

Björnerstedt, Jonas; Westermarck, Andreas

Working Paper

## Stationary equilibria in bargaining with externalities

Working Paper, No. 2006:29

**Provided in Cooperation with:**

Department of Economics, Uppsala University

*Suggested Citation:* Björnerstedt, Jonas; Westermarck, Andreas (2008) : Stationary equilibria in bargaining with externalities, Working Paper, No. 2006:29, Uppsala University, Department of Economics, Uppsala,  
<https://nbn-resolving.de/urn:nbn:se:uu:diva-211154>

This Version is available at:

<https://hdl.handle.net/10419/83130>

**Standard-Nutzungsbedingungen:**

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

**Terms of use:**

*Documents in EconStor may be saved and copied for your personal and scholarly purposes.*

*You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.*

*If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.*

# Stationary Equilibria in Bargaining with Externalities\*

Jonas Björnerstedt<sup>†</sup>

Andreas Westermark<sup>‡</sup>

January 7, 2008

## Abstract

This paper studies infinite-horizon bargaining between a seller and multiple buyers when externalities are present. We extend the analysis in Jehiel & Moldovanu (1995a) by allowing for both pure and mixed equilibria. A characterization of the stationary subgame perfect equilibria in generic games is presented. Equilibria with delay exist only for strong positive externalities. Since each buyer receives a positive payoff when the seller makes an agreement with some other buyer, positive externalities induce a war of attrition between buyers.

**Keywords :** Bargaining, externalities, delay.

**JEL Classification :** C72, C78, D62.

---

\*Our work has benefitted greatly from discussions with Sven-Olof Fridolfsson, Lars Persson, Johan Stennek and from comments by a referee and an associate editor. We are also grateful for comments by participants at seminars/workshops at ESEM 2006 in Vienna and the International Conference on Game Theory 2006 at Stony Brook. The authors gratefully acknowledge financial support from the Jan Wallander and Tom Hedelius Foundation and Westermark also from the Swedish Council for Working Life and Social Research.

<sup>†</sup>Swedish Competition Authority, SE-103 85 Stockholm, Sweden. E-mail: jonas.bjornerstedt@kkv.se.

<sup>‡</sup>Corresponding author. Department of Economics, Uppsala University, Box 513, SE-751 20 Uppsala, Sweden. Fax (+46) 18 471 15 94, phone (+46) 18 471 15 94. E-mail: andreas.westermark@nek.uu.se.

# 1 Introduction

Many agreements in society are determined through bargaining. It is not uncommon that these agreements impose externalities on other potential buyers. For instance, if an exclusive patent right is sold to one of several manufacturing firms, externalities are imposed on the other firms, when these firms compete with each other and when the patent right affects the cost structure of the buying firm.

Another large strand of literature concerns the issue of delay in bargaining. One focus has been on asymmetric information; see e.g., Admati & Perry (1987), Cramton (1992) and others. The possibility of delay with perfect information has also been analyzed by e.g. Fernandez & Glazer (1990) and Cai (2000). In a couple of seminal papers, Jehiel & Moldovanu (1995a and 1995b) analyzed delay in bargaining with externalities with perfect information. Both games with a finite and an infinite horizon were analyzed, and the results depended on whether a final deadline existed. With an infinite horizon, only pure strategy equilibria are analyzed. The focus is on “simple” pure strategy equilibria. Since stationary equilibria in pure strategies do not always exist, they also analyze strategies with bounded recall, i.e., strategies that are more complicated than stationary ones, but where players have limited memory capacity.

In this paper, we analyze infinite-horizon bargaining between a seller and many buyers with perfect information in the presence of externalities. We analyze delay and provide a characterization of generic stationary subgame perfect equilibria in both mixed and pure stationary strategies.

The bargaining model is a generalization of the infinite horizon model in Jehiel & Moldovanu (1995a), allowing both for different discount factors and relative probabilities of bidding. In the model, the seller owns an indivisible object that can be sold to one of many potential buyers. The purchase of the good by one buyer imposes externalities on the other potential buyers. The externalities imposed on a buyer that does not acquire the good can vary between buyers and depend on the identity of the buyer that obtains the good. Bargaining takes place as follows. In each period, one of the buyers is randomly selected to bargain with the seller. The proposer is then randomly selected among the chosen buyer and the seller. The selected proposer offers a price which the respondent accepts or rejects. The game ends in case of acceptance, otherwise the negotiation proceeds to the next stage where a buyer is once more randomly drawn and so on. We assume there to be no deadline; thus, there is an infinite horizon.

Generically, there are four types of equilibria. When the externalities are greater than the surplus in the agreement, there are equilibria exhibiting hold-up characteristics, where the seller agrees with some set of buyers with a positive probability less than one. As in Jehiel &

Moldovanu (1995a), there is a hold-up problem. As the externalities are larger than the payoffs, all buyers would prefer that some other buyer makes an agreement with the seller. The hold-up is substantial in the sense that the expected time of agreement does not converge to zero as the time for a round of negotiation decreases to zero. There are also equilibria where the seller agrees with only one buyer with probability one – a buyer is singled out by the seller. Moreover, there are equilibria where the seller agrees with one buyer with probability one and with another with a positive mixed probability, converging to zero as discount factors converge to one. This equilibrium is in some respects similar to equilibria in two-person noncooperative bargaining games with outside options. Finally, there are equilibria where the seller agrees with one buyer with probability one and with several buyers with a positive probability smaller than one. An important condition determining which equilibrium types exist is whether the surpluses of agreement exceed or are smaller than the externalities. When the externalities dominate the surpluses, both single out and hold-up equilibrium types can exist. When surpluses dominate, hold-up equilibria can be shown not to exist.

Allowing for mixed equilibria thus gives different results than in the infinite horizon model in Jehiel & Moldovanu (1995a), more in line with their deadline model where delay only occurs with positive externalities. They also get delay when there are negative externalities, both when there is a deadline and with an infinite horizon. With an infinite horizon as in Jehiel & Moldovanu (1995a), there is cyclical delay when externalities are negative. These equilibria are nonstationary and are hence ruled out in this paper.

In section 2, the model is described and existence is proven. Section 3 defines genericity, section 4 characterizes the equilibria, section 5 describes conditions for when equilibria are unique and finally section 6 concludes the paper. All proofs are relegated to the appendix.

## 2 The Model

One seller bargains with a set  $N$  of buyers with  $n = |N| > 1$  on the sale of an indivisible good. The surplus of selling to buyer  $i$  is  $\pi_i > 0$ , with all other buyers  $j$  receiving their externality  $e_{j,i}$ . For notational convenience, we also define  $e_{i,i} = 0$ . We assume that in each round, all buyer seller pairs meet with equal probability.<sup>1</sup> Let  $r_S$  and  $r_B$  denote the discount rates for the seller and the buyers, respectively. We assume that  $r_i \in [\underline{r}_i, \bar{r}_i]$  with  $\underline{r}_i > 0$  and  $\bar{r}_i < \infty$  for  $i \in \{S, B\}$ .

---

<sup>1</sup>Arbitrary matching probabilities complicate the notation without qualitatively affecting the results. Moreover, the result on delay in proposition 2 does not depend on the fact that the matching probabilities are identical. The proposition must be modified so that the equilibrium probabilities for making acceptable offers are adjusted to take into account the fact that the matching probabilities are asymmetric; specifically, the equilibrium probabilities for making acceptable offers must be adjusted so that the actual agreement probabilities in proposition 2 remain unchanged. See also Jehiel & Moldovanu (1995b) for a motivation for assuming symmetric matching probabilities.

Let  $\Delta$  denote the amount of time that passes between two consecutive rounds. The seller has discount factor  $\delta_S = e^{-r_B \Delta}$  and the buyers discount factor  $\delta_B = e^{-r_B \Delta}$  between rounds. We let  $\rho = \frac{r_B}{r_S}$  denote the relative discount factor.<sup>2</sup> The seller makes a bid with probability  $\eta$ , and the buyer with probability  $1 - \eta$ . Let  $\Omega \subset R_+^{n+3} \times R^{n(n-1)}$  denote the set of parameters.

Let  $v_{S,i}$  and  $w_{S,i}$  denote the value for the seller when bidding and receiving a bid from buyer  $i$ , and  $v_{i,S}$  and  $w_{i,S}$  denote the value for buyer  $i$  when bidding and receiving a bid. Let  $p_{S,i}$  be the probability that the seller gives an acceptable bid to  $i$  when bidding and  $p_{i,S}$  the probability that  $i$  gives an acceptable bid. Defining  $p_i = \eta p_{S,i} + (1 - \eta) p_{i,S}$ , the value equations are

$$\begin{aligned} v_{S,i} &= (1 - p_{S,i}) w_{S,i} + p_{S,i} (\pi_i - w_{i,S}), \\ w_{S,i} &= \delta_S \left( \frac{\eta}{n} \sum_{j \in N} v_{S,j} + \frac{1 - \eta}{n} \sum_{j \in N} w_{S,j} \right), \\ v_{i,S} &= p_{i,S} (\pi_i - w_{S,i}) + (1 - p_{i,S}) w_{i,S}, \\ w_{i,S} &= \delta_B \frac{1}{n} ((1 - \eta) v_{i,S} + \eta w_{i,S}) + \delta_B \sum_{j \in N \setminus \{i\}} \frac{p_j}{n} e_{i,j} + \delta_B \sum_{j \in N \setminus \{i\}} \frac{1 - p_j}{n} w_{i,S}. \end{aligned} \tag{1}$$

To understand the equations, first consider  $v_{S,i}$ . When negotiating with  $i$  and giving an acceptable offer (with probability  $p_{S,i}$ ), it is sufficient to offer  $w_{i,S}$  to  $i$ . Since  $w_{S,i}$  is the continuation value in case of disagreement, the value  $v_{S,i}$  follows.  $v_{i,S}$  is determined by similar reasoning. To understand  $w_{S,i}$ , note that when rejecting a proposal by  $i$ ,  $S$  gets  $v_{S,j}$  with probability  $\frac{\eta}{n}$  and  $w_{S,j}$  with  $\frac{1 - \eta}{n}$ . Finally, consider  $w_{i,S}$ . When  $i$  rejects a proposal,  $i$  is selected to bargain with  $S$  with probability  $\frac{1}{n}$  giving  $(1 - \eta) v_{i,S} + \eta w_{i,S}$ . If some other player  $j$  is selected,  $i$  will receive  $e_{i,j}$  if  $S$  and  $j$  agree in the next period and  $w_{i,S}$  otherwise.

In characterizing equilibrium types, we divide the buyers into three sets  $A$ ,  $M$  and  $R$  where agreement occurs with probability one for  $a \in A$ , with a positive probability of less than one for  $m \in M$  and with zero probability for  $r \in R$ . For it to be profitable for the seller or  $a$  to make an acceptable offer, it is necessary that the following deviation condition is satisfied

$$\pi_a - w_{a,S} \geq w_{S,a}. \tag{2}$$

Similarly, for the seller and  $m$  to bid with  $0 < p_m < 1$ , the proposer must be indifferent between bidding and not

$$\pi_m - w_{m,S} = w_{S,m}. \tag{3}$$

---

<sup>2</sup>The analysis is in principle unaffected by a generalization to arbitrary sequences of discount factors. However, to keep the model as simple as possible, we proceed with the above definition.

It is also necessary that it is profitable to make unacceptable offers in negotiations with buyers where  $p_r = 0$ , i.e., that

$$\pi_r - w_{r,S} \leq w_{S,r}. \quad (4)$$

Existence is established along the lines in Westermarck (2003); see also Gomes (2005).

**Proposition 1** *There exists a stationary subgame perfect equilibrium for all  $\omega \in \Omega$ .*

All proofs are relegated to the appendix.

### 3 Genericity

One of the results of this paper is that many equilibrium types exist only for special parameter configurations.<sup>3</sup> More specifically, the set of  $\pi_i$  and  $e_{i,j}$  that supports these equilibrium types has strictly lower dimensionality than the full parameter space as  $\delta_B \rightarrow 1$  and  $\delta_S \rightarrow 1$ . This point is illustrated by the following example.

**Example 1** *Consider the case with two buyers, and conjecture an equilibrium where  $p_1 = p_2 = 1$ . Using this in (1) gives*

$$w_{a,S} = \delta_B \frac{(1 - \eta)(\pi_a - w_{S,a}) + e_{a,j}}{2 - \delta_B \eta},$$

for  $j \neq a$  and

$$w_{S,a} = \frac{\delta_S \eta}{2(1 - \delta_S(1 - \eta)) - \eta \delta_B} \frac{1}{2} ((2 - \delta_B)(\pi_1 + \pi_2) - \delta_B(e_{1,2} + e_{2,1})).$$

Using these in the condition for acceptance (2) gives

$$\pi_a - \frac{\delta_B}{2 - \delta_B} e_{a,j} \geq \frac{\delta_S \eta}{2(1 - \delta_S(1 - \eta)) - \eta \delta_B} \frac{1}{2} ((2 - \delta_B)(\pi_1 + \pi_2) - \delta_B(e_{1,2} + e_{2,1})),$$

for  $j \neq a$ . In the limit we have, setting  $a = 1, j = 2$  and  $a = 2, j = 1$  in the expression above

$$\pi_1 + e_{2,1} \geq \pi_2 + e_{1,2}, \quad (5)$$

$$\pi_2 + e_{1,2} \geq \pi_1 + e_{2,1}.$$

This implies that  $\pi_1 + e_{2,1} = \pi_2 + e_{1,2}$ , thus imposing an additional restriction on the parameters for this equilibrium to exist. Thus, agreement with both sellers with probability one cannot be a

---

<sup>3</sup>The concept of equilibrium types is formalized in detail below.

generic equilibrium. For equilibria with agreement with both buyers with probability 1 to exist for  $\delta_B, \delta_S$  close to one, the seller must essentially be indifferent between with whom the seller agrees. If not, the seller could simply wait for the best buyer. Moreover, when the discount factors are close to one, the region where the equilibrium exists is arbitrarily small.

Now, let us define generic equilibrium types. We classify equilibrium types according to how  $N$  is partitioned in the sets  $A$ ,  $M$  and  $R$ , respectively. Specifically, let  $\sigma$  denote a stationary strategy profile. Let  $\Phi(\sigma) = (|A|, |M|, |R|)$  denote the *equilibrium type* of  $\sigma$ , given that  $\sigma$  induces  $p_a = 1$  for  $a \in A$ ,  $p_m \in (0, 1)$  for  $m \in M$  and  $p_r = 0$  for  $r \in R$ . Let  $\Sigma(\omega, \Delta)$  denote the correspondence from the set of parameters  $\omega \in \Omega$  and  $\Delta$  to the (possibly empty) set of stationary equilibria for these parameters. Define

$$\Omega(u, \Delta) = \{\omega \in \Omega : \exists \sigma \in \Sigma(\omega, \Delta) \text{ such that } \Phi(\sigma) = u\}$$

as the set of parameter values generating the equilibrium type  $u$ , given  $\Delta$ . Let  $\lambda$  denote a Lebesgue measure of subsets of  $\Omega$ . We define genericity in terms of whether equilibria in the limit exist on a subset of  $\Omega$  with a positive measure.

**Definition 1** *The equilibrium type  $u$  is said to be generic if  $\lim_{\Delta \rightarrow 0} \lambda(\Omega(u, \Delta)) > 0$ .*

Note that genericity for equilibrium types is not defined in the strong sense that the equilibrium exists for almost all parameter values. It is sufficient that it exists on a set with a positive measure. Genericity is here defined for equilibrium types as  $\Delta \rightarrow 0$ . By continuity, the measure of non-generic equilibria can be made arbitrarily small for  $\Delta$  sufficiently close to zero.

## 4 Equilibrium Characterization

In this section, we characterize the generic equilibrium types. In general, there is a large number of equilibrium types. Specifically, any partition of the set of buyers in three sets  $A$ ,  $M$  and  $R$  is a possible equilibrium type. However, as will be shown in Proposition 5, the following four cases

Hold-up	$\Phi = (0,  M , n -  M )$ for $ M  > 1$
Single out	$\Phi = (1, 0, n - 1)$
Outside option	$\Phi = (1, 1, n - 2)$
Type IV	$\Phi = (1,  M , n -  M  - 1)$ for $ M  > 1$ .

are the generic equilibrium types. As we will see below, differences between surpluses and externalities are crucial for which equilibrium types that exist. Let  $D$  be the  $n \times n$  matrix where  $d_{i,j} = \pi_i - e_{i,j}$  for  $j \neq i$  and  $d_{i,i} = 0$ . For  $S \subset N$  and  $T \subset N$ , let  $D_{S,T}$  be the submatrix of  $D$  with rows from  $S$  and columns

from  $T$ . Similarly, let  $\pi$  be the  $n$  dimensional vector with  $\pi_i$  as the  $i$ 'th element and  $\pi_S$  as the subvector with elements from  $S$ . Similarly, with  $J$  and  $j$  denoting an  $n \times n$  matrix of ones and an  $n$  vector of ones, respectively, we define  $J_{S,T}$  and  $j_S$  as above.

**Definition 2** Say that  $D$  satisfies surplus dominance (SD) if  $d_{i,j} > 0$  for all  $i, j \in N$  such that  $j \neq i$ . Moreover,  $D$  satisfies externality dominance (ED) if  $d_{i,j} < 0$  for all  $i, j \in N$  such that  $j \neq i$ .

It is useful to rewrite the deviation conditions as in the following Lemma.

**Lemma 1** In equilibrium, the deviation conditions (2), (3) and (4) can be written as

$$w_{S,i} \leq \pi_a - \frac{\sum_{j \in A} e_{a,j} + \sum_{j \in M} p_j e_{a,j}}{n \frac{1-\delta_B}{\delta_B} + |A| - 1 + \sum_{j \in M} p_j} \quad (6)$$

$$w_{S,i} = \pi_m - \frac{\sum_{j \in A} e_{m,j} + \sum_{j \in M} p_j e_{m,j}}{n \frac{1-\delta_B}{\delta_B} + |A| + \sum_{j \in M \setminus \{m\}} p_j} \quad (7)$$

$$w_{S,i} \geq \pi_r - \frac{\sum_{j \in A} e_{r,j} + \sum_{j \in M} p_j e_{r,j}}{n \frac{1-\delta_B}{\delta_B} + |A| + \sum_{j \in M} p_j}, \quad (8)$$

where  $0 \leq p_m \leq 1$ .

Note that, in the limit, as long as the denominators in (6) - (8) are well-defined, these conditions can be rewritten in terms of differences between surpluses and externalities,  $d_{i,j}$ . Before showing genericity, we establish some conditions that guarantee existence of the three first equilibrium types. First, we focus on equilibria where  $A$  is empty. The following proposition illustrates conditions for hold-up equilibria to exist.

**Proposition 2** If  $D_{M,M}$  is invertible,

$$D_{M,M}^{-1} \cdot \pi_M \ll 0 \quad (9)$$

and

$$\pi_R \ll D_{R,M} \cdot D_{M,M}^{-1} \cdot \pi_M, \quad (10)$$

there is a  $\bar{\Delta} > 0$  such that for all  $\Delta < \bar{\Delta}$  there exists an equilibrium with  $0 < p_m < 1$  for  $m \in M \subseteq N$  with  $|M| > 1$  and  $p_r = 0$  for  $r \in R = N \setminus M$ . If either (9) or (10) is strictly violated, there is a  $\bar{\Delta} > 0$  such that for all  $\Delta < \bar{\Delta}$  no equilibrium exists.

The equilibrium is inefficient as, in the limit (as  $\Delta \rightarrow 0$ ), the expected amount of time that passes until agreement is

$$-\frac{1}{r_B j_M} \frac{1}{D_{M,M}^{-1} \cdot \pi_M}. \quad (11)$$



The equilibrium payoff for the seller is  $v_{S,i} = w_{S,i} = 0$  for all  $i$ . This follows by noting that when the seller is proposing to either  $m \in M$  or  $r \in R$ , we have  $v_{S,m} = w_{S,m}$  and  $v_{S,r} = w_{S,r}$ , respectively, and using the value equation for  $w_{S,i}$  in (1). From the deviation condition (3), the equilibrium payoff for buyers in  $M$  is then  $v_{m,S} = w_{m,S} = \pi_m$ . Finally, the expected payoff for each buyer  $r \in R$  is a probability-weighted average of externalities imposed when the seller agrees with  $m \in M$ ; see (8). Using  $w_{S,i} = 0$  in (7), the equilibrium probabilities are  $p_M = -n \frac{1-\delta_B}{\delta_B} D_{M,M}^{-1} \cdot \pi_M$  and expression (10) implies that the reject condition (8) holds.

The reason for delay is the following. Since the externalities are larger than the surpluses, if buyers could choose between getting the entire surplus of agreement or getting the externality, they prefer the externality. This generates a hold-up problem, which precludes agreement with probability one. Moreover, the delay is substantial and hence inefficiencies arise. When  $D$  satisfies SD, there is no hold up equilibrium. The reason is that, from (3) and (7),  $w_{m,S}$  is (almost) a weighted average of externalities.<sup>4</sup> If  $\pi_m > e_{m,j}$  for all  $j \in N \setminus \{m\}$  then, for  $\delta_B$  close to one,  $w_{m,S} < \pi_m$ , implying that the seller gains by making an acceptable offer with probability one; see (2). In the case with two buyers, the equilibrium can easily be illustrated.

**Example 2** *Hold-up equilibrium when  $n = |M| = 2$ . From proposition 2 above, the equilibrium probabilities are*

$$\begin{aligned} p_1 &= 2 \frac{1-\delta_B}{\delta_B} \frac{\pi_2}{e_{2,1} - \pi_2}, \\ p_2 &= 2 \frac{1-\delta_B}{\delta_B} \frac{\pi_1}{e_{1,2} - \pi_1}. \end{aligned}$$

*The probabilities are positive when  $e_{2,1} > \pi_2$  and  $e_{1,2} > \pi_1$ . Moreover, limit equilibrium delay is*

$$\frac{1}{r_B} \frac{1}{\frac{\pi_1}{e_{1,2} - \pi_1} + \frac{\pi_2}{e_{2,1} - \pi_2}}.$$

*Note that equilibrium delay increases in externalities, since there is an increase in the payoff for buyer  $i$  when the seller agrees with the other buyer.*

As pointed out by Jehiel & Moldovanu (1995a), a model with this property is a situation where a single individual must pay for a public good; see Bliss & Nalebuff (1984) for an analysis of such a model with imperfect information. This interpretation naturally leads to positive

---

<sup>4</sup>By (7), the weights sum to less than one.

externalities, since if one agents pays for a public good, all other agents benefit. In such a setup, it is reasonable that externalities can be larger than own surpluses.<sup>5</sup>

Now consider equilibria where  $A$  is nonempty. The following proposition describes conditions for single out equilibria to exist.

**Proposition 3** *If*

$$\pi_r - e_{r,a} < \frac{\eta\pi_a}{\eta + \rho(1-\eta)} \quad (12)$$

*for all  $r \neq a$  then there is a  $\bar{\Delta} > 0$  such that for all  $\Delta < \bar{\Delta}$ , there exists an equilibrium with  $p_a = 1$  for some  $a \in N$ , and  $p_r = 0$  for all  $r \neq a$ . If (12) is strictly violated, there is a  $\bar{\Delta} > 0$  such that for all  $\Delta < \bar{\Delta}$  no equilibrium exists.*

Since  $p_a = 1$  and  $p_r = 0$  for all other buyers  $r$ , we can think of the equilibrium as a situation where the seller only bargains with  $a$  and the surplus consists of  $\pi_a$ , giving the seller  $\frac{\eta}{\eta + \rho(1-\eta)}\pi_a$ . Note that if  $\rho = 1$ , the seller gets a share of the surplus corresponding to the probability of being selected as proposer, i.e.,  $\eta$ . If the seller were to deviate and instead agree with  $r$ , the payoff would be  $\pi_r - e_{r,a}$ . Such deviations are unprofitable by (12). Note that if  $D$  satisfies ED, then, for all  $a \in N$  we have  $e_{r,a} > \pi_r$  for all  $r \neq a$  and hence, single out equilibria exist. The following proposition establishes conditions on parameters for the outside option equilibria.

**Proposition 4** *If*

$$\infty > \frac{(\pi_m - e_{m,a}) - \frac{\eta}{\eta + \rho(1-\eta)}\pi_a}{(\pi_a - e_{a,m}) - (\pi_m - e_{m,a})} > 0, \quad (13)$$

$$\pi_m > e_{m,a} \quad (14)$$

*and*

$$\pi_r - e_{r,a} < \pi_m - e_{m,a}, \quad (15)$$

*for all  $r \neq a, m$  there is a  $\bar{\Delta} > 0$  such that for all  $\Delta < \bar{\Delta}$ , there exists an equilibrium with  $p_a = 1$ ,  $p_m > 0$  for some  $a, m \in N$  and  $p_r = 0$  for all  $r \neq a, m$ . If any of (13), (14) or (15) is strictly violated, there is a  $\bar{\Delta} > 0$  such that for all  $\Delta < \bar{\Delta}$  no equilibrium exists.*

*We have  $p_m \rightarrow 0$  as  $\Delta \rightarrow 0$ .*

The inequalities in (13) ensure that the probability  $p_m \in (0, 1)$ . As  $\Delta \rightarrow 0$  (and hence  $\delta_B, \delta_S \rightarrow 1$ ), it can be shown that  $p_m$  converges to zero. The relationship between the equilibria in

---

<sup>5</sup> Another motivation for large externalities is given by the discussion (although the example involves negative externalities) on the Ukrainian nuclear arsenal in Jehiel, Moldovanu & Stacchetti (1996).

Propositions 3 and 4 can best be understood in relation to bargaining with outside options where the outside option is to agree with  $m$  with probability one. From the proof of the proposition, the equilibrium payoff of  $S$  in proposition 4 is  $\pi_m - e_{m,a}$ . When making a comparison with the equilibria in proposition 3, the pure strategy equilibria only exist when the Rubinstein-Ståhl split  $\frac{\eta}{\eta+\rho(1-\eta)}\pi_a$  is greater than the "outside option" of agreeing with  $m$  first:  $\pi_m - e_{m,a}$ . If not, then, in case (13) holds, the seller gets the outside option payoff. Moreover, (14) states that the payoff of  $S$  is positive and (15) that  $S$  does not want to deviate and agree with  $r$ . Note that if  $D$  satisfies ED, then (14) is violated and hence, outside option equilibria do not exist. To understand why  $p_m \in (0, 1)$  in equilibrium, first consider the case when  $p_m = 1$ . From the discussion of example 1 above, if  $\pi_a - e_{a,m} > \pi_m - e_{m,a}$ ,  $S$  will never want to agree with  $m$ . Thus,  $S$  gains by reducing the probability  $p_m$ . In the case where  $p_m = 0$ , the payoff to  $S$  is  $\frac{\eta}{\eta+\rho(1-\eta)}\pi_a$ . As (12) is violated from (13),  $S$  gains by agreeing with  $m$  to obtain  $\pi_m - e_{m,a}$ . Thus, to ensure that neither of these deviations are profitable, by continuity we have  $p_m \in (0, 1)$ . As shown by the following example, none of the equilibria in propositions (2) - (4) need to exist.

**Example 3** *Nonexistence of a hold-up, single out or outside option equilibrium. We assume that  $5 > \pi_i > 0$ , equal discount factors and*

$$D = \begin{pmatrix} 0 & 2 & 2 \\ 1 & 0 & 4 \\ \frac{5}{2} & 1 & 0 \end{pmatrix}. \quad (16)$$

*Assume that  $\eta < 0.2$ . It is easily verified that (12) is violated for all  $a \in N$  and all  $r \neq a$ . Moreover, we have  $\pi_i > e_{i,j}$  for all  $i \in N$  and  $j \neq i$  implying that (9) is violated. Thus, there are no single out and hold-up equilibria. To check whether there are outside option equilibria, since (15) holds and there is agreement with some  $a$  with probability 1, buyer  $m$  must be the buyer solving*

$$m = \arg \max_i \pi_i - e_{i,a} = \arg \max_i d_{i,a}.$$

*Thus, if  $a = 1$  then  $m = 3$ , if  $a = 2$  then  $m = 1$  and if  $a = 3$  then  $m = 2$ . In addition, (13) must hold. Since  $\eta < 0.2$ , the numerator of the ratio in (13) is positive. However, for all possible choices of  $a$ , the denominator is negative since  $d_{m,a} > d_{a,m}$ , implying that there is no outside option equilibrium. As can easily be checked, the following candidate is an equilibrium (when  $\delta_B, \delta_S \rightarrow 1$ );  $p_1 = 1$ ,  $p_2 = \frac{1}{2}$  and  $p_3 = \frac{1}{2}$ . To check this, use expression (7) with  $a = 1$  and take limits.*

The next proposition shows that the equilibrium types in propositions 2 - 4 and equilibria

of Type IV are the only generic equilibrium types.

**Proposition 5** *The generic equilibrium types are the following;*

1. *Hold-up:*  $u_H = (0, i, n - i)$  for all  $1 < i \leq n$ .
2. *Single out:*  $u_S = (1, 0, n - 1)$  for all  $i \in N$ .
3. *Outside option:*  $u_O = (1, 1, n - 2)$  for all  $i \in N$  and  $j \neq i$ .
4. *Type IV:*  $u_{IV} = (1, i, n - i - 1)$  for some  $1 < i \leq n$ .

The equilibrium types ruled out in proposition 5 are those where there is agreement with certainty with more than one buyer, i.e.,  $|A| > 1$ . The reason that these equilibrium types are not generic is that, in the limit, the condition for acceptance (2) holds with equality for all buyers in  $A$ , rendering additional restrictions on the parameter space; see also the discussion in Example 1. This is in contrast to the fact that equilibrium types can exist when there are multiple buyers who agree with a positive probability of less than one (i.e.,  $|M| > 1$ ). The reason why these structures are generic is that even though the condition for making acceptable offers, (3), also holds with equality, the probabilities are not constrained to be one, implying enough degrees of freedom for adjustment of probabilities when parameters are changed. The reason why there cannot be an equilibrium type when  $A$  is empty and where the seller agrees with a unique buyer  $m$  with  $p_m \in (0, 1)$  is that it takes at least two buyers in  $M$  for a war of attrition to occur.

The equilibrium types in the above proposition only exist generically. From the non-generic case of example 1, when equality holds in equations (5), it can be shown that none of the generic equilibrium types exist. For a further discussion, see Björnstedt & Westermarck (2006).

The reason for the delay in Jehiel & Moldovanu (1995a) with positive externalities and a finite horizon is that the price that buyers are willing to pay increases, the closer to the deadline one gets. In the last period, all buyers are willing to pay their valuation (Jehiel & Moldovanu assumes  $\eta = 1$ ). In the period just before the deadline, buyers are not willing to pay as much, since if some other buyer gets the good in the last period, the buyer ends up with a positive payoff, due to positive externalities. Thus, prices increase the closer to the deadline one gets, thus inducing the seller to wait. The argument here is slightly different. Although there is no deadline, buyers want to wait to agree since if some other buyer ends up with the good, that buyer receives a positive payoff since externalities are positive. Thus, there is a war of attrition between buyers. Jehiel & Moldovanu (1995b) also get a delay when there are negative

externalities. The reason is illustrated by their Example 3.1 with three buyers. In that example, buyer 3 suffers no externalities at all, while the first two buyers suffer large externalities if the third buyer obtains the object. The seller would then like to threaten the first two buyers with selling to buyer 3 to obtain a higher price from the first two buyers. However, this threat is not credible until the last period. Buyers 1 and 2 are then willing to pay a fairly high amount in the period before the last. Buyers 1 and 2 then face a war of attrition, each trying to wait for the other to buy the object. As shown by our result, there is no delay with negative externalities and thus, the existence of a deadline is crucial for such a war of attrition to occur. With an infinite horizon as in Jehiel & Moldovanu (1995a), there is cyclical delay. The intuition for this is very similar to the deadline effect with a finite horizon. These equilibria are nonstationary and are hence ruled out in this paper.

Jehiel & Moldovanu (1995a) introduce the well-defined buyer property, i.e., there is agreement with probability one with a single buyer. The single out equilibrium trivially satisfies this criterion. From example 2, it is easily seen that the hold-up equilibrium does not satisfy the property. The outside option equilibrium satisfies the well-defined buyer property in the limit, as  $p_m \rightarrow 0$ . For the equilibria of type IV, the property is violated as all probabilities do not converge to zero. To see this, suppose by contradiction that all probabilities  $p_m$  converge to zero in the limit. Condition (7) for  $m \in M$  can be rewritten as (with  $w_{S,m} = w_{S,i}$  for all  $i$ ), in the limit

$$w_{S,m} = \frac{\pi_m - e_{m,a} + \sum_{j \in M \setminus \{m\}} p_j (\pi_m - e_{m,j})}{1 + \sum_{j \in M \setminus \{m\}} p_j}. \quad (17)$$

If  $p_m \rightarrow 0$  for all  $m \in M$ , we get  $\pi_m - e_{m,a} = w_{S,m}$  in the limit and hence, for  $k, m \in M$  we have

$$\pi_k - e_{k,a} = \pi_m - e_{m,a},$$

thus establishing non-genericity. Generically, there is thus agreement with positive probability with at least two buyers in the limit, violating the well-defined buyer property. Note that, for generic equilibria, we can then write (6) and (8) as, in the limit,

$$w_{S,i} \leq \frac{\sum_{j \in A} (\pi_a - e_{a,j}) + \sum_{j \in M} p_j (\pi_a - e_{a,j})}{\sum_{j \in M} p_j} \quad (18)$$

$$w_{S,i} \geq \frac{\sum_{j \in A} (\pi_r - e_{r,j}) + \sum_{j \in M} p_j (\pi_r - e_{r,j})}{1 + \sum_{j \in M} p_j}. \quad (19)$$

It is possible to establish restrictions on parameters that ensure the existence of equilibria of type IV. From example 3 above, any pure strategy equilibrium is bilaterally inefficient, since for

all  $a$ , there is some  $j$  such that  $d_{j,a} > d_{a,j}$ . We then let

$$i^*(j, K) = \arg \max_{i \in K} d_{i,j}.$$

**Definition 3** We say that  $D$  is bilaterally inefficient in  $K$  if we have  $d_{i^*(j, K), j} > d_{j, i^*(j, K)}$  for all  $j \in K$ .

Note that this condition implies that the denominator (13) is negative and hence, only bilaterally inefficient outside option equilibria can exist.

**Proposition 6** If  $\eta > 0$ ,  $D$  satisfies SD,  $D$  is bilaterally inefficient in  $K$  and

$$\max_{k \in K} \min_{j \in K \setminus \{k\}} d_{k,j} > \max_{r \in R} \max_{j \in K} d_{r,j} \quad (20)$$

holds then there is an equilibrium where  $|A| = 1$  and  $|M| > 1$ . If  $D$  satisfies ED, there is no equilibrium of type IV.

Which equilibrium types that exist depend on whether  $D$  satisfies SD or ED. From the discussion following proposition 2, using SD or ED in conditions (12) and (14), and using proposition 6 gives the following result.

**Corollary 1** When  $D$  satisfies ED, only hold-up and single out equilibria can exist. When  $D$  satisfies SD, no hold-up equilibrium exists, implying that there is some buyer  $i$  with  $p_i = 1$ .

Finally, consider the case when all externalities are zero. Then, the equilibrium allocation entails no delay and is Walrasian in the limit, in the sense that the buyer with the largest valuation buys the good.<sup>6</sup> With an appropriate renumbering of buyers, let 1 denote the buyer with the largest surplus and let 2 denote the buyer with the second largest surplus. We have the following corollary to propositions 3-5.<sup>7</sup>

**Corollary 2** When  $e_{i,j} = 0$  for all  $i, j \in N$ , only the equilibria in propositions 3 and 4 exist. We always have  $p_1 = 1$ . If  $p_m > 0$  for  $m \neq 1$ , then  $m = 2$ .

---

<sup>6</sup>This observation is also mentioned by Jehiel & Moldovanu (1995b).

<sup>7</sup>Hendon & Tranæs (1991) analyze a model where there is one seller that sells an indivisible good to one of two buyers. All players have the same discount factor  $\delta$  and selection probabilities are symmetric. The first buyer has valuation 1 and the second buyer has valuation  $R > 1$ . There are no externalities. Basically, their model is a special case of our model and the results in their paper are in line with corollary 2.

## 5 Uniqueness and multiplicity

The following condition is useful to show uniqueness of equilibrium. Suppose that externalities are negative and that there is some player  $k$  such that  $\pi_k - e_{k,l} > \pi_i - e_{i,j}$  for all  $l, i, j \in N$ . First, there is no equilibrium as in proposition 6. To see this, note that expression (18) holds with equality in the limit and that, from expression (17), probabilities serve as a weighting function of the payoff differences  $\pi_i - e_{i,j}$ ; see also (3). Then, it is easily seen that if there is some player  $k$  such that  $\pi_k - e_{k,l} > \pi_i - e_{i,j}$  for all  $l, i, j \in N$ , there is no equilibrium as in proposition 6. From (17) - (19) we have  $k \in A \cup M$  and  $w_{S,k} > w_{S,j}$  for all  $j \in A \cup M$  such that  $j \neq k$ . Since, from the proof of Proposition 5, (18) holds with equality in the limit, we have  $w_{S,a} = w_{S,m}$  for all  $m \in M$ , a contradiction. Second, in equilibrium there must be agreement with player  $k$  with probability one. Suppose that this is not the case. Then condition (12) is violated for  $k = r$ , since

$$\pi_a < \pi_a - e_{a,k} < \pi_k - e_{k,a}.$$

Moreover, condition (15) is violated for  $k = r$ . If  $k = m$  in proposition (4), then condition (13) is violated, since the numerator is positive, while the denominator is negative. Moreover, since  $a = k$ , there is a unique player

$$m = \arg \max_{j \in N \setminus \{k\}} \{\pi_j - e_{j,a}\},$$

such that condition (15) holds. Thus, there are only two possible equilibria. Either, we have  $p_k = 1$  and  $p_r = 0$  for all  $r \in N \setminus \{k\}$  or  $p_k = 1$ ,  $p_m \in (0, 1)$  and  $p_r = 0$  for all  $r \in N \setminus \{k, m\}$ . Moreover, if (12) holds then (13) is violated and vice versa. Using propositions (3) and (4), there is a unique equilibrium for  $\Delta$  close to zero, unless  $\pi_m - e_{m,k} = \frac{\eta}{\eta + \rho(1-\eta)} \pi_k$ . We have

**Proposition 7** *Suppose that all externalities are negative, there is some player  $k$  such that  $\pi_k - e_{k,l} > \pi_i - e_{i,j}$  for all  $l, i, j \in N$  and  $\pi_m - e_{m,k} \neq \frac{\eta}{\eta + \rho(1-\eta)} \pi_k$ , then there is some  $\bar{\Delta} > 0$  such that, for all  $\Delta < \bar{\Delta}$ , there is a unique equilibrium.*

Other conditions also have implications for the number of equilibria. If  $D$  satisfies ED, there are multiple equilibria. Condition (12) is satisfied for all  $i \in N$ , implying that there are  $n$  single out equilibria. Moreover, if all externalities are negative, there is one single out equilibrium at most, as can be seen from (12). If this equilibrium exists, we must have  $a = \arg \max_i \pi_i$ , since any other choice of  $a$  violates (12).

## 6 Concluding remarks

In the paper, a model where a seller bargains with multiple buyers when externalities are present is analyzed. Restricting the attention to generic equilibrium types and allowing for mixed strategies makes the analysis of bargaining with externalities fairly simple. We characterize the equilibria and show that delay only occurs when externalities are positive, in contrast to Jehiel & Moldovanu (1995a). The reason for the delay is that buyers prefer the externalities to agreeing with the seller, thus generating a war of attrition. The paper by Gomes & Jehiel (2005) also has a model allowing for externalities. Specifically, their model has a finite number of agents and a finite number of states with an exogenous rule prescribing how states can be changed. Their setup allows for more transitions than the model presented here. Consider an example with two buyers. Then, there are three possible states,  $a_S, a_1$  and  $a_2$ , where  $a_S$  is the state where the seller owns the good and  $a_i$  the state where buyer  $i$  owns the good. The only allowed transitions where ownership changes<sup>8</sup> in our model are  $a_S \rightarrow a_1$  and  $a_S \rightarrow a_2$ . The transition  $a_1 \rightarrow a_2$  where one buyer sells the good to the other buyer is not allowed. Note that this implicitly violates assumption 3 on page 633. One way of thinking about this is that resale is not allowed in our model, while it is allowed in Gomes & Jehiel (2005).

---

<sup>8</sup>Naturally, in their paper, the transition where the owner keeps the good, i.e.,  $a_i \rightarrow a_i$  is also allowed.



## A Appendix

### Proof of Proposition 1:

We now show existence of equilibrium. To do this, we demonstrate that the following strategy profile is an equilibrium. First, respondents always accept proposals equal to their continuation payoff. Second, the firm and workers optimally choose whether to make acceptable offers or not, allowing for mixed strategies. Thus, the firm and worker, respectively, choose  $p_{S,i} \in [0, \eta]$  and  $p_{i,S} \in [0, 1 - \eta]$ . A probability  $p_{S,i} \in (0, \eta)$  or  $p_{i,S} \in (0, 1 - \eta)$  is interpreted to imply that the firm or worker makes an acceptable offer with some positive probability less than one.

Here, let  $p_S = (p_{S,i})_{i \in N}$ ,  $p_B = (p_{i,S})_{i \in N}$  and  $p = (p_S, p_B)$ . Let  $P_S = \{p_S \in \mathbb{R}_+^n \mid p_{S,i} \leq \eta\}$  and  $P_B = \{p_B \in [0, 1 - \eta]^n\}$  denote the set of possible probabilities for the seller and buyer  $i$ , respectively.

Now we define a mapping that enables us to find an equilibrium. Let  $Q = [0, k_2] \times [k_1, k_2]^n$  where

$$k_1 = -n \max\{e_{i,k}, 0\} - \sum_{i \in N} \pi_i$$

$$k_2 = \sum_{i \in N} \pi_i + n \max\{e_{i,k}, 0\}.$$

and  $X = \times_{i \in N} ([0, \eta] \times [0, 1 - \eta])$ . Furthermore, let  $E = Q \times X$ . Note that  $E$  is compact and convex. For some  $q \in Q$  and  $x \in X$  we define

$$\mu_S(q, x) = \max_{p_S \in P_S} \sum_{i \in N} p_{S,i} [\delta_S \pi_i - \delta_S q_i] + \left(1 - \sum_{i \in N} p_{S,i}\right) \delta_S q_S \quad (21)$$

and

$$\begin{aligned} \mu_i(q, x) = & \max_{p_{i,S} \in P_B} p_{i,S} [\delta_B \pi_i - \delta_B q_S] + x_{S,i} \delta_B q_i + \left(1 - \sum_{j \neq i} [x_{S,j} + x_{j,S}]\right) \delta_B q_i \\ & - [x_{S,i} + p_{i,S}] \delta_B q_i + \sum_{j \neq i} [x_{S,j} + x_{j,S}] \delta_B e_{i,j}. \end{aligned} \quad (22)$$

Note that these are continuation payoffs before the proposer has been selected. Also, let

$$\alpha_S(q, x) = \arg \max_{p_S \leq P_S} \sum_{i \in N} p_{S,i} [\delta_S \pi_i - \delta_S q_i] + \left(1 - \sum_{i \in N} p_{S,i}\right) \delta_S q_S, \quad (23)$$

let  $\alpha_{S,i}(q, x)$  be the  $i$ 'th element of  $\alpha_S(q, x)$  and let

$$\begin{aligned} \alpha_{i,S}(q, x) = \arg \max_{p_{i,S} \in P_B} & p_{i,S} [\delta_B \pi_i - \delta_B q_S] + x_{S,i} \delta_B q_i + \left( 1 - \sum_{j \neq i} [x_{S,j} + x_{j,S}] \right) \delta_B q_i \\ & - [x_{S,i} + p_{i,S}] \delta_B q_i + \sum_{j \neq i} [x_{S,j} + x_{j,S}] \delta_B e_{i,j}. \end{aligned} \quad (24)$$

Finally, we let

$$\Phi(q, x) = ((\mu_i(q, x))_{i \in N}, (\alpha_{S,i}(q, x), \alpha_{i,S}(q, x))_{i \in N}).$$

Thus,  $\Phi$  is a mapping consisting of continuation payoffs and optimal choices of the players, given  $q$  and  $x$ .

We claim that a fixed point of  $\Phi$  gives equilibrium continuation payoffs and strategies. To see that a fixed point of  $\Phi$  gives equilibrium continuation payoffs and strategies, first note that, if  $w_{S,i} = \mu_S(q, x) = q_S$ ,  $w_{i,S} = \mu_i(q, x) = q_i$ ,  $p_{S,i} = \alpha_{S,i}(q, x) = x_{S,i}$ , and  $p_{i,S} = \alpha_{i,S}(q, x) = x_{i,S}$  for all  $i \in N$ , then (21) and (22) are the same as the value equations (for respondent payoffs). Second, (21), (22), (23) and (24), respectively, imply that the seller and the buyers optimally choose whether to make an acceptable offer or not. Moreover, by construction, respondents choose optimally.

By the maximum theorem under convexity,  $\alpha_S(q, x)$  and  $\alpha_{i,S}(q, x)$  are upper-hemicontinuous, convex-valued and compact-valued correspondences on  $E$  and  $\mu_i(q, x)$  is a continuous function on  $E$  for all  $i \in N$ . Thus,  $\Phi$  is an upper-hemicontinuous, convex-valued and compact-valued correspondence. Then the Kakutani fixed-point theorem implies that a fixed point exists. ■

Now, let us turn to equilibrium characterization in section 4.

Note that since the right-hand side of the value equation for  $w_{S,i}$  is the same for all  $i$ , we have  $w_{S,i} = w_{S,j}$  for all  $i$  and  $j$ . Hence, we can write

$$w_{S,i} = \frac{\delta_S \eta}{1 - \delta_S (1 - \eta)} \frac{1}{n} \sum_{j \in N} v_{S,j}. \quad (25)$$

In negotiations with  $r \in R$ ,  $m \in M$  and  $a \in A$  we have, from value equations (1) and the indifference condition (3),

$$v_{S,a} = \pi_a - w_{a,S} \quad (26)$$

$$v_{a,S} = \pi_a - w_{S,a},$$

$$v_{S,m} = w_{S,m} \quad (27)$$

$$v_{m,S} = \pi_m - w_{S,m}$$

and

$$v_{S,r} = w_{S,r} \quad (28)$$

$$v_{r,S} = w_{r,S}.$$

Using (27), (28) in (3) and (26) gives

$$\begin{aligned} w_{r,S} &= \frac{\sum_{j \in A} e_{r,j} + \sum_{j \in M} p_j e_{r,j}}{n \frac{1-\delta_B}{\delta_B} + |A| + \sum_{j \in M} p_j} \\ w_{m,S} &= \frac{(1-\eta)(\pi_m - w_{S,m}) + \sum_{j \in A} e_{m,j} + \sum_{j \in M} p_j e_{m,j}}{n \frac{1-\delta_B}{\delta_B} + 1 - \eta + |A| + \sum_{j \in M \setminus \{m\}} p_j} \\ w_{a,S} &= \frac{(1-\eta)(\pi_a - w_{S,a}) + \sum_{j \in A} e_{a,j} + \sum_{j \in M} p_j e_{a,j}}{n \frac{1-\delta_B}{\delta_B} - \eta + |A| + \sum_{j \in M} p_j}. \end{aligned} \quad (29)$$

We also have the following result.

**Lemma 2** *Any equilibrium satisfies*

$$w_{S,i} = \frac{n \frac{1-\delta_B}{\delta_B} \sum_{a \in A} \pi_a + \sum_{a \in A} \left( (|A| - 1 + \sum_{m \in M} p_m) \pi_a - \sum_{j \in A} e_{a,j} - \sum_{m \in M} p_m e_{a,m} \right)}{\left( \frac{1-\delta_S}{\delta_S} \frac{n}{\eta} L + |A| n \frac{1-\delta_B}{\delta_B} \right) + (|A| + \sum_{m \in M} p_m - 1) |A|} \quad (30)$$

if  $|A| > 0$  and  $w_{S,i} = 0$  if  $|A| = 0$ .

**Proof:** From (25) we have, using (26), (27), (28) and  $w_{S,i} = w_{S,j}$ ,

$$w_{S,i} = \frac{\delta_S \eta}{1 - \delta_S (1 - \eta)} \frac{1}{n} \left( \sum_{a \in A} (\pi_a - w_{a,S}) + (n - |A|) w_{S,i} \right). \quad (31)$$

If  $|A| = 0$  then

$$w_{S,i} = 0 \quad (32)$$

and if  $|A| > 0$

$$w_{S,i} = \frac{1}{\frac{1-\delta_S}{\delta_S} \frac{n}{\eta} + |A|} \sum_{a \in A} (\pi_a - w_{a,S}). \quad (33)$$

We can rewrite  $w_{a,S}$  in (29) as

$$w_{a,S} = \frac{H}{L} - \frac{(1-\eta)}{L} w_{S,i}$$

where

$$H = (1 - \eta) \pi_a + \sum_{j \in A} e_{a,j} + \sum_{j \in M} p_j e_{a,j},$$

$$L = n \frac{1 - \delta_B}{\delta_B} - \eta + |A| + \sum_{j \in M} p_j.$$

Using this in (33) gives

$$w_{S,i} = \frac{\sum_{a \in A} (L \pi_a - H)}{\left( \frac{1 - \delta_S}{\delta_S} \frac{n}{\eta} + |A| \right) L - (1 - \eta) |A|}. \quad (34)$$

Using the definition of  $H$  and  $L$  in (34) gives (30). ■

**Proof of Lemma 1.** Using (29) in the deviation conditions (2), (3) and (4) and solving for  $w_{S,i}$  establishes (6) - (8). ■

**Proof of Proposition 2:**

**Step 1:** Using  $w_{S,i} = 0$  from Lemma 2 when  $|A| = 0$  in (7) gives

$$n \frac{1 - \delta_B}{\delta_B} \pi_m = \sum_{j \in M \setminus \{m\}} e_{m,j} p_j - \pi_m \sum_{j \in M \setminus \{m\}} p_j. \quad (35)$$

Note that the condition  $|M| > 1$  in the proposition follows, since when  $|M| = 1$  we have  $(1 - \delta_B) \pi_m = 0$ , contradicting  $\pi_m > 0$ . In matrix form, the above expression is

$$n \frac{1 - \delta_B}{\delta_B} \pi_M = -D_{M,M} \cdot p_M. \quad (36)$$

Since the matrix on the right-hand side is invertible by assumption, we get

$$p_M = -n \frac{1 - \delta_B}{\delta_B} D_{M,M}^{-1} \cdot \pi_M. \quad (37)$$

Thus, from assumption (9),  $p_M \gg 0$  for all  $\delta_B, \delta_S < 1$ . Since  $\delta_B = e^{-r_B \Delta}$ ,  $\Delta \rightarrow 0$  implies  $p_M \rightarrow 0$ . Thus, there exists a  $\bar{\Delta} > 0$  such that  $p_m < 1$  for all  $m \in M$  and  $\Delta < \bar{\Delta}$ .

For the reject condition (8) to hold, using  $w_{S,r} = 0$ , we get

$$n \frac{1 - \delta_B}{\delta_B} \pi_r \leq \sum_{k \in M} p_k e_{r,k} - \pi_r \sum_{k \in M} p_k. \quad (38)$$

In matrix form, using the solution for probabilities (37), this condition becomes

$$\pi_R \leq D_{R,M} \cdot D_{M,M}^{-1} \cdot \pi_M$$

which holds by condition (10) in the proposition. If either (9) or (10) is strictly violated, then there is a  $\bar{\Delta}$  such that no equilibrium exists for all  $\Delta < \bar{\Delta}$ , since either  $p_M \ll 0$  or (38) is strictly violated for  $\Delta$  small.

**Step 2: Pareto Inefficiency**

Consider a strategy profile  $\sigma'$  where the seller accepts 0 and offers  $\pi_m$  when meeting  $m \in M$ . Similarly, any buyer  $m$  offers zero and accepts  $\pi_m$ . Moreover, the acceptance probabilities are  $p'_j = kp_j$  for all  $j \in N$  where  $k > 1$  and  $k$  is chosen such that  $p'_j < 1$  for all  $j \in N$ . Then,  $v'_{S,i} = w'_{S,i} = 0$ . Furthermore, using the value equations (1), we can write

$$v_{r,S} = w'_{r,S} = \frac{1}{n \frac{1-\delta_B}{\delta_B} + \sum_{j \in M} p'_j} \sum_{j \in M} p'_j e_{r,j} = \frac{k}{n \frac{1-\delta_B}{\delta_B} + k \sum_{j \in M} p_j} \sum_{j \in M} p_j e_{r,j}.$$

By (38) we have  $\sum_{j \in M} p_j e_{i,j} > 0$  and hence,  $w'_{r,S} > w_{r,S}$ . Moreover, using the value equations (1),

$$w'_{m,S} = \frac{k}{n \frac{1-\delta_B}{\delta_B} + (1-\eta)kp_{m,S} + \sum_{j \in N \setminus \{m\}} kp_j} \left( (1-\eta)p_{m,S}\pi_m + \sum_{j \in N \setminus \{m\}} p_j e_{m,j} \right).$$

Setting  $k = 1$  gives  $w'_{m,S} = \pi_m > 0$ , implying  $(1-\eta)p_{m,S}\pi_m + \sum_{j \in N \setminus \{m\}} p_j e_{m,j} > 0$ . Hence  $w'_{m,S} > \pi_m = w_{m,S}$  for  $k > 1$ . Moreover,

$$v'_{m,S} = kp_{m,S}\pi_m + (1 - kp_{m,S})w'_{m,S} > v_{m,S}.$$

A small increase in probabilities above the equilibrium probabilities thus increases the payoff for all buyers, while leaving the payoff for the seller constant. By continuity of (1), there is some strategy profile improving the payoff for all players.

**Step 3: Delay**

The expected amount of time that passes until an agreement is reached is

$$\Delta \frac{\sum_{m \in M} p_m}{n} \frac{1}{\left(1 - \frac{\sum_{m \in M} p_m}{n}\right)} \left( \sum_{k=1}^{\infty} k \left(1 - \frac{\sum_{m \in M} p_m}{n}\right)^k \right).$$

As

$$\sum_{k=1}^{\infty} k \left(1 - \frac{\sum_{m \in M} p_m}{n}\right)^k = \frac{1 - \frac{\sum_{m \in M} p_m}{n}}{\left(\frac{\sum_{m \in M} p_m}{n}\right)^2}$$

and using (37) and  $\delta_B = e^{-r_B \Delta}$ , delay is,

$$\frac{1}{r_B} \log \delta_B \frac{1}{\frac{1-\delta_B}{\delta_B} j_M \cdot D_{M,M}^{-1} \cdot \pi_M}.$$

Using L'Hôspitals rule, limit delay is given by (11). ■

**Proof of Proposition 3:** Combining (6) with (30) and setting  $|A| = 1$  and  $|M| = 0$  gives

$$w_{S,i} = \frac{n \frac{1-\delta_B}{\delta_B}}{\frac{1-\delta_S}{\delta_S} \frac{n}{\eta} \left( n \frac{1-\delta_B}{\delta_B} + 1 - \eta \right) + n \frac{1-\delta_B}{\delta_B}} \pi_a \leq \pi_a, \quad (39)$$

which is true as the denominator is larger than the numerator. Using (30) in (8) gives the reject condition as

$$\frac{\eta \pi_a}{\frac{1-\delta_S}{\delta_S} \frac{\delta_B}{\delta_S} \left( n \frac{1-\delta_B}{\delta_B} + 1 - \eta \right) + \eta} \geq \pi_r - \frac{e_{r,a}}{n \frac{1-\delta_B}{\delta_B} + 1}. \quad (40)$$

Note that, by L'Hôspitals rule,  $\lim_{\Delta \rightarrow 0} \frac{1-\delta_S}{1-\delta_B} = \frac{r_S}{r_B} = \rho$ . From condition (12) in the statement of the proposition, there exists a  $\bar{\Delta} > 0$  such that, for all  $\Delta < \bar{\Delta}$ , (40) holds. As  $w_{S,a} > 0$ , the seller also makes a non-negative profit. Thus, there is some  $\bar{\Delta} > 0$  such that, for all  $\Delta < \bar{\Delta}$ , the conditions for the equilibrium to exist are satisfied. If (12) is strictly violated, then (40) is violated for  $\Delta$  small, implying that there is a  $\bar{\Delta}$  such that no equilibrium exists for  $\Delta < \bar{\Delta}$ . ■

**Proof of Proposition 4:** From indifference (7) and using (30) with  $|A| = 1$  and  $|M| = 1$ , we have

$$\pi_m - \frac{e_{m,a}}{n \frac{1-\delta_B}{\delta_B} + 1} = \frac{n \frac{1-\delta_B}{\delta_B} \pi_a + \pi_a p_m - p_m e_{a,m}}{\left( \frac{1-\delta_S}{\delta_S} \frac{n}{\eta} \left( n \frac{1-\delta_B}{\delta_B} + 1 - \eta + p_m \right) + n \frac{1-\delta_B}{\delta_B} \right) + p_m}.$$

Solving for  $p_m$  gives

$$p_m = n \frac{1-\delta_B}{\delta_B} V(\delta_S, \delta_B), \quad (41)$$

$$V(\delta_S, \delta_B) = \frac{1}{\eta} \frac{(\pi_m - e_{m,a}) \left( \frac{1-\delta_S}{\delta_S} n + \frac{1-\delta_S}{1-\delta_B} \frac{\delta_B}{\delta_S} (1-\eta) + \eta \right) - \eta \pi_a}{\pi_a - e_{a,m} - \left( \pi_m - \frac{e_{m,a}}{n \frac{1-\delta_B}{\delta_B} + 1} \right) \left( \frac{1-\delta_S}{\delta_S} \frac{n}{\eta} + 1 \right)}.$$

Note that  $V(\delta_S, \delta_B)$  is continuous. Let

$$V = \lim_{\delta_B, \delta_S \rightarrow 1} V(\delta_S, \delta_B) = \frac{1}{\eta} \frac{(\pi_m - e_{m,a}) (\rho (1-\eta) + \eta) - \eta \pi_a}{\pi_a - e_{a,m} - (\pi_m - e_{m,a})}. \quad (42)$$

By condition (13),  $0 < V < \infty$ . Since  $p_m = n \frac{1-\delta_B}{\delta_B} V(\delta_S, \delta_B)$ , since  $\lim_{t \rightarrow \infty} p_m = 0$  and by the definition of  $\delta_B$ , there exists a  $\Delta_1 > 0$  such that for all  $\Delta < \Delta_1$  we have  $0 < p_m < 1$ .

The reject condition (8) is, using the indifference condition (7), in the limit,

$$\pi_m - e_{m,a} \geq \pi_r - e_{r,a}.$$

By (15), this expression strictly holds. Using  $\delta_S = e^{-rs\Delta}$  and  $\delta_B = e^{-r_B\Delta}$ , there exists a  $\Delta_2 > 0$  such that (4) holds for all  $\Delta < \Delta_2$ .

Using the indifference condition (7), the acceptance condition (6) is

$$\pi_m - \frac{e_{m,a}}{n \frac{1-\delta_B}{\delta_B} + 1} \leq \pi_a - \frac{p_m e_{a,m}}{n \frac{1-\delta_B}{\delta_B} + p_m}.$$

This expression can be rewritten as

$$((U + UV(\delta_S, \delta_B))\pi_a - UV(\delta_S, \delta_B)e_{a,m})(U + 1) - ((U + 1)\pi_m - e_{m,a})(U + UV(\delta_S, \delta_B)) \geq 0$$

where

$$U = n \frac{1 - \delta_B}{\delta_B}.$$

In the limit, using the definition of  $V$ ,

$$\rho(1 - \eta)(\pi_m - e_{m,a}) \geq 0.$$

By condition (14), the condition for acceptance holds strictly. Since  $\delta_i = e^{-r_i\Delta}$ , from (14), there exists a  $\Delta_3 > 0$  such that (2) holds for all  $\Delta < \Delta_3$ .

To ensure that all conditions hold, we choose  $\bar{\Delta} = \min\{\Delta_1, \Delta_2, \Delta_3\}$ .

If any of (13), (14) or (15) is strictly violated then either  $p_m < 0$ ,  $w_{S,i} < 0$  or (8) is violated for  $\Delta$  small, implying that there is a  $\bar{\Delta}$  such that no equilibrium exists for  $\Delta < \bar{\Delta}$ . ■

**Proof of Proposition 5:** To prove the proposition, we need to show both that the equilibrium types stated in the proposition are generic and that any other equilibrium type only exists non-generically. We begin by showing that the equilibrium types stated in the proposition are generic. Then, we continue to show that any other equilibrium type only exists non-generically.

**Lemma 3** *The hold-up equilibria in proposition 2 are generic.*

**Proof:** To show that the hold-up equilibrium type is generic, consider the case where  $e_{j,i} = e_j$  for all  $i \neq j$  and assume that  $e_i > \pi_i$  for all  $i \in N$ . In addition, let all  $m$  have identical pies and externalities  $\pi_m = \alpha$  and  $e_m = \beta$  for all  $m \in M$  and similarly  $\pi_r = \theta$  and  $e_r = \tau$  for all  $r \in R$ .

Let  $\bar{\omega}$  denote this parameter vector. We also assume  $\beta > \alpha$  and

$$\frac{\theta}{\tau - \theta} < \frac{|M|}{|M| - 1} \frac{\alpha}{\beta - \alpha}. \quad (43)$$

Then, the invertibility condition in proposition 2 is satisfied, using  $D_{M,M} = (\alpha - \beta) (J_{M,M} - I_M)$

$$-D_{M,M}^{-1} \cdot \pi_M = \frac{1}{|M| - 1} \frac{\alpha}{\beta - \alpha} j_M.$$

This is positive as, by assumption,  $\beta > \alpha$  and hence, condition (9) is satisfied. Furthermore, using  $D_{R,M} = (\theta - \tau) J_{R,M}$

$$\pi_R = \theta j_R \ll (\tau - \theta) \frac{|M|}{|M| - 1} \frac{\alpha}{\beta - \alpha} j_R.$$

Then, since (43) holds, condition (10) is satisfied. Since the invertibility condition is satisfied for the parameter vector  $\bar{\omega}$  and the determinant is a continuous function of  $\omega \in \Omega$ , there exists a ball  $B(\bar{\omega})$  with radius  $\varepsilon$  around the parameter vector  $\bar{\omega}$  such that the matrix  $D_{M,M}$  is invertible and conditions (9) and (10) still hold. Fix  $\omega$  and let  $\Delta(\omega)$  denote the value of  $\Delta$  such that the probabilities are smaller than one for all  $\Delta < \Delta(\omega)$ . Let  $\bar{\Delta} = \inf_{\omega \in B(\bar{\omega})} \Delta(\omega)$ . By continuity, we can choose  $\varepsilon$  such that  $\bar{\Delta} > 0$ . Then, a hold-up equilibrium exists for all  $\omega \in B(\bar{\omega})$  for all  $\bar{\Delta} > \Delta > 0$ , establishing that  $u_H$  is generic. ■

**Lemma 4** *The single out equilibria in proposition 3 are generic.*

**Proof:** Consider the single out equilibrium type  $u_S$ . Suppose  $k > \frac{\eta}{\eta + \bar{\rho}(1 - \eta)}$  and that

$$\pi_r - e_{r,a} < k\pi_a. \quad (44)$$

for all  $r \neq a$  for some  $a$ . Then there is some  $\rho \in (\underline{\rho}, \bar{\rho})$  such that (12) holds. Since  $\rho < \bar{\rho}$  then (12) holds for all  $r_S$  and  $r_B$  where  $\rho' = \frac{r_S}{r_B} \leq \rho$ . Given  $\rho$ , condition (12) in Proposition 3 holds for some parameter value  $\bar{\omega} \in \Omega$ . Then there exists a closed ball  $B(\bar{\omega})$  with radius  $\varepsilon$  around the parameter vector  $\bar{\omega}$  such that the condition holds for all  $\omega \in B(\bar{\omega})$ .

Also, from the proof of Proposition 3, the condition for acceptance (??) for  $a$  holds for all  $\delta_B$  and  $\delta_S$ . Rewriting (40) we get

$$\pi_r - \frac{\delta_B e_{r,a}}{n(1 - \delta_B) + \delta_B} \leq \frac{\delta_S \eta}{\eta \delta_S + \frac{1 - \delta_S}{1 - \delta_B} ((1 - \eta) \delta_B + n(1 - \delta_B))} \pi_a. \quad (45)$$



Rearranging and letting

$$L_1(\delta_B, \delta_S) = \left( (\delta_S - \delta_B) \rho' + \delta_B \left( \rho' - \frac{1 - \delta_S}{1 - \delta_B} \right) \right) (1 - \eta) \delta_B \\ + \left( (\delta_S \eta + \delta_S \rho' (1 - \eta)) - \delta_B \frac{1 - \delta_S}{1 - \delta_B} \right) (n(1 - \delta_B))$$

we get

$$n \frac{1 - \delta_B}{\delta_B} \pi_r + L_1(\delta_B, \delta_S) \frac{1}{\delta_B} \frac{\eta}{\eta + \rho' (1 - \eta)} \pi_a \leq \frac{\eta}{\eta + \rho' (1 - \eta)} \pi_a - (\pi_r - e_{r,a}) \quad (46)$$

Note that (46) converges to (12) with  $\rho = \rho'$ .

For a given  $\omega$ , let  $\Delta(\omega)$  denote the smallest  $\Delta$  such that (46) holds for all  $\Delta < \Delta(\omega)$  and all  $r \neq a$ . Let  $\bar{\Delta} = \inf_{\omega \in B(\bar{\omega})} \Delta(\omega)$ . By continuity of (??) and (12) and since  $\lim_{\frac{1 - \delta_S^t}{1 - \delta_B^t} = \rho'} = \rho'$ ,  $\varepsilon$  can be chosen such that  $\bar{\Delta} > 0$ . Then  $B(\bar{\omega}) \subseteq \Omega(u_S, \Delta)$  for all  $\Delta < \bar{\Delta}$ . Then  $\lim_{t \rightarrow \infty} \lambda(\Omega(u_S, \Delta)) \geq \lambda(B(\bar{\omega})) > 0$  establishing that  $u_S$  is generic. ■

**Lemma 5** *The outside option equilibria in proposition 4 are generic.*

**Proof:** Suppose, without loss of generality, that the denominator of (13) is positive, that  $k < \frac{\eta}{\eta + \underline{\rho}(1 - \eta)}$  and that

$$\frac{(\pi_m - e_{m,a}) - k\pi_a}{(\pi_a - e_{a,m}) - (\pi_m - e_{m,a})} > 0$$

and that conditions (14)-(15) hold. Since  $\rho \geq \underline{\rho} = \frac{\bar{r}_S}{r_B}$ , (13) holds for all  $r_S$  and  $r_B$  such that  $\rho = \frac{r_S}{r_B}$ . Given that condition (13) holds for some parameter value  $\bar{\omega} \in \Omega$ , there exists a closed ball  $B^m(\bar{\omega})$  with radius  $\varepsilon^m$  around the parameter vector  $\bar{\omega}$  such that the condition holds for all  $\omega \in B^m(\bar{\omega})$ .

Since condition (13) holds, from the proof of Proposition 4 we have  $p_m = n \frac{1 - \delta_B}{\delta_B} V(\delta_S, \delta_B) > 0$ . Moreover,  $p_m$  converges to zero. By continuity of  $V(\delta_S, \delta_B)$  in  $\delta_B, \delta_S$  and using  $\delta_i = e^{-r_i \Delta}$ , there is some  $\Delta(\omega)$  such that  $p_m$  is smaller than one for all  $\Delta < \Delta(\omega)$ . Let  $\Delta^m = \inf_{\omega \in B^m(\bar{\omega})} \Delta(\omega)$ . A similar argument using (14) and (15) establishes the existence of  $\Delta^a$  ( $\Delta^r$ ) and  $B^a(\bar{\omega})$  ( $B^r(\bar{\omega})$ ), where  $\Delta^a$  ( $\Delta^r$ ) is the smallest  $\Delta(\omega)$  such that, for all  $a$  ( $r$ ), the condition for acceptance (rejection) (2) ((4)) holds for all  $\Delta < \Delta(\omega)$  for all  $\omega \in B^a(\bar{\omega})$  ( $\omega \in B^r(\bar{\omega})$ ).

Letting  $\bar{\Delta} = \min\{\Delta^a, \Delta^m, \Delta^r\}$  and  $B(\bar{\omega}) = B^a(\bar{\omega}) \cap B^m(\bar{\omega}) \cap B^r(\bar{\omega})$ . By continuity of (41) and the solutions for the values in Proposition 4 and since  $\lim_{\frac{1 - \delta_S^t}{1 - \delta_B^t} = \rho} = \rho$ ,  $\varepsilon$  can be chosen such that  $\bar{\Delta} > 0$ . Then, for  $\Delta < \bar{\Delta}$  and  $\omega \in B(\bar{\omega})$ , conditions (2) and (4) hold with  $0 < p_m < 1$ . Moreover,  $w_{S,i} > 0$ . Then  $B(\bar{\omega}) \subseteq \Omega(u_O, \Delta)$  for all  $\Delta < \bar{\Delta}$  and hence,  $\lim_{t \rightarrow \infty} \lambda(\Omega(u_O, \Delta)) \geq \lambda(B(\bar{\omega})) > 0$  establishing that  $u_O$  is generic. ■

**Lemma 6** *The equilibria of type IV are generic.*

**Proof:** Consider some sequence  $\{\Delta^t\}$  such that  $\Delta^t \rightarrow 0$ . Note first that, using (1) together with  $v_{S,m} = w_{S,m}$  and  $v_{S,r} = w_{S,r}$  it is easily seen that the condition for acceptance is

$$v_{S,a} - w_{S,i} = \frac{1 - \delta_S}{1 - \delta_S \frac{n-1}{n} - \delta_S \frac{1-\eta}{n}} v_{S,a} = (1 - \delta_S) \delta_S \frac{\eta}{n} w_{S,i} \geq 0. \quad (47)$$

Hence, (18) holds with equality in the limit;

$$w_{S,i} = \frac{\sum_{j \in A} (\pi_a - e_{a,j}) + \sum_{j \in M} p_j (\pi_a - e_{a,j})}{\sum_{j \in M} p_j}. \quad (48)$$

Let  $K = A \cup M$  be any subset of  $N$  consisting of at least three elements. Define  $\bar{\omega}$  as follows. Renumber the members from 1 to  $|M| + 1$  and let  $d_{i,i-1} = 1$  for  $i = 2, \dots, |M|$  and  $d_{i,i+1} = 4$  for  $i = 2, \dots, |M|$  and  $d_{|M|+1,1} = \frac{5}{2}$  and  $d_{|M|+1,2} = 1$ . All other off-diagonal elements are 2. Suppose that, for buyers  $r = |M| + 2, \dots, n$ , we have

$$\pi_r - e_{r,j} < \frac{1}{2}.$$

From the first row in  $D_{K,K}$  we have, using (29), that  $w_{S,i} = 2$ . From (29), we then have

$$p_1 = 2p_2 \quad (49)$$

$$p_i = 2p_{i+2} \text{ for } i = 1, \dots, |M| - 1.$$

Define

$$F(p_K, w_{s,i}) = D_{K,K} \cdot p_K - w_{s,i} (J_{K,K} - I_K) \cdot p_K.$$

Let  $\Phi$  be the matrix consisting of  $F_{p_i}(p_K, w_{s,i})$  and  $F_{w_{s,i}}(p_K, w_{s,i})$ . We have  $\Phi_{i,i-1} = -1$  and  $\Phi_{i,i+1} = 2$  for  $i = 2, \dots, |M|$ ,  $\Phi_{|M|+1,1} = \frac{1}{2}$  and  $\Phi_{|M|+1,2} = -1$ . Moreover,  $\Phi_{i,|M|+2} = P - p_i$ , where  $P = \sum_{j=1}^{|M|+1} p_j$ . All other elements are zero. Since  $P - p_1 > 0$ ,  $\text{rank}(\Phi) = |M| + 1$ .

Using theorem H.2.2 in Mas Colell (1985), it follows that there is a locally parametrizable solution set with one degree of freedom. Hence, we can set  $p_1 = 1$ . Thus, there exists a ball  $B(\bar{\omega})$  with radius  $\varepsilon$  around the parameter vector  $\bar{\omega}$ , such that there is a solution to

$$F(p_K, w_{s,i}) = 0, \quad (50)$$

with  $p_1 = 1$  for all  $\omega \in B(\bar{\omega})$ .

Suppose  $K = N$ . Since  $\eta$  is small, condition (12) is violated for all  $a$  and some  $r$  in  $B(\bar{\omega})$ .

Further, since  $D_{K,K}$  satisfies SD, there is no hold-up equilibrium for any  $\omega \in B(\bar{\omega})$ . Moreover, since  $\eta$  is small and  $D_{K,K}$  is bilaterally inefficient, from  $\Phi$  it is easily seen that there is a ball  $B^{IV}$  centered at  $\omega^{IV} \in \text{int}B(\bar{\omega})$ , such that condition (13) is violated. Hence, we can choose the radius  $\varepsilon^{IV} > 0$  such that  $B^{IV}(\omega^{IV}) \subset B(\bar{\omega})$ . Thus, for  $\omega \in B(\omega^{IV})$  the only equilibrium candidate is the one that solves (50). Since an equilibrium exists from proposition 1, this candidate must be an equilibrium. Since  $w_{S,i} > 0$ , the condition for acceptance (47) holds for any  $\eta > 0$ . Moreover, the solution to (17) and (48) is also independent of  $\eta$ , and hence an equilibrium exists for any  $\eta > 0$ .

Suppose  $K \subset N$ . The solution to (50) only depends on  $D_{K,K}$ , and hence, from above, there is a solution and the condition for acceptance (47) holds. Consider  $r \in R$ . Combining (7) and (8) gives

$$\pi_m - \frac{\sum_{j \in A} e_{m,j} + \sum_{j \in M} p_j e_{m,j}}{n \frac{1-\delta_B}{\delta_B} + |A| + \sum_{j \in M \setminus \{m\}} p_j} \geq \pi_r - \frac{\sum_{j \in A} e_{r,j} + \sum_{j \in M} p_j e_{r,j}}{n \frac{1-\delta_B}{\delta_B} + |A| + \sum_{j \in M} p_j}. \quad (51)$$

For a given  $\omega$ , let  $\Delta(\omega)$  denote the smallest  $\Delta$  such that (51) holds for all  $\Delta < \Delta(\omega)$  and all  $r > |M| + 1$ . Let  $\bar{\Delta} = \inf_{\omega \in B(\omega^{IV})} \Delta(\omega)$ . By continuity,  $\varepsilon$  can be chosen such that  $\bar{\Delta} > 0$ . Then,  $B(\omega^{IV}) \subseteq \Omega(u_{IV}, \Delta)$  for all  $\Delta < \bar{\Delta}$  and hence,  $\lim_{t \rightarrow \infty} \lambda(\Omega(u_{IV}, \Delta)) \geq \lambda(B(\omega^{IV})) > 0$ , establishing that  $u_{IV}$  is generic. ■

**Lemma 7** *Equilibria with  $|A| > 1$  and  $|M| = 0$  are non-generic.*

**Proof:** To show that equilibria with  $|A| > 1$  are non-generic, note that  $w_{S,i}$  in (30) is well defined for  $\delta_B = \delta_S = 1$ . For  $\omega \in \Omega$  and  $\Delta \geq 0$  let  $\psi : \Omega \times [0, \infty) \rightarrow R_+$  be the correspondence satisfying (30) and (6) with  $|M| = 0$ . The correspondence  $\psi$  is upper-hemicontinuous (uhc), see Border 1985: If for  $t = 1, 2, \dots$  we have  $p^t \in \psi(\omega^t, \Delta^t)$  and  $(\omega^t, \Delta^t) \rightarrow (\omega, \Delta)$  as  $t \rightarrow \infty$ , and  $p = \lim_{t \rightarrow \infty} p^t$  then, since (30) and (6) with  $|M| = 0$  define closed sets, we have  $p \in \psi(\omega, \Delta)$ , establishing that  $\psi$  is uhc.

For  $\Delta \geq 0$ , let the correspondence  $\varphi(\Delta)$  be the set of  $\omega$ , such that  $\psi(\omega, \Delta)$  is non-empty.  $\varphi(\Delta)$  is uhc: Let  $\Delta^t \rightarrow \Delta$  and  $\omega^t \rightarrow \omega$  such that  $\omega^t \in \varphi(\Delta^t)$ . Then there exists a  $p^t$  such that  $p^t \in \psi(\omega^t, \Delta^t)$  and since  $\psi$  is uhc  $p^t \rightarrow p \in \psi(\omega, \Delta)$ . Thus  $\omega \in \varphi(\Delta)$ .

Consider a sequence  $\{\Delta^t\}$  such that  $\Delta^t \rightarrow 0$ . Using the solution for  $w_{S,i}$  from (30) with  $|M| = 0$  in (6) when  $|A| > 1$ , gives

$$\pi_a - \frac{1}{|A| - 1} \sum_{k \in A} e_{a,k} \geq \frac{1}{|A|} \sum_{h \in A} \left( \pi_h - \frac{1}{|A| - 1} \sum_{k \in A} e_{h,k} \right). \quad (52)$$

Since (52) holds for all  $a$ , it holds for the  $a$  that minimizes the left-hand side. As the minimal element is weakly greater than the average over all  $a$ , then  $\pi_a - \frac{1}{|A| - 1} \sum_{j \in A} e_{a,j}$  is the same for

all  $a$ . Then  $\varphi(0)$  is defined as, for all  $a \in A$  and  $r \in R$ ,

$$\pi_a - \frac{1}{|A|-1} \sum_{k \in A} e_{a,k} = K$$

$$\pi_r - w_{S,r} \leq \frac{\sum_{k \in A} e_{r,k}}{|A|}.$$

Thus,  $\lambda(\varphi(0)) = 0$ . Suppose that  $\lim_{t \rightarrow \infty} \lambda(\varphi(\Delta^t)) > 0$ . Then there exists a sequence  $(\omega^t, \Delta^t) \rightarrow (\omega, 0)$  such that  $\omega^t \in \varphi(\Delta^t)$  for all  $\Delta^t$  but  $\omega \notin \varphi(0)$ . This contradicts the upper-hemicontinuity of  $\varphi$ , establishing non-genericity. ■

**Lemma 8** *Equilibria with  $|A| > 1$  and  $|M| \geq 1$  are non-generic.*

**Proof:**

First note that when  $|A| > 1$ , then  $w_{S,i}$  in (30) is well defined for  $\delta_B = \delta_S = 1$ . Then for  $\omega \in \Omega$  and  $\Delta \geq 0$  let  $\psi : \Omega \times [0, \infty) \rightarrow [0, 1]^M \times R_+$  be the correspondence satisfying (30) and (6) - (7). For  $\Delta \geq 0$ , let the correspondence  $\varphi(\Delta)$  be the set of  $\omega$ , such that  $\psi(\omega, \Delta)$  is non-empty. Both  $\psi$  and  $\varphi$  are uhc from an argument similar to the previous Lemma.

Consider a sequence  $\{\Delta^t\}$  such that  $\Delta^t \rightarrow 0$ . Using (30) in (6) when  $|A| > 1$  and  $\delta_B = \delta_S = 1$ , gives

$$\pi_a - \frac{\sum_{j \in A \setminus \{a\}} e_{a,j} + \sum_{m \in M} p_m e_{a,m}}{|A| - 1 + \sum_{m \in M} p_m} \geq \frac{1}{|A|} \sum_{a \in A} \left( \pi_a - \frac{\sum_{j \in A \setminus \{a\}} e_{a,j} + \sum_{m \in M} p_m e_{a,m}}{|A| - 1 + \sum_{m \in M} p_m} \right). \quad (53)$$

Using the same argument following (52), the left-hand side of (53) is the same for all  $a$ . Thus, for all  $a, b \in A$ , letting  $P_1 = |A| - 1 + \sum_{m \in M} p_m$ ,

$$\pi_a P_1 - \left( \sum_{j \in A \setminus \{a\}} e_{a,j} + \sum_{m \in M} p_m e_{a,m} \right) = \pi_b P_1 - \left( \sum_{j \in A \setminus \{b\}} e_{b,j} + \sum_{m \in M} p_m e_{b,m} \right). \quad (54)$$

Combining (54) for some  $a \in A$  with the system of  $|M|$  equations obtained from substituting (30) in (7), setting  $\delta_B = 1$  and rearranging, we can define the following system  $0 = F(p_M, \omega)$ , where

$F : [0, 1]^{|M|} \times \Omega \rightarrow \mathbb{R}^{|M|+1}$  is given by the  $1 + |M|$  equations, letting  $P_m = |A| + \sum_{j \in M \setminus \{m\}} p_j$ ,

$$\begin{aligned} & \pi_a P_1 - \left( \sum_{j \in A \setminus \{a\}} e_{a,j} + \sum_{m \in M} p_m e_{a,m} \right) - \left( \pi_b P_1 - \left( \sum_{j \in A \setminus \{b\}} e_{b,j} + \sum_{m \in M} p_m e_{b,m} \right) \right), \\ & \left( \pi_m P_1 - \frac{1}{|A|} \sum_{k \in A} \left( \pi_k P_1 - \left( \sum_{j \in A \setminus \{k\}} e_{k,j} + \sum_{m \in M} p_m e_{k,m} \right) \right) \right) P_m \\ & - \left( \sum_{j \in A} e_{m,j} + \sum_{j \in M} p_j e_{m,j} \right) P_1. \end{aligned}$$

The derivative of the above system with respect to  $\pi_a$  and  $\pi_m$  is

$$Z = \begin{pmatrix} z_{aa} & 0 & \cdots & \cdots & 0 \\ z_{ma} & z_{mm} & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ z_{ma} & 0 & \cdots & 0 & z_{mm} \end{pmatrix},$$

where  $z_{mm} = P_1 P_m$ ,  $z_{ma} = \frac{z_{mm}}{|A|}$  and  $z_{aa} = P_1 \left(1 - \frac{1}{|A|}\right)$ . Since  $|A| > 1$  and  $p_m \geq 0$ , we have  $P_1 > 0$  and  $P_m > 0$ . Then since  $\det(Z) = z_{aa} (z_{mm})^{|M|} \neq 0$ ,  $Z$  is invertible. Using the Transversality Theorem 8.3.1 in Mas Colell (1985), the equation system is regular on the set  $\hat{\Omega}$  with  $\lambda(\hat{\Omega}) = 1$ . Using Proposition H.2.2 in Mas Colell (1985), there is no solution when the system is regular since the number of equations is larger than  $|M|$ . Since the probabilities must also satisfy (54) for all  $a, b \in A$ , the set of parameter values for which an equilibrium exists is  $\Omega^* \subseteq \Omega \setminus \hat{\Omega}$ , establishing  $\lambda(\varphi(0)) = 0$ . Suppose that  $\lim_{t \rightarrow \infty} \lambda(\varphi(\Delta^t)) > 0$ . Then, there exists a sequence  $(\omega^t, \Delta^t) \rightarrow (\omega, 0)$  such that  $\omega^t \in \varphi(\Delta^t)$  for all  $\Delta^t$  but  $\omega \notin \varphi(0)$ . This contradicts the upper-hemicontinuity of  $\varphi$ , establishing non-genericity. ■

The proof of Proposition 5 then follows by Lemmas 3-8.

■

**Proof of proposition 6.** To prove existence, set  $N = K$  and note that for  $\eta$  close to zero, condition (12) is violated for all  $a$  and some  $r$ . Moreover, since  $D_{K,K}$  satisfies SD there is no hold-up equilibrium. Condition (13) is also violated. By genericity  $|A| = 1$  and  $|M| \geq 2$ . Since an equilibrium exists from proposition 1, there is such an equilibrium. In the limit, the equilibrium probabilities  $p_M$  and value  $w_{S,i}$  solves (17) and (48). Since  $D_{K,K}$  has all off diagonal elements positive  $w_{S,i} > 0$  by (17), establishing that it is strictly profitable to make acceptable offers when  $a$  and  $S$  meet for any  $\eta > 0$ , see (47). Moreover, the solution to (17) and (48) is

independent of  $\eta$ , establishing that there is an equilibrium for any  $\eta > 0$ .

To prove existence for an arbitrary  $N \supset K$ , first note that the deviation conditions only depend on elements in  $D$  corresponding to the  $K$  columns. The remaining columns can then be arbitrarily chosen. Finally, if the rows corresponding to  $R = N \setminus K$  satisfy (20), the condition for rejection (51) holds strictly. To see this, first note that the payoff of  $S$  in (17) is a weighted average of payoff differences. This average is at least equal to the left-hand side of (20). Similarly, the payoff when deviating and agreeing with some  $r \in R$  is at most equal to the right-hand side of (20). Hence, from (20), the condition for rejection (51) holds and an equilibrium exists. If  $D$  satisfies ED then, (17),  $w_{S,i} < 0$ , implying that  $S$  gains by never selling the object. ■

**Proof of Corollary 2:** To see this, note that there cannot be  $m, n$  such that (17) holds, since we generically have  $\pi_n \neq \pi_m$ .

Moreover, the conditions for existence in proposition 2 are violated since  $e_{i,j} = 0$ . Furthermore, if a single-out equilibrium exists, then there must be agreement with the buyer with the largest surplus. Otherwise, condition (12) is violated since  $\frac{\eta}{\eta + \rho(1-\eta)} \leq 1$ . Finally, suppose that (12) is violated for the buyer with the highest valuation. Then, we see that (13) in proposition 4 is satisfied for  $a = 1$  and  $m = 2$  and (15) is satisfied for  $m = 2$  and  $r \neq 1$ , establishing the existence of an outside option equilibrium. Further, it is the only outside option equilibrium that exists. First, if  $r = 1$  in proposition 4, condition (15) is violated. Thus, the only other possible outside option equilibrium is when  $m = 1$  and  $a = 2$ . Since  $\pi_1 > \pi_2$ , (13) is violated. ■

## References

- [1] Admati, A., Perry, M., 1987. Strategic Delay In Bargaining. *Review of Economic Studies* 54, 345-64.
- [2] Border, K., 1985. *Fixed Point Theorems with Application to Economics and Game Theory*. Cambridge University Press, Cambridge.
- [3] Björnerstedt, J., & Westermarck, A., 2006, The Inefficiency of Price Quantity Bargaining, Uppsala University working paper 2006:6, revised version.
- [4] Cai, H., 2000. Delay in Multilateral Bargaining under Complete Information. *Journal of Economic Theory* 93, 260-76.
- [5] Cramton, P., 1992. Strategic Delay in Bargaining with Two-Sided Uncertainty. *Review of Economic Studies* 59, 205-225.
- [6] Fernandez, R., Glazer, J., 1991. Striking for a Bargain Between Two Completely Informed Agents. *American Economic Review* 81, 240-52.
- [7] Gomes, A., 2005. Multilateral Contracting with Externalities. *Econometrica* 73, 1329-50.
- [8] Gomes, A., & Jehiel, P., 2005, Dynamic Processes of Social and Economic Interactions: On the Persistence of Inefficiencies, *Journal of Political Economy* v113, no3, p 626-667.
- [9] Hendon, E., and Tranæs, T., 1991, Sequential Bargaining in a Market with One Seller and Two Different Buyers, *Games and Economic Behavior* 3, 453-466.
- [10] Jehiel, P., Moldovanu, B., 1995a. Cyclical Delay in Bargaining with Externalities. *Review of Economic Studies* 62, 619-637.
- [11] Jehiel, P., Moldovanu, B., 1995b. Negative Externalities May Cause Delay in Negotiation. *Econometrica* 63,, 1321-1335.
- [12] Jehiel, P., Moldovanu, B., Stacchetti, E., 1996. How (Not) to Sell Nuclear Weapons. *American Economic Review* 86, 814-29.
- [13] Mas Colell, A., 1985. *The Theory of General Economic Equilibrium*. Cambridge University Press, Cambridge.
- [14] Westermarck, A., 2003. Bargaining, Binding Contracts and Competitive Wages. *Games and Economic Behavior* 43, 296-311.