrow \infty}^{(K)} \underline{u}_n$, there is a sequence $\{(x_n, \alpha_n)\}$ such that $(x, \alpha) = \lim_{n \rightarrow \infty} (x_n, \alpha_n)$, and $(x_n, \alpha_n) \in \underline{u}_n$ and $\alpha_n > 0$ for $n = 1, 2, \dots$. Thus $(x_n, \alpha_n) \in u_n$, $n = 1, 2, \dots$ and hence $(x, \alpha) \in \liminf_{n \rightarrow \infty} u_n$.

Let $(x, \alpha) \in \limsup_{n \rightarrow \infty} u_n$. Then there is a sequence $\{(x_n, \alpha_n)\}$ such that for each $n \in \mathbb{N}$, $(x_n, \alpha_n) \in u_n$, and $\bar{d}((x_n, \alpha_n), (x, \alpha)) \rightarrow 0$. So for each $n \in \mathbb{N}$, $x_n \in \langle u_n \rangle_0$, and $d(x_n, x) \rightarrow 0$, and hence $x \in \limsup_{n \rightarrow \infty} \langle u_n \rangle_0 = \langle u \rangle_0$. Clearly $(x, \alpha) \in \limsup_{n \rightarrow \infty} \underline{u}_n = \underline{u}$. Thus $(x, \alpha) \in u$. \square

For each $u, v \in P_{USC}^1(X)$, $\underline{u} = \text{end } \overleftarrow{u}$ and so $H_{\text{end}}(u, v) = H_{\text{end}}(\overleftarrow{u}, \overleftarrow{v})$.

For each $u \in F_{USC}^1(X)$, $[u]_0 = \langle \overrightarrow{u} \rangle_0$ and $\text{end } u = (\overrightarrow{u})$.

Let $u, v \in F_{USC}^1(X)$. Then $\overrightarrow{u}, \overrightarrow{v} \in P_{USC}^1(X)$, and $H_{\text{send}}(u, v) = H_{\text{send}}(\overrightarrow{u}, \overrightarrow{v})$, $H_{\text{end}}(u, v) = H_{\text{end}}(\overrightarrow{u}, \overrightarrow{v})$, and $H([u]_0, [v]_0) = H(\langle \overrightarrow{u} \rangle_0, \langle \overrightarrow{v} \rangle_0)$. Thus (i), (ii)

and (iii) in Theorem 3.2 imply that (the following (3) is part of (1))

$$H_{\text{end}}(u, v) \leq H_{\text{send}}(u, v), \quad (3)$$

$$H([u]_0, [v]_0) \leq H_{\text{send}}(u, v), \quad (4)$$

$$\text{if } H_{\text{end}}(u, v) < 1, \text{ then } H_{\text{send}}(u, v) \leq H_{\text{end}}(u, v) + H([u]_0, [v]_0). \quad (5)$$

Proposition 3.3. *Let $u, u_n, n = 1, 2, \dots$, be fuzzy sets in $F_{USC}^1(X)$.*

(i) $H_{\text{send}}(u_n, u) \rightarrow 0$ if and only if $H_{\text{end}}(u_n, u) \rightarrow 0$ and $H([u_n]_0, [u]_0) \rightarrow 0$.

(ii) $\lim_{n \rightarrow \infty}^{(K)} \text{send } u_n = \text{send } u$ if and only if $\lim_{n \rightarrow \infty}^{(\Gamma)} u_n = u$ and $\lim_{n \rightarrow \infty}^{(K)} [u_n]_0 = [u]_0$.

Proof. (i) follows immediately from (3), (4) and (5). (i) can be seen as a special case of the clause (iv) in Theorem 3.2. (ii) can be seen as a special case of the clause (v) in Theorem 3.2. \square

The following Theorem 3.4 is an already known conclusion¹, which is useful in this paper. It can be proved in a similar fashion to Theorem 4.1 in [14]. In Theorem 4.1 in [14], we include the case that $C = \emptyset$. In Theorem 3.4, we exclude the case that $C = \emptyset$.

Theorem 3.4. *Suppose that C, C_n are sets in $C(X)$, $n = 1, 2, \dots$. Then $H(C_n, C) \rightarrow 0$ implies that $\lim_{n \rightarrow \infty}^{(K)} C_n = C$.*

Remark 3.5. From Theorem 3.4, we obtain that for $u, u_n, n = 1, 2, \dots$ in $F_{USC}^1(X)$: (i) $H_{\text{end}}(u_n, u) \rightarrow 0$ implies that $\lim_{n \rightarrow \infty}^{(K)} \underline{u_n} = \underline{u}$; (ii) $H_{\text{send}}(u_n, u) \rightarrow 0$ implies that $\lim_{n \rightarrow \infty}^{(K)} u_n = u$.

So for $u, u_n, n = 1, 2, \dots$ in $F_{USC}^1(X)$: (iii) $H_{\text{end}}(u_n, u) \rightarrow 0$ implies that $\lim_{n \rightarrow \infty}^{(\Gamma)} u_n = u$; (iv) $H_{\text{send}}(u_n, u) \rightarrow 0$ implies that $\lim_{n \rightarrow \infty}^{(K)} \text{send } u_n = \text{send } u$.

The converses of the implications in clauses (i), (ii), (iii) and (iv) in Remark 3.5 are false. Let $u = [0, +\infty)_{F(\mathbb{R})}$ and for $n = 1, 2, \dots$, let $u_n = [0, n]_{F(\mathbb{R})}$. Then $\lim_{n \rightarrow \infty}^{(K)} \text{send } u_n = \text{send } u$, but $H_{\text{end}}(u_n, u) = 1 \not\rightarrow 0$. So combined with Proposition 3.3, the converses of the implications in (iii) and (iv) are false, and thus the converses of the implications in (i) and (ii) are false.

¹Theorem 3.4 may come from [1]. However we cannot obtain [1]. A reviewer kindly recommended [1] and the symbol χ_S in Paragraph 2 of Section 2.

Remark 3.6. Let $U \in C(X)$ and let $\{u(\alpha) : \alpha \in [0, 1]\}$ be a subset of $C(X)$ with $u(\alpha) \supseteq u(\beta)$ for all $0 \leq \alpha \leq \beta \leq 1$.

- (i) If $\alpha \in [0, 1)$ and $\lim_{\gamma \rightarrow \alpha+} H(u(\gamma), U) = 0$, then $U = \overline{\bigcup_{\gamma > \alpha} u(\gamma)}$;
- (ii) If $\alpha \in (0, 1]$ and $\lim_{\beta \rightarrow \alpha-} H(u(\beta), U) = 0$, then $U = \bigcap_{\beta < \alpha} u(\beta)$.

Assume that $\alpha \in [0, 1)$ and $\lim_{\gamma \rightarrow \alpha+} H(u(\gamma), U) = 0$. Choose a decreasing sequence $\{\gamma_n\}$ in $(\alpha, 1]$ with $\gamma_n \rightarrow \alpha+$. Then $\lim_{n \rightarrow \infty} H(u(\gamma_n), U) = 0$. Thus by Theorem 3.4, $U = \lim_{n \rightarrow \infty}^{(K)} u(\gamma_n) = \overline{\bigcup_{n \in \mathbb{N}} u(\gamma_n)} = \overline{\bigcup_{\gamma > \alpha} u(\gamma)}$, and hence (i) is true.

Assume that $\alpha \in (0, 1]$ and $\lim_{\beta \rightarrow \alpha-} H(u(\beta), U) = 0$. Choose an increasing sequence $\{\beta_n\}$ in $[0, \alpha)$ with $\beta_n \rightarrow \alpha-$. Then $\lim_{n \rightarrow \infty} H(u(\beta_n), U) = 0$. Thus by Theorem 3.4, $U = \lim_{n \rightarrow \infty}^{(K)} u(\beta_n) = \bigcap_{n \in \mathbb{N}} u(\beta_n) = \bigcap_{\beta < \alpha} u(\beta)$, and hence (ii) is true.

4. Level characterizations of Γ -convergence

In this section, we investigate the level characterizations of the Γ -convergence. It is shown that under some conditions, the Γ -convergence of fuzzy sets can be decomposed to the Kuratowski convergence of certain α -cuts.

Throughout this paper, we suppose that the measure on \mathbb{R} is the Lebesgue measure.

Rojas-Medar and Román-Flores ([23]) have presented the following important and useful property of the Γ -convergence.

Theorem 4.1. [23] *Suppose that $u, u_n, n = 1, 2, \dots$, are fuzzy sets in $F_{USC}^1(X)$. Then $\lim_{n \rightarrow \infty}^{(\Gamma)} u_n = u$ if and only if for all $\alpha \in (0, 1]$,*

$$\{u > \alpha\} \subseteq \liminf_{n \rightarrow \infty} [u_n]_\alpha \subseteq \limsup_{n \rightarrow \infty} [u_n]_\alpha \subseteq [u]_\alpha. \quad (6)$$

Theorem 4.2 is Theorem 2.1 in [14]. Of course, the conclusion that $\limsup_{n \rightarrow \infty} C_n$ are closed sets in (X, d) in Theorem 4.2 can also be deduced from the fact that $\limsup_{n \rightarrow \infty} C_n = \bigcap_{n=1}^{\infty} \overline{\bigcup_{m \geq n} C_m}$.

Theorem 4.2. [19] *Let (X, d) be a metric space and let $\{C_n\}$ be a sequence of sets in X . Then $\liminf_{n \rightarrow \infty} C_n$ and $\limsup_{n \rightarrow \infty} C_n$ are closed sets in (X, d) .*

Theorem 4.3. *Suppose that $u, u_n, n = 1, 2, \dots$, are fuzzy sets in $F_{USC}^1(X)$. Then $\lim_{n \rightarrow \infty}^{(\Gamma)} u_n = u$ if and only if for all $\alpha \in (0, 1]$,*

$$\overline{\{u > \alpha\}} \subseteq \liminf_{n \rightarrow \infty} [u_n]_\alpha \subseteq \limsup_{n \rightarrow \infty} [u_n]_\alpha \subseteq [u]_\alpha.$$

Proof. By Theorem 4.2, for each $\alpha \in [0, 1]$, $\liminf_{n \rightarrow \infty} [u_n]_\alpha$ is a closed set in (X, d) . So the desired result follows from Theorem 4.1. \square

Remark 4.4. Suppose that $u, u_n, n = 1, 2, \dots$, are fuzzy sets in $F_{USC}^1(X)$. If $\lim_{n \rightarrow \infty}^{(\Gamma)} u_n = u$, then (i) $[u]_0 = \bigcup_{\alpha > 0} \overline{\liminf_{n \rightarrow \infty} [u_n]_\alpha}$, (ii) $[u]_0 \subseteq \liminf_{n \rightarrow \infty} [u_n]_0$, and (iii) $[u]_0 \subsetneq \liminf_{n \rightarrow \infty} [u_n]_0$ may happen.

By Theorem 4.1, $[u]_0 = \overline{\bigcup_{\alpha > 0} \{u > \alpha\}} \subseteq \overline{\bigcup_{\alpha > 0} \liminf_{n \rightarrow \infty} [u_n]_\alpha} \subseteq \overline{\bigcup_{\alpha > 0} [u]_\alpha} = [u]_0$. So (i) is true. By Theorem 4.2, $\liminf_{n \rightarrow \infty} [u_n]_0$ is closed in (X, d) , so $\bigcup_{\alpha > 0} \liminf_{n \rightarrow \infty} [u_n]_\alpha \subseteq \liminf_{n \rightarrow \infty} [u_n]_0$, and then (ii) follows from (i). We show (iii) by the following example. Let $u = \widehat{0}_{F(\mathbb{R})}$ and for $n = 1, 2, \dots$, define $u_n \in F_{USCB}^1(\mathbb{R})$ as

$$u_n(x) = \begin{cases} 1, & x = 0, \\ 1/n, & x \in (0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

Then $H_{\text{end}}(u_n, u) \rightarrow 0$, and therefore from Remark 3.5 $\lim_{n \rightarrow \infty}^{(\Gamma)} u_n = u$. And $[u]_0 = \{0\} \subsetneq [0, 1] = \liminf_{n \rightarrow \infty} [u_n]_0$.

Since $\lim_{n \rightarrow \infty}^{(\Gamma)} u_n = u$ implies that $[u]_0 \subseteq \liminf_{n \rightarrow \infty} [u_n]_0$, then by Proposition 3.3, $\lim_{n \rightarrow \infty}^{(K)} u_n = \text{send } u$ if and only if $\lim_{n \rightarrow \infty}^{(\Gamma)} u_n = u$ and $[u]_0 \supseteq \limsup_{n \rightarrow \infty} [u_n]_0$.

Let $u \in F(X)$. Denote

$$\begin{aligned} D(u) &:= \{\alpha \in (0, 1) : [u]_\alpha \not\subseteq \overline{\{u > \alpha\}}\}, \\ P(u) &:= \{\alpha \in (0, 1) : \overline{\{u > \alpha\}} \subsetneq [u]_\alpha\}. \end{aligned}$$

A number α in $P(u)$ is called a platform point of u . Clearly $P(u) \subseteq D(u)$. $P(u) \subsetneq D(u)$ could happen. See Example 4.5.

Example 4.5. let $u \in F(\mathbb{R})$ be defined by

$$u(x) = \begin{cases} 1, & x \in (0, 1), \\ 0.6, & x \in [1, 3], \\ 0, & x \in \mathbb{R} \setminus (0, 3]. \end{cases}$$

Then $P(u) = \emptyset$ and $D(u) = \{0.6\}$. So $P(u) \subsetneq D(u)$.

The symbol l^2 denotes the **Hilbert space** $l^2 := \{(x_i)_{i=1}^{+\infty} : \sum_{i=1}^{+\infty} x_i^2 < +\infty\}$. $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ denote the norm and inner product on l^2 , respectively. For each $x = (x_i)_{i=1}^{+\infty}$ and $y = (y_i)_{i=1}^{+\infty}$ in l^2 , $\langle x, y \rangle = \sum_{i=1}^{+\infty} x_i \cdot y_i$ and $\|x\| =$

$\sqrt{\langle x, x \rangle}$. \mathbb{R}^m can be seen as a subspace of l^2 . $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ are also used to denote the Euclidean norm and Euclidean inner product on \mathbb{R}^m , respectively.

A set S is said to be countable if it is finite or countably infinite. In this sense, “countable” means the same as “at most countable”. Theorem 5.1 in [14] says that $D(u)$ is countable when $u \in F(\mathbb{R}^m)$.

Theorem 4.6. *Let $u \in F(l^2)$. Then the set $D(u)$ is countable.*

Proof. The proof is similar to that of Theorem 5.1 in [14]. A sketch of the proof is given as follows. We use \mathbf{S}^1 to denote the set $\{e \in l^2 : \|e\| = 1\}$.

For each $t \in l^2$ and $r > 0$, define $S_{u,t,r}(\cdot, \cdot) : \mathbf{S}^1 \times [0, 1] \rightarrow \{-\infty\} \cup \mathbb{R}$ by

$$S_{u,t,r}(e, \alpha) = \begin{cases} -\infty, & \text{if } [u]_\alpha \cap \overline{B(t, r)} = \emptyset, \\ \sup\{\langle e, x - t \rangle : x \in [u]_\alpha \cap \overline{B(t, r)}\}, & \text{if } [u]_\alpha \cap \overline{B(t, r)} \neq \emptyset, \end{cases}$$

where $B(t, r) := \{x \in l^2 : \|x - t\| < r\}$ and $\overline{B(t, r)} := \{x \in l^2 : \|x - t\| \leq r\}$.

Let $t \in l^2$, $r > 0$ and $e \in \mathbf{S}^1$. Define a set $D(u, t, r, e)$ as follows: $\alpha \in D(u, t, r, e)$ if and only if (i) $\alpha \in (0, 1)$ and $S_{u,t,r}(e, \alpha) \in \mathbb{R}$, (ii) $S_{u,t,r}(e, \beta) = -\infty$ for all $\beta > \alpha$ or $-\infty < \lim_{\beta \rightarrow \alpha+} S_{u,t,r}(e, \beta) < \lim_{\beta \rightarrow \alpha-} S_{u,t,r}(e, \beta)$. From the monotonicity of $S_{u,t,r}(e, \cdot)$, $D(u, t, r, e)$ is countable. Then proceed similarly to the proof of Lemma A.1 in [14] (Delete the “,” and add “and all $e, f \in \mathbf{S}^1$,” at the end of line 9 of page 82 in [14] and the narrative of this proof will become clearer), in more detail, replacing \mathbb{S}^{m-1} by \mathbf{S}^1 in this proof, we obtain that $D(u, t, r) = \bigcup_{e \in \mathbf{S}^1} D(u, t, r, e)$ is countable.

Here we mention that the narrative of the proof of formula (A.6) in Theorem 5.1 in [14] can be slightly simplified. The detailed operations are performed as follows: replace Lines 1 and 2 from the bottom in Page 82 and Lines 1 and 2 in Page 83 in [14] by

- Since $2\langle a, b \rangle = \|a\|^2 + \|b\|^2 - \|a - b\|^2$ for each $a, b \in \mathbb{R}^m$, then

$$\langle e, x - q \rangle = \frac{\langle y - q, x - q \rangle}{\|y - q\|} = \frac{\|x - q\|^2 + \|y - q\|^2 - \|x - y\|^2}{2\|y - q\|}$$

Note that $2\langle a, b \rangle = \|a\|^2 + \|b\|^2 - \|a - b\|^2$ for each $a, b \in l^2$. So proceed similarly to this slightly simplified proof of Theorem 5.1 in [14], in more detail, replacing \mathbb{R}^m by l^2 , the definition of \mathbb{Q}^m in Page 82 by $l_{\mathbb{Q}}^2 := \{(z_i)_{i=1}^{+\infty} \in l^2 : z_i \in \mathbb{Q} \text{ for each } i \in \mathbb{N}\}$, and \mathbb{Q}^m by $l_{\mathbb{Q}}^2$ in this proof, we obtain that $D(u)$ is countable. Here we mention that $l_{\mathbb{Q}}^2$ is countable. Of course, this fact can also be shown by proceeding similarly to the original proof of Theorem 5.1 in [14]. \square

The following Theorem 4.7 is a generalization of Theorem 5.1 in [14] and Theorem 4.6. Its proof is based on the well-known conclusion:

- each separable metric space is homeomorphic to a subspace of the Hilbert space l^2 .

Theorem 4.7. *Let (X, d) be a metric space and $u \in F(X)$. If $([u]_0, d)$ is separable, then the set $D(u)$ is countable.*

Proof. Let f be a homeomorphism from $([u]_0, d)$ to a subspace of l^2 . Consider $u_f \in F(l^2)$ defined by

$$u_f(t) = \begin{cases} u(f^{-1}(t)), & t \in f([u]_0), \\ 0, & t \in l^2 \setminus f([u]_0). \end{cases}$$

Then by Theorem 4.6, $D(u_f)$ is countable. To show that $D(u)$ is countable, it suffices to show that $D(u) = D(u_f)$.

For $S \subseteq [u]_0$, let $\overline{S}^{[u]_0}$ denote the topological closure of S in $([u]_0, d)$. For $W \subseteq f([u]_0)$, let \overline{W}^{l^2} denote the topological closure of W in l^2 , and let $\overline{W}^{f([u]_0)}$ denote the topological closure of W in $f([u]_0)$, here we see $f([u]_0)$ as a subspace of l^2 . As described previously, for $S \subseteq X$, we use \overline{S} to denote the topological closure of S in X .

We claim that

- (i) $D(u) = \{\alpha \in (0, 1) : \text{there exists an } x \in \{u > 0\} \text{ such that } x \in [u]_\alpha \text{ and } x \notin \overline{\{u > \alpha\}}\}$;
- (ii) $D(u_f) = \{\alpha \in (0, 1) : \text{there exists a } y \in \{u_f > 0\} \text{ such that } y \in [u_f]_\alpha \text{ and } y \notin \overline{\{u_f > \alpha\}}^{l^2}\}$;
- (iii) $D(u_f) = \{\alpha \in (0, 1) : \text{there exists an } x \in \{u > 0\} \text{ such that } f(x) \in [u_f]_\alpha \text{ and } f(x) \notin \overline{\{u_f > \alpha\}}^{l^2}\}$;
- (iv) Let $\alpha \in [0, 1]$ and $x \in [u]_0$. Then
 - (iv-1) $x \in [u]_\alpha \Leftrightarrow f(x) \in [u_f]_\alpha$, and
 - (iv-2) $x \notin \overline{\{u > \alpha\}} \Leftrightarrow x \notin \overline{\{u > \alpha\}}^{[u]_0} \Leftrightarrow f(x) \notin \overline{\{u_f > \alpha\}}^{f([u]_0)} \Leftrightarrow f(x) \notin \overline{\{u_f > \alpha\}}^{l^2}$.

Clearly (i) and (ii) are true. Let $y \in l^2$. Then $y \in \{u_f > 0\}$ if and only if there exists a unique $x \in \{u > 0\}$ such that $y = f(x)$. So by (ii), (iii) is true.

Note that for each $x \in [u]_0$, $u(x) = u_f(f(x))$. So (iv-1) is true.

Now we show (iv-2). Since $x \in [u]_0$, the first “ \Leftrightarrow ” holds. Since $f(x) \in f([u]_0)$, the third “ \Leftrightarrow ” holds. Note that $f(\{u > \alpha\}) = \{u_f > \alpha\}$. As f is a homeomorphism from $([u]_0, d)$ to $f([u]_0)$, it follows that $f(\overline{\{u > \alpha\}}^{[u]_0}) = \overline{\{u_f > \alpha\}}^{f([u]_0)}$. Thus the second “ \Leftrightarrow ” holds. So (iv-2) is true.

From (i), (iii) and (iv), $D(u) = D(u_f)$. \square

Corollary 4.8. *Let (X, d) be a separable metric space and $u \in F(X)$. Then the set $D(u)$ is countable.*

Proof. As $([u]_0, d)$ is a subspace of the separable metric space (X, d) , it follows that $([u]_0, d)$ is separable. Thus, by Theorem 4.7, $D(u)$ is countable. \square

Remark 4.9. It is well-known that both \mathbb{R}^m and l^2 are separable metric spaces. Thus both Theorem 5.1 in [14] and Theorem 4.6 are special cases of Corollary 4.8, which is a corollary of Theorem 4.7. So Theorem 4.7 is a generalization of Theorem 5.1 in [14] and Theorem 4.6.

Remark 4.10. Theorems 4.6, 4.7 and Corollary 4.8 remain true if $D(u)$ is replaced by $P(u)$, since $P(u) \subseteq D(u)$ for each $u \in F(X)$.

Let $u \in F(X)$. Define $F(u) := \{\alpha \in (0, 1) : \{u > \alpha\} \neq [u]_\alpha\}$. Clearly $(0, 1) \setminus F(u) \subseteq (0, 1) \setminus D(u)$; that is, $D(u) \subseteq F(u)$. $D(u) \subsetneq F(u)$ may happen. For instance, let u be the fuzzy set defined in Example 4.5. Then $F(u) = (0, 1) \supsetneq \{0.6\} = D(u)$.

For each $u \in F(X)$, $P(u) \subseteq D(u) \subseteq F(u)$. We claim that if $u \in F_{USC}^1(X)$, then $P(u) = D(u) = F(u)$. Let $u \in F_{USC}^1(X)$. Then for each $\alpha \in [0, 1]$, $\overline{\{u > \alpha\}} \subseteq [u]_\alpha$. Thus $F(u) \subseteq P(u)$, and so $P(u) = D(u) = F(u)$.

Theorem 4.11. *Suppose that $u, u_n, n = 1, 2, \dots$, are fuzzy sets in $F_{USC}^1(X)$. Then the following statements are true.*

- (i) *If there is a dense set P in $(0, 1)$ such that $[u]_\alpha = \lim_{n \rightarrow \infty}^{(K)} [u_n]_\alpha$ for $\alpha \in P$, then $u = \lim_{n \rightarrow \infty}^{(\Gamma)} u_n$.*
- (ii) *If $u = \lim_{n \rightarrow \infty}^{(\Gamma)} u_n$, then $[u]_\alpha = \lim_{n \rightarrow \infty}^{(K)} [u_n]_\alpha$ for all $\alpha \in (0, 1) \setminus P(u)$.*

Proof. The proof of (i) is similar to “(ii) \Rightarrow (i)” in the proof of Theorem 6.2 in [14]. Here we mention the following basic facts. Let $\alpha \in (0, 1)$. Then there exist two sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in P with $\alpha_n \rightarrow \alpha+$ and $\beta_n \rightarrow \alpha-$. Indeed, we can get two such sequences by choosing α_n in $P \cap (\alpha, \alpha + (1 - \alpha)/n)$ and β_n in $P \cap ((1 - 1/n)\alpha, \alpha)$ for each $n \in \mathbb{N}$. Let $\alpha = 1$. Then there exists

a sequence $\{\gamma_n\}$ in P with $\gamma_n \rightarrow 1-$. Indeed, we can get a such sequence by choosing γ_n in $P \cap (1 - 1/n, 1)$ for each $n \in \mathbb{N}$.

(ii) follows immediately from Theorem 4.3. \square

The following theorem gives some conditions under which the Γ -convergence of fuzzy sets can be decomposed to the Kuratowski convergence of certain α -cuts. For simplicity, we use “a.e.” to denote “almost everywhere”.

Theorem 4.12. *Suppose that $u, u_n, n = 1, 2, \dots$, are fuzzy sets in $F_{USC}^1(X)$. If $([u]_0, d)$ is separable, then the following are equivalent:*

- (i) $\lim_{n \rightarrow \infty}^{(\Gamma)} u_n = u$;
- (ii) $\lim_{n \rightarrow \infty}^{(K)} [u_n]_\alpha = [u]_\alpha$ holds a.e. on $\alpha \in (0, 1)$;
- (iii) $\lim_{n \rightarrow \infty}^{(K)} [u_n]_\alpha = [u]_\alpha$ holds for all $\alpha \in (0, 1) \setminus P(u)$;
- (iv) There is a dense subset P of $(0, 1) \setminus P(u)$ such that $\lim_{n \rightarrow \infty}^{(K)} [u_n]_\alpha = [u]_\alpha$ holds for $\alpha \in P$;
- (v) There is a countable dense subset P of $(0, 1) \setminus P(u)$ such that $\lim_{n \rightarrow \infty}^{(K)} [u_n]_\alpha = [u]_\alpha$ holds for $\alpha \in P$.

Proof. Suppose that $u, u_n, n = 1, 2, \dots$, are fuzzy sets in $F_{USC}^1(X)$. We claim that

- (a) $(ii) \Rightarrow (i) \Rightarrow (iii) \Rightarrow (iv) \Leftrightarrow (v)$.
 - (b) If $(0, 1) \setminus P(u)$ is dense in $(0, 1)$, then $(i) \Leftrightarrow (iii) \Leftrightarrow (iv) \Leftrightarrow (v)$.
 - (c) If $P(u)$ is a set of measure zero, then $(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv) \Leftrightarrow (v)$.
- (ii) \Rightarrow (i) follows from Theorem 4.11 (i). (i) \Rightarrow (iii) is Theorem 4.11 (ii). Clearly (iii) \Rightarrow (iv) \Leftrightarrow (v). So (a) is true. If $(0, 1) \setminus P(u)$ is dense in $(0, 1)$, then a dense subset P of $(0, 1) \setminus P(u)$ is also a dense subset of $(0, 1)$. So by Theorem 4.11 (i), (iv) \Rightarrow (i). Combined with (a), we have that (b) is true.

Assume that $P(u)$ is a set of measure zero. Then (iii) \Rightarrow (ii). At this time, $(0, 1) \setminus P(u)$ is dense in $(0, 1)$. So from (a) and (b), we obtain that (c) is true.

By Theorem 4.7 and Remark 4.10, we know that if $([u]_0, d)$ is separable, then $P(u)$ is countable, and hence $P(u)$ is a set of measure zero. So by (c), we obtain the desired results. \square

Remark 4.13. By Lemma 5.4, if $u \in F_{USCG}^1(X)$ then $P(u) = P_0(u)$ is countable. So by affirmation (c) in the proof of Theorem 4.12, Theorem 4.12 remains true if “ $([u]_0, d)$ is separable” is replaced by “ $u \in F_{USCG}^1(X)$ ”.

Here we mention that the condition “ $u \in F_{USCG}^1(X)$ ” implies the condition “ $([u]_0, d)$ is separable”. Let $u \in F_{USCG}^1(X)$. Then for each $\alpha \in (0, 1]$, $([u]_\alpha, d)$ is separable, because each compact metric space is separable. Since

$[u]_0 = \overline{\bigcup_{n=1}^{+\infty} [u]_{1/n}}$, it follows that $([u]_0, d)$ is separable (For $n = 1, 2, \dots$, let A_n be a countable dense set of $([u]_{1/n}, d)$. Then $\bigcup_{n=1}^{+\infty} A_n$ is a countable dense set of $([u]_0, d)$).

5. Level characterizations of endograph metric convergence

In this section, we discuss the level characterizations of endograph metric convergence. It is shown that under some condition, the H_{end} metric convergence of fuzzy sets can be decomposed to the Hausdorff metric convergence of certain α -cuts.

Let u be a fuzzy set in $F_{USC}^1(X)$. Denote

$$P_0(u) := (0, 1) \setminus \{\alpha \in (0, 1) : \lim_{\beta \rightarrow \alpha} H([u]_\beta, [u]_\alpha) = 0\}.$$

If $\alpha \in P_0(u)$, then $\lim_{\beta \rightarrow \alpha} H([u]_\beta, [u]_\alpha)$ may not exist. In this paper the limit quantities are allowed to take the value $+\infty$.

Proposition 5.1. *Let $u \in F_{USC}^1(X)$. Then $P(u) \subseteq P_0(u)$.*

Proof. If $\alpha \in P(u)$, then $\overline{\{u > \alpha\}} \subsetneq [u]_\alpha$, i.e. $\xi := H(\overline{\{u > \alpha\}}, [u]_\alpha) > 0$. Thus $\lim_{\gamma \rightarrow \alpha+} H([u]_\gamma, [u]_\alpha) \geq \xi > 0$. Hence $\alpha \in P_0(u)$. So $P(u) \subseteq P_0(u)$. \square

$P(u) \subsetneq P_0(u)$ could happen. See Example 5.2.

Example 5.2. Let $u \in F_{USC}^1(\mathbb{R}^2)$ be defined by putting

$$[u]_\alpha = \{0\} \cup \{z : \arg z \in [\alpha, 1]\} \text{ for each } \alpha \in [0, 1],$$

here we write each $(x, y) \in \mathbb{R}^2$ as a complex number $z = x + iy$. Then $P(u) = \emptyset$ and $P_0(u) = (0, 1)$. So $P(u) \subsetneq P_0(u)$.

This example also shows that for $u \in F_{USC}^1(X)$, $P_0(u)$ need not be countable even X is a separable metric space.

Below Lemma 5.3 and Lemma 5.4(i)(ii)(v) may be known. Here we give their proofs for the completeness of this paper.

Lemma 5.3. *For $n = 1, 2, \dots$, let $U_n \in K(X)$ and $V_n \in K(X)$.*

(i) If $U_1 \supseteq U_2 \supseteq \dots \supseteq U_n \supseteq \dots$, then $U = \bigcap_{n=1}^{+\infty} U_n \in K(X)$ and $H(U_n, U) \rightarrow 0$ as $n \rightarrow +\infty$.

(ii) If $V_1 \subseteq V_2 \subseteq \dots \subseteq V_n \subseteq \dots$ and $V = \overline{\bigcup_{n=1}^{+\infty} V_n} \in K(X)$, then $H(V_n, V) \rightarrow 0$ as $n \rightarrow +\infty$.

Proof. Here we give a proof using Theorem 2.3. We only show (ii). (i) can be shown in a similar way. Clearly for $n = 1, 2, \dots$, $V_n \in K(V)$. Then by Theorem 2.3, $\{V_n\}$ has a subsequence $\{V_{n_k}\}$ which converges to $C \in K(V)$. Then $H(V_n, C) \rightarrow 0$ since for $V_{n_{k_1}} \subseteq V_n \subseteq V_{n_{k_2}}$, $H(V_n, C) \leq \max\{H^*(V_{n_{k_2}}, C), H^*(C, V_{n_{k_1}})\} = \max\{H(V_{n_{k_2}}, C), H(C, V_{n_{k_1}})\}$. Thus by Theorem 3.4, $C = \lim_{n \rightarrow \infty}^{(K)} V_n = V$. So (ii) is true. \square

Lemma 5.4. *Let $u \in F_{USCG}^1(X)$.*

- (i) *For each $\alpha \in (0, 1]$, $\lim_{\beta \rightarrow \alpha-} H([u]_\beta, [u]_\alpha) = 0$.*
- (ii) *For each $\alpha \in (0, 1)$, $\lim_{\gamma \rightarrow \alpha+} H([u]_\gamma, \overline{\{u > \alpha\}}) = 0$.*
- (iii) *$P(u) = P_0(u)$.*
- (iv) *$P_0(u)$ is countable.*
- (v) *If $u \in F_{USCB}^1(X)$, then $\lim_{\gamma \rightarrow 0+} H([u]_\gamma, [u]_0) = 0$.*

Proof. Assume that (i) is not true. Then there is an $\alpha \in (0, 1]$, an $\varepsilon > 0$ and an increasing sequence $\{\beta_n\}$ in $[0, \alpha)$ with $\beta_n \rightarrow \alpha-$ such that $H([u]_{\beta_n}, [u]_\alpha) > \varepsilon$. By Lemma 5.3 (i), $H([u]_{\beta_n}, [u]_\alpha) \rightarrow 0$. This is a contradiction. So (i) is true. Similarly, by Lemma 5.3 (ii), (ii) and (v) are true.

Now we show (iii). Given $\alpha \in (0, 1) \setminus P(u)$. Then $[u]_\alpha = \overline{\{u > \alpha\}}$. By (i) and (ii), we have that $\lim_{\delta \rightarrow \alpha} H([u]_\delta, [u]_\alpha) = 0$. Thus $\alpha \in (0, 1) \setminus P_0(u)$. So $(0, 1) \setminus P(u) \subseteq (0, 1) \setminus P_0(u)$. This means that $P_0(u) \subseteq P(u)$. Combined with Proposition 5.1, we obtain that $P(u) = P_0(u)$. So (iii) is proved.

(iv) is Lemma 6.12 in [15]. (iv) can also be shown in such a way: let $u \in F_{USCG}^1(X)$, then by Remark 4.13, $([u]_0, d)$ is separable, and thus by Theorem 4.7 and Remark 4.10, $P(u)$ is countable. So by (iii), $P_0(u)$ is countable. \square

Proposition 5.5. (i) *If $u, v \in F_{USCG}^1(X)$, then $H([u]_\alpha, [v]_\alpha)$ is left-continuous at each $\alpha \in (0, 1]$.* (ii) *If $u, v \in F_{USCB}^1(X)$, then $H([u]_\alpha, [v]_\alpha)$ is right-continuous at $\alpha = 0$.*

Proof. Let $u, v \in F_{USCG}^1(X)$. For each $\alpha \in (0, 1]$, by Lemma 5.4(i), $\lim_{\beta \rightarrow \alpha-} (H([u]_\alpha, [u]_\beta) + H([v]_\alpha, [v]_\beta)) = 0$. Note that $H([u]_\alpha, [v]_\alpha)$ is finite at each $\alpha \in (0, 1]$ and for each $\alpha, \beta \in (0, 1]$, $|H([u]_\alpha, [v]_\alpha) - H([u]_\beta, [v]_\beta)| \leq H([u]_\alpha, [u]_\beta) + H([v]_\alpha, [v]_\beta)$. So for each $\alpha \in (0, 1]$, $\lim_{\beta \rightarrow \alpha-} H([u]_\beta, [v]_\beta) = H([u]_\alpha, [v]_\alpha)$, and (i) is true. Similarly, it can be shown that (ii) is true by using Lemma 5.4(v). \square

Proposition 5.6. *Let $u, v \in F_{USC}^1(X)$ and $\varepsilon \in (0, 1]$. Let $\alpha, \beta \in [0, 1]$ with $\alpha - \beta \geq \varepsilon$. If $H^*(\text{end } u, \text{end } v) < \varepsilon$, then $H^*([u]_\alpha, [v]_\beta) \leq H^*(\text{end } u, \text{end } v) < \varepsilon$.*

Proof. Set $a := H^*([u]_\alpha, \overline{\{v > \beta\}})$ and $b := \sup\{\bar{d}((y, \alpha), \text{end } v) : y \in [u]_\alpha\}$.

We claim the following property (i) if $b < \varepsilon$, then $a \leq b$.

Assume that $b < \varepsilon$. Let $y \in [u]_\alpha$. Notice that $\bar{d}((y, \alpha), \text{end } v) = \inf\{\bar{d}((y, \alpha), (z, \gamma)) : (z, \gamma) \in \text{end } v\} \leq b < \varepsilon$, and that for $(z, \gamma) \in \text{end } v$ with $\gamma \leq \beta$, $\bar{d}((y, \alpha), (z, \gamma)) \geq |\alpha - \beta| \geq \varepsilon$. Thus $\bar{d}((y, \alpha), \text{end } v) = \inf\{\bar{d}((y, \alpha), (z, \gamma)) : (z, \gamma) \in \text{end } v \text{ with } \gamma > \beta\}$. Since for each $(z, \gamma) \in \text{end } v$ with $\gamma > \beta$, $\bar{d}((y, \alpha), (z, \gamma)) \geq d(y, z) \geq d(y, \overline{\{v > \beta\}})$, it follows that $\bar{d}((y, \alpha), \text{end } v) \geq d(y, \overline{\{v > \beta\}})$. Since $y \in [u]_\alpha$ is arbitrary, we have that $a \leq b$. So (i) is true.

If $H^*(\text{end } u, \text{end } v) < \varepsilon$, then $b < \varepsilon$, and hence by (i), $H^*([u]_\alpha, [v]_\beta) \leq a \leq b \leq H^*(\text{end } u, \text{end } v) < \varepsilon$. So the proof is complete. \square

Theorem 5.7. Let $u, u_n, n = 1, 2, \dots$, be fuzzy sets in $F_{USC}^1(X)$.

(i) The following are equivalent:

- (i-1) $\lim_{n \rightarrow \infty} H^*(\text{end } u, \text{end } u_n) = 0$;
- (i-2) For each $\alpha \in [0, 1)$ and $\xi \in (0, 1 - \alpha]$, $\lim_{n \rightarrow \infty} H^*([u]_{\alpha+\xi}, [u_n]_\alpha) = 0$;
- (i-3) There is a dense subset P of $[0, 1)$ such that for each $\alpha \in P$ and $\xi \in (0, 1 - \alpha]$, $\lim_{n \rightarrow \infty} H^*([u]_{\alpha+\xi}, [u_n]_\alpha) = 0$.

(ii) The following are equivalent:

- (ii-1) $\lim_{n \rightarrow \infty} H^*(\text{end } u_n, \text{end } u) = 0$;
- (ii-2) for each $\alpha \in (0, 1]$ and $\zeta \in (0, \alpha]$, $\lim_{n \rightarrow \infty} H^*([u_n]_\alpha, [u]_{\alpha-\zeta}) = 0$;
- (ii-3) There is a dense subset P of $(0, 1]$ such that for each $\alpha \in P$ and $\zeta \in (0, \alpha]$, $\lim_{n \rightarrow \infty} H^*([u_n]_\alpha, [u]_{\alpha-\zeta}) = 0$.

Proof. We only prove (i). (ii) can be proved similarly.

To show (i-1) \Rightarrow (i-2), assume that $\lim_{n \rightarrow \infty} H^*(\text{end } u, \text{end } u_n) = 0$. Let $\alpha \in [0, 1)$ and $\xi \in (0, 1 - \alpha]$. Then for each $\varepsilon \in (0, \xi)$, there exists an $N(\varepsilon)$ such that for all $n \geq N$, $H^*(\text{end } u, \text{end } u_n) < \varepsilon$, and hence by Proposition 5.6, $H^*([u]_{\alpha+\xi}, [u_n]_\alpha) < \varepsilon$. Since $\varepsilon \in (0, \xi)$ is arbitrary, it follows that $\lim_{n \rightarrow \infty} H^*([u]_{\alpha+\xi}, [u_n]_\alpha) = 0$. So (i-1) \Rightarrow (i-2).

Assume that (i-2) is true. To show that (i-1) is true, let $\varepsilon > 0$. Select a $k \in \mathbb{N}$ with $1/k \leq \varepsilon$. From (i-2), we have that for $l = 1, \dots, k$, $\lim_{n \rightarrow \infty} H^*([u]_{l/k}, [u_n]_{(l-1)/k}) = 0$. So there is an $N(\varepsilon)$ such that for all $n \geq N$ and $l = 1, \dots, k$,

$$H^*([u]_{l/k}, [u_n]_{(l-1)/k}) \leq \varepsilon. \quad (7)$$

Let $(x, \alpha) \in \text{end } u$ and $n \in \mathbb{N}$. If $\alpha \leq \varepsilon$, then $\bar{d}((x, \alpha), \text{end } u_n) \leq \bar{d}((x, \alpha), (x, 0)) \leq \varepsilon$. If $\alpha > \varepsilon$, then we can choose an $l \in \{1, \dots, k-1\}$ such that $l/k < \alpha \leq (l+1)/k$, and thus

$$\bar{d}((x, \alpha), \text{end } u_n) \leq \bar{d}((x, \alpha), [u_n]_{(l-1)/k} \times \{(l-1)/k\})$$

$$= \inf\{d(x, y) + |\alpha - \frac{l-1}{k}| : y \in [u_n]_{(l-1)/k}\} = d(x, [u_n]_{(l-1)/k}) + |\alpha - \frac{l-1}{k}|.$$

So by (7), for $n \geq N$,

$$H^*(\text{end } u, \text{end } u_n) \leq \varepsilon \vee (\max\{H^*([u]_{l/k}, [u_n]_{(l-1)/k}), l = 1, \dots, k-1\} + 2/k) \leq 3\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have that $\lim_{n \rightarrow \infty} H^*(\text{end } u, \text{end } u_n) = 0$. So (i-2) \Rightarrow (i-1).

Let $\alpha \in [0, 1)$ and $\xi \in (0, 1 - \alpha]$. Choose $\beta \in P \cap [\alpha, \alpha + \xi]$. Then

$$H^*([u]_{\alpha+\xi}, [u_n]_{\alpha}) \leq H^*([u]_{\alpha+\xi}, [u_n]_{\beta}).$$

Using this fact, we obtain that (i-3) \Rightarrow (i-2).

(i-2) \Rightarrow (i-3) is obvious. Thus (i) is proved. \square

Remark 5.8. Let $u, u_n, n = 1, 2, \dots$, be fuzzy sets in $F_{USC}^1(X)$. Clearly, (i-2) in Theorem 5.7 is equivalent to the following (i-2)', and (ii-2) in Theorem 5.7 is equivalent to the following (ii-2)':

(i-2)' For each $\alpha \in [0, 1)$ there is a sequence $\{\xi_m\}$ in $(0, 1 - \alpha]$ with $\xi_m \rightarrow 0+$ satisfying that $\lim_{n \rightarrow \infty} H^*([u]_{\alpha+\xi_m}, [u_n]_{\alpha}) = 0$;

(ii-2)' For each $\alpha \in (0, 1]$ there is a sequence $\{\zeta_m\}$ in $(0, \alpha]$ with $\zeta_m \rightarrow 0+$ satisfying that $\lim_{n \rightarrow \infty} H^*([u_n]_{\alpha}, [u]_{\alpha-\zeta_m}) = 0$.

Similarly, we can give (i-3)' and (ii-3)' which are equivalent to (i-3) and (ii-3) in Theorem 5.7, respectively.

Corollary 5.9. Let $u, u_n, n = 1, 2, \dots$, be fuzzy sets in $F_{USC}^1(X)$. Then the following are equivalent:

(i) $\lim_{n \rightarrow \infty} H_{\text{end}}(u_n, u) = 0$;

(ii) For each $\alpha \in (0, 1)$, $\lim_{n \rightarrow \infty} H^*([u]_{\alpha+\xi}, [u_n]_{\alpha}) = 0$ when $\xi \in (0, 1 - \alpha]$, and $\lim_{n \rightarrow \infty} H^*([u_n]_{\alpha}, [u]_{\alpha-\zeta}) = 0$ when $\zeta \in (0, \alpha]$;

(iii) There is a dense subset P of $(0, 1)$ such that for each $\alpha \in P$, $\lim_{n \rightarrow \infty} H^*([u]_{\alpha+\xi}, [u_n]_{\alpha}) = 0$ when $\xi \in (0, 1 - \alpha]$, and $\lim_{n \rightarrow \infty} H^*([u_n]_{\alpha}, [u]_{\alpha-\zeta}) = 0$ when $\zeta \in (0, \alpha]$.

Proof. Note that $\lim_{n \rightarrow \infty} H_{\text{end}}(u_n, u) = 0$ if and only if $\lim_{n \rightarrow \infty} H^*(\text{end } u, \text{end } u_n) = 0$ and $\lim_{n \rightarrow \infty} H^*(\text{end } u_n, \text{end } u) = 0$. So Theorem 5.7 imply that (i) \Rightarrow (ii) and (iii) \Rightarrow (i). Clearly (ii) \Rightarrow (iii). So (i) \Leftrightarrow (ii) \Leftrightarrow (iii). \square

Lemma 5.10. Let $u, u_n, n = 1, 2, \dots$, be fuzzy sets in $F_{USC}^1(X)$, and let P be a dense subset of $(0, 1)$.

(i) If for each $\alpha \in P$, $\lim_{n \rightarrow \infty} H^*(\overline{\{u > \alpha\}}, [u_n]_{\alpha}) = 0$, then $\lim_{n \rightarrow \infty} H^*(\text{end } u, \text{end } u_n) =$

- 0;
(ii) If for each $\alpha \in P$, $\lim_{n \rightarrow \infty} H^*([u_n]_\alpha, [u]_\alpha) = 0$, then $\lim_{n \rightarrow \infty} H^*(\text{end } u_n, \text{end } u) = 0$;
(iii) If for each $\alpha \in P$, $\lim_{n \rightarrow \infty} H^*(\overline{\{u > \alpha\}}, [u_n]_\alpha) = 0$ and $\lim_{n \rightarrow \infty} H^*([u_n]_\alpha, [u]_\alpha) = 0$, then $\lim_{n \rightarrow \infty} H_{\text{end}}(u_n, u) = 0$.

Proof. The desired results follow immediately from Theorem 5.7. \square

Lemma 5.11. Let $u, u_n, n = 1, 2, \dots$, be fuzzy sets in $F_{USC}^1(X)$.

- (i) Let $\alpha \in [0, 1]$. If $\lim_{n \rightarrow \infty} H^*(\text{end } u, \text{end } u_n) = 0$, and $\lim_{\gamma \rightarrow \alpha+} H([u]_\gamma, \overline{\{u > \alpha\}}) = 0$, then $\lim_{n \rightarrow \infty} H^*(\overline{\{u > \alpha\}}, [u_n]_\alpha) = 0$.
(ii) Let $\alpha \in (0, 1]$. If $\lim_{n \rightarrow \infty} H^*(\text{end } u_n, \text{end } u) = 0$, and $\lim_{\beta \rightarrow \alpha-} H([u]_\alpha, [u]_\beta) = 0$, then $\lim_{n \rightarrow \infty} H^*([u_n]_\alpha, [u]_\alpha) = 0$.

Proof. We only prove (i). (ii) can be proved similarly.

Let $\varepsilon > 0$. Since $\lim_{\gamma \rightarrow \alpha+} H([u]_\gamma, \overline{\{u > \alpha\}}) = 0$, then there is a $\gamma(\alpha) \in (\alpha, 1]$ such that $H(\overline{\{u > \alpha\}}, [u]_\gamma) < \varepsilon/2$. By Theorem 5.7 (i), $\lim_{n \rightarrow \infty} H^*(\text{end } u, \text{end } u_n) = 0$ implies that $\lim_{n \rightarrow \infty} H^*([u]_\gamma, [u_n]_\alpha) = 0$. Then there is an $N \in \mathbb{N}$ such that for all $n \geq N$, $H^*([u]_\gamma, [u_n]_\alpha) < \varepsilon/2$. Hence for all $n \geq N$,

$$H^*(\overline{\{u > \alpha\}}, [u_n]_\alpha) \leq H(\overline{\{u > \alpha\}}, [u]_\gamma) + H^*([u]_\gamma, [u_n]_\alpha) < \varepsilon.$$

From the arbitrariness of $\varepsilon > 0$, we thus have $\lim_{n \rightarrow \infty} H^*(\overline{\{u > \alpha\}}, [u_n]_\alpha) = 0$. \square

The assumption that $\lim_{\gamma \rightarrow \alpha+} H([u]_\gamma, \overline{\{u > \alpha\}}) = 0$ in (i) of Lemma 5.11 can not be omitted. The assumption that $\lim_{\beta \rightarrow \alpha-} H([u]_\alpha, [u]_\beta) = 0$ in (ii) of Lemma 5.11 also can not be omitted. The following Examples 5.12 and 5.13 are counterexamples.

Example 5.12. Let u be a fuzzy set in $F_{USC}^1(\mathbb{R})$ defined by putting

$$[u]_\alpha = \begin{cases} (-\infty, \frac{\frac{2}{3}}{\alpha - \frac{1}{3}}], & \alpha \in (\frac{1}{3}, 1], \\ (-\infty, +\infty), & \alpha \in [0, \frac{1}{3}]. \end{cases}$$

For $n = 1, 2, \dots$, let u_n be a fuzzy set in $F_{USC}^1(\mathbb{R})$ given by putting

$$[u_n]_\alpha = \begin{cases} (-\infty, \frac{1 - \frac{1}{3} \frac{n-1}{n}}{\alpha - \frac{1}{3} \frac{n-1}{n}}], & \alpha \in (\frac{1}{3} \frac{n-1}{n}, 1], \\ (-\infty, +\infty), & \alpha \in [0, \frac{1}{3} \frac{n-1}{n}]. \end{cases}$$

Clearly $\lim_{\gamma \rightarrow \frac{1}{3}+} H([u]_\gamma, \overline{\{u > \frac{1}{3}\}}) = \lim_{\gamma \rightarrow \frac{1}{3}+} H([u]_\gamma, (-\infty, +\infty)) = +\infty \neq 0$.

It can be seen that $H_{\text{end}}(u, u_n) \rightarrow 0$. However for each $n \in \mathbb{N}$,

$$H^*(\overline{\{u > \frac{1}{3}\}}, [u_n]_{\frac{1}{3}}) = H^*([u]_{\frac{1}{3}}, [u_n]_{\frac{1}{3}}) = H^*((-\infty, +\infty), (-\infty, \frac{3 - \frac{n-1}{n}}{1 - \frac{n-1}{n}})) = +\infty,$$

and so $H^*(\overline{\{u > \frac{1}{3}\}}, [u_n]_{\frac{1}{3}}) \not\rightarrow 0$.

Example 5.13. Let u be a fuzzy set in $F_{USC}^1(\mathbb{R})$ defined by putting

$$[u]_\alpha = \begin{cases} \{1\}, & \alpha = 1, \\ \{1\} \cup (-\infty, -\frac{1}{1-\alpha}], & \alpha \in [0, 1), \end{cases}$$

For $n = 1, 2, \dots$, let u_n be a fuzzy set in $F_{USC}^1(\mathbb{R})$ given by putting

$$[u_n]_\alpha = \begin{cases} [u]_\alpha, & \alpha \in [0, 1 - \frac{1}{n}], \\ [u]_{1-\frac{1}{n}}, & \alpha \in [1 - \frac{1}{n}, 1], \end{cases} \quad n = 1, 2, \dots$$

Clearly $\lim_{\beta \rightarrow 1-} H([u]_1, [u]_\beta) = +\infty \neq 0$.

We can see that $H_{\text{end}}(u, u_n) \rightarrow 0$. However for each $n \in \mathbb{N}$,

$$H^*([u_n]_1, [u]_1) = H^*(\{1\} \cup (-\infty, -n], \{1\}) = +\infty,$$

and so $H^*([u_n]_1, [u]_1) \not\rightarrow 0$.

Theorem 5.14. Let u be a fuzzy set in $F_{USCG}^1(X)$ and let u_n , $n = 1, 2, \dots$, be fuzzy sets in $F_{USC}^1(X)$.

(i) The following are equivalent:

- (i-1) $\lim_{n \rightarrow \infty} H^*(\text{end } u, \text{end } u_n) = 0$;
- (i-2) For each $\alpha \in (0, 1)$, $\lim_{n \rightarrow \infty} H^*(\overline{\{u > \alpha\}}, [u_n]_\alpha) = 0$;
- (i-3) There is a dense subset P of $(0, 1)$ such that for each $\alpha \in P$, $\lim_{n \rightarrow \infty} H^*(\overline{\{u > \alpha\}}, [u_n]_\alpha) = 0$.

(ii) The following are equivalent:

- (ii-1) $\lim_{n \rightarrow \infty} H^*(\text{end } u_n, \text{end } u) = 0$;
- (ii-2) For each $\alpha \in (0, 1]$, $\lim_{n \rightarrow \infty} H^*([u_n]_\alpha, [u]_\alpha) = 0$;
- (ii-3) There is a dense subset P of $(0, 1]$ such that for each $\alpha \in P$, $\lim_{n \rightarrow \infty} H^*([u_n]_\alpha, [u]_\alpha) = 0$.

Proof. The desired results follow from Lemmas 5.4, 5.10 and 5.11 (or Lemmas 5.4 and 5.11 and Theorem 5.7). The proof is routine. \square

Theorem 5.15. Let u be a fuzzy set in $F_{USC}^1(X)$ and let u_n , $n = 1, 2, \dots$, be fuzzy sets in $F_{USC}^1(X)$. Then the following are equivalent:

- (i) $\lim_{n \rightarrow \infty} H_{\text{end}}(u_n, u) = 0$;
- (ii) For each $\alpha \in (0, 1)$, $\lim_{n \rightarrow \infty} H^*(\overline{\{u > \alpha\}}, [u_n]_\alpha) = 0$ and $\lim_{n \rightarrow \infty} H^*([u_n]_\alpha, [u]_\alpha) = 0$;
- (iii) There is a dense subset P of $(0, 1)$ such that for each $\alpha \in P$, $\lim_{n \rightarrow \infty} H^*(\overline{\{u > \alpha\}}, [u_n]_\alpha) = 0$ and $\lim_{n \rightarrow \infty} H^*([u_n]_\alpha, [u]_\alpha) = 0$;
- (iv) For each $\alpha \in (0, 1) \setminus P_0(u)$, $\lim_{n \rightarrow \infty} H([u]_\alpha, [u_n]_\alpha) = 0$;
- (v) There is a dense subset P of $(0, 1) \setminus P_0(u)$ such that for each $\alpha \in P$, $\lim_{n \rightarrow \infty} H([u]_\alpha, [u_n]_\alpha) = 0$;
- (vi) There is a countable dense subset P of $(0, 1) \setminus P_0(u)$ such that for each $\alpha \in P$, $\lim_{n \rightarrow \infty} H([u]_\alpha, [u_n]_\alpha) = 0$;
- (vii) $H([u_n]_\alpha, [u]_\alpha) \rightarrow 0$ holds a.e. on $\alpha \in (0, 1)$.

Proof. By Theorem 5.14, (i) \Leftrightarrow (ii) \Leftrightarrow (iii). Since for each $\alpha \in (0, 1) \setminus P_0(u)$, $\overline{\{u > \alpha\}} = [u]_\alpha$, then (ii) \Rightarrow (iv). Clearly (iv) \Rightarrow (v) \Leftrightarrow (vi). Since $\{u > \alpha\} \subseteq [u]_\alpha$, (vi) \Rightarrow (iii). By Lemma 5.4, $P_0(u)$ is at most countable, and therefore (iv) \Rightarrow (vii). Since $\{u > \alpha\} \subseteq [u]_\alpha$, (vii) \Rightarrow (iii). So the proof is completed.

The desired conclusion can also be deduced from Theorems 5.14 and 5.20 as follows. By Theorem 5.14, (i) \Leftrightarrow (ii) \Leftrightarrow (iii). From Theorem 5.20, (i) \Leftrightarrow (iv) \Leftrightarrow (v) \Leftrightarrow (vi) \Leftrightarrow (vii). So Theorem 5.15 is proved. \square

Theorem 5.16. Let u , u_n , $n = 1, 2, \dots$, be fuzzy sets in $F_{USC}^1(X)$. If there is a dense subset P of $(0, 1)$ such that $H([u_n]_\alpha, [u]_\alpha) \rightarrow 0$ for each $\alpha \in P$, then $H_{\text{end}}(u_n, u) \rightarrow 0$.

Proof. We proceed by contradiction. If $H_{\text{end}}(u_n, u) \not\rightarrow 0$, then $H^*(\text{end } u_n, \text{end } u) \not\rightarrow 0$ or $H^*(\text{end } u, \text{end } u_n) \not\rightarrow 0$.

Suppose that $H^*(\text{end } u_n, \text{end } u) \not\rightarrow 0$. Then there is an $\varepsilon > 0$ and a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $H^*(\text{end } u_{n_k}, \text{end } u) > \varepsilon$ for each $k \in \mathbb{N}$. So for each $k \in \mathbb{N}$, there exists an $(x_{n_k}, \alpha_{n_k}) \in \text{end } u_{n_k}$ such that

$$\overline{d}((x_{n_k}, \alpha_{n_k}), \text{end } u) > \varepsilon. \quad (8)$$

With no loss of generality we can assume that $\{\alpha_{n_k}\}$ converges to an element α in $[0, 1]$. By (8), $\alpha_{n_k} > \varepsilon$ for each $k \in \mathbb{N}$, and so $\alpha \geq \varepsilon$. Pick $\beta \in P$ satisfying $\alpha \in (\beta, \beta + \varepsilon/2)$. Then there exists K such that $\alpha_{n_k} \in (\beta, \beta + \varepsilon/2)$ for all $k \geq K$. Thus for each $k \geq K$,

$$\overline{d}((x_{n_k}, \alpha_{n_k}), \text{end } u)$$

$$\begin{aligned}
&\leq \bar{d}((x_{n_k}, \beta), \text{end } u) + \bar{d}((x_{n_k}, \alpha_{n_k}), (x_{n_k}, \beta)) < \bar{d}((x_{n_k}, \beta), [u]_\beta \times \{\beta\}) + \varepsilon/2 \\
&= \bar{d}(x_{n_k}, [u]_\beta) + \varepsilon/2 \leq H([u_{n_k}]_\beta, [u]_\beta) + \varepsilon/2.
\end{aligned} \tag{9}$$

Note that $H([u_{n_k}]_\beta, [u]_\beta) \rightarrow 0$, thus (8) contradicts (9). So the supposition is false.

For $H^*(\text{end } u, \text{end } u_n) \not\rightarrow 0$, we can similarly derive a contradiction. \square

Remark 5.17. It can be seen that Theorem 5.16 can also be deduced from Lemma 5.10 (iii). Fan (Lemma 1 in [6]) proved a result of Theorem 5.16 type.

Theorem 5.18. Let $u, u_n, n = 1, 2, \dots$, be fuzzy sets in $F_{USC}^1(X)$. If $H_{\text{end}}(u_n, u) \rightarrow 0$, then $H([u_n]_\alpha, [u]_\alpha) \rightarrow 0$ for each $\alpha \in (0, 1) \setminus P_0(u)$.

Proof. Let $\alpha \in (0, 1) \setminus P_0(u)$. Given $\varepsilon > 0$. Then there exists a $\delta(\alpha, \varepsilon) \in (0, \varepsilon/2)$ such that $[\alpha - \delta, \alpha + \delta] \subset [0, 1]$ and $H([u]_\beta, [u]_\alpha) < \varepsilon/2$ for all $\beta \in [\alpha - \delta, \alpha + \delta]$.

From $H_{\text{end}}(u_n, u) \rightarrow 0$, there exists an $N(\delta)$ such that

$$H_{\text{end}}(u_n, u) < \delta \tag{10}$$

for all $n \geq N$. Thus by Proposition 5.6, $H^*([u_n]_\alpha, [u]_{\alpha-\delta}) < \delta < \varepsilon/2$. So, for each $n \geq N$,

$$\begin{aligned}
&H^*([u_n]_\alpha, [u]_\alpha) \\
&\leq H^*([u_n]_\alpha, [u]_{\alpha-\delta}) + H([u]_\alpha, [u]_{\alpha-\delta}) \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned} \tag{11}$$

Similarly, it follows from (10) and Proposition 5.6 that $H^*([u]_{\alpha+\delta}, [u_n]_\alpha) < \delta < \varepsilon/2$, and then, for each $n \geq N$,

$$\begin{aligned}
&H^*([u]_\alpha, [u_n]_\alpha) \\
&\leq H([u]_\alpha, [u]_{\alpha+\delta}) + H^*([u]_{\alpha+\delta}, [u_n]_\alpha) \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned} \tag{12}$$

Combined with (11) and (12), we obtain that $H([u]_\alpha, [u_n]_\alpha) < \varepsilon$ for each $n \geq N$. Since $\varepsilon > 0$ is arbitrary, it follows that $H([u]_\alpha, [u_n]_\alpha) \rightarrow 0$. \square

Remark 5.19. Note that for each $\alpha \in (0, 1) \setminus P_0(u)$, $\lim_{\lambda \rightarrow \alpha} H([u]_\alpha, [u]_\lambda) = 0$ and $[u]_\alpha = \overline{\{u > \alpha\}}$. Thus Theorem 5.18 can also be deduced from Lemma 5.11.

The following theorem gives some conditions under which the H_{end} convergence of fuzzy sets can be decomposed to the Hausdorff metric convergence of certain α -cuts.

Theorem 5.20. *Let u be a fuzzy set in $F_{USCG}^1(X)$ and let u_n , $n = 1, 2, \dots$, be fuzzy sets in $F_{USC}^1(X)$. Then the following are equivalent:*

- (i) $H_{\text{end}}(u_n, u) \rightarrow 0$;
- (ii) $H([u_n]_\alpha, [u]_\alpha) \rightarrow 0$ holds a.e. on $\alpha \in (0, 1)$;
- (iii) $H([u_n]_\alpha, [u]_\alpha) \rightarrow 0$ for all $\alpha \in (0, 1) \setminus P_0(u)$;
- (iv) There is a dense subset P of $(0, 1) \setminus P_0(u)$ such that $H([u_n]_\alpha, [u]_\alpha) \rightarrow 0$ for $\alpha \in P$;
- (v) There is a countable dense subset P of $(0, 1) \setminus P_0(u)$ such that $H([u_n]_\alpha, [u]_\alpha) \rightarrow 0$ for $\alpha \in P$.

Proof. Suppose that u , u_n , $n = 1, 2, \dots$, are fuzzy sets in $F_{USC}^1(X)$. We claim that

- (a) $(ii) \Rightarrow (i) \Rightarrow (iii) \Rightarrow (iv) \Leftrightarrow (v)$.
 - (b) If $(0, 1) \setminus P_0(u)$ is dense in $(0, 1)$, then $(i) \Leftrightarrow (iii) \Leftrightarrow (iv) \Leftrightarrow (v)$.
 - (c) If $P_0(u)$ is a set of measure zero, then $(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv) \Leftrightarrow (v)$.
- $(ii) \Rightarrow (i)$ follows from Theorem 5.16. $(i) \Rightarrow (iii)$ is Theorem 5.18. Clearly $(iii) \Rightarrow (iv) \Leftrightarrow (v)$. So (a) is true. If $(0, 1) \setminus P_0(u)$ is dense in $(0, 1)$, then a dense subset P of $(0, 1) \setminus P_0(u)$ is also a dense subset of $(0, 1)$. So by Theorem 5.16, $(iv) \Rightarrow (i)$. Combined with (a), we have that (b) is true.

Assume that $P_0(u)$ is a set of measure zero. Then $(iii) \Rightarrow (ii)$. At this time, $(0, 1) \setminus P_0(u)$ is dense in $(0, 1)$. So from (a) and (b), we obtain that (c) is true.

By Lemma 5.4, for each $u \in F_{USCG}^1(X)$, $P_0(u)$ is countable, and then $P_0(u)$ is a set of measure zero. So by (c), we obtain the desired result. \square

Remark 5.21. Clearly, Theorem 5.15 implies Theorem 5.20.

6. Relations among metrics on $F_{USC}^1(X)$

In this section, we discuss the relation among the H_{end} metric, the H_{send} metric, the d_∞ metric, and the d_p^* metric on $F_{USC}^1(X)$.

For $u, v \in F_{USC}^1(X)$, the d_p distance given by

$$d_p(u, v) = \left(\int_0^1 H([u]_\alpha, [v]_\alpha)^p d\alpha \right)^{1/p}$$

is well-defined if and only if $H([u]_\alpha, [v]_\alpha)$ is a measurable function of α on $[0, 1]$ (here we see $H([u]_\alpha, [v]_\alpha)$ as a function of α from $[0, 1]$ to $\mathbb{R} \cup \{+\infty\}$). In the sequel, we suppose that the d_p distance satisfying $p \geq 1$.

Since $H([u]_\alpha, [v]_\alpha)$ could be a non-measurable function of α on $[0, 1]$ (see Example 2.13 in [16]), we introduce the d_p^* distance on $F_{USC}^1(X)$, $p \geq 1$, in [15], which is defined by

$$d_p^*(u, v) := \inf \left\{ \left(\int_0^1 f(\alpha)^p d\alpha \right)^{1/p} : f \text{ is a measurable function from } [0, 1] \text{ to } \mathbb{R} \cup \{+\infty\}; \right. \\ \left. f(\alpha) \geq H([u]_\alpha, [v]_\alpha) \text{ for } \alpha \in [0, 1] \right\}$$

for $u, v \in F_{USC}^1(X)$.

For each $u, v \in F_{USC}^1(X)$, if $d_p(u, v)$ is well-defined then clearly $d_p^*(u, v) = d_p(u, v)$. In [15, 16] and references therein, we have given several conclusions on the well-definedness of the d_p distance including: the d_p distance is well-defined on $F_{USC}^1(\mathbb{R}^m)$, and the d_p distance is well-defined on $F_{USCG}^1(X)$.

Let $u, v \in F_{USCG}^1(X)$. By Proposition 5.5(i), the function $H([u]_\alpha, [v]_\alpha)$ of α is left-continuous on $(0, 1]$. So from Proposition 6.1 below, $H([u]_\alpha, [v]_\alpha)$ is a measurable function of α on $[0, 1]$. Proposition 6.1 may be known. Here we give a proof for the completeness of this paper. A conclusion stronger than Proposition 6.1 was given in [16] and the references therein.

Proposition 6.1. *If a function $f : [0, 1] \rightarrow \mathbb{R} \cup \{+\infty\}$ is left-continuous on $(0, 1]$, then f is a measurable function on $[0, 1]$.*

Proof. Let $r \in \mathbb{R}$. Denote the set $\{x \in [0, 1] : f(x) > r\}$ by $\{f > r\}$. If $\{f > r\} \setminus \{0\} = \emptyset$, then $\{f > r\} \setminus \{0\}$ is a measurable set. Suppose that $\{f > r\} \setminus \{0\} \neq \emptyset$. For each $x \in \{f > r\} \setminus \{0\}$, let $\widehat{x} = \bigcup \{[a, b] : x \in [a, b] \subseteq \{f > r\} \setminus \{0\}\}$. Clearly \widehat{x} is an interval. Note that for each interval A with $x \in A \subseteq \{f > r\} \setminus \{0\}$, $A = \bigcup \{[a, b] : x \in [a, b] \subseteq A\} \subseteq \widehat{x}$. So \widehat{x} is the largest interval in $\{f > r\} \setminus \{0\}$ which contains x . For each $x \in X$, if $\alpha \in \widehat{x}$, then there exists a $\delta(\alpha) > 0$ with $[\alpha - \delta(\alpha), \alpha] \subseteq \{f > r\} \setminus \{0\}$ since f is left-continuous at α . So \widehat{x} is a positive length interval. For each $x, y \in \{f > r\} \setminus \{0\}$, if $\widehat{x} \cap \widehat{y} \neq \emptyset$, then $\widehat{x} \cup \widehat{y}$ is an interval with $\{x, y\} \subset \widehat{x} \cup \widehat{y} \subseteq \{f > r\} \setminus \{0\}$, and hence $\widehat{x} = \widehat{x} \cup \widehat{y} = \widehat{y}$. Thus $\{f > r\} \setminus \{0\}$ is a union of disjoint positive length intervals. Clearly the set S of all these disjoint positive length intervals is countable (For each $B \in S$, choose a q_B in $B \cap \mathbb{Q}$. Then $q_{B_1} \neq q_{B_2}$ for each B_1 and B_2 in S

with $B_1 \neq B_2$. Thus $\overline{S} = \overline{\{q_B : B \in S\}} \leq \overline{\mathbb{Q}}$ and so S is countable, where \overline{C} denotes the cardinality of a set C . So $\{f > r\} \setminus \{0\}$ is a countable union of intervals and hence is a measurable set. Thus $\{f > r\}$ is a measurable set as $\{f > r\} = \{f > r\} \setminus \{0\}$ or $\{f > r\} = (\{f > r\} \setminus \{0\}) \cup \{0\}$. Since $r \in \mathbb{R}$ is arbitrary, it follows that f is a measurable function on $[0, 1]$. \square

d_p^* is an extended metric but may not be a metric on $F_{USC}^1(X)$. See also Remark 3.3 in [15]. We can see that the d_p distance is a metric on $F_{USCB}^1(X)$. The d_p distance on $F_{USCG}^1(\mathbb{R}^m)$ is an extended metric but not a metric, and the d_p distance on $F_{USC}^1(\mathbb{R}^m)$ is an extended metric but not a metric. The d_p distance on $F_{USCG}^1(\mathbb{R}^m)$ could take the value $+\infty$. Let $u \in F_{USCG}^1(\mathbb{R}^m)$ be defined by putting

$$[u]_\alpha = \{x \in \mathbb{R}^m : \|x\| \leq n\} \text{ for each } n \in \mathbb{N} \text{ and } \alpha \in (1/(n+1), 1/n].$$

Denote the origin of \mathbb{R}^m by o . Then $d_p(u, \widehat{o}_{F(\mathbb{R}^m)}) = (\sum_{n=1}^{+\infty} n^p \cdot (1/n - 1/(n+1)))^{1/p} = +\infty$.

For simplicity, in this paper, we call the d_p^* distance on $F_{USC}^1(X)$ the d_p^* metric, and call the d_p distance on $F_{USC}^1(\mathbb{R}^m)$ or $F_{USCG}^1(X)$ the d_p metric.

Clearly for $u, v \in F_{USC}^1(X)$,

$$d_\infty(u, v) \geq d_p^*(u, v). \quad (13)$$

The proof of (13) is routine. Set $d_\infty(u, v) = \xi \in \mathbb{R} \cup \{+\infty\}$. Define $f : [0, 1] \rightarrow \mathbb{R} \cup \{+\infty\}$ by $f(\alpha) = \xi$ for each $\alpha \in [0, 1]$. Hence f is a measurable function from $[0, 1]$ to $\mathbb{R} \cup \{+\infty\}$ and $f(\alpha) \geq H([u]_\alpha, [v]_\alpha)$ for $\alpha \in [0, 1]$. So $d_p^*(u, v) \leq \left(\int_0^1 f(\alpha)^p d\alpha \right)^{1/p} = \xi$. Thus (13) is true.

Theorem 6.2. *Let $u \in F_{USC}^1(X)$ and for each positive integer n , let $u_n \in F_{USC}^1(X)$. If $d_p^*(u_n, u) \rightarrow 0$, then $H_{\text{end}}(u_n, u) \rightarrow 0$.*

Proof. We prove by contradiction. If $H_{\text{end}}(u_n, u) \not\rightarrow 0$, then there is an $\varepsilon > 0$ and a subsequence $\{v_n\}$ of $\{u_n\}$ such that for each $n \in \mathbb{N}$,

$$H_{\text{end}}(v_n, u) \geq \varepsilon. \quad (14)$$

By (14), for each $n \in \mathbb{N}$, $v_n \neq u$ and hence $d_p^*(v_n, u) \neq 0$. From the definition of d_p^* , there exists a sequence $\{f_n\}$ of measurable function from $[0, 1]$ to $\mathbb{R} \cup \{+\infty\}$ such that for each $n \in \mathbb{N}$,

$$H([v_n]_\alpha, [u]_\alpha) \leq f_n(\alpha) \text{ for all } \alpha \in [0, 1], \quad (15)$$

$$\left(\int_0^1 f_n(\alpha)^p d\alpha\right)^{1/p} \leq \frac{n+1}{n} d_p^*(v_n, u). \quad (16)$$

Since $d_p^*(v_n, u) \rightarrow 0$, by (16), we have $\left(\int_0^1 f_n(\alpha)^p d\alpha\right)^{1/p} \rightarrow 0$. Thus there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\{f_{n_k}(\alpha)\}$ converges to 0 a.e. on $\alpha \in [0, 1]$. Hence by (15), $H([v_{n_k}]_\alpha, [u]_\alpha) \rightarrow 0$ holds a.e. on $\alpha \in (0, 1)$. By Theorem 5.16, this implies $H_{\text{end}}(v_{n_k}, u) \rightarrow 0$, which contradicts (14). \square

Theorem 4.1 in [16] says that for $u \in F_{USCG}^1(X)$ and $v \in F_{USC}^1(X)$, $H([u]_\alpha, [v]_\alpha)$ is a measurable function of α on $[0, 1]$. So $d_p^*(u, v) = d_p(u, v)$ for $u \in F_{USCG}^1(X)$ and $v \in F_{USC}^1(X)$.

Theorem 6.3. *Suppose that $u \in F_{USCG}^1(X)$ and $u_n \in F_{USC}^1(X)$, $n = 1, 2, \dots$, and that there is a measurable function F on $[0, 1]$ such that $\int_0^1 F^p(\alpha) d\alpha < +\infty$ and $H([u_n]_\alpha, [u]_\alpha) \leq F(\alpha)$ for $n = 1, 2, \dots$. If $H_{\text{end}}(u_n, u) \rightarrow 0$, then $d_p(u_n, u) \rightarrow 0$.*

Proof. By Theorem 5.20, $H_{\text{end}}(u_n, u) \rightarrow 0$ if and only if $H([u_n]_\alpha, [u]_\alpha) \rightarrow 0$ holds a.e. on $\alpha \in (0, 1)$. So the desired result follows from the Lebesgue's Dominated Convergence Theorem. \square

Proposition 6.4. *Let $u \in F_{USCB}^1(X)$ and for each positive integer n , let $u_n \in F_{USC}^1(X)$. Then $H_{\text{send}}(u_n, u) \rightarrow 0$ if and only if $H([u_n]_0, [u]_0) \rightarrow 0$ and $d_p(u_n, u) \rightarrow 0$.*

Proof. From Proposition 3.3, $H_{\text{send}}(u_n, u) \rightarrow 0$ if and only if $H([u_n]_0, [u]_0) \rightarrow 0$ and $H_{\text{end}}(u_n, u) \rightarrow 0$. To prove the desired result, we only need to show that

$$H([u_n]_0, [u]_0) \rightarrow 0 \text{ and } d_p(u_n, u) \rightarrow 0 \Leftrightarrow H([u_n]_0, [u]_0) \rightarrow 0 \text{ and } H_{\text{end}}(u_n, u) \rightarrow 0.$$

From Theorem 6.2, “ \Rightarrow ” is true. To show “ \Leftarrow ”, suppose that $H([u_n]_0, [u]_0) \rightarrow 0$ and $H_{\text{end}}(u_n, u) \rightarrow 0$. Then there exists an $N \in \mathbb{N}$ such that $\bigcup_{n \geq N} [u_n]_0$ is bounded. Hence there is an $M > 0$ such that for $n \geq N$ and $\alpha \in [0, 1]$, $H([u_n]_\alpha, [u]_\alpha) \leq d_\infty(u_n, u) < M$. Thus by Theorem 6.3, $d_p(u_n, u) \rightarrow 0$. So “ \Leftarrow ” is true. \square

However, for a sequence $\{u_n\}$ and an element u in $F_{USCG}(X)$, $H_{\text{send}}(u_n, u) \rightarrow 0$ does not necessarily imply $d_p(u_n, u) \rightarrow 0$. See the following Example 6.5.

Example 6.5. Define $u \in F_{USCG}^1(\mathbb{R})$ by putting

$$[u]_\alpha = [0, 1/\alpha] \text{ for } \alpha \in (0, 1].$$

For each $n = 1, 2, \dots$, define $u_n \in F_{USCG}^1(\mathbb{R})$ by putting

$$[u_n]_\alpha = \begin{cases} [0, 1/\alpha], & \alpha \in (1/n, 1], \\ [0, \frac{1}{\alpha} + n^2], & \alpha \in (0, 1/n]. \end{cases}$$

Then $H_{\text{send}}(u_n, u) \rightarrow 0$ and $d_p(u_n, u) = n^{2-1/p} \not\rightarrow 0$.

In [15], we obtain that the Skorokhod metric convergence imply the sendograph metric convergence on $F_{USC}^1(X)$ (see Theorem 8.1 in [15]), and that Skorokhod metric convergence need not imply the d_p convergence on a subset of $F_{USCG}^1(X)$ (see the end of Section 5 in [15]). From the above conclusions in [15], we can also deduce that the sendograph metric convergence need not imply the d_p convergence on $F_{USCG}^1(X)$.

7. Characterizations of compactness in $(F_{USCG}^1(X), H_{\text{end}})$ and $(F_{USCB}^1(X), H_{\text{send}})$

Based on the conclusions in previous sections, we give characterizations of total boundedness, relative compactness and compactness in $(F_{USCG}^1(X), H_{\text{end}})$ and $(F_{USCB}^1(X), H_{\text{send}})$.

- A subset Y of a topological space Z is said to be *compact* if for every set I and every family of open sets, $O_i, i \in I$, such that $Y \subset \bigcup_{i \in I} O_i$ there exists a finite family $O_{i_1}, O_{i_2}, \dots, O_{i_n}$ such that $Y \subseteq O_{i_1} \cup O_{i_2} \cup \dots \cup O_{i_n}$. In the case of a metric topology, the criterion for compactness becomes that any sequence in Y has a subsequence convergent in Y .
- A *relatively compact* subset Y of a topological space Z is a subset with compact closure. In the case of a metric topology, the criterion for relative compactness becomes that any sequence in Y has a subsequence convergent in X .
- Let (X, d) be a metric space. A set U in X is *totally bounded* if and only if, for each $\varepsilon > 0$, it contains a finite ε approximation, where an ε approximation to U is a subset S of U such that $d(x, S) < \varepsilon$ for each $x \in U$. It is known that U in X is totally bounded if and only if, for each $\varepsilon > 0$, there is a finite *weak ε -net* of U , where a *weak ε -net* of U is a subset S of X satisfying $d(x, S) < \varepsilon$ for each $x \in U$.

Let (X, d) be a metric space. A set U is compact in (X, d) implies that U is relatively compact in (X, d) , which in turn implies that U is totally bounded in (X, d) .

We use (\tilde{X}, \tilde{d}) to denote the completion of (X, d) . We see (X, d) as a subspace of (\tilde{X}, \tilde{d}) . Let $S \subseteq \tilde{X}$. The symbol \tilde{S} is used to denote the closure of S in (\tilde{X}, \tilde{d}) .

As defined in Section 2, we have $K(\tilde{X})$, $C(\tilde{X})$, $F_{USC}^1(\tilde{X})$, $F_{USCG}^1(\tilde{X})$, etc. according to (\tilde{X}, \tilde{d}) . If there is no confusion, we also use H to denote the Hausdorff metric on $C(\tilde{X})$ induced by \tilde{d} . We also use H to denote the Hausdorff metric on $C(\tilde{X} \times [0, 1])$ induced by \tilde{d} . We also use H_{end} to denote the endograph metric on $F_{USC}^1(\tilde{X})$ defined by using H on $C(\tilde{X} \times [0, 1])$.

Let $U \subseteq X$. If U is compact in (X, d) , then U is compact in (\tilde{X}, \tilde{d}) . We see $(K(X), H)$ as a subspace of $(K(\tilde{X}), H)$.

Define j from $F(X)$ to $F(\tilde{X})$ as follows: for $u \in F(X)$, $j(u) \in F(\tilde{X})$ is given by

$$j(u)(t) = \begin{cases} u(t), & t \in X, \\ 0, & t \in \tilde{X} \setminus X. \end{cases}$$

Let $u \in F_{USCG}^1(X)$. Then $j(u) \in F_{USCG}^1(\tilde{X})$ because $[j(u)]_\alpha = [u]_\alpha \subseteq K(\tilde{X})$ for each $\alpha \in (0, 1]$. Clearly for each $u, v \in F_{USCG}^1(X)$, $H_{\text{end}}(u, v) = H_{\text{end}}(j(u), j(v))$. In the sequel, we treat $(F_{USCG}^1(X), H_{\text{end}})$ as a subspace of $(F_{USCG}^1(\tilde{X}), H_{\text{end}})$ by identifying u in $F_{USCG}^1(X)$ with $j(u)$ in $F_{USCG}^1(\tilde{X})$.

7.1. Characterizations of compactness in $(K(X), H)$

In this subsection, we give characterizations of total boundedness, relative compactness and compactness in $(K(X), H)$. The results in this subsection are basis for contents in the sequel.

Theorem 7.1. *Let (X, d) be a complete metric space and let $\{C_n\}$ be a Cauchy sequence in $(K(X), H)$. Put $D = \overline{\bigcup_{n=1}^{+\infty} C_n}$. For $n = 1, 2, \dots$, put $D_n = \bigcup_{l=1}^n C_l$. Then $D \in K(X)$ and $H(D_n, D) \rightarrow 0$ as $n \rightarrow \infty$.*

Proof. Let $k, j \in \mathbb{N}$ with $k > j$. Then $H^*(D_j, D_k) = 0$, and for each $i \in \mathbb{N}$ with $1 \leq i \leq j$, $H^*(C_i, D_j) = 0$. Thus $H(D_k, D_j) = H^*(D_k, D_j) = \max\{H^*(C_i, D_j) : i = 1, \dots, k\} = \max\{H^*(C_i, D_j) : i = j + 1, \dots, k\} \leq \max\{H^*(C_i, C_j) : i = j + 1, \dots, k\}$. Hence

$$H(D_k, D_j) \leq \max\{H(C_i, C_j) : i = j + 1, \dots, k\}.$$

So $\{D_n\}$ is a Cauchy sequence in $(K(X), H)$. From Theorem 2.3, $(K(X), H)$ is complete, and thus there is an $E \in K(X)$ such that $H(D_n, E) \rightarrow 0$ as $n \rightarrow \infty$. By Theorem 3.4, $E = \lim_{n \rightarrow \infty}^{(K)} D_n = D$. \square

Theorem 7.2. *Let (X, d) be a metric space and $\mathcal{D} \subseteq K(X)$. Then \mathcal{D} is totally bounded in $(K(X), H)$ if and only if $\mathbf{D} = \bigcup\{C : C \in \mathcal{D}\}$ is totally bounded in (X, d) .*

Proof. If $\mathcal{D} = \emptyset$, then the desired result follows immediately. Suppose that $\mathcal{D} \neq \emptyset$.

Necessity. To prove that \mathbf{D} is totally bounded it suffices to show that each sequence in \mathbf{D} has a Cauchy subsequence.

Given a sequence $\{x_n\}$ in \mathbf{D} . Suppose that $x_n \in C_n \in \mathcal{D}$ for $n = 1, 2, \dots$. Since \mathcal{D} is totally bounded in $(K(X), H)$, then $\{C_n\}$ has a Cauchy subsequence $\{C_{n_k}\}$ in $(K(X), H)$. As $\{C_{n_k}\}$ is also a Cauchy sequence in $(K(\tilde{X}), H)$, by Theorem 7.1, $\overline{\bigcup_{k=1}^{+\infty} C_{n_k}}$ is in $K(\tilde{X})$. Thus $\{x_{n_k}\}$ has a Cauchy subsequence, and hence so does $\{x_n\}$.

Sufficiency. If \mathbf{D} is totally bounded in X , then $\tilde{\mathbf{D}}$ is in $K(\tilde{X})$. So, by Theorem 2.3, $(K(\tilde{\mathbf{D}}), H)$ is compact, and thus $\mathcal{D} \subseteq K(\tilde{\mathbf{D}})$ is totally bounded in $(K(\tilde{\mathbf{D}}), H)$. So \mathcal{D} is obviously totally bounded in $(K(X), H)$. \square

Theorem 7.3. [8] *Let (X, d) be a metric space and $\mathcal{D} \subseteq K(X)$. Then \mathcal{D} is relatively compact in $(K(X), H)$ if and only if $\mathbf{D} = \bigcup\{C : C \in \mathcal{D}\}$ is relatively compact in (X, d) .*

Proof. If $\mathcal{D} = \emptyset$, then the desired result follows immediately. Suppose that $\mathcal{D} \neq \emptyset$.

Necessity. To prove that \mathbf{D} is relatively compact it suffices to show that each sequence in \mathbf{D} has a convergent subsequence in (X, d) .

Given a sequence $\{x_n\}$ in \mathbf{D} . Suppose that $x_n \in C_n \in \mathcal{D}$ for $n = 1, 2, \dots$. Since \mathcal{D} is relatively compact in $(K(X), H)$, then $\{C_n\}$ has a subsequence $\{C_{n_k}\}$ which converges to an element C in $(K(X), H)$. Clearly $\{C_{n_k}\}$ converges to C in $(K(\tilde{X}), H)$. Hence, by Theorem 7.1, $\overline{\bigcup_{k=1}^{+\infty} C_{n_k}}$ is in $K(\tilde{X})$ (Indeed, $\overline{\bigcup_{k=1}^{+\infty} C_{n_k}}$ is in $K(X)$). So $\{x_{n_k}\}$ has a subsequence which converges to x in $\overline{\bigcup_{k=1}^{+\infty} C_{n_k}}$, and thus $x \in C = \lim_{k \rightarrow \infty}^{(K)} C_{n_k} \subseteq X$.

Sufficiency. If \mathbf{D} is relatively compact in X , then $\overline{\mathbf{D}}$ is in $K(X)$, and therefore $(K(\overline{\mathbf{D}}), H)$ is compact. Thus $\mathcal{D} \subseteq K(\overline{\mathbf{D}})$ is relatively compact in $(K(\overline{\mathbf{D}}), H)$. So clearly \mathcal{D} is relatively compact in $(K(X), H)$. \square

Lemma 7.4. *Let (X, d) be a metric space and $\mathcal{D} \subseteq K(X)$. If \mathcal{D} is compact in $(K(X), H)$, then $\mathbf{D} = \bigcup\{C : C \in \mathcal{D}\}$ is compact in (X, d) .*

Proof. If $\mathcal{D} = \emptyset$, then the desired result follows immediately. Suppose that $\mathcal{D} \neq \emptyset$. To show that \mathbf{D} is compact, we only need to show that each sequence in \mathbf{D} has a subsequence which converges to a point in \mathbf{D} .

Given a sequence $\{x_n\}$ in \mathbf{D} . Suppose that $x_n \in C_n \in \mathcal{D}$ for $n = 1, 2, \dots$. Since \mathcal{D} is compact, then $\{C_n\}$ has a subsequence $\{C_{n_k}\}$ converges to $C \in \mathcal{D}$. Clearly $\{C_{n_k}\}$ converges to C in $(K(\tilde{X}), H)$. Hence, by Theorem 7.1, $\overline{\bigcup_{k=1}^{+\infty} C_{n_k}}$ is in $K(\tilde{X})$ (Indeed, $\overline{\bigcup_{k=1}^{+\infty} C_{n_k}}$ is in $K(\mathbf{D})$). So $\{x_{n_k}\}$ has a subsequence which converges to x in $\overline{\bigcup_{k=1}^{+\infty} C_{n_k}}$. Thus $x \in C = \lim_{k \rightarrow \infty}^{(K)} C_{n_k} \subseteq \mathbf{D}$. \square

Remark 7.5. The converse of the implication in Lemma 7.4 does not hold. Let $(X, d) = \mathbb{R}$ and $\mathcal{D} = \{[0, x] : x \in (0.3, 1]\} \subset K(\mathbb{R})$. Then $\mathbf{D} = [0, 1] \in K(\mathbb{R})$. But \mathcal{D} is not compact in $(K(\mathbb{R}), H)$.

Theorem 7.6. *Let (X, d) be a metric space and $\mathcal{D} \subseteq K(X)$. Then the following are equivalent:*

- (i) \mathcal{D} is compact in $(K(X), H)$;
- (ii) $\mathbf{D} = \bigcup\{C : C \in \mathcal{D}\}$ is relatively compact in (X, d) and \mathcal{D} is closed in $(K(X), H)$;
- (iii) $\mathbf{D} = \bigcup\{C : C \in \mathcal{D}\}$ is compact in (X, d) and \mathcal{D} is closed in $(K(X), H)$.

Proof. Note that \mathcal{D} is compact in $(K(X), H)$ if and only if \mathcal{D} is relatively compact and closed in $(K(X), H)$. Then from Theorem 7.3 we have (i) \Leftrightarrow (ii). Clearly (iii) \Rightarrow (ii). We shall complete the proof by showing that (i) \Rightarrow (iii), which can be deduced by Lemma 7.4. \square

Remark 7.7. Theorem 7.3 is Proposition 5 in [8]. We cannot find the proof of Proposition 5 in [8]. Some of the results in this subsection may be known. We cannot find the results of this subsection other than Theorem 7.3 in the references that we can obtain. So we give our proofs here.

7.2. Characterizations of compactness in $(F_{USCG}^1(X), H_{\text{end}})$

In this subsection, we give characterizations of total boundedness, relative compactness and compactness in $(F_{USCG}^1(X), H_{\text{end}})$.

Suppose that U is a subset of $F_{USCG}^1(X)$ and $\alpha \in [0, 1]$. For writing convenience, we denote

- $U(\alpha) := \bigcup_{u \in U} [u]_\alpha$, and
- $U_\alpha := \{[u]_\alpha : u \in U\}$.

Theorem 7.8. *Let U be a subset of $F_{USCG}^1(X)$. Then U is totally bounded in $(F_{USCG}^1(X), H_{\text{end}})$ if and only if $U(\alpha)$ is totally bounded in (X, d) for each $\alpha \in (0, 1]$.*

Proof. Necessity. Suppose that U is totally bounded in $(F_{USCG}^1(X), H_{\text{end}})$. Let $\alpha \in (0, 1]$. To show that $U(\alpha)$ is totally bounded in X , we only need to show that each sequence in $U(\alpha)$ has a Cauchy subsequence.

Given a sequence $\{x_n\} \subset U(\alpha)$. Suppose that $x_n \in [u_n]_\alpha$, $u_n \in U$, $n = 1, 2, \dots$. Then $\{u_n\}$ has a Cauchy subsequence $\{u_{n_l}\}$. So given $\varepsilon \in (0, \alpha)$, there is a $K(\varepsilon) \in \mathbb{N}$ such that $H_{\text{end}}(u_{n_l}, u_{n_K}) < \varepsilon$ for all $l \geq K$. Thus by Proposition 5.6,

$$H^*([u_{n_l}]_\alpha, [u_{n_K}]_{\alpha-\varepsilon}) < \varepsilon \quad (17)$$

for all $l \geq K$. From (17) and the arbitrariness of ε , $\bigcup_{l=1}^{+\infty} [u_{n_l}]_\alpha$ is totally bounded in (X, d) . Thus $\{x_n\}$, which is a subsequence of $\{x_n\}$, has a Cauchy subsequence, and so does $\{x_n\}$.

In the following, we give a detailed proof for the above conclusion that $\bigcup_{l=1}^{+\infty} [u_{n_l}]_\alpha$ is totally bounded in (X, d) . To show that $\bigcup_{l=1}^{+\infty} [u_{n_l}]_\alpha$ is totally bounded in (X, d) , it suffices to show that for each $\lambda > 0$, there exists a finite weak λ -net of $\bigcup_{l=1}^{+\infty} [u_{n_l}]_\alpha$.

Let $\lambda > 0$. Set $\varepsilon = \min\{\lambda/2, \alpha/2\}$. Then $\varepsilon \in (0, \alpha)$. Hence there is a $K(\varepsilon)$ such that (17) holds for all $l \geq K$. Since $\bigcup_{l=1}^K [u_{n_l}]_{\alpha-\varepsilon}$ is compact, there is a finite ε approximation $\{z_j\}_{j=1}^m$ to $\bigcup_{l=1}^K [u_{n_l}]_{\alpha-\varepsilon}$. We claim that $\{z_j\}_{j=1}^m$ is a finite weak λ -net of $\bigcup_{l=1}^{+\infty} [u_{n_l}]_\alpha$.

Let $z \in \bigcup_{l=1}^{+\infty} [u_{n_l}]_\alpha$. If $z \in \bigcup_{l=1}^K [u_{n_l}]_\alpha$, then clearly $d(z, \{z_j\}_{j=1}^m) < \varepsilon < \lambda$. If $z \in \bigcup_{l=K+1}^{+\infty} [u_{n_l}]_\alpha$, then by (17), there exists a $y_z \in [u_{n_K}]_{\alpha-\varepsilon}$ such that $d(z, y_z) < \varepsilon$, and hence $d(z, \{z_j\}_{j=1}^m) \leq d(z, y_z) + d(y_z, \{z_j\}_{j=1}^m) < 2\varepsilon \leq \lambda$. Thus $d(z, \{z_j\}_{j=1}^m) < \lambda$ for each $z \in \bigcup_{l=1}^{+\infty} [u_{n_l}]_\alpha$. So $\{z_j\}_{j=1}^m$ is a finite weak λ -net of $\bigcup_{l=1}^{+\infty} [u_{n_l}]_\alpha$.

Sufficiency. Suppose that $U(\alpha)$ is totally bounded in (X, d) for each $\alpha \in (0, 1]$. Then $U(\alpha)$ is relatively compact in (\tilde{X}, d) for each $\alpha \in (0, 1]$. So from the sufficiency part of Theorem 7.9, we obtain that U is relatively compact in $(F_{USCG}^1(\tilde{X}), H_{\text{end}})$. Then U is totally bounded in $(F_{USCG}^1(\tilde{X}), H_{\text{end}})$, and so U is totally bounded in $(F_{USCG}^1(X), H_{\text{end}})$. \square

Theorem 7.9. *Let U be a subset of $F_{USCG}^1(X)$. Then U is relatively compact in $(F_{USCG}^1(X), H_{\text{end}})$ if and only if $U(\alpha)$ is relatively compact in (X, d) for each $\alpha \in (0, 1]$.*

Proof. Necessity. Suppose that U is relatively compact. Given $\alpha \in (0, 1]$. To show that $U(\alpha)$ is relatively compact in X , we only need to show that each sequence in $U(\alpha)$ has a convergent subsequence in (X, d) .

Let $\{x_n\}$ be a sequence in $U(\alpha)$. Suppose that $x_n \in [u_n]_\alpha$, $u_n \in U$, $n = 1, 2, \dots$. Then there is a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and $u \in F_{USCG}^1(X)$ such that $H_{\text{end}}(u_{n_k}, u) \rightarrow 0$. So, by Theorem 5.20, $H([u_{n_k}]_\alpha, [u]_\alpha) \rightarrow 0$ holds a.e. on $\alpha \in (0, 1)$, and therefore there is a $\beta \in (0, \alpha)$ such that $H([u_{n_k}]_\beta, [u]_\beta) \rightarrow 0$. Hence by Theorem 7.3, $\bigcup_{k=1}^{+\infty} [u_{n_k}]_\beta$ is relatively compact in X . Thus $\{x_{n_k}\}$ has a convergent subsequence in X , and so does $\{x_n\}$.

Sufficiency. Suppose that $U(\alpha)$ is relatively compact in X for each $\alpha \in (0, 1]$. Note that $U(\alpha) = \bigcup \{D : D \in U_\alpha\}$ for each $\alpha \in [0, 1]$. So, by Theorem 7.3, we obtain that U_α is relatively compact in $(K(X), H)$ for each $\alpha \in (0, 1]$. This means that the following affirmation (a) is true.

- (a) Given a sequence $\{w_n : n = 1, 2, \dots\}$ in U and $\alpha \in (0, 1]$. Then the corresponding sequence $\{[w_n]_\alpha : n = 1, 2, \dots\}$ has a convergent subsequence in $(K(X), H)$.

To show that U is relatively compact in $(F_{USCG}^1(X), H_{\text{end}})$, we only need to show that each sequence in U has a convergent subsequence in $(F_{USCG}^1(X), H_{\text{end}})$. Suppose that $\{u_n\}$ is a sequence in U . Based on the above affirmation (a) and Theorem 5.20, and proceeding similarly to the proof of the ‘‘Sufficiency part’’ of Theorem 7.1 in [14], it can be shown that $\{u_n\}$ has a subsequence $\{v_n\}$ which converges to an element v in $(F_{USCG}^1(X), H_{\text{end}})$.

A sketch of the proof of the existence of $\{v_n\}$ and v is given as follows.

First, we construct a subsequence $\{v_n\}$ of $\{u_n\}$ such that $[v_n]_q$ converges to an element u_q in $(K(X), H)$ for all $q \in \mathbb{Q}'$, where $\mathbb{Q}' = \mathbb{Q} \cap (0, 1]$. For $\alpha \in (0, 1]$, we define $v_\alpha = \bigcap_{q < \alpha, q \in \mathbb{Q}'} u_q$. Then we define a fuzzy set v in X by using $\{v_\alpha, \alpha \in (0, 1]\}$. At last, we show that $v \in F_{USCG}^1(X)$, $[v]_\alpha = v_\alpha$ for each $\alpha \in (0, 1]$, and $H_{\text{end}}(v_n, v) \rightarrow 0$. \square

Theorem 7.10. *Let U be a subset of $F_{USCG}^1(X)$. Then the following are equivalent:*

- (i) U is compact in $(F_{USCG}^1(X), H_{\text{end}})$;
- (ii) $U(\alpha)$ is relatively compact in (X, d) for each $\alpha \in (0, 1]$ and U is closed in $(F_{USCG}^1(X), H_{\text{end}})$;
- (iii) $U(\alpha)$ is compact in (X, d) for each $\alpha \in (0, 1]$ and U is closed in $(F_{USCG}^1(X), H_{\text{end}})$.

Proof. By Theorem 7.9, (i) \Leftrightarrow (ii). Obviously (iii) \Rightarrow (ii). We shall complete the proof by showing that (i) \Rightarrow (iii). To do this, suppose that (i) is true. To verify (iii), from the equivalence of (i) and (ii), we only need to show that $U(\alpha)$ is closed in (X, d) for each $\alpha \in (0, 1]$.

Let $\alpha \in (0, 1]$ and let $\{x_n\}$ be a sequence in $U(\alpha)$ with $x_n \rightarrow x$. Suppose that $x_n \in [u_n]_\alpha$ and $u_n \in U$ for $n = 1, 2, \dots$. Then there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and $u \in U$ such that $H_{\text{end}}(u_{n_k}, u) \rightarrow 0$. So by Remark 3.5 $\lim_{n \rightarrow \infty}^{(r)} u_{n_k} = u$ and therefore by Theorem 4.3, $\limsup_{n \rightarrow \infty} [u_{n_k}]_\alpha \subseteq [u]_\alpha$. Hence $x \in [u]_\alpha$, and thus $x \in U(\alpha)$.

We can also show $x \in [u]_\alpha \subseteq U(\alpha)$ in the following way. From Theorem 5.18, $H([u_{n_k}]_\alpha, [u]_\alpha) \rightarrow 0$ holds for $\alpha \in (0, 1) \setminus P_0(u)$. If $\alpha \in (0, 1) \setminus P_0(u)$, then $x \in [u]_\alpha$. If $\alpha \in \{1\} \cup P_0(u)$, then for all $\beta \in (0, \alpha) \setminus P_0(u)$, $x \in [u]_\beta$. Thus $x \in [u]_\alpha$. \square

7.3. Characterizations of compactness in $(P_{USCB}^1(X), H_{\text{send}})$ and $(F_{USCB}^1(X), H_{\text{send}})$

In this subsection, we give the characterizations of totally bounded sets, relatively compact sets and compact sets in $(P_{USCB}^1(X), H_{\text{send}})$. Then, by treating $(F_{USCB}^1(X), H_{\text{send}})$ as a subspace of $(P_{USCB}^1(X), H_{\text{send}})$, we give the characterizations of totally bounded sets and compact sets in $(F_{USCB}^1(X), H_{\text{send}})$. The characterization of relatively compact sets in $(F_{USCB}^1(X), H_{\text{send}})$ has already been given in [8].

Suppose that U is a subset of $P_{USC}^1(X)$ and $\alpha \in [0, 1]$. For writing convenience, we denote

- $U(\alpha) := \bigcup_{u \in U} \langle u \rangle_\alpha$, and
- $U_\alpha := \{\langle u \rangle_\alpha : u \in U\}$.

Theorem 7.11. *Suppose that U is a subset of $P_{USCB}^1(X)$. Then U is totally bounded in $(P_{USCB}^1(X), H_{\text{send}})$ if and only if $U(0)$ is totally bounded in (X, d) .*

Proof. Necessity. Suppose that U is totally bounded. By clause (ii) of Theorem 3.2, U_0 is totally bounded in $(K(X), H)$. From Theorem 7.2, this is equivalent to $U(0)$ is totally bounded in (X, d) .

Sufficiency. Suppose that $U(0)$ is totally bounded. To show that U is totally bounded in $(P_{USCB}^1(X), H_{\text{send}})$, we only need to prove that each sequence in U has a Cauchy subsequence with respect to H_{send} .

Let $\{u_n\}$ be a sequence in U . Note that $U(\alpha)$ is totally bounded for each $\alpha \in [0, 1]$. Then by Theorem 7.8, $\{\overleftarrow{u_n}\}$ has a Cauchy subsequence $\{v_n\}$ in $(F_{USCB}^1(X), H_{\text{end}})$. From Theorem 7.2, $\{v_n\}$ has a subsequence $\{w_n\}$ such that $\{[w_n]_0\}$ is a Cauchy sequence in $(K(X), H)$. Thus by clauses (iii) of Theorem 3.2, $\{w_n\}$ is a Cauchy sequence in $(P_{USCB}^1(X), H_{\text{send}})$. \square

Theorem 7.12. *Suppose that U is a subset of $F_{USCB}^1(X)$. Then U is totally bounded in $(F_{USCB}^1(X), H_{\text{send}})$ if and only if $U(0)$ is totally bounded in (X, d) .*

Proof. Note that U is totally bounded in $(F_{USCB}^1(X), H_{\text{send}})$ if and only if \overrightarrow{U} is totally bounded in $(P_{USCB}^1(X), H_{\text{send}})$, and that $U(0) = \overrightarrow{U}(0)$. So the desired result follows from Theorem 7.11. \square

Theorem 7.13. *Suppose that U is a subset of $P_{USCB}^1(X)$. Then U is relatively compact in $(P_{USCB}^1(X), H_{\text{send}})$ if and only if $U(0)$ is relatively compact in X .*

Proof. Necessity. Suppose that U is relatively compact. Then by clause (ii) of Theorem 3.2, U_0 is relatively compact in $(K(X), H)$. By Theorem 7.3, $U(0)$ is relatively compact in X .

Sufficiency. To prove that U is relatively compact, it suffices to show that each sequence in U has a convergent subsequence in $(P_{USCB}^1(X), H_{\text{send}})$.

Let $\{u_n\}$ be a sequence in U . Since $U(0)$ is relatively compact in X , then $U(\alpha)$ is relatively compact in X for each $\alpha \in [0, 1]$. By Theorems 7.9 and 7.3, there is a subsequence $\{u_{n_k}\}$ of $\{u_n\}$, a $u \in F_{USCG}^1(X)$ and a $u_0 \in K(X)$ such that $H_{\text{end}}(\overleftarrow{u_{n_k}}, u) \rightarrow 0$ and $H(\langle u_{n_k} \rangle_0, u_0) \rightarrow 0$.

By Theorem 3.4 and Remark 3.5, $\lim_{k \rightarrow \infty}^{(\Gamma)} \overleftarrow{u_{n_k}} = u$ and $\lim_{k \rightarrow \infty}^{(K)} \langle u_{n_k} \rangle_0 = u_0$. By Remark 4.4, $[u]_0 \subseteq \liminf_{k \rightarrow \infty} [\overleftarrow{u_{n_k}}]_0 \subseteq \liminf_{k \rightarrow \infty} \langle u_{n_k} \rangle_0 = \lim_{k \rightarrow \infty}^{(K)} \langle u_{n_k} \rangle_0 = u_0$. So we can define $w \in P_{USCB}^1(X)$ by putting

$$\langle w \rangle_\alpha = \begin{cases} [u]_\alpha, & \alpha \in (0, 1], \\ u_0, & \alpha = 0. \end{cases}$$

Then $u = \overleftarrow{w}$, $H_{\text{end}}(u_{n_k}, w) = H_{\text{end}}(\overleftarrow{u_{n_k}}, u) \rightarrow 0$ and $H(\langle u_{n_k} \rangle_0, \langle w \rangle_0) = H(\langle u_{n_k} \rangle_0, u_0) \rightarrow 0$ for $n = 1, 2, \dots$. Thus from (iii) or (iv) of Theorem 3.2, $\{u_{n_k}\}$ converges to w in $(P_{USCB}^1(X), H_{\text{send}})$. \square

- $u \in F_{USC}^1(X)$ is said to be right-continuous at 0 if $\lim_{\delta \rightarrow 0+} H([u]_\delta, [u]_0) = 0$.
- $U \subseteq F_{USC}^1(X)$ is said to be equi-right-continuous at 0 if for each $\varepsilon > 0$, there is a $\delta \in (0, 1]$ such that $H([u]_\delta, [u]_0) < \varepsilon$ for all $u \in U$.

Let $\delta \in (0, 1]$ and $\xi \in [0, \delta]$. Then $H([u]_\xi, [u]_0) = H^*([u]_0, [u]_\xi) \leq H^*([u]_0, [u]_\delta) = H([u]_\delta, [u]_0)$. So $U \subseteq F_{USC}^1(X)$ is equi-right-continuous at 0 if and only if for each $\varepsilon > 0$, there is a $\delta \in (0, 1]$ such that $H([u]_\xi, [u]_0) < \varepsilon$ for all $u \in U$ and $\xi \in [0, \delta]$.

Theorem 7.14 below is presented in [8].

Theorem 7.14. [8] Suppose that U is a subset of $F_{USCB}^1(X)$. Then U is relatively compact in $(F_{USCB}^1(X), H_{\text{send}})$ if and only if $U(0)$ is relatively compact in X and U is equi-right-continuous at 0.

$\overrightarrow{F_{USCB}^1(X)}$ need not be a closed set of $P_{USCB}^1(X)$. For instance, $F_{USCB}^1(D)$ given in Example 8.4 is not a closed set of $P_{USCB}^1(D)$. We can see that $\overrightarrow{F_{USCB}^1(X)}$ is a closed set of $P_{USCB}^1(X)$ if and only if X has only one element.

For a set U in $F_{USCB}^1(X)$, suppose that

- (a) U is relatively compact in $(F_{USCB}^1(X), H_{\text{send}})$;
 - (b) \overrightarrow{U} is relatively compact in $(P_{USCB}^1(X), H_{\text{send}})$;
 - (c) The topological closure of \overrightarrow{U} in $(P_{USCB}^1(X), H_{\text{send}})$ is a subset of $\overrightarrow{F_{USCB}^1(X)}$.
- Then (a) holds if and only if (b) and (c) hold.

$\overrightarrow{F_{USCB}^1(X)}$ is closed in $P_{USCB}^1(X)$ if and only if for each set U in $F_{USCB}^1(X)$, (c) holds.

The following Proposition 7.15 illustrates the role of the condition “ U is equi-right-continuous at 0” in the characterization of the relative compactness for a set U in $(F_{USCB}^1(X), H_{\text{send}})$ given in Theorem 7.14.

Proposition 7.15. Let U be a subset of $F_{USCB}^1(X)$. Suppose the following conditions (i), (ii) and (iii):

- (i) U is relatively compact in $(F_{USCB}^1(X), H_{\text{send}})$;
- (ii) U is equi-right-continuous at 0;
- (iii) The topological closure of \overrightarrow{U} in $(P_{USCB}^1(X), H_{\text{send}})$ is a subset of $\overrightarrow{F_{USCB}^1(X)}$.

Then the condition (i) implies the condition (ii), and the condition (ii) implies the condition (iii). If \vec{U} is relatively compact in $(P_{USCB}^1(X), H_{\text{send}})$, then the conditions (i), (ii) and (iii) are equivalent to each other.

Proof. By Theorem 7.14, we have that (i) \Rightarrow (ii).

Now we show that (ii) \Rightarrow (iii). Suppose that $\{\vec{u}_n\}$ is a sequence in \vec{U} which converges to an element u in $(P_{USCB}^1(X), H_{\text{send}})$. Then by Theorem 3.2(iv), $\lim_{n \rightarrow \infty} H(\langle \vec{u}_n \rangle_0, \langle u \rangle_0) = 0$; that is, $\lim_{n \rightarrow \infty} H([u_n]_0, \langle u \rangle_0) = 0$.

Let $\varepsilon > 0$. Since $\vec{u} \in F_{USCB}^1(X)$, by Lemma 5.4(v), $\lim_{\alpha \rightarrow 0+} H([\vec{u}]_\alpha, [\vec{u}]_0) = 0$. Note that U is equi-right-continuous at 0. Then there is a $\delta > 0$ such that for all $\alpha \in [0, \delta]$ and $n \in \mathbb{N}$, $H([u_n]_0, [u_n]_\alpha) \leq \varepsilon/3$ and $H([\vec{u}]_\alpha, [\vec{u}]_0) \leq \varepsilon/3$. By Theorem 3.2(iv), $\{u_n\}$ converges to \vec{u} in $(F_{USCB}^1(X), H_{\text{end}})$. Then by Theorem 5.20, $H([u_n]_\alpha, [\vec{u}]_\alpha) \rightarrow 0$ holds a.e. on $\alpha \in (0, 1)$. So we can choose an $\alpha_0 \in (0, \delta)$ with $H([u_n]_{\alpha_0}, [\vec{u}]_{\alpha_0}) \rightarrow 0$. Thus there is an N such that for all $n \geq N$, $H([u_n]_{\alpha_0}, [\vec{u}]_{\alpha_0}) \leq \varepsilon/3$, and hence $H([u_n]_0, [\vec{u}]_0) \leq H([u_n]_0, [u_n]_{\alpha_0}) + H([u_n]_{\alpha_0}, [\vec{u}]_{\alpha_0}) + H([\vec{u}]_{\alpha_0}, [\vec{u}]_0) \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have $\lim_{n \rightarrow \infty} H([u_n]_0, [\vec{u}]_0) = 0$.

Thus $\langle u \rangle_0 = [\vec{u}]_0$, and hence by Proposition 3.1, $u \in \overrightarrow{F_{USCB}^1(X)}$. So (ii) \Rightarrow (iii).

If \vec{U} is relatively compact in $(P_{USCB}^1(X), H_{\text{send}})$, then clearly (iii) \Rightarrow (i), and thus the conditions (i), (ii) and (iii) are equivalent to each other. \square

Remark 7.16. For conditions (i), (ii) and (iii) in Proposition 7.15, (ii) does not imply (i); (iii) does not imply (ii).

Let $\{u_n\}$ be a sequence of fuzzy sets in $F_{USCB}^1(\mathbb{R})$ defined by $u_n := [0, n]_{F(\mathbb{R})}$, $n = 1, 2, \dots$. Then $\{u_n\}$ is equi-right-continuous at 0 but $\{u_n\}$ is not relatively compact in $(F_{USCB}^1(X), H_{\text{send}})$. So (ii) does not imply (i).

Let $\{v_n\}$ be a sequence of fuzzy sets in $F_{USCB}^1(\mathbb{R})$ defined by

$$v_n(x) = \begin{cases} 1, & x \in [0, n], \\ 1/n, & x \in [-n, 0], \\ 0, & x \in \mathbb{R} \setminus [-n, n], \end{cases} \quad n = 1, 2, \dots$$

Then $\{\vec{v}_n\}$ has no limit in $(P_{USCB}^1(X), H_{\text{send}})$ and hence is closed in $(P_{USCB}^1(X), H_{\text{send}})$. However $\{v_n\}$ is not equi-right-continuous at 0. So (iii) does not imply (ii).

Remark 7.17. Theorem 7.13 and Proposition 7.15 imply Theorem 7.14. This is because by Proposition 7.15 we can obtain that for a subset U of $F_{USCB}^1(X)$, U is relatively compact in $(F_{USCB}^1(X), H_{\text{send}})$ if and only if \vec{U} is relatively compact in $(P_{USCB}^1(X), H_{\text{send}})$ and U is equi-right-continuous at 0.

Theorem 7.18. *Suppose that U is a subset of $P_{USCB}^1(X)$. Then the following are equivalent:*

- (i) U is compact in $(P_{USCB}^1(X), H_{\text{send}})$;
- (ii) U is closed in $(P_{USCB}^1(X), H_{\text{send}})$, and $U(0)$ is relatively compact in (X, d) ;
- (iii) U is closed in $(P_{USCB}^1(X), H_{\text{send}})$, and $U(0)$ is compact in (X, d) .

Proof. By Theorem 7.13, we obtain that (i) \Leftrightarrow (ii). Clearly (iii) \Rightarrow (ii). We shall complete the proof by showing that (i) \Rightarrow (iii). To do this, suppose that U is compact in $(P_{USCB}^1(X), H_{\text{send}})$. Then U is closed in $(P_{USCB}^1(X), H_{\text{send}})$. To verify (iii), we only need to show that $U(0)$ is compact in (X, d) .

By clause (ii) of Theorem 3.2, U_0 is compact in $(K(X), H)$. Thus by Lemma 7.4, $U(0) = \bigcup\{D : D \in U_0\}$ is compact in (X, d) . \square

Theorem 7.19. *Suppose that U is a subset of $F_{USCB}^1(X)$. Then the following are equivalent:*

- (i) U is compact in $(F_{USCB}^1(X), H_{\text{send}})$;
- (ii) U is closed in $(F_{USCB}^1(X), H_{\text{send}})$, $U(0)$ is relatively compact in X and U is equi-right-continuous at 0;
- (iii) U is closed in $(F_{USCB}^1(X), H_{\text{send}})$, $U(0)$ is compact in X and U is equi-right-continuous at 0.

Proof. By Theorem 7.14, we obtain that (i) \Leftrightarrow (ii). Clearly (iii) \Rightarrow (ii). We shall complete the proof by showing that (i) \Rightarrow (iii). To do this, suppose that U is compact in $(F_{USCB}^1(X), H_{\text{send}})$. Since (i) implies (ii), to verify (iii) we only need to show that $U(0)$ is compact in X .

Note that U is compact in $(F_{USCB}^1(X), H_{\text{send}})$ if and only if \vec{U} is compact in $(P_{USCB}^1(X), H_{\text{send}})$. Thus by Theorem 7.18, $U(0) = \vec{U}(0)$ is compact in X . \square

Remark 7.20. For a subset U of $F_{USCB}^1(X)$,

- (a) U satisfies (i) of Theorem 7.19 if and only if \vec{U} satisfies (i) of Theorem 7.18;
- (b) U satisfies (ii) of Theorem 7.19 if and only if \vec{U} satisfies (ii) of Theorem 7.18;
- (c) U satisfies (iii) of Theorem 7.19 if and only if \vec{U} satisfies (iii) of Theorem 7.18.

Clauses (b) and (c) can be obtained by using Proposition 7.15 and Theorem 7.13. Theorem 7.18 and clauses (a), (b) and (c) imply Theorem 7.19.

Since U is compact in $(F_{USCB}^1(X), H_{\text{send}})$ if and only if \vec{U} is compact in $(P_{USCB}^1(X), H_{\text{send}})$, we can use Theorem 7.18 to judge the compactness of a set U in $(F_{USCB}^1(X), H_{\text{send}})$.

8. Completions of $(F_{USCB}^1(X), H_{\text{send}})$, $(F_{USCG}^1(X), H_{\text{end}})$ and $(F_{USCB}^1(X), d_\infty)$

In this section, we show that $(P_{USCB}^1(\tilde{X}), H_{\text{send}})$ is a completion of $(F_{USCB}^1(X), H_{\text{send}})$, $(F_{USCG}^1(\tilde{X}), H_{\text{end}})$ is a completion of $(F_{USCB}^1(X), H_{\text{end}})$, and $(F_{USCB}^1(\tilde{X}), d_\infty)$ is a completion of $(F_{USCB}^1(X), d_\infty)$.

We can see that for $x, y \in (X, d)$,

$$d(x, y) = d_\infty(\widehat{x}, \widehat{y}) = d_p(\widehat{x}, \widehat{y}) = H_{\text{send}}(\widehat{x}, \widehat{y}) = H_{\text{send}}(\vec{\widehat{x}}, \vec{\widehat{y}}), \quad (18)$$

$$H_{\text{end}}(\vec{\widehat{x}}, \vec{\widehat{y}}) = H_{\text{end}}(\widehat{x}, \widehat{y}) = \min\{d(x, y), 1\}. \quad (19)$$

Theorem 8.1. *Let (X, d) be a metric space. Then the following are equivalent:*

- (i) (X, d) is complete;
- (ii) $(F_{USCG}^1(X), H_{\text{end}})$ is complete.

Proof. (i) \Rightarrow (ii). Assume that (i) is true. To show that (ii) is true, we only need to show that each Cauchy sequence in $(F_{USCG}^1(X), H_{\text{end}})$ is a convergent sequence in $(F_{USCG}^1(X), H_{\text{end}})$. Let $\{u_n\}$ be a Cauchy sequence in $(F_{USCG}^1(X), H_{\text{end}})$. Then $U := \{u_n, n = 1, 2, \dots\}$ is total bounded in $(F_{USCG}^1(X), H_{\text{end}})$. By Theorem 7.8, for each $\alpha \in (0, 1]$, $U(\alpha)$ is total bounded in (X, d) . Since (X, d) is complete, it follows that for each $\alpha \in (0, 1]$, $U(\alpha)$ is relatively compact in (X, d) . Thus by Theorem 7.9, U is relatively compact in $(F_{USCG}^1(X), H_{\text{end}})$. So $\{u_n\}$ has a convergent subsequence in $(F_{USCG}^1(X), H_{\text{end}})$, and thus $\{u_n\}$ is convergent in $(F_{USCG}^1(X), H_{\text{end}})$. So (ii) is true.

(ii) \Rightarrow (i). Assume that (ii) is true. Let $\{x_n\}$ be a Cauchy sequence in (X, d) . Then by (19), $\{\widehat{x}_n\}$ is a Cauchy sequence in $(F_{USCG}^1(X), H_{\text{end}})$. So $\{\widehat{x}_n\}$ is a convergent sequence in $(F_{USCG}^1(X), H_{\text{end}})$. Hence by Remark 8.2 (i), there exists an $x \in X$ such that $H_{\text{end}}(\widehat{x}_n, \widehat{x}) \rightarrow 0$. Then by (19), $d(x_n, x) \rightarrow 0$. Thus $\{x_n\}$ is a convergent sequence in (X, d) . So (X, d) is complete; that is, (i) is true. \square

Remark 8.2. Define $\widehat{X} := \{\widehat{x} : x \in X\}$. Then $\vec{\widehat{X}} := \{\vec{\widehat{x}} : x \in X\}$.
(i) If S satisfies $\widehat{X} \subseteq S \subseteq F_{USC}^1(X)$, then \widehat{X} is closed in (S, H_{end}) .

(ii) (i) remains true if H_{end} is replaced by each one of d_∞ , H_{send} and d_p^* .

(iii) If S satisfies $\overrightarrow{X} \subseteq S \subseteq P_{USC}^1(X)$, then \overrightarrow{X} is closed in (S, H_{send}) .

We claim that for $u \in F_{USC}^1(X)$, if for each $\alpha \in (0, 1]$, $[u]_\alpha$ is a singleton, then $u \in \widehat{X}$. To show $u \in \widehat{X}$, we only need to show that $[u]_\alpha = [u]_\beta$ for all $\alpha, \beta \in (0, 1]$. This is true since otherwise there exists α, β such that $0 < \alpha < \beta \leq 1$ and $[u]_\alpha \supsetneq [u]_\beta$, which contradicts that $[u]_\alpha$ and $[u]_\beta$ are singletons.

To show (i), we only need to show that \widehat{X} is closed in $(F_{USC}^1(X), H_{\text{end}})$. Let $x, p, q \in X$ and $\alpha \in [0, 1]$. Then $\overline{d}((p, \alpha), \text{end } \widehat{x}) = \min\{\overline{d}((p, \alpha), (x, \alpha)), \alpha\} = \min\{d(p, x), \alpha\}$. And $\max\{d(p, x), d(q, x)\} \geq \frac{1}{2}d(p, q)$ since $d(p, x) + d(q, x) \geq d(p, q)$. Given $u \in F_{USC}^1(X) \setminus \widehat{X}$. Then there exist $p_1, q_1 \in X$ and $\alpha_1 \in (0, 1]$ such that $p_1 \neq q_1$ and $p_1, q_1 \in [u]_{\alpha_1}$. Thus for each $\widehat{x} \in \widehat{X}$, $H_{\text{end}}(u, \widehat{x}) \geq \max\{\overline{d}((p_1, \alpha_1), \text{end } \widehat{x}), \overline{d}((q_1, \alpha_1), \text{end } \widehat{x})\} = \max\{\min\{d(p_1, x), \alpha_1\}, \min\{d(q_1, x), \alpha_1\}\} \geq \min\{\frac{1}{2}d(p_1, q_1), \alpha_1\}$. This implies that u is not in the closure of \widehat{X} in $(F_{USC}^1(X), H_{\text{end}})$. Since $u \in F_{USC}^1(X) \setminus \widehat{X}$ is arbitrary, \widehat{X} is closed in $(F_{USC}^1(X), H_{\text{end}})$. So (i) is true.

By (1), each one of the d_∞ convergence and the H_{send} convergence implies the H_{end} convergence on $F_{USC}^1(X)$. Theorem 6.2 says that the d_p^* convergence implies the H_{end} convergence on $F_{USC}^1(X)$. So (ii) follows from (i).

To show (iii), we only need to show that \overrightarrow{X} is closed in $(P_{USC}^1(X), H_{\text{send}})$. Given $u \in P_{USC}^1(X) \setminus \overrightarrow{X}$. If $u \in \overrightarrow{F_{USC}^1(X)} \setminus \overrightarrow{X}$. Then $\langle u \rangle_0$ has at least two points. If $u \in \overrightarrow{F_{USC}^1(X)} \setminus \overrightarrow{X}$, then there exists $\alpha \in (0, 1]$ such that $\langle u \rangle_\alpha$ has at least two points. So there exist $p_1, q_1 \in X$ and $\alpha_1 \in [0, 1]$ such that $p_1 \neq q_1$ and $p_1, q_1 \in \langle u \rangle_{\alpha_1}$. Thus for each $\overrightarrow{x} \in \overrightarrow{X}$, $H_{\text{send}}(u, \overrightarrow{x}) \geq \max\{\overline{d}((p_1, \alpha_1), \overrightarrow{x}), \overline{d}((q_1, \alpha_1), \overrightarrow{x})\} = \max\{d(p_1, x), d(q_1, x)\} \geq \frac{1}{2}d(p_1, q_1)$. This implies that u is not in the closure of \overrightarrow{X} in $(P_{USC}^1(X), H_{\text{send}})$. Since $u \in P_{USC}^1(X) \setminus \overrightarrow{X}$ is arbitrary, \overrightarrow{X} is closed in $(P_{USC}^1(X), H_{\text{send}})$. So (iii) is true.

(ii) \Rightarrow (i) in Theorem 8.1 can also be shown as follows. (X, d) is complete if and only if (X, d^*) is complete, where $d^*(x, y) = \min\{d(x, y), 1\}$ for $x, y \in X$. Note that $H_{\text{end}}(\widehat{x}, \widehat{y}) = d^*(x, y)$. So the desired result follows from the fact that (X, d^*) is isometric to the closed subspace $(\widehat{X}, H_{\text{end}})$ of $(F_{USC}^1(X), H_{\text{end}})$.

Proposition 8.3. *Let (X, d) be a metric space. Then the following are equivalent:*

- (i) (X, d) is complete;
- (ii) $(F_{USCB}^1(X), d_\infty)$ is complete.

Proof. By (18), (X, d) is isometric to (\widehat{X}, d_∞) , which by Remark 8.2 (ii) is a closed subspace of $(F_{USCB}^1(X), d_\infty)$. So (ii) \Rightarrow (i) is proved.

To show (i) \Rightarrow (ii), suppose that (X, d) is complete. Let $\{u_n\}$ be a Cauchy sequence in $(F_{USCB}^1(X), d_\infty)$. Then for each $\alpha \in [0, 1]$, $\{[u_n]_\alpha\}$ is a Cauchy sequence in $(K(X), H)$, and hence by Theorem 2.3, there is a $u(\alpha) \in K(X)$ such that $H([u_n]_\alpha, u(\alpha)) \rightarrow 0$. We have that:

- (a-1) For $0 \leq \beta \leq \alpha \leq 1$, $u(\alpha) \subseteq u(\beta)$ since by Theorem 3.4, $u(\alpha) = \lim_{n \rightarrow \infty}^{(K)} [u_n]_\alpha \subseteq \lim_{n \rightarrow \infty}^{(K)} [u_n]_\beta = u(\beta)$;
- (a-2) $\lim_{n \rightarrow \infty} \sup_{\alpha \in [0, 1]} H([u_n]_\alpha, u(\alpha)) = 0$; that is, $\{H([u_n]_\alpha, u(\alpha))\}$ converges uniformly to 0 on $\alpha \in [0, 1]$;
- (a-3) For each $\alpha \in (0, 1]$, $\lim_{\beta \rightarrow \alpha^-} H(u(\beta), u(\alpha)) = 0$;
- (a-4) $\lim_{\gamma \rightarrow 0^+} H(u(\gamma), u(0)) = 0$.

Given $\varepsilon > 0$. As $\{u_n\}$ is a Cauchy sequence in $(F_{USCB}^1(X), d_\infty)$, there is an $N(\varepsilon) \in \mathbb{N}$ such that for all $n, m \geq N$, $\sup_{\alpha \in [0, 1]} H([u_n]_\alpha, [u_m]_\alpha) \leq \varepsilon$. Let $\alpha \in [0, 1]$ and $n \geq N$. Then for each $l \in \mathbb{N}$, $H([u_n]_\alpha, u(\alpha)) \leq H([u_n]_\alpha, [u_{n+l}]_\alpha) + H([u_{n+l}]_\alpha, u(\alpha)) \leq \varepsilon + H([u_{n+l}]_\alpha, u(\alpha))$. So $H([u_n]_\alpha, u(\alpha)) \leq \varepsilon$ as $\lim_{l \rightarrow \infty} H([u_{n+l}]_\alpha, u(\alpha)) = 0$. As $\alpha \in [0, 1]$ and $n \geq N$ are arbitrary, we obtain that $\sup_{\alpha \in [0, 1]} H([u_n]_\alpha, u(\alpha)) \leq \varepsilon$ for all $n \geq N$. Since $\varepsilon > 0$ is arbitrary, we have that (a-2) is true.

Let $\alpha \in (0, 1]$. Given $\varepsilon > 0$, by (a-2), there is an N such that for $n \geq N$, $\sup_{\xi \in [0, 1]} H([u_n]_\xi, u(\xi)) \leq \varepsilon/3$. By Lemma 5.4(i), $\lim_{\beta \rightarrow \alpha^-} H([u_N]_\alpha, [u_N]_\beta) = 0$. Then there is a $\delta > 0$ such that for all $\beta \in [\alpha - \delta, \alpha]$, $H([u_N]_\alpha, [u_N]_\beta) \leq \varepsilon/3$. Thus for all $\beta \in [\alpha - \delta, \alpha]$, $H(u(\alpha), u(\beta)) \leq H(u(\alpha), [u_N]_\alpha) + H([u_N]_\alpha, [u_N]_\beta) + H([u_N]_\beta, u(\beta)) \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows that (a-3) is true. Similarly, by using (a-2) and the fact that for $n \in \mathbb{N}$, $\lim_{\gamma \rightarrow 0^+} H([u_n]_\gamma, [u_n]_0) = 0$, we can show that (a-4) is true.

If there is a $u \in F_{USCB}^1(X)$ such that $[u]_\alpha = u(\alpha)$ for all $\alpha \in [0, 1]$, then by (a-2), $d_\infty(u_n, u) \rightarrow 0$, and so the proof is complete. As $\{u(\alpha) : \alpha \in [0, 1]\} \subseteq K(X)$, to prove the existence of a such $u \in F_{USCB}^1(X)$, by Proposition 2.1, we only need to show that $\{u(\alpha), \alpha \in [0, 1]\}$ has the following properties:

- (b) for each $\alpha \in (0, 1]$, $u(\alpha) = \bigcap_{\beta < \alpha} u(\beta)$, and (c) $u(0) = \overline{\bigcup_{\gamma > 0} u(\gamma)}$.

By Remark 3.6 and (a-1), (b) and (c) follow from (a-3) and (a-4), respectively. This completes the proof.

(b) and (c) can also be shown in the following way. From (a-1), $\overline{\bigcup_{\gamma > 0} u(\gamma)} \subseteq$

$u(0)$ and then $\overline{\bigcup_{\gamma>0} u(\gamma)} \in K(X)$. By Lemma 5.3, for each $\alpha \in (0, 1]$, $\lim_{\beta \rightarrow \alpha-} H(u(\beta), \bigcap_{\beta < \alpha} u(\beta)) = 0$, and $\lim_{\gamma \rightarrow 0+} H(u(\gamma), \overline{\bigcup_{\gamma>0} u(\gamma)}) = 0$ (see the proof of Lemma 5.4(i)(v)). Combined with (a-3) and (a-4), we obtain that (b) and (c) are true. \square

Even if (X, d) is complete, $(F_{USCB}^1(X), H_{\text{send}})$ need not be complete. See Example 8.4 below.

Example 8.4. We use ρ_1 to denote the induced metric on $\{0, 1\}$ by the Euclidean metric on \mathbb{R} . Let D denote the metric space $(\{0, 1\}, \rho_1)$. Then D is a subspace of \mathbb{R} and D is complete. For $n = 1, 2, \dots$, let $u_n \in F_{USCB}^1(D)$ be defined by

$$u_n(x) = \begin{cases} 1, & x = 0, \\ 1/n, & x = 1. \end{cases}$$

Define $u \in P_{USCB}^1(D) \setminus \overrightarrow{F_{USCB}^1(D)}$ by putting

$$\langle u \rangle_\alpha = \begin{cases} \{0\}, & \alpha \in (0, 1], \\ \{0, 1\}, & \alpha = 0. \end{cases}$$

Then $\{\overrightarrow{u_n}\}$ converges to u in $(P_{USCB}^1(D), H_{\text{send}})$. Thus $\{u_n\}$ is a Cauchy sequence in $(F_{USCB}^1(D), H_{\text{send}})$ and has no limit in $(F_{USCB}^1(D), H_{\text{send}})$. So $(F_{USCB}^1(D), H_{\text{send}})$ is not complete.

We can see that $(F_{USCB}^1(X), H_{\text{send}})$ is complete if and only if X has only one element.

Theorem 8.5 below discusses the completeness of $(P_{USCB}^1(X), H_{\text{send}})$ and then Theorem 8.6 below gives the completion of $(F_{USCB}^1(X), H_{\text{send}})$.

Theorem 8.5. *Let (X, d) be a metric space. Then the following are equivalent:*

- (i) X is complete;
- (ii) $(P_{USCB}^1(X), H_{\text{send}})$ is complete.

Proof. (i) \Rightarrow (ii). Let $\{u_n\}$ be a Cauchy sequence in $(P_{USCB}^1(X), H_{\text{send}})$. Then $\{\overleftarrow{u_n}\} \subseteq F_{USCB}^1(X)$, and by (i) of Theorem 3.2, $H_{\text{end}}(\overleftarrow{u_n}, \overleftarrow{u_m}) = H_{\text{end}}(u_n, u_m) \leq H_{\text{send}}(u_n, u_m)$ for $n, m = 1, 2, \dots$. Hence $\{\overleftarrow{u_n}\}$ is a Cauchy sequence in $(F_{USCB}^1(X), H_{\text{end}})$. From Theorem 8.1, there is a $u \in F_{USCB}^1(X)$ such that $\{\overleftarrow{u_n}\}$ converges to u in $(F_{USCB}^1(X), H_{\text{end}})$.

By (ii) of Theorem 3.2, $H(\langle u_n \rangle_0, \langle u_m \rangle_0) \leq H_{\text{send}}(u_n, u_m)$ for $n, m = 1, 2, \dots$. So $\{\langle u_n \rangle_0\}$ is a Cauchy sequence in $(K(X), H)$. From Theorem 2.3, there is a $u_0 \in K(X)$ such that $\{\langle u_n \rangle_0\}$ converges to u_0 in $(K(X), H)$.

By Theorem 3.4 and Remark 3.5, $\lim_{n \rightarrow \infty}^{(\Gamma)} \overleftarrow{u}_n = u$ and $\lim_{n \rightarrow \infty}^{(K)} \langle u_n \rangle_0 = u_0$. By Remark 4.4, $[u]_0 \subseteq \liminf_{n \rightarrow \infty} [\overleftarrow{u}_n]_0 \subseteq \liminf_{n \rightarrow \infty} \langle u_n \rangle_0 = \lim_{n \rightarrow \infty}^{(K)} \langle u_n \rangle_0 = u_0$. So we can define $w \in P_{USCB}^1(X)$ by putting

$$\langle w \rangle_\alpha = \begin{cases} [u]_\alpha, & \alpha > 0, \\ u_0, & \alpha = 0. \end{cases}$$

Then $u = \overleftarrow{w}$, $H_{\text{end}}(u_n, w) = H_{\text{end}}(\overleftarrow{u}_n, u)$ and $H(\langle u_n \rangle_0, \langle w \rangle_0) = H(\langle u_n \rangle_0, u_0)$ for $n = 1, 2, \dots$. Thus from (iii) or (iv) of Theorem 3.2, $\{u_n\}$ converges to w in $(P_{USCB}^1(X), H_{\text{send}})$.

(ii) \Rightarrow (i). Note that $d(x, y) = H_{\text{send}}(\overrightarrow{x}, \overrightarrow{y})$. So the desired result follows from the fact that (X, d) is isometric to $(\overrightarrow{X}, H_{\text{send}})$, which by Remark 8.2 (iii) is a closed subspace of $(P_{USCB}^1(X), H_{\text{send}})$. \square

Theorem 8.6. $(P_{USCB}^1(\tilde{X}), H_{\text{send}})$ is a completion of $(F_{USCB}^1(X), H_{\text{send}})$.

Proof. From Theorem 8.5, $(P_{USCB}^1(\tilde{X}), H_{\text{send}})$ is complete. To show that $(P_{USCB}^1(\tilde{X}), H_{\text{send}})$ is a completion of $(F_{USCB}^1(X), H_{\text{send}})$, we only need to verify that $\overrightarrow{F_{USCB}^1(X)}$ is dense in $(P_{USCB}^1(\tilde{X}), H_{\text{send}})$.

We claim the following affirmations (a) and (b):

- (a) For each $u \in P_{USCB}^1(\tilde{X})$ and each $\varepsilon > 0$, there exists a $v \in F_{USCB}^1(\tilde{X})$ such that $H_{\text{send}}(u, \overrightarrow{v}) \leq \varepsilon$.
- (b) For each $v \in F_{USCB}^1(\tilde{X})$ and each $\varepsilon > 0$, there exists a $w \in F_{USCB}^1(X)$ such that $d_\infty(v, w) \leq \varepsilon$ (so by (1), $H_{\text{end}}(v, w) \leq H_{\text{send}}(v, w) \leq \varepsilon$, and by (13), $d_p(v, w) = d_p^*(v, w) \leq \varepsilon$).

If affirmations (a) and (b) are true, then for each $u \in P_{USCB}^1(\tilde{X})$ and each $\varepsilon > 0$, there is a $v \in F_{USCB}^1(\tilde{X})$ and $w \in F_{USCB}^1(X)$ such that $H_{\text{send}}(u, \overrightarrow{v}) \leq \varepsilon/2$ and $H_{\text{send}}(\overrightarrow{v}, \overrightarrow{w}) = H_{\text{send}}(v, w) \leq \varepsilon/2$. Thus $H_{\text{send}}(u, \overrightarrow{w}) \leq H_{\text{send}}(u, \overrightarrow{v}) + H_{\text{send}}(\overrightarrow{v}, \overrightarrow{w}) \leq \varepsilon$. So for each $u \in P_{USCB}^1(\tilde{X})$ and each $\varepsilon > 0$, there is a $w \in F_{USCB}^1(X)$ such that $H_{\text{send}}(u, \overrightarrow{w}) \leq \varepsilon$. This means that $\overrightarrow{F_{USCB}^1(X)}$ is dense in $(P_{USCB}^1(\tilde{X}), H_{\text{send}})$. So to show the desired result, it suffices to prove affirmations (a) and (b).

Let $u \in P_{USCB}^1(\tilde{X})$ and $\varepsilon > 0$. Define $u_\varepsilon \in F_{USCB}^1(\tilde{X})$ by putting

$$[u_\varepsilon]_\alpha = \begin{cases} \langle u \rangle_\alpha, & \alpha \in (\varepsilon, 1], \\ \langle u \rangle_0, & \alpha \in [0, \varepsilon]. \end{cases}$$

Then $H_{\text{send}}(u, \vec{u}_\varepsilon) \leq \varepsilon$. So affirmation (a) is proved.

Let $v \in F_{USCB}^1(\tilde{X})$ and $\varepsilon > 0$. We can choose a finite subset C_0 of X such that $H(C_0, [v]_0) < \varepsilon$ as follows. Since $[v]_0 \in K(\tilde{X})$, there is a finite subset A of $[v]_0$ such that $H(A, [v]_0) < \varepsilon/2$. As X is dense in (\tilde{X}, \tilde{d}) , there is a subset C_0 of X such that $H(C_0, A) < \varepsilon/2$. Thus $H(C_0, [v]_0) \leq H(C_0, A) + H(A, [v]_0) < \varepsilon$.

Define

$$C_\alpha := \{x \in C_0 : d(x, [v]_\alpha) \leq \varepsilon\}, \quad \alpha \in (0, 1]. \quad (20)$$

We affirm that $\{C_\alpha : \alpha \in [0, 1]\}$ has the following properties

- (i) $C_\alpha \neq \emptyset$ for all $\alpha \in [0, 1]$.
- (ii) $H(C_\alpha, [v]_\alpha) \leq \varepsilon$ for all $\alpha \in [0, 1]$.
- (iii) $C_\alpha = \bigcap_{\beta < \alpha} C_\beta$ for all $\alpha \in (0, 1]$.
- (iv) $C_0 = \bigcup_{\alpha > 0} C_\alpha = \overline{\bigcup_{\alpha > 0} C_\alpha}$.

For each $y \in [v]_\alpha$, there exists $z_y \in C_0$ such that $d(y, z_y) = d(y, C_0) < \varepsilon$. Hence $z_y \in C_\alpha$ and thus $C_\alpha \neq \emptyset$. So (i) is true.

To show (ii), let $\alpha \in [0, 1]$. If $\alpha = 0$, then (ii) is true since $H(C_0, [v]_0) < \varepsilon$. Assume that $0 < \alpha \leq 1$. From (20), $H^*(C_\alpha, [v]_\alpha) \leq \varepsilon$. So to show $H(C_\alpha, [v]_\alpha) \leq \varepsilon$, it suffices to show that $H^*([v]_\alpha, C_\alpha) < \varepsilon$.

Let $y \in [v]_\alpha$. From $d(y, z_y) \geq d(y, C_\alpha) \geq d(y, C_0)$ and $d(y, z_y) = d(y, C_0)$, we have that $d(y, z_y) = d(y, C_\alpha) = d(y, C_0)$. Thus

$$H^*([v]_\alpha, C_\alpha) = \sup_{y \in [v]_\alpha} d(y, C_\alpha) = \sup_{y \in [v]_\alpha} d(y, C_0) \leq \sup_{y \in [v]_0} d(y, C_0) = H^*([v]_0, C_0) \leq H(C_0, [v]_0) < \varepsilon.$$

So (ii) is proved.

To show (iii), let $\alpha_0 \in (0, 1]$. From the definition of $\{C_\alpha : \alpha \in [0, 1]\}$, we can see that $C_\xi \subseteq C_\eta$ for $0 \leq \eta \leq \xi \leq 1$. Thus $C_{\alpha_0} \subseteq \bigcap_{\beta < \alpha_0} C_\beta$.

By Lemma 5.4 (i), $\lim_{\beta \rightarrow \alpha_0^-} H([v]_{\alpha_0}, [v]_\beta) = 0$. Note that for each $x \in X$ and $\xi, \eta \in [0, 1]$, $|d(x, [v]_\xi) - d(x, [v]_\eta)| \leq H([v]_\xi, [v]_\eta)$. Thus for each $x \in X$, $d(x, [v]_{\alpha_0}) = \lim_{\beta \rightarrow \alpha_0^-} d(x, [v]_\beta)$. Let $x \in \bigcap_{\beta < \alpha_0} C_\beta$. Then $d(x, [v]_{\alpha_0}) = \lim_{\beta \rightarrow \alpha_0^-} d(x, [v]_\beta) \leq \varepsilon$, which means that $x \in C_{\alpha_0}$. Since $x \in \bigcap_{\beta < \alpha_0} C_\beta$ is arbitrary, we have $\bigcap_{\beta < \alpha_0} C_\beta \subseteq C_{\alpha_0}$. Hence $C_{\alpha_0} = \bigcap_{\beta < \alpha_0} C_\beta$. So (iii) is true.

Since $C_\alpha \subseteq C_0$ for all $\alpha \in [0, 1]$ and C_0 is finite, it follows that $\bigcup_{\alpha > 0} C_\alpha = \overline{\bigcup_{\alpha > 0} C_\alpha} \subseteq C_0$. So to show (iv), we only need to show that $C_0 \subseteq \bigcup_{\alpha > 0} C_\alpha$.

Since $[v]_0 = \overline{\cup_{\alpha>0}[v]_\alpha}$, it follows that for each $x \in X$, $d(x, [v]_0) = d(x, \cup_{\alpha>0}[v]_\alpha) = \inf_{\alpha>0} d(x, [v]_\alpha)$. Let $x \in C_0$. Then $d(x, [v]_0) < \varepsilon$. So there exists $\alpha > 0$ such that $d(x, [v]_\alpha) < \varepsilon$, which means that $x \in C_\alpha$. Thus $C_0 \subseteq \cup_{\alpha>0} C_\alpha$ and (iv) is proved.

Now, define $w \in F(X)$ by putting $[w]_\alpha = C_\alpha$ for all $\alpha \in [0, 1]$. Then by (i), (iii) and (iv), $w \in F_{USCB}^1(X)$. From (ii), we have $d_\infty(v, w) \leq \varepsilon$, and then affirmation (b) is proved. This completes the proof. \square

Proposition 8.7. $(F_{USCB}^1(\tilde{X}), d_\infty)$ is a completion of $(F_{USCB}^1(X), d_\infty)$.

Proof. By Proposition 8.3, $(F_{USCB}^1(\tilde{X}), d_\infty)$ is complete. So from affirmation (b) in the proof of Theorem 8.6, we have the desired result. \square

Corollary 8.8. $(P_{USCB}^1(\tilde{X}), H_{\text{send}})$ is a completion of $(P_{USCB}^1(X), H_{\text{send}})$.

Proof. Note that $(F_{USCB}^1(X), H_{\text{send}})$ can be seen as a subspace of $(P_{USCB}^1(X), H_{\text{send}})$, and $(P_{USCB}^1(X), H_{\text{send}})$ is a subspace of $(P_{USCB}^1(\tilde{X}), H_{\text{send}})$. So the desired result follows from Theorem 8.6. \square

Theorem 8.9. $(F_{USCG}^1(\tilde{X}), H_{\text{end}})$ is a completion of $(F_{USCB}^1(X), H_{\text{end}})$.

Proof. From Theorem 8.1, $(F_{USCG}^1(\tilde{X}), H_{\text{end}})$ is complete. To prove the desired result, it suffices to show that $F_{USCB}^1(X)$ is dense in $(F_{USCG}^1(\tilde{X}), H_{\text{end}})$. By affirmation (b) in the proof of Theorem 8.6, to verify this it is enough to show that for each $u \in F_{USCG}^1(\tilde{X})$ and each $\varepsilon > 0$, there is a $v \in F_{USCB}^1(\tilde{X})$ such that $H_{\text{end}}(u, v) \leq \varepsilon$.

Let $u \in F_{USCG}^1(\tilde{X})$ and $\varepsilon > 0$. Define $u^\varepsilon \in F_{USCB}^1(\tilde{X})$ by putting

$$[u^\varepsilon]_\alpha = \begin{cases} [u]_\alpha, & \alpha \in (\varepsilon, 1], \\ [u]_\varepsilon, & \alpha \in [0, \varepsilon]. \end{cases}$$

Then $H_{\text{end}}(u, u^\varepsilon) \leq \varepsilon$. \square

Corollary 8.10. $(F_{USCG}^1(\tilde{X}), H_{\text{end}})$ is a completion of $(F_{USCG}^1(X), H_{\text{end}})$.

Proof. Since $F_{USCB}^1(X) \subseteq F_{USCG}^1(X) \subseteq F_{USCG}^1(\tilde{X})$, the desired result follows from Theorem 8.9. \square

Remark 8.11. If X is complete, then by Theorem 8.6, $(P_{USCB}^1(\tilde{X}), H_{\text{send}})$ is a completion of $(F_{USCB}^1(X), H_{\text{send}})$, and thus $(P_{USCB}^1(X), H_{\text{send}})$ is complete. So Theorem 8.6 implies that $(P_{USCB}^1(X), H_{\text{send}})$ is complete when X is complete. Similarly, Proposition 8.7 implies that $(F_{USCB}^1(X), d_\infty)$ is complete when X is complete. Theorem 8.9 implies that $(F_{USCG}^1(X), H_{\text{end}})$ is complete when X is complete.

9. Conclusions

In this paper, we discuss the properties and relations of H_{end} metric and H_{send} metric on fuzzy sets in a metric space X .

To aid discussion, we introduce the sets $P_{USC}^1(X)$ and $P_{USCB}^1(X)$. $P_{USCB}^1(X)$ is a subset of $P_{USC}^1(X)$. The $F_{USC}^1(X)$ and $F_{USCB}^1(X)$ can be viewed as the subsets of $P_{USC}^1(X)$ and $P_{USCB}^1(X)$, respectively. We define the H_{send} distance and the H_{end} distance on $P_{USC}^1(X)$, and give the relations among the H_{send} distance, the H_{end} distance and the Kuratowski convergence on $P_{USC}^1(X)$. Then, as corollaries, we obtain the relations among the H_{send} metric, the H_{end} metric and the Γ -convergence on $F_{USC}^1(X)$.

We give the level characterizations of H_{end} convergence and Γ -convergence on $F_{USC}^1(X)$. By using the above results including the level characterizations of the H_{end} convergence, we give the relations among the H_{end} metric, the H_{send} metric and the d_p^* metric.

Based on above results, we give characterizations of compactness and completions of two kinds of fuzzy set spaces $(F_{USCG}^1(X), H_{\text{end}})$ and $(F_{USCB}^1(X), H_{\text{send}})$, respectively. We also investigate characterizations of compactness and completions of $(P_{USCB}^1(X), H_{\text{send}})$. $(F_{USCB}^1(X), H_{\text{send}})$ can be treated as a subspace of $(P_{USCB}^1(X), H_{\text{send}})$.

Note that \mathbb{R}^m is complete and for a set V in \mathbb{R}^m , the following are equivalent: (i) V is bounded; (ii) V is totally bounded; (iii) V is relatively compact. We can obtain the characterizations of compactness and completions of $(F_{USCB}^1(\mathbb{R}^m), H_{\text{send}})$, $(F_{USCG}^1(\mathbb{R}^m), H_{\text{end}})$ and $(P_{USCB}^1(\mathbb{R}^m), H_{\text{send}})$ by using that of $(F_{USCB}^1(X), H_{\text{send}})$, $(F_{USCG}^1(X), H_{\text{end}})$ and $(P_{USCB}^1(X), H_{\text{send}})$ given in this paper.

The results in this paper have potential applications in fuzzy set research involving H_{end} metric and H_{send} metric.

Acknowledgement

The author would like to thank the Area Editor and the three anonymous referees for their invaluable comments and suggestions which greatly improves the readability of this paper.

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