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"Core Convergence with Asymmetric Information"

by

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Core Convergence with Asymmetric Information

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Abstract

We analyze the ex ante incentive compatible core for replicated private information economies. We show that any allocation in the core when the economy is replicated sufficiently often is approximately Walrasian for the associated Arrow-Debreu economy.

1   Introduction

The Debreu-Scarf core convergence theorem is a fundamental result in general equilibrium analysis: under suitable convexity assumptions, any allocation that is not an Arrow-Debreu equilibrium will be blocked in a sufficiently large replica economy. The theorem suggests why trade among many agents will lead to a system of prices that agents take as given when minimal assumptions on the stability of allocations are imposed.

Our aim in this paper is to prove a core convergence theorem for exchange economies in the presence of asymmetric information. The particular manner in which we model the asymmetry of information follows the development in McLean and Postlewaite (2002a, 2003). Agents' utility functions will depend on an underlying but unobserved state of nature θ, and each agent will receive a private signal that is correlated with the state of nature. A replication of this initial economy consists of a set of agents whose utility functions and initial endowments are the same as those in the underlying initial economy, but whose private signals are independent conditional on θ. No agent's information is redundant in this replication procedure: regardless of the number of replications, each agent still has information that can not be inferred from the aggregate information of other agents.

If the state θ on which the agents' utilities depend were observable prior to consumption, then agents could exchange state contingent goods prior to the realization

of θ. However, in our model θ is *not* observable; all information about the realization of θ is embodied in the vector of agents' types. Consequently, agents can trade bundles contingent on the realized vector of types but not contingent on θ. Since agents types are independent conditional on θ, by the law of large numbers, the vector of agents' types will (with high probability) provide a highly accurate prediction of the realized θ when the number of agents gets large. Hence, in this case one might hope that allocations contingent on the agents' information might approximate desirable allocations contingent on θ.

The main difficulty in formalizing this idea arises from the observation that allocations contingent on agents' information may not be incentive compatible. There are several core concepts one might employ. In this paper, we study the *ex ante incentive compatible core*, in which decisions are made at the ex ante stage and incentive constraints are taken into account. Our main theorem shows that if an asymmetric information exchange economy is replicated sufficiently many times, any ex ante incentive compatible core allocation is approximately competitive in the sense that, for almost all agents, the utility from the core allocation is close to the utility they receive at a certain complete information competitive equilibrium allocation. Thus, asymmetric information economies asymptotically behave like complete information economies as far as core-type stability is concerned.

Several complications arise in the analysis of the core when an economy with asymmetrically informed agents is replicated. First, while the core with complete information is nonempty under quite general circumstances, Vohra (1999) and Forges, Mertens and Vohra (2000) show that the ex ante incentive compatible core may be empty in well-behaved pure exchange economies. However, in McLean and Postlewaite (2003), we showed that when agents are sufficiently "informationally small," the ex ante incentive compatible core is nonempty. Further, they show that the informational size of agents will converge to zero when asymmetric information economies are replicated in a natural manner, and, consequently, the ex ante incentive compatible $\varepsilon-$core will be nonempty after a suitable number of replications.

The second complication in investigating the ex ante incentive compatible core with replication is technical. A key step in the proof of the Debreu-Scarf theorem is the argument that any core allocation must satisfy an "equal treatment" property. The equal treatment property states that all replicas of a given type must receive the same bundle in any core allocation. This property greatly simplifies the analysis since the dimensionality of the space of allocations goes to infinity when the number of replications goes to infinity, but attention can be restricted to allocations that are feasible for the initial economy.

When agents are asymmetrically informed, the argument for equal treatment of different agents of the same type breaks down.[1] Consequently, analysis cannot be

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[1]Forges, Heifetz and Minelli (2001) analyze the ex ante incentive compatible core in a model that differs somewhat from the model in this paper. In their model, they provide an example that shows

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restricted to feasible allocations for the initial economy. We show, however, that ex ante incentive compatible core allocations satisfy an "asymptotic equal treatment property". This property states (approximately) the following. Given $\varepsilon > 0$, there exists a sufficiently large replicated economy such that, for each type of agent, all but a fraction ε of the replications of that type must receive bundles that differ in utility by no more than ε in any ex ante incentive compatible core allocation.

To prove our main results, we first associate with each asymmetric information economy the complete information Arrow Debreu economy E^1 with state contingent commodities in which the state θ is observed prior to consumption. The r-replication of this complete information economy is denoted E^r. We use the approximate equal treatment equal property to prove our main result in the following steps.

1. The asymptotic equal treatment property assures that for sufficiently large r, an ex ante incentive compatible core allocation of the r-replicated asymmetric information ecomomy can be approximated in utility by a certain equal treatment allocation x^r for the economy of E^r.

2. Next we show that, for sufficiently large r, the special equal treatment alocation x^r will be an allocation in the ε -core of E^r.

3. Finally, we show that, for sufficiently large r, ε -core allocations of E^r are close in utility to a Walras allocation of E^1.

Our proof relies on the increasing numbers of agents in four different ways. First, we need large numbers to assure that core allocations of the asymmetric information economy satisfy an approximately equal treatment property. Second, as in the complete information case, large numbers of agents are necessary to form the coalitions that block noncompetitive allocations. Third, we need large numbers to assure that allocations contingent on agents' information can be approximated by allocations contingent on the true state. Finally, large numbers are necessary that agents in the asymmetric information economy are informationally small, ensuring that any blocking allocation ignoring incentive constraints can be approximated by an incentive compatible blocking allocation.

2 Basic Notation

Our notation follows that in McLean and Postlewaite (2003) whenever possible. Throughout the paper, let $J_q = \{1, .., q\}$ for each positive integer q and let $||\cdot||$ denote the 1-norm unless specified otherwise. Let $N = \{1, 2, ..., n\}$ denote the set of *economic agents*. Let $\Theta = \{\theta_1, .., \theta_m\}$ denote the (finite) *state space* and let $T_1, T_2, ..., T_n$ be finite sets where T_i represents the set of possible *signals* that agent i might receive.

that allocations in the ex ante incentive compatible core may not exhibit equal treatment.

For each $S \subseteq N$, let $T_S \equiv \prod_{i \in S} T_i$. Elements of T_S will be written t_S. For notational simplicity, we will simply write T for T_N and t for t_N. If $t \in T$, then we will often write $t = (t_{N \setminus S}, t_S)$. If X is a finite set, define

$$\Delta_X := \{\rho \in \Re^{|X|} | \rho(x) \geq 0, \sum_{x \in X} \rho(x) = 1\}$$

and

$$\Delta_X^0 := \{\rho \in \Re^{|X|} | \rho(x) > 0, \sum_{x \in X} \rho(x) = 1\}.$$

In our model, nature chooses an element $\theta \in \Theta$. The state of nature is unobservable but each agent i receives a "signal" t_i that is correlated with nature's choice of θ. More formally, let $(\tilde{\theta}, \tilde{t}_1, \tilde{t}_2, ..., \tilde{t}_n)$ be an (n+1)-dimensional random vector taking values in $\Theta \times T$ with associated distribution $P \in \Delta_{\Theta \times T}$ where

$$P(\theta, t_1, .., t_n) = \text{Prob}\{\tilde{\theta} = \theta, \tilde{t}_1 = t_1, ..., \tilde{t}_n = t_n\}.$$

We will make the following assumption regarding the marginal distributions[2]:

For each $\theta \in \Theta$,
$$P(\theta) = \text{Prob}\{\tilde{\theta} = \theta\} > 0$$

and for each $t = (t_1, .., t_n) \in T$,

$$P(t) = \text{Prob}\{\tilde{t}_1 = t_1, .., \tilde{t}_n = t_n\} > 0.$$

If $t \in T$, let $P_\Theta(\cdot|t) \in \Delta_\Theta$ denote the induced conditional probability measure on Θ. Let $\chi_\theta \in \Delta_\Theta$ denote the degenerate measure that assigns probability one to state θ.

2.1 Economies

The *consumption set* of each agent is $\Re_+^\ell$ and for each $\theta \in \Theta, w_i \in \Re_{++}^\ell$ denotes the (state independent) initial endowment of agent i in state θ. The preferences of agent i are given by a utility function $u_i : \Re_+^\ell \times \Theta \rightarrow \Re$ where $u_i(\cdot, \theta)$ is the utility function of agent i in state θ.[3] The following assumptions are maintained throughout the paper:

(i) $u_i(\cdot, \theta)$ is continuous and strictly concave
(ii) $u_i(0, \theta) = 0$

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[2]The assumption that $P(t) > 0$ for all $t \in T$ is relaxed in McLean and Postlewaite (2002b).

[3]We note that this formulation differs from that of Forges, Heifetz and Minelli (2001), who also investigate convergence of the ex ante core. We discuss their paper in the section on related literature at the end of this paper.

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(iii) $u_i(\cdot, \theta)$ is (strongly) monotonic: if $x, y \in \Re_+^\ell$, $x \geq y$ and $x \neq y$, then $u_i(x, \theta) > u_i(y, \theta)$.

The collection $(\{u_i, w_i\}_{i \in N}, \tilde{\theta}, \tilde{t}, P)$ will be called a *private information economy* (PIE for short). It will be assumed that the data defining the PIE is common knowledge. A *private information economy allocation* $z = (z_1, z_2, ..., z_n)$ for the PIE is a collection of functions $z_i \colon T \to \Re_+^\ell$ satisfying $\sum_{i \in N}(z_i(t) - w_i) \leq 0$ for all $t \in T$. We will not distinguish between $w_i \in \Re_{++}^\ell$ and the constant allocation that assigns the bundle w_i to agent i for all $t \in T$.

For each $\pi \in \Delta_\Theta$, the collection $(\pi, \{u_i, w_i\}_{i \in N})$ defines an associated Arrow-Debreu economy with state contingent commodities. A commodity vector for agent i in this Arrow-Debreu economy is a vector of state contingent bundles in $\Re_+^{\ell m}$ and is written as

$$(x_i(\theta_1), .., x_i(\theta_m)).$$

The initial endowment of agent i is the vector $\hat{w}_i = (w_i, .., w_i) \in R_{++}^{\ell m}$ and the utility of agent i is the function $v_i : \Re_+^{\ell m} \to \Re$ defined for each $(x_i(\theta_1), .., x_i(\theta_m)) \in \Re_+^{\ell m}$ as follows:

$$v_i(x_i(\theta_1), .., x_i(\theta_m)) := \sum_{k=1}^m u_i(x_i(\theta_k); \theta_k)\pi(\theta_k).$$

The Arrow Debreu economy with commodity bundles, endowments and utilities defined in this manner will be denoted $E(\pi)$. As discussed in the introduction, this economy is of interest because if the state θ were observable after it is realized, agents could exchange state contingent goods prior to realization. However, in our model θ is *not* observable; all information about the realization of θ is embodied in the vector of agents' types, t. Consequently, agents can trade bundles contingent on the realized vector of types t, but not contingent on θ.

We will refer to $E(\pi)$ as the $\pi-$*auxiliary economy*, or auxiliary economy for short. In addition, each PIE $(\{u_i, w_i\}_{i \in N}, \tilde{\theta}, \tilde{t}, P)$ gives rise to a natural auxiliary economy $E(P_\Theta)$ where P_Θ is the marginal of P on Θ. Since each u_i is concave, a standard argument establishes that the auxiliary economy has a nonempty core.

For each $\varepsilon \geq 0$, we define $\varepsilon-$blocking in the auxiliary economy $E(\pi)$. An allocation $(y_i)_{i \in S}$ is feasible for S in $E(\pi)$ if $\sum_{i \in S} y_i(\theta) = \sum_{i \in S} w_i$ for each $\theta \in \Theta$. A coalition $S \subseteq N$ can $\varepsilon-$block the allocation $(x_i)_{i \in N}$ if there exists an allocation $(y_i)_{i \in S}$ that is feasible for S satisfying the following condition:

$$\sum_{\theta \in \Theta} u_i(y_i(\theta), \theta)\pi(\theta) > \sum_{\theta \in \Theta} u_i(x_i(\theta), \theta)\pi(\theta) + \varepsilon$$

for all $i \in S$. The $\varepsilon-$ core of $E(\pi)$ consists of those allocations that are efficient and which are not $\varepsilon-$blocked by any $S \subseteq N$ with $S \neq N$.

3 Incentive Compatible Cores

3.1 Notions of Blocking

Let $e = (\{u_i, w_i\}_{i \in N}, \widetilde{\theta}, \widetilde{t}, P)$ be a PIE. In order to define the core of an economy with incomplete information, it is necessary to define "improve upon" or "block" taking incentive compatibility into account. For each $S \subseteq N$, let the set of *S-feasible allocations* for the PIE e be defined as

$$A_S = \{(z_i)_{i \in S} | z_i : T_S \to \Re_+^\ell \text{ and } \sum_{i \in S}(z_i(t_S) - w_i) \leq 0 \text{ for all } t_S \in T_S\}.$$

An S-feasible allocation $(z_i)_{i \in S}$ is *incentive compatible* if

$$\sum_{\theta \in \Theta} \sum_{t_{S \setminus i} \in T_{S \setminus i}} u_i(z_i(t_{S \setminus i}, t_i), \theta) P(\theta, t_{S \setminus i} \mid t_i) \geq \sum_{\theta \in \Theta} \sum_{t_{S \setminus i} \in T_{S \setminus i}} u_i(z_i(t_{S \setminus i}, t_i'), \theta) P(\theta, t_{S \setminus i} \mid t_i)$$

for each $t_i, t_i' \in T_i$ and $i \in S$.

The set of incentive compatible, S-feasible allocations will be denoted A_S^*.

Definition 1: Let $e = (\{u_i, w_i\}_{i \in N}, \widetilde{\theta}, \widetilde{t}, P)$ be a PIE and let $(z_i)_{i \in N} \in A_N$.

(i) (*Ex ante blocking*) A coalition $S \subseteq N$ can $X-$block $(z_i)_{i \in N}$ if there exists $(x_i)_{i \in S} \in A_S$ satisfying the following condition:

$$\sum_{t_S \in T_S} \sum_{\theta \in \Theta} u_i(x_i(t_S), \theta) P(\theta, t_S) > \sum_{t_S \in T_S} \sum_{t_{N \setminus S} \in T_{N \setminus S}} \sum_{\theta \in \Theta} u_i(z_i(t_{N \setminus S}, t_S), \theta) P(\theta, t_{N \setminus S}, t_S)$$

for all $i \in S$.

(ii) (*ex ante incentive compatible blocking*) A coalition $S \subseteq N$ can $ICX-$block $(z_i)_{i \in N}$ if there exists $(x_i)_{i \in S} \in A_S^*$ satisfying the following condition:

$$\sum_{t_S \in T_S} \sum_{\theta \in \Theta} u_i(x_i(t_S), \theta) P(\theta, t_S) > \sum_{t_S \in T_S} \sum_{t_{N \setminus S} \in T_{N \setminus S}} \sum_{\theta \in \Theta} u_i(z_i(t_{N \setminus S}, t_S), \theta) P(\theta, t_{N \setminus S}, t_S)$$

for all $i \in S$.

(iii) (*ex ante incentive compatible $\varepsilon-$blocking*) Suppose $\varepsilon \geq 0$. A coalition $S \subseteq N$ can $\varepsilon ICX-$block $(z_i)_{i \in N}$ if there exists $(x_i)_{i \in S} \in A_S^*$ satisfying the following condition:

$$\sum_{t_S \in T_S} \sum_{\theta \in \Theta} u_i(x_i(t_S), \theta) P(\theta, t_S) > \sum_{t_S \in T_S} \sum_{t_{N \setminus S} \in T_{N \setminus S}} \sum_{\theta \in \Theta} u_i(z_i(t_{N \setminus S}, t_S), \theta) P(\theta, t_{N \setminus S}, t_S) + \varepsilon$$

for all $i \in S$.

Definition 2: Let $e = (\{u_i, w_i\}_{i \in N}, \widetilde{\theta}, \widetilde{t}, P)$ be a PIE.

(i) An N-feasible, incentive compatible allocation $(z_i)_{i \in N} \in A_N^*$ is an *Ex Ante Incentive Compatible Core Allocation* for e if $(z_i)_{i \in N}$ cannot be ICX-blocked by any $S \subseteq N$.

(ii) An N-feasible, incentive compatible allocation $(z_i)_{i \in N} \in A_N^*$ is an *Ex Ante Incentive Compatible $\varepsilon-$Core Allocation* for e if $(z_i)_{i \in N}$ cannot be ICX-blocked by N and $(z_i)_{i \in N}$ cannot be εICX-blocked by any $S \neq N$.

4   The Replica Problem: Notation and Definitions

Recall that $J_r = \{1, 2, ..., r\}$ and define $N_r = N \times J_r$. Given the collection $\{w_i, u_i\}_{i \in N}$ and a positive integer r, let $\{w_{is}, u_{is}\}_{(i,s) \in N_r}$ denote the r replication of $\{w_i, u_i\}_{i \in N}$ satisfying:

(1) $w_{is} = w_i$ for all $i \in N$ and all $s \in J_r$

(2) $u_{is}(z, \theta) = u_i(z, \theta)$ for all $z \in \Re_+^{\ell}, i \in N$ and $s \in J_r$.

For any positive integer r, let $T^r = T \times \cdots \times T$ denote the r-fold Cartesian product and let $t^r = (t^r(1), .., t^r(r))$ denote a generic element of T^r where $t^r(s) = (t_1^r(s), .., t_n^r(s)) \in T$. If $P^r \in \Delta_{\Theta \times T^r}$, then $e^r = (\{w_{is}, u_{is}\}_{(i,s) \in N_r}, \tilde{\theta}, \tilde{t}^r, P^r)$ is a PIE with nr agents.

Definition 3: A sequence of replica economies $\{(\{w_{is}, u_{is}\}_{(i,s) \in N_r}, \tilde{\theta}, \tilde{t}^r, P^r)\}_{r=1}^{\infty}$ is a *strongly conditionally independent sequence* if there exists a $P \in \Delta_{\Theta \times T}^*$ such that

(a) For each r, each $s \in J_r$ and each $(\theta, t_1, .., t_n) \in \Theta \times T$,

$$\text{Prob}\{\tilde{\theta} = \theta, \tilde{t}_1^r(s) = t_1, \tilde{t}_2^r(s) = t_2, ..., \tilde{t}_n^r(s) = t_n\} = P(\theta, t_1, t_2, ..., t_n)$$

(b) For each r and each θ, the nr random variables

$$\tilde{t}_1^r(1), \tilde{t}_2^r(1), ..., \tilde{t}_n^r(1), ..., \tilde{t}_1^r(r), \tilde{t}_2^r(r), ..., \tilde{t}_n^r(r)$$

are independent given $\tilde{\theta} = \theta$.

(c) For every $\theta, \hat{\theta}$ with $\theta \neq \hat{\theta}$, there exists a $t \in T$ such that $P(t|\theta) \neq P(t|\hat{\theta})$.

A strongly conditionally independent sequence is a sequence of PIE's with nr agents containing r "copies" of each agent $i \in N$. Each copy of an agent i is identical, i.e., has the same endowment and the same utility function. Furthermore, the realizations of agents' types are independent given the true value of $\tilde{\theta}$. Given a profile of types $(t_1(1), ..., t_n(1), ..., t_1(r), ..., t_n(r)) \in T^r$, it follows that

$$\begin{aligned} \text{Prob}\{\tilde{t}_i^r(s) &= t_i(s), \forall i \in N, \forall s \in J_r | \tilde{\theta} = \theta\} \\ &= P^r(t_1(1), ..., t_n(1), ..., t_1(r), ..., t_n(r)|\theta) \\ &= \prod_{i \in N} \prod_{s \in J_r} P(t_i(s)|\theta) \end{aligned}$$

As r increases, each agent is becoming "small" in the economy in terms of endowment, and we can show that each agent is also becoming informationally small in the sense of McLean and Postlewaite (2002a). Note that, for large r, an agent may have a small amount of private information regarding the preferences of everyone through his information about $\tilde{\theta}$.

Given an auxiliary economy $E(\pi)$, we can also define the *r-replicated auxiliary economy* $E^r(\pi)$ consisting of r replicas of type i where agent (i, s) has endowment w_i and utility

$$v_{i,s}(x_{i,s}(\theta_1), ..., x_{i,s}(\theta_m)) := \sum_{k=1}^{m} u_i(x_{i,s}(\theta_k); \theta_k)\pi(\theta_k).$$

Note that, in this notation, $E^1(\pi) = E(\pi)$ and we will use these interchangeably. The $\varepsilon-$core of the r-replicated auxiliary economy is defined in the obvious way. Note that the Debreu-Scarf Theorem can be applied to the r-replicated auxiliary economy: core allocations of $E^r(\pi)$ are equal treatment allocations and the intersection of the "projections" of the cores of the replications coincides with the set of Walras equilibria of the auxiliary economy $E^1(\pi)$.

5   The Core Convergence Results

5.1   Walras Allocations of the Auxiliary Economy are Close in Utility to Epsilon Core Allocations in a Large Replica Economy

In McLean and Postlewaite (2003) (see the proof of Theorem 3 in that paper), it is shown that, for large replica economies, the ex ante incentive compatible $\varepsilon-$core is nonempty. In particular, they prove the following result:[4]

Theorem A: Let $\{(\{w_{is}, u_{is}\}_{(i,s)\in N_r}, \widetilde{\theta}, \widetilde{t}^r, P^r)\}_{r=1}^{\infty}$ be a strongly conditionally independent sequence with $P_{\Theta}^r \equiv \pi$ and let $(x_i)_{i\in N}$ be a Walras equilibrium allocation of the auxiliary economy $E(\pi)$. Then for every $\varepsilon > 0$, there exists an integer $\hat{r} > 0$ such that for all $r > \hat{r}$ there exists an allocation $(\xi_{is}^r)_{(i,s)\in N_r}$ in the ex ante IC $\varepsilon-$core of the PIE $(\{w_{is}, u_{is}\}_{(i,s)\in N_r}, \widetilde{\theta}, \widetilde{t}^r, P^r)$, satisfying

$$\left| \sum_{t^r \in T^r} \sum_{\theta \in \Theta} u_i(\xi_{i,s}^r(t^r), \theta) P^r(\theta, t^r) - \sum_{\theta \in \Theta} u_i(x_i(\theta), \theta)\pi(\theta) \right| \leq \varepsilon$$

for all $(i, s) \in N_r$.

Theorem A is the first step toward a core convergence result. We must now investigate the extent to which an incentive compatible core allocation of a large replica PIE is close in utility to some Walras allocation of the underlying auxiliary economy. We break the analysis into two parts that are presented in the next two subsections.

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[4]Actually, we proved the theorem under the weaker hypothesis that the sequence is a conditionally independent sequence. See McLean and Postlewaite (2003) for definitions and details.

5.2 Core Allocations in a Large Replica Economy are Close in Utility to Epsilon Core Allocations of the Replicated Auxiliary Economy

Theorem B: Let $\{(\{w_{is}, u_{is}\}_{(i,s)\in N_r}, \widetilde{\theta}, \widetilde{t}^r, P^r)\}_{r=1}^{\infty}$ be a strongly conditionally independent sequence with $P_{\Theta}^r \equiv \pi$. Then for every $\varepsilon > 0$, there exists an integer $\hat{r} > 0$ such that for all $r > \hat{r}$ and for each allocation $(\xi_{is}^r)_{(i,s)\in N_r}$ in the ex ante IC core of the PIE $(\{w_{is}, u_{is}\}_{(i,s)\in N_r}, \widetilde{\theta}, \widetilde{t}^r, P^r)$, there exists an equal treatment allocation $(x_{is}^r)_{(i,s)\in N_r}$ in the $\varepsilon-$core of the r-replicated auxiliary economy $E^r(\pi)$ satisfying

$$\#\{s \in J_r| \le \left| \sum_{t^r \in T^r} \sum_{\theta \in \Theta} u_i(\xi_{i,s}^r(t^r), \theta)P^r(\theta, t^r) - \sum_{\theta \in \Theta} u_i(x_{is}^r(\theta), \theta)\pi(\theta) \right| \le \varepsilon\} \ge (1-\varepsilon)r$$

for each $i \in N$.

Theorem B is an immediate consequence of the following two propositions whose proofs are found in Section 7 below.

Proposition 1: (Asymptotic equal treatment for most agents) Let $\{(\{w_{is}, u_{is}\}_{(i,s)\in N_r}, \widetilde{\theta}, \widetilde{t}^r, P^r)\}_{r=1}^{\infty}$ be a strongly conditionally independent sequence with $P_{\Theta}^r \equiv \pi$. For every $\varepsilon > 0$ there exists an $\hat{r}$ such that, for all $r > \hat{r}$ and for each allocation $(x_{is}^r)_{(i,s)\in N_r}$ in the incentive compatible core of the PIE $(\{w_{is}, u_{is}\}_{(i,s)\in N_r}, \widetilde{\theta}, \widetilde{t}^r, P^r)$ the following holds: if

$$\bar{x}_i^r(\theta) := \frac{1}{r} \sum_{s\in J_r} \sum_{t^r \in T^r} x_{is}^r(t^r) P^r(t^r|\theta)$$

then

$$\#\{s \in J_r : \left| \sum_{t^r \in T^r} \sum_{\theta \in \Theta} u_i(x_{i,s}^r(t^r), \theta)P^r(\theta, t^r) - \sum_{\theta \in \Theta} u_i(\bar{x}_i^r(\theta), \theta)P(\theta) \right| \le \varepsilon\} \ge (1-\varepsilon)r$$

for each $i \in N$.

Proposition 2: Let $\{(\{w_{is}, u_{is}\}_{(i,s)\in N_r}, \widetilde{\theta}, \widetilde{t}^r, P^r)\}_{r=1}^{\infty}$ be a strongly conditionally independent sequence with $P_{\Theta}^r \equiv \pi$. For every $\varepsilon > 0$ there exists an $\hat{r}$ such that, for all $r > \hat{r}$ and for each allocation $(x_{is}^r)_{(i,s)\in N_r}$ in the ex ante incentive compatible core of the PIE $(\{w_{is}, u_{is}\}_{(i,s)\in N_r}, \widetilde{\theta}, \widetilde{t}^r, P^r)$ the following holds: if

$$\xi_{is}^r(\theta) := \frac{1}{r} \sum_{s\in J_r} \sum_{t^r \in T^r} x_{is}^r(t^r) P^r(t^r|\theta)$$

for each $(i, s) \in N_r$, then the (equal treatment) allocation $(\xi_{is}^r)_{(i,s)\in N_r}$ belongs to the $\varepsilon-$core of the r-replicated auxiliary economy $E^r(\pi)$.

As we mentioned in the introduction, incentive compatible core allocations will not necessarily satisfy an equal treatment property. However, Proposition 1 states that, for sufficiently large replica economies, incentive compatible core allocations will satisfy an approximate equal treatment property for most agents. The proof is somewhat involved but we will explain the main ideas here.

In the complete information case, one demonstrates that a core allocation x in a replica economy is an equal treatment allocation by first constructing the coalition of the worst-off agent of each type $i \in N$. The type i agent in this coalition is then given the average of the bundles of all type i agents in the allocation x. Strict concavity guarantees that for any type for which different agents of that type received different bundles, the average is strictly preferred to the least preferred bundle for agents of that type. In the environment with asymmetric information, it is not sufficient to construct such an average allocation for the coalition of the worst-off agents, since the allocation may not be incentive compatible.

To solve this problem, we consider a coalition consisting of the εr agents of each type who are the worst-off of that type (where r is the number of replicas). For any fixed ε, for large r number of agents in this coalition will be large, but a small proportion of all agents. If each of the εr agents of some type get utility from an incentive compatible core allocation x that is more than the average utility for that type minus ε, then we can construct an allocation for the coalition consisting of the εr worst-off agents of each type that yields higher utility then the allocation x. Since this coalition consists of a large number of agents, we can use the approximation theorem in McLean and Postlewaite (2003) to find an incentive compatible allocation with which this coalition can block x.

This shows that for sufficiently many replications, we can ensure that the set of agents whose utility in a core allocation x is more than the average of the utilities for their type minus ε will be small. What remains is to show that the set of agents of some type whose utility is more than ε *above* the average of the utilities for their type is also small. We show that given ε, if there are more than εr agents whose utility is more than ε *above* the average of the utilities for their type, then there must be $\varepsilon^3 r$ agents whose utility is more than $\varepsilon^3 r$ *below* the average of utility for that type. It then follows from the first part of the proof that the allocation x is blocked.

5.3 Core Allocations in a Large Replica Economy are Close in Utility to Walras Allocations of the Auxiliary Economy

Suppose we replicate a private information economy r times. From Theorem B, we know that for any ex ante incentive compatible core allocation of the r-replicated economy, there is an equal treatment allocation x in the $\varepsilon-$core of the r-replicated auxiliary economy $E^r(\pi)$ that gives $(1-\varepsilon)r$ agents utility within ε of the utility they

get in the incentive compatible core allocation. As mentioned above, the auxiliary economy $E(\pi)$ is, in a sense, the Arrow-Debreu economy of interest; if it were possible, agents would trade bundles contingent on the state θ. The Debreu-Scarf theorem applies to replications of the auxiliary economy $E(\pi)$, hence allocations in the core for the r-replicated economy $E^r(\pi)$ will be approximately Walrasian for this economy. For small ε, allocations in the ε-core of $E^r(\pi)$ will also be approximately Walrasian. Combining these observations, we conclude that, when r is large, allocations in the ex ante incentive compatible core for the r-replicated private information economy will give most agents utility that is close to that of some Walrasian equilibrium allocation for the auxiliary economy $E(\pi)$. The next theorem formalizes this.

Theorem C: Let $\{(\{w_{is}, u_{is}\}_{(i,s)\in N_r}, \widetilde{\theta}, \widetilde{t}^r, P^r)\}_{r=1}^{\infty}$ be a strongly conditionally independent sequence with $P_{\Theta}^r \equiv \pi$. Then for every $\varepsilon > 0$, there exists an integer $\hat{r} > 0$ such that for all $r > \hat{r}$ and for each allocation $(\xi_{is}^r)_{(i,s)\in N \times J_r}$ in the ex ante IC core of the PIE $(\{w_{is}, u_{is}\}_{(i,s)\in N_r}, \widetilde{\theta}, \widetilde{t}^r, P^r)$, there exists a Walras equilibrium allocation $(x_i^r)_{i\in N}$ of the auxiliary economy $E(\pi)$ satisfying

$$\#\{s \in J_r | \le \left| \sum_{t\in T}\sum_{\theta\in\Theta} u_i(\xi_{i,s}^r(t^r), \theta)P^r(\theta, t^r) - \sum_{\theta\in\Theta} u_i(x_i^r(\theta), \theta)\pi(\theta) \right| \le \varepsilon\} \ge (1-\varepsilon)r$$

for each $i \in N$.

6   Related literature

Forges, Heifetz and Minelli (2001) (hereafter FHM) examine the ex ante incentive compatible core in a model that is related to the model in this paper. FHM uses the information structure and replication process in Gul and Postlewaite (1992). In this framework, there is no state of the world θ, and in the initial economy before replication, each agent's utility depends on the types of all agents in the economy. When the economy is replicated, the utilities of agents in any cohort, depend only on the types of the agents in their cohort, and the types of the agents in different cohorts are independent. In this model, FHM show that equal treatment may fail, and that there will be noncompetitive allocations that are in the core of all replicated economies. They then show that if one restrict each agent's utility to depend only on his own type, then a core convergence theorem obtains.

The model in this paper differs from that in FHM is several ways. First, our model is essentially a "common value" model in that agents' types are purely informational. Agents' utilities depend on the bundle they get and the state of the world; an agent's type, and all other agents' types, are of interest only insofar as they provide information about the state θ. This paper and FHM can then be seen as complements in that we show core convergence in "common-value" economies while FHM shows core convergence in private-value economies.

In addition, there is an important technical difference in the models in FHM and in the present paper. Feasibility is FHM is defined on average across the realizations of agents' types. While FHM show that asymptotically the ex post infeasibility of the allocations they consider goes to zero, full feasibility is likely to fail for any finitely replication. The allocations we consider, on the other hand, are fully feasible for all replications and all realizations of agents' types.

Serrano, Vohra and Volij (2001) consider an interim core concept where evaluation takes place after agents receive their private information. They show that when the true state of the world is verifiable ex post, core convergence may fail. One can also consider ex post replication, that is, replicating an economy *after* agents' types have been realized. Such replication leads to non-exclusive information and non-empty cores. Einy, Moreno and Shitovitz (2001a, 2001b) consider convergence of the core with ex post replication.

These are the papers that are closest to the present work. In addition, there is a large literature that studies the core in the presence of asymmetric information. Forges, Minelli and Vohra (2000) survey this literature and the interested reader is directed to that paper for a review of various notions of the core in economies with asymmetrically informed agents, and work employing alternative core concepts.

7 Proofs:

7.1 A Preliminary Lemma

Lemma 1: Let $\{(\{w_{is}, u_{is}\}_{(i,s) \in N_r}, \widetilde{\theta}, \widetilde{t}^r, P^r)\}_{r=1}^{\infty}$ be a strongly conditionally independent sequence with $P_{\Theta}^r \equiv \pi$ and let $(\zeta_i)_{i \in N}$ be an allocation of the auxiliary economy $E(\pi)$. Then for every $\varepsilon > 0$, there exists an integer $\hat{r} > 0$ such that, for all $r > \hat{r}$, there exists an incentive compatible allocation $(z_{is}^r)_{(i,s) \in N_r}$ for the PIE $(\{w_{is}, u_{is}\}_{(i,s) \in N_r}, \widetilde{\theta}, \widetilde{t}^r, P^r)$ which satisfies

$$\sum_{t^r \in T^r} \sum_{\theta \in \Theta} u_i(z_{is}^r(t^r); \theta) P^r(\theta, t^r) \geq \sum_{\theta \in \Theta} u_i(\zeta_i(\theta); \theta) P(\theta) - \varepsilon$$

for each $(i, s) \in N_r$.

Proof: The proof is a synthesis of results found in our earlier papers McLean and Postlewaite 2002(a) and 2003. After noting that a strongly conditionally independent sequence is a conditionaly independent sequence as defined in those papers, the proof of Lemma 1 is identical to that of Steps 1 and 2 in the proof of Theorem 3 in McLean and Postlewaite (2003). While the proof of Theorem 3 in that paper asssumes that $(\zeta_i)_{i \in N}$ is a Walras equilibrium of $E(\pi)$, the conclusion at Step 2 is valid for any allocation $(\zeta_i)_{i \in N}$ of the auxiliary economy $E(\pi)$.

7.2 Proof of Proposition 1

To prove Proposition 1, we will prove Lemma 2 below from which Proposition 1 immediately follows. First we introduce some notation. An allocation $(\zeta_{is}^r)_{(i,s)\in N_r}$ for the PIE $(\{w_{is}, u_{is}\}_{(i,s)\in N_r}, \widetilde{\theta}, \widetilde{t}^r, P^r)$ will be written simply as (ζ_{is}^r) and $U_i(\zeta_{is}^r)$ will denote the utility to agent (i,s) associated with the bundle ζ_{is}^r. That is,

$$U_i(\zeta_{is}^r) = \sum_{t^r \in T^r} \sum_{\theta \in \Theta} u_i(\zeta_{is}^r(t^r); \theta) P^r(\theta, t^r).$$

Similarly, an allocation $(\xi_{is}^r)_{(i,s)\in N_r}$ for the PIE $E^r(\pi)$ will be written simply as (ξ_{is}^r) and $v_i(\zeta_{is}^r)$ will denote the utility to agent (i,s) associated with the bundle ξ_{is}^r. That is,

$$v_i(\xi_{is}^r) = \sum_{\theta \in \Theta} u_i(\xi_{is}^r(\theta), \theta) \pi(\theta).$$

Lemma 2: Suppose that $0 < \varepsilon < 1/2$. Let $\{(\{w_{is}, u_{is}\}_{(i,s)\in N_r}, \widetilde{\theta}, \widetilde{t}^r, P^r)\}_{r=1}^{\infty}$ be a strongly conditionally independent sequence with $P_{\Theta}^r \equiv \pi$. Then there exists an $\hat{r}$ such that, for all $r > \hat{r}$ and for each allocation (x_{is}^r) in the ex ante incentive compatible core of the PIE $(\{w_{is}, u_{is}\}_{(i,s)\in N_r}, \widetilde{\theta}, \widetilde{t}^r, P^r)$ the following holds: if $(\bar{x}_i^r)_{i\in N}$ is the allocation for $E(\pi)$ defined as

$$\bar{x}_i^r(\theta) = \frac{1}{r} \sum_{t^r \in T^r} \sum_{s \in J_r} x_{is}^r(t^r) P^r(t^r|\theta)$$

for each $\theta \in \Theta$, then

$$\#\{s \in J_r | v_i((1-\varepsilon)\bar{x}_i^r) \le U_i(x_{is}^r) \le (1+\varepsilon)v_i(\bar{x}_i^r)\} \ge (1-2\varepsilon)r$$

for each $i \in N$.

Proof: **Part 1**: In this part of the proof, we will show that, for each $\varepsilon > 0$, exists an $\hat{r}$ such that, for all $r > \hat{r}$

$$\#\{s \in J_r | v_i((1-\varepsilon)\bar{x}_i^r) > U_i(x_{is}^r)\} < \varepsilon r$$

for each $i \in N$. Suppose not. Then there exists $\varepsilon > 0$ and a subsequence of positive integers $\{r_k\}$ such that, for each k, there exists an $i^{r_k} \in N$ and an IC core allocation $(x_{is}^{r_k})$ of the PIE $(\{w_{is}, u_{is}\}_{(i,s)\in N_{r_k}}, \widetilde{\theta}, \widetilde{t}^{r_k}, P^{r_k})$ such that

$$\#\{s \in J_{r_k} | v_{i^{r_k}}((1-\varepsilon)\bar{x}_{i^{r_k}}^{r_k}) > U_i(x_{i^{r_k},s}^{r_k})\} \ge \varepsilon r_k.$$

For sufficiently large k, we will construct a coalition C^{r_k} and a feasible allocation for C^{r_k} that blocks the PIE allocation (x_{is}^r) in the ex ante incentive compatible sense. Since (x_{is}^r) is an ex ante incentive compatible core allocation, this contradiction then yields the result.

We will abuse notation slightly and simply write r instead of r_k. Since N is finite, we will assume WLOG that $i^r = 1$ for all r. Furthermore, we assume that the agents of each type $j \in N$ are numbered so that $U_j(x_{js}^r) \leq U_j(x_{js'}^r)$ for $s' > s$.

The allocation $(\bar{x}_i^r)$ is a feasible allocation for $E(\pi)$. Since $(x_{i,s}^r)$ is an ex ante IC core allocation, the concavity assumption implies that $(\bar{x}_i^r)$ is individually rational for $E(\pi)$ and $w_i \neq 0$ implies that $v_i(\bar{x}_i^r) \geq v_i(w_i) > 0$ for each $i \in N$. Choosing a (second) subsequence if necessary, we will assume that $\bar{x}_i^r(\theta) \to x_i^*(\theta)$ for each i and θ. Note that (x_i^*) is a feasible, individually rational allocation for the auxiliary economy $E(\pi)$ so that $v_i(x_i^*) \geq v_i(w_i) > 0$ for each $i \in N$. [5]For $i = 1$ in particular, this means that

$$\sum_{\theta \in \Theta} u_1(x_1^*(\theta), \theta)\pi(\theta) = v_1(x_1^*) > 0.$$

Hence, the normalization and monotonicity assumptions imply that there exists a $\hat{\theta} \in \Theta$ such that $x_1^*(\hat{\theta}) \neq 0$ and $\pi(\hat{\theta}) > 0$. Therefore, monotonicity implies that

$$\left[u_j(x_j^*(\hat{\theta}) + x_1^*(\hat{\theta}), \hat{\theta}) - u_j(x_j^*(\hat{\theta}), \hat{\theta}) \right] \pi(\hat{\theta}) > 0$$

for each $j \neq 1$ and we conclude that $v_j(x_j^* + x_1^*) - v_j(x_j^*) > 0$ for each $j \neq 1$. Suppose that $0 < \eta < 1$. Then strict concavity and the normalization assumption imply that

$$v_j(x_j^*) = v_j(\frac{1-\eta}{1-\eta}x_j^*) \geq (1-\eta)v_j(\frac{1}{1-\eta}x_j^*)$$

so that

$$v_j(\frac{1}{1-\eta}x_j^*) \leq v_j(x_j^*) + \frac{\eta}{1-\eta}v_j(x_j^*).$$

Applying concavity again, we obtain

$$v_j(x_j^* + \frac{\varepsilon}{2(n-1)}x_1^*) \geq \frac{\varepsilon}{2(n-1)}v_j(x_j^* + x_1^*) + (1 - \frac{\varepsilon}{2(n-1)})v_j(x_j^*)$$

so that

$$v_j(x_j^*) + \frac{\varepsilon}{2(n-1)}\left[v_j(x_j^* + x_1^*) - v_j(x_j^*) \right] \leq v_j(x_j^* + \frac{\varepsilon}{2(n-1)}x_i^*).$$

Hence, there exists $\eta^* > 0$ such that $\eta^* < \varepsilon$ and $\frac{\eta^*}{1-\eta^*}v_j(x_j^*) < \frac{\varepsilon}{2(n-1)}\left[v_j(x_j^* + x_1^*) - v_j(x_j^*) \right]$ for each $j \neq 1$ from which it follows that

$$v_j(\frac{1}{1-\eta^*}x_j^*) < v_j(x_j^* + \frac{\varepsilon}{2(n-1)}x_1^*).$$

---

[5]The monotonicity and normalization assumptions imply that $v_i(w_i) = \sum_\theta u_i(w_i, \theta)\pi(\theta) > 0$ for each $i \in N$.

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Note that η^* does not depend on r.

Now consider the coalition $C^r = \{(i,s)|i \in N, s \leq \lfloor \eta^* r \rfloor\}$ consisting of the $\lfloor \eta^* r \rfloor$ "worst off" agents of each type. (Here, $\lfloor x \rfloor$ denotes the the largest integer leass than or equal to x.) Define an allocation for the auxiliary economy $E(\pi)$ as follows: for each $\theta \in \Theta$,

$$y_1(\theta) = (1 - \frac{\varepsilon}{2})x_1^*(\theta)$$

and

$$y_j(\theta) = x_j^*(\theta) + \frac{\varepsilon}{2(n-1)}x_1^*(\theta)$$

if $j \neq 1$. We now apply the approximation Lemma 1 to the auxiliary economy allocation $(y_i)_{i \in N}$ and the strongly conditionally independent sequence of PIE's in which the stage r PIE consists of the $n \lfloor \eta^* r \rfloor$ agents in C^r. In particular, there exists an $\hat{r}$ such that, for all $r > \hat{r}$, there exists an incentive compatible allocation $(z_{i,s}^r)_{(i,s) \in C^r}$ for the PIE consisting of the $n \lfloor \eta^* r \rfloor$ agents in C^r such that, for each $(i,s) \in C^r$,

$$U_i(z_{i,s}^r) > v_i(y_i) - \gamma$$

where γ is chosen so that

$$0 < \gamma \leq \min \left\{ \frac{v_1((1-\frac{\varepsilon}{2})x_1^*) - v_1((1-\varepsilon)x_1^*)}{2}, \min_{j \neq i} \left\{ \frac{v_j(x_j^* + \frac{\varepsilon}{2(n-1)}x_1^*) - v_j(\frac{1}{1-\eta^*}x_j^*)}{2} \right\} \right\}.$$

Since $\eta^* < \varepsilon$, it follows that $U_1(x_{1s}^r) < v_1((1-\varepsilon)\bar{x}_1^r)$ if $1 \leq s \leq \lfloor \eta^* r \rfloor$. We conclude that for all sufficiently large r and for each $(1,s) \in C^r$,

$$
\begin{aligned}
U_1(x_{1s}^r) &< v_1((1-\varepsilon)\bar{x}_1^r) \\
&< \frac{v_1((1-\frac{\varepsilon}{2})x_1^*) + v_1((1-\varepsilon)x_1^*)}{2} \\
&\leq v_1((1-\frac{\varepsilon}{2})x_1^*) - \gamma \\
&< U_1(z_{1s}^r).
\end{aligned}
$$

To complete the proof, fix r and let $m = \lfloor \eta^* r \rfloor$. For each $j \neq 1$ and for each $s \geq m+1$, recall that $U_j(x_{j,m+1}^r) \leq U_j(x_{j,s}^r)$. Therefore, monotonicity and concavity imply that

$$
\begin{aligned}
U_j(x_{j,m+1}^r) &\leq \frac{1}{r-m}\sum_{s \geq m+1} U_j(x_{js}^r + \frac{1}{r-m}\sum_{\sigma \leq m} x_{j\sigma}^r) \\
&\leq U_j(\frac{1}{r-m}\sum_{s \in J_r} x_{js}^r) \\
&\leq v_j(\frac{r}{r-m}\bar{x}_j^r) \\
&\leq v_j(\frac{1}{1-\eta^*}\bar{x}_j^r).
\end{aligned}
$$

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If $j \neq 1$, then for sufficiently large r and for each $(j,s) \in C^r$, it follows that

$$
\begin{aligned}
U_j(x_{js}^r) &\leq U_j(x_{j,\lfloor \eta^* r \rfloor + 1}^r) \\
&\leq v_j\left(\frac{1}{1-\eta^*}\bar{x}_j^r\right) \\
&< \frac{v_j\left(\frac{1}{1-\eta^*}x_j^*\right) + v_j\left(x_j^* + \frac{\varepsilon}{2(n-1)}x_1^*\right)}{2} \\
&\leq v_j\left(x_j^* + \frac{\varepsilon}{2(n-1)}x_1^*\right) - \gamma \\
&< U_j(z_{js}^r).
\end{aligned}
$$

In summary, the PIE allocation (z_{is}^r) is feasible for C^r and allows C^r to block (x_{is}^r) in the ex ante incentive compatible sense. This contradicts the assumption that (x_{is}^r) is an ex ante incentive compatible core allocation of the PIE $(\{w_{is}, u_{is}\}_{(i,s)\in N_r}, \tilde{\theta}, \tilde{t}^r, P^r)$. Hence, for every $\varepsilon > 0$, exists an $\hat{r}$ such that, for all $r > \hat{r}$

$$
\#\{s \in J_r | v_i((1-\varepsilon)\bar{x}_i^r) > U_i(x_{is}^r)\} < \varepsilon r
$$

for each $i \in N$.

Part 2: In this part of the proof, we will show that, for each $\varepsilon > 0$, exists an $\hat{r}$ such that, for all $r > \hat{r}$,

$$
\#\{s \in J_r | U_i(x_{is}^r) > (1+\varepsilon)v_i(\bar{x}_i^r)\} < \varepsilon r
$$

for each $i \in N$. Suppose that there exists $\varepsilon > 0$ and a subsequence of positive integers $\{r_k\}$ such that, for each k, there exists an $i^{r_k} \in N$ such that

$$
\#\{s \in J_{r_k} | U_{i^{r_k}}(x_{i^{r_k},s}^{r_k}) > (1+\varepsilon)v_{i^{r_k}}(\bar{x}_{i^{r_k}}^{r_k})\} \geq \varepsilon r_k.
$$

We will again abuse notation slightly and simply write r instead of r_k. Since N is finite, we will assume WLOG that $i^r = 1$ for all r. Furthermore, we again assume that the agents of each type j are numbered so that , $U_j(x_{js}^r) \leq U_j(x_{js'}^r)$ for $s' > s$. To begin, fix r and define the sets

$$
\begin{aligned}
L &= \{(1,s) | U_1(x_{1s}^r) < v_1((1-\varepsilon^3)\bar{x}_1^r)\} \\
M &= \{(1,s) | v_1((1-\varepsilon^3)\bar{x}_1^r) \leq U_1(x_{1s}^r) \leq (1+\varepsilon)v_1(\bar{x}_1^r)\} \\
H &= \{(1,s) | U_1(x_{1s}^r) > (1+\varepsilon)v_1(\bar{x}_1^r)\}
\end{aligned}
$$

and let $|S|$ = number of agents in $S = L, M, H$.

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Claim: If $|H| > \varepsilon r$, then $|L| \geq \varepsilon^3 r$

Proof: Denote by $\bar{x}_i^r(S)$ the average bundle for group S, that is

$$\bar{x}_1^r(S)(\theta) = \frac{1}{|S|} \sum_{s \in S} \left[\sum_{t^r \in T^r} x_{1,s}^r(t^r) P^r(t^r|\theta) \right].$$

In the remainder of the proof, we will suppress the superscript r. Therefore,

$$
\begin{aligned}
v_1(\bar{x}_1) &= v_1\left(\frac{|L|}{r}\bar{x}_1(L) + \frac{|M|}{r}\bar{x}_1(M) + \frac{|H|}{r}\bar{x}_1(H)\right) \\
&> \frac{|L|}{r}v_1(\bar{x}_1(L)) + \frac{|M|}{r}v_1(\bar{x}_1(M)) + \frac{|H|}{r}v_1(\bar{x}_1(H)).
\end{aligned}
$$

The strict inequality is a consequence of strict concavity and the fact that not all the $x_{1,s}$ are the same. Since $U_1(x_{1,s}) > v_1((1 - \varepsilon^3)\bar{x}_1)$ for all $(1,s) \in M$ and $U_1(x_{1s}^r) > (1 + \varepsilon)v_1(\bar{x}_1^r)$ for all $(1,s) \in M$, concavity implies that $v_1(\bar{x}_1(M)) \geq v_1((1 - \varepsilon^3)\bar{x}_1)$ and $v_1(\bar{x}_1(H)) > (1 + \varepsilon)v_1(\bar{x}_1^r)$. Therefore,

$$
\begin{aligned}
&\frac{|L|}{r}v_1(\bar{x}_1(L)) + \frac{|M|}{r}v_1(\bar{x}_1(M)) + \frac{|H|}{r}v_1(\bar{x}_1(H)) \\
&> \frac{|L|}{r}v_1(\bar{x}_1(L)) + \frac{|M|}{r}v_1((1 - \varepsilon^3)\bar{x}_1) + (1 + \varepsilon)\frac{|H|}{r}v_1(\bar{x}_1) \\
&\geq \frac{|L|}{r}v_1(\bar{x}_1(L)) + \frac{|M|}{r}(1 - \varepsilon^3)v_1(\bar{x}_1) + \frac{|H|}{r}v_1(\bar{x}_1) + \varepsilon\frac{|H|}{r}v_1(\bar{x}_1) \\
&= \frac{|L|}{r}v_1(\bar{x}_1(L)) + \frac{|M|}{r}v_1(\bar{x}_1) + \frac{|H|}{r}v_1(\bar{x}_1) + [\varepsilon\frac{|H|}{r} - \varepsilon^3\frac{|M|}{r})]v_1(\bar{x}_1).
\end{aligned}
$$

Combining these observations, it follows that

$$v_1(\bar{x}_1) > \frac{|L|}{r}v_1(\bar{x}_1(L)) + \frac{|M|}{r}v_1(\bar{x}_1) + \frac{|H|}{r}v_1(\bar{x}_1) + [\varepsilon\frac{|H|}{r} - \varepsilon^3\frac{|M|}{r})]v_1(\bar{x}_1).$$

Now suppose that $|L| < \varepsilon^3 r$ and $|H| > \varepsilon r$. Then

$$\varepsilon\frac{|H|}{r} > \varepsilon^2 > \frac{|L|}{r\varepsilon}.$$

Since $\varepsilon < 1/2$, it follows that $\frac{1}{\varepsilon} - \frac{|M|}{r} > 1$ so that

$$\frac{|L|}{r}v_1(\bar{x}_1(L)) + \frac{|M|}{r}v_1(\bar{x}_1) + \frac{|H|}{r}v_1(\bar{x}_1) + [\varepsilon\frac{|H|}{r} - \varepsilon^3\frac{|M|}{r})]v_1(\bar{x}_1)$$
$$> \frac{|L|}{r}v_1(\bar{x}_1(L)) + \frac{|M|}{r}v_1(\bar{x}_1) + \frac{|H|}{r}v_1(\bar{x}_1) + [\varepsilon^2 - \varepsilon^3\frac{|M|}{r})]v_1(\bar{x}_1)$$
$$= \frac{|L|}{r}v_1(\bar{x}_1(L)) + \frac{|M|}{r}v_1(\bar{x}_1) + \frac{|H|}{r}v_1(\bar{x}_1) + \varepsilon^2 v_1(\bar{x}_1)[1 - \varepsilon\frac{|M|}{r}]$$
$$\geq \frac{|L|}{r}v_1(\bar{x}_1(L)) + \frac{|M|}{r}v_1(\bar{x}_1) + \frac{|H|}{r}v_1(\bar{x}_1) + \frac{|L|}{r}v_1(\bar{x}_1)[\frac{1}{\varepsilon} - \frac{|M|}{r}]$$
$$\geq \frac{|L|}{r}v_1(\bar{x}_1(L)) + \frac{|M|}{r}v_1(\bar{x}_1) + \frac{|H|}{r}v_1(\bar{x}_1) + \frac{|L|}{r}v_1(\bar{x}_1)$$
$$> v_i(\bar{x}_i)$$

a contradiction. Hence, $|H| > \varepsilon r$ implies that $|L| \geq \varepsilon^3 r$ and the proof of the claim is complete.

¿From the claim, we now conclude the following: there exists an $\varepsilon > 0$ and a sequence of positive integers $\{r\}$ such that, for each r,

$$\#\{s \in J_r | v_1((1 - \varepsilon^3)\bar{x}_1^r) > U_1(x_{1s}^r)\} \geq \varepsilon^3 r$$

We can now duplicate the proof of Part 1 (with ε^3 in place of ε) and, for sufficiently large r, construct a coalition that can block (x_{is}), contradictng the assumption that (x_{is}) is an ex ante incentive compatible core allocation of the PIE $(\{w_{is}, u_{is}\}_{(i,s)\in N_r}, \tilde{\theta}, \tilde{t}^r, P^r)$. Hence, for every $\varepsilon > 0$, exists an $\hat{r}$ such that, for all $r > \hat{r}$

$$\#\{s \in J_r | U_i(x_{is}^r) > (1 + \varepsilon)v_i(\bar{x}_i^r)\} < \varepsilon r$$

for each $i \in N$.

Part 3: Combining the conclusions of Parts 1 and 2, it follows that, for every $\varepsilon > 0$, exists an $\hat{r}$ such that, for all $r > \hat{r}$,

$$\#\{s \in J_r | v_i((1 - \varepsilon)\bar{x}_i^r) \leq U_i(x_{is}^r)\} \geq (1 - \varepsilon)r$$

and

$$\#\{s \in J_r | U_i(x_{is}^r) \leq (1 + \varepsilon)v_i(\bar{x}_i^r)\} \geq (1 - \varepsilon)r$$

for each $i \in N$. Therefore, for all $r > \hat{r}$,

$$\#\{s \in J_r | v_i((1 - \varepsilon)\bar{x}_i^r) \leq U_i(x_{is}^r) \leq (1 + \varepsilon)v_i(\bar{x}_i^r)\} \geq (1 - 2\varepsilon)r.$$

7.3 Proof of Proposition 2

Let

$$\xi_{is}^r(\theta) = \frac{1}{r} \sum_{t^r \in T^r} \sum_{s \in J_r} x_{is}^r(t^r) P^r(t^r|\theta) := x_i^r(\theta)$$

for each i and s. We claim that $(\xi_{is}^r)_{(i,s) \in N_r}$ belongs to the $\varepsilon-$core of the r-replicated auxiliary economy $E^r(\pi)$ for all sufficiently large r. Suppose not. Extracting a subsequence if necessary, there exists for each r a coalition $C^r \subseteq N \times J_r$ and an allocation $(y_{is}^r)_{(i,s) \in C^r}$ that is feasible for C^r such that

$$\sum_\theta u_i(y_{i,s}^r(\theta), \theta) \pi(\theta) \geq \sum_\theta u_i(\xi_{i,s}^r(\theta), \theta) \pi(\theta) + \varepsilon$$
$$= \sum_\theta u_i(x_i^r(\theta), \theta) \pi(\theta) + \varepsilon$$

for each $(i,s) \in C^r$. Let $I^r = \{i \in N | (i,s) \in C^r\}$ for some $s \in J_r$. Since N is finite, we will assume (extracting another subsequence if necessary) that there exists a $Q \subseteq N$ such that $I^r = Q$ for all r. For each $i \in Q$, let $K_i^r = \{s \in J_r | (i,s) \in C^r\}$. Next, for each $i \in Q$, define

$$y_i^r(\theta) = \frac{1}{|K_i^r|} \sum_{s \in K_i^r} y_{i,s}^r(\theta)$$

and note that

$$\sum_{i \in Q} y_i^r(\theta) = \frac{1}{|K_i^r|} \sum_{i \in Q} \sum_{s \in K_i^r} y_{i,s}^r(\theta) = \sum_{i \in Q} w_i.$$

Concavity implies that

$$\sum_\theta u_i(y_i^r(\theta), \theta) \pi(\theta) \geq \sum_\theta u_i(x_i^r(\theta), \theta) \pi(\theta) + \varepsilon$$

for each $i \in Q$. Extracting further subsequences if necessary, we conclude that there exist allocations (y_i^*) and (x_i^*) such that $(y_i^r) \to (y_i^*)$ and $(x_i^r) \to (x_i^*)$ from which it follows that for each $i \in Q$,

$$\sum_\theta u_i(y_i^*(\theta), \theta) \pi(\theta) \geq \sum_\theta u_i(x_i^*(\theta), \theta) \pi(\theta) + \varepsilon.$$

Applying Proposition 1, it follows that for sufficiently large r and for each $i \in Q$, there exists $Z_i^r \subseteq J_r$ such that $|Z_i^r| = \lceil (1 - \varepsilon)r \rceil$ and

$$\left| \sum_{t^r \in T^r} \sum_{\theta \in \Theta} u_i(x_{i,s}^r(t^r), \theta) P^r(\theta, t^r) - \sum_{\theta \in \Theta} u_i(x_i^r(\theta), \theta) P(\theta) \right| \leq \frac{\varepsilon}{8}$$

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for each $s \in Z_i^r$. Now (renumbering the agents of each type in Q if necessary), consider a strongly conditionally independent sequence where the set of agents in the rth replica is given by

$$\Pi^r = \bigcup_{i \in Q} \{(i,s)|s \in Z_i^r\}.$$

Since (y_i^*) is an allocation for $E(\pi)$, we can apply Lemma1 to this special sequence of restricted PIE's and find incentive compatible PIE allocations $(z_{is}^r)_{(i,s) \in \Pi^r}$ such that for all sufficiently large r and for each $(i,s) \in \Pi^r$,

$$\sum_{t \in T} \sum_{\theta \in \Theta} u_i(z_{i,s}^r(t), \theta) P(\theta, t) \geq \sum_\theta u_i(y_i^*(\theta), \theta) \pi(\theta) - \varepsilon/2.$$
$$\geq \sum_\theta u_i(x_i^*(\theta), \theta) \pi(\theta) + \varepsilon/2.$$

Therefore, continuity implies that

$$\sum_{t \in T} \sum_{\theta \in \Theta} u_i(z_{i,s}^r(t), \theta) P(\theta, t) \geq \sum_{\theta \in \Theta} u_j(x_i^r(\theta), \theta) P(\theta, t) + \varepsilon/4$$

for each $(i,s) \in \Pi^r$ and for sufficiently large r. From the definition of Z_i^r, we conclude that

$$\sum_{t \in T} \sum_{\theta \in \Theta} u_i(z_{i,s}^r(t), \theta) P(\theta, t) \geq \sum_{t \in T} \sum_{\theta \in \Theta} u_j(x_{i,s}^r(t), \theta) P(\theta, t) + \varepsilon/8$$

for each $(i,s) \in \Pi^r$ and for sufficiently large r. This contradicts the assumption that $(\xi_{i,s}^r)$ is an allocation in the ex ante incentive compatible core of the PIE e^r.

7.4 Proof of Theorem C

Theorem C is a consequence of Theorem B and the following claim

Claim: For every $\alpha > 0$, there exists an $\eta > 0$ and an integer $\hat{r}$ such that, for all $r > \hat{r}$ and for each equal treatment allocation (x_{is}^r) in the $\eta-$core of $E^r(\pi)$, there exists a Walras equilibrium (y_i) of the auxiliary economy $E^1(\pi)$ satisfying

$$|v_i(x_{is}^r) - v_i(y_i)| < \alpha$$

for each $(i,s) \in N_r$.

Proof: Suppose not. Then there exists an $\alpha > 0$ and a strictly increasing sequence of positive integers $(r_k)_{k \geq 1}$ such that, for each k, there exists an equal treatment allocation $(x_{is}^{r_k})$ in the $\frac{1}{k}-$ core of $E^{r_k}(\pi)$ with $x_{is}^{r_k} = x_i^{r_k}$ for each i and s, and an type i_k such $|v_{i_k}(x_{i_k}^r) - v_{i_k}(y_{i_k})| \geq \alpha$ for all Walras equilibria (y_i) of the auxiliary economy $E(\pi)$. Let A_k denote the "projection" of the set of equal treatment allocations in the $\frac{1}{k}$- core of $E^{r_k}(\pi)$ onto the space of feasible allocations of $E(\pi)$. Therefore, $(x_i^{r_k}) \in A_k$

for each k. Choosing a subsequence if necessary, we may assume that $(x_i^{r_k}) \to (x_i^*)$ where (x_i^*) is a feasible allocation for $E(\pi)$. Since $A_{k+1} \subseteq A_k$, it follows that, for each m, $(x_i^{r_k}) \in A_m$ whenever $k \geq m$. Hence, $(x_i^*) \in A_m$ since A_m is closed. Therefore, $(x_i^*) \in \bigcap_{m=1}^{\infty} A_m$. Next, we show that (x_i^*) is a Walras equilibrium of $E(\pi)$. This requires only a slight modification of the proof of the Debreu-Scarf Theorem and we will follow Proposition 5.2 in Hildenbrand and Kirman (1988). Let

$$\psi(i) = \{z \in \Re^{\ell m} | v_i(z + w_i) > v_i(x_i^*)\}.$$

If

$$conv \left[\bigcup_{i \in N} \psi(i) \right] \cap int\ \Re_-^{\ell m} \neq \emptyset,$$

then there exists a positive integer M, positive integers $\beta_1, ...\beta_K$ summing to M, types $i_1, ..i_K$ in N and bundles $\zeta_1, .., \zeta_K$ with $\zeta_j \in \psi(i_j)$ for each j, such that

$$v_{i_j}(\zeta_j + w_{i_j}) > v_{i_j}(x_{i_j}^*)$$

for each j and

$$\sum_{j=1}^{K} \beta_j \left(\zeta_j + w_{i_j} \right) = \sum_{j=1}^{K} \beta_j w_{i_j}.$$

Choose k large enough so that

$$v_{i_j}(\zeta_j + w_{i_j}) > v_{i_j}(x_{i_j}^*) + \frac{1}{k}$$

for each j. Since $\sum_{j=1}^{K} (r_k \beta_j) \left(\zeta_j + w_{i_j} \right) = \sum_{j=1}^{K} (r_k \beta_j) w_{i_j}$, it follows that a coalition consisting of $r_k \beta_j$ agents of type i_j for each $j = 1, ..., K$ can $\frac{1}{k}$−block the (r_k−replication of) allocation (x_i^*). This contradicts the conclusion that $(x_i^*) \in \bigcap_{m=1}^{\infty} A_m$ and we conclude that $conv \left[\bigcup_{i \in N} \psi(i) \right] \cap int\ \Re_-^{\ell} = \emptyset$. Completing the proof of the Debreu Scarf theorem, it follows that (x_i^*) is a Walras equilibrium of $E(\pi)$. Since $(x_i^{r_k}) \underset{k \to \infty}{\to} (x_i^*)$, this contradicts the assumption that for each k there is an agent i_k such that $|v_{i_k}(x_{i_k}^r) - v_{i_k}(y_{i_k})| \geq \alpha$ for all Walras equilibria (y_i) of the auxiliary economy $E(\pi)$. This completes the proof of the Claim.

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