Unawareness, Beliefs, and Speculative Trade*

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First draft: June 6, 2005
This draft: November 6, 2009

Abstract

We define a generalized state-space model with interactive unawareness and probabilistic beliefs. Such models are desirable for potential applications of asymmetric unawareness. We compare unawareness with probability zero belief. Applying our unawareness belief structures, we show that the common prior assumption is too weak to rule out speculative trade in all states. Yet, we prove a generalized "No-trade" theorem according to which there can not be common certainty of strict preference to trade. Moreover, we show a generalization of the "No-agreeing-to-disagree" theorem.

Keywords: Unawareness, awareness, zero probability, type-space, common prior, agreement, speculative trade, interactive epistemology.

JEL-Classifications: C70, C72, D80, D82.

*Aviad acknowledges support from the Open University of Israel, Research Grant No. 46106. Martin acknowledges financial support from the Spanish Ministerio de Educación y Ciencia via a Ramon y Cajal Fellowship and Research Grant SEJ2004-07861, as well as the Fundación Ramón Areces. Burkhard received financial support from the NSF SES-0647811, DFG SFB/TR 15, Minerva Stiftung, and IGA-UCD. We thank Leandro Chaves Rêgo, Eddie Dekel, Enrique Kawamura, Li Jing, Salvatore Modica, Klaus Nehring and participants in the Workshop on Unawareness, Stanford University, 2006, NASEM 2006, LOFT 2006, LAMES 2006, SAET 2007, TARK 2007, Stony Brook 2007, ESEM 2007, NBER GE conference 2007, GAMES 2008 and seminars at Bocconi, UCLA, St. Louis, Tucson, and Venice for helpful comments. Some of the material was circulated previously as a working paper titled "Unawareness, Beliefs and Games".

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1 Introduction

Unawareness is probably the most common and most important kind of ignorance. Business people invest most of their time not in updating prior beliefs and crossing out states of the world that they previously assumed to be possible. Rather, their efforts are mostly aimed at exploring unmapped terrain, trying to figure out business opportunities that they could not even have spelled out before. More broadly, every book we read, every new acquaintance we make, expands our horizon and our language, by fusing it with the horizons of those we encounter, turning the world more intelligible and more meaningful to us than it was before (Gadamer, 1960).

With this in mind, we should not be surprised that the standard state-spaces aimed at modeling knowledge or certainty are not adequate for capturing unawareness (Dekel, Lipman and Rustichini, 1998). Indeed, more elaborate models are needed (Fagin and Halpern, 1988, Modica and Rustichini, 1994, 1999, Halpern, 2001). In all of these models, the horizon of propositions the individual has in her disposition to talk about the world is always a genuine part of the description of the state of affairs.

Things become even more intricate when several players are involved. Each player may not only have different languages, but may also form a belief on the extent to which other players are aware of the issues that she herself has in mind. And the complexity continues further, because the player may be uncertain as to the sub-language that each other player attributes to her or to others; and so on.

Heifetz, Meier and Schipper (2006) showed how an unawareness structure consisting of a *lattice of spaces* is adequate for modeling mutual unawareness. Every space in the lattice captures one particular horizon of meanings or propositions. Higher spaces capture wider horizons, in which states correspond to situations described by a richer vocabulary. The join of several spaces – the lowest space at least as high as every one of them – corresponds to the fusion of the horizons of meanings expressible in these spaces.

In a companion work (Heifetz, Meier and Schipper, 2008), we showed the precise sense in which such unawareness structures are adequate and general enough for modeling mutual unawareness. We put forward an axiom system, which extends to the multi-player case a variant of the axiom system of Modica and Rustichini (1999). We then showed how the collections of all maximally-consistent sets of formulas in our system form a canonical unawareness structure.[1] In a parallel work, Halpern and Rêgo (2008) devised

[1]Each space in the lattice of this canonical unawareness structure consists of the maximally consistent sets of formulas in a sub-language generated by a subset of the atomic propositions.

another sound and complete axiomatization for our class of unawareness structures.[2]

In this paper we extend unawareness structures so as to encompass probabilistic beliefs (Section 2) rather than only knowledge or ignorance. The definition of types (Definition 1), and the way beliefs relate across different spaces of the lattice, is a non-trivial modification of the coherence conditions for knowledge operators in unawareness structures, as formulated in Heifetz, Meier and Schipper (2006). We show that we obtain all properties of unawareness suggested in the literature.

In Section 3 we define the notion of a common prior. Conceptually, a prior of a player is a convex combination of (the beliefs of) her types (see e.g. Samet, 1998). If the priors of the different players coincide, we have a common prior. A prior of a player induces a prior on each particular space in the lattice, and if the prior is common to the players, the induced prior on each particular space is common as well.

What are the implications of the existence of a common prior? First, we extend an example from Heifetz, Meier and Schipper (2006) and show that *speculative trade* is compatible with the existence of a common prior (Section 1.1). This need not be surprising if one views unawareness as a particular kind of "delusion", since we know that with deluded beliefs, speculative trade is possible even with a common prior (Geanakoplos, 1989). Nevertheless, we show that under a mild positivity condition, a common prior is *not compatible* with common certainty of *strict preference* to carry out speculative trade. That is, even though types with limited awareness are, in a particular sense, deluded, a common prior precludes the possibility of common certainty of the event that based on private information players are willing to engage in a zero-sum bet with *strictly positive* subjective gains to everybody. This is so because unaware types are "deluded" only concerning aspects of the world outside their vocabulary, while a common prior captures a prior agreement on the likelihood of whatever the players do have a common vocabulary. An implication of this generalized no-trade theorem is that arbitrary small transaction fees rule out speculative trade under unawareness. We complement this result by generalizing Aumann's (1976) "No-Agreeing-to-disagree" result to unawareness belief structures.

To what extend could unawareness be modeled by probability zero belief in appli-

[2]The precise connection between Fagin and Halpern (1988), Modica and Rustichini (1999), Halpern (2001) and Heifetz, Meier and Schipper (2006) is understood from Halpern and Rêgo (2008) and Heifetz, Meier, and Schipper (2008). The connection between Heifetz, Meier and Schipper (2006, 2008) and Galanis (2009a) is explored in Galanis (2009b). The connection with the models of Ewerhart (2001), Li (2009) and Feinberg (2009) are yet to be explored.

cations? First, assigning probability zero to an event is still compatible with realizing what could happen if the probability zero event were nevertheless to obtain. This is conceptually distinct from being completely unaware of the event, and it is the latter concept that we want to model here. Second, if a modeler aims nevertheless to model "unawareness" of an event as zero probability belief of that event, then this is impossible to do in a standard type-space. According to the symmetry axiom of awareness (see Proposition 3), a decision maker is unaware of an event if and only if she is unaware of its negation. So a decision maker being unaware of an event would have to assign probability zero to the event *and* probability zero to the negation of the event. Because of additivity, a probability measure in a standard type-space can never assign both zero to an event and its complement. In Section 2.12 we show how to extend our definitions of types in unawareness structures so as to characterize unawareness of an event as probability zero belief of the *existence* of that event, where we interpret the event that some event E exists as the set of states where this event does or does not happen, i.e. as the event $E \cup \neg E$. While such a characterization is trivial in a standard type-space because every measurable event exists in every state, in our lattice of spaces the event that an event E exists is non-trivial. We show that in the extended model a decision maker assigns probability zero to the existence of an event if and only if she assigns probability zero to the event and probability zero to its negation. Yet, unawareness modeled as probability zero lacks transparency. If in the extended model a decision maker assigns probability zero to an event, then it is not clear whether she does so because of being unaware of the event or because she is aware of the event but assigns probability zero to the event obtaining. We view this as a drawback of the probability zero model of "unawareness". In any case, we demonstrate that no matter whether the applied economist chooses to model unawareness as unawareness proper or by probability zero belief, our (extended) unawareness structures provide the appropriate modeling tool.

In the following section we present our interactive unawareness belief structure. In Section 3 we define a common prior and investigate agreement and speculation under unawareness. Section 4 contains an informal discussion of the common prior and the related literature. Some further properties of our unawareness belief structures are relegated to the appendix. Proofs are relegated to the appendix as well. In a separate appendix, Meier and Schipper (2009), we extend the "No-trade" theorem to infinite unawareness structures.

1.1 Introductory Example - Speculation under Unawareness

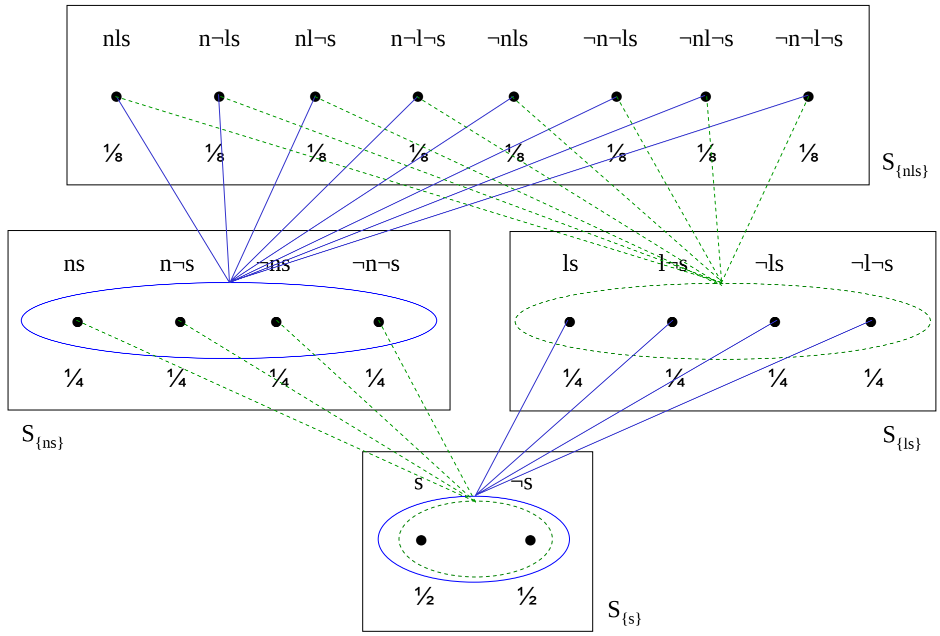
The purpose of the following example is threefold: First, it shall motivate the study of unawareness and speculation under unawareness. Second, it should illustrate informally some features of our model. Third, it is a counter example to the standard "No Trade" theorems in the context of unawareness.

Consider a probabilistic version of the speculative trade example of Heifetz, Meier and Schipper (2006). There is an owner, o, of a firm and a potential buyer, b, whose awareness differ. The owner is aware that there may be a costly lawsuit $[l]$ involving the firm, but he is unaware of a potential novelty $[n]$ enhancing the value of the firm. In contrast, the buyer is aware that there might be an innovation, but he is unaware of the lawsuit. Both are aware that the firm may face high sales $[s]$ or not in future.

Both agents can only reason and form beliefs about contingencies of which they are aware of respectively. The information structure is given in Figure 1. There are four state-spaces of different expressive power. The description of each state is printed above the state. While the upmost space, $S_{\{nls\}}$, contains all contingencies, the space $S_{\{ls\}}$ misses the novelty, $S_{\{ns\}}$ misses the law suit, and $S_{\{s\}}$ is capable of expressing only events pertaining to the sales. At any state in the upmost space $S_{\{nls\}}$, the buyer's belief has full support on the lower space $S_{\{ns\}}$ (as given by the solid ellipse and lines) and the seller's belief has full support on $S_{\{ls\}}$ (dashed ellipse and lines). Thus the buyer forms beliefs about sales and the novelty but is unaware of the law suit, and the seller forms beliefs about sales and the law suit but is unaware of the novelty. At any state in $S_{\{ns\}}$ the seller's belief has full support on the lower space $S_{\{s\}}$. That is, the buyer is certain that the seller is unaware of the novelty. Analogously, the seller is certain that the buyer is unaware of the law suit since at any state in $S_{\{ls\}}$ the belief of the buyer has full support on the space $S_{\{s\}}$. Figure 1 provides an example of an unawareness structure developed in this paper. The probability distribution given in each space illustrates an example of a common prior in unawareness structures, a projective system of probability measures. I.e., the prior on a lower space is the marginal of the prior in the upmost space. The beliefs of both agents are consistent with the common prior.

Suppose that the status quo value of the firm with high sales is 100 dollars, but only 80 dollars with low sales. If the potential innovation obtains, this would add 20 dollars to the value of the firm, whereas the potential lawsuit would cost the firm 20 dollars. According to the beliefs at state (nls), the buyer's expected value of the firm is 100, whereas the seller's expected value of the firm is 80 dollars. However, the buyer (resp. seller) is certain that the seller's (resp. buyer's) expected value is 90 dollars.

Figure 1: Information Structure in the Speculative Trade Example



We assume that both players are rational in the sense of maximizing their respective payoff given their belief and awareness. The buyer (resp. seller) prefers to buy (resp. sell) at price x if her expected value of the firm is at least (resp. at most) x. The buyer (resp. seller) strictly prefers to buy (resp. sell) at price x if her expected value of the firm is strictly above (resp. strictly below) x.

Note that despite the fact that both agents' beliefs are consistent with the common prior, at state (nls) and at the price 90 dollars, there is common certainty of preference to trade, but each player strictly prefers to trade. This is impossible in standard state-space structures with a common prior. In standard "No Trade" theorems, if there is common certainty of willingness to trade, then agents are necessarily indifferent to trade (Milgrom and Stokey, 1982).

Despite this counter example to the "No Trade" theorems, we can prove in section 3 a generalized "No-trade" theorem according to which, if there is a common prior, then there can not be common certainty of strict preference to trade.

2 Model

2.1 State-Spaces

Let $\mathcal{S} = \{S_\alpha\}_{\alpha\in\mathcal{A}}$ be a complete lattice of disjoint *state-spaces*, with the partial order $\preceq$ on $\mathcal{S}$. If S_α and S_β are such that $S_\alpha \succeq S_\beta$ we say that "S_α is more expressive than S_β – states of S_α describe situations with a richer vocabulary than states of S_β ".[3] Denote by $\Omega = \bigcup_{\alpha\in\mathcal{A}} S_\alpha$ the union of these spaces. Each $S \in \mathcal{S}$ is a measurable space, with a σ-field $\mathcal{F}_S$.

Spaces in the lattice can be more or less "rich" in terms of facts that may or may not obtain in them. The partial order relates to the "richness" of spaces. The upmost space of the lattice may be interpreted as the "objective" state-space. Its states encompass full descriptions.

2.2 Projections

For every S and S' such that $S' \succeq S$, there is a measurable surjective projection $r_S^{S'} : S' \to S$, where r_S^S is the identity. ("$r_S^{S'}(\omega)$ is the restriction of the description ω to the more limited vocabulary of S.") Note that the cardinality of S is smaller than or equal to the cardinality of S'. We require the projections to commute: If $S'' \succeq S' \succeq S$ then $r_S^{S''} = r_S^{S'} \circ r_{S'}^{S''}$. If $\omega \in S'$, denote $\omega_S = r_S^{S'}(\omega)$. If $D \subseteq S'$, denote $D_S = \{\omega_S : \omega \in D\}$.

Projections "translate" states in "more expressive" spaces to states in "less expressive" spaces by "erasing" facts that can not be expressed in a lower space.

2.3 Events

Denote $g(S) = \{S' : S' \succeq S\}$. For $D \subseteq S$, denote $D^\uparrow = \bigcup_{S'\in g(S)} \left(r_S^{S'}\right)^{-1}(D)$. ("All the extensions of descriptions in D to at least as expressive vocabularies.")

An *event* is a pair (E, S), where $E = D^\uparrow$ with $D \subseteq S$, where $S \in \mathcal{S}$. D is called the *base* and S the *base-space* of (E, S), denoted by $S(E)$. If $E \neq \emptyset$, then S is uniquely determined by E and, abusing notation, we write E for (E, S). Otherwise, we write $\emptyset^S$ for $(\emptyset, S)$. Note that not every subset of Ω is an event.

[3]Here and in what follows, phrases within quotation marks hint at intended interpretations, but we emphasize that these interpretations are not part of the definition of the set-theoretic structure.

Some fact may obtain in a subset of a space. Then this fact should be also "expressible" in "more expressive" spaces. Therefore the event contains not only the particular subset but also its inverse images in "more expressive" spaces.

Let Σ be the set of *measurable events* of Ω, i.e., $D^{\uparrow}$ such that $D \in \mathcal{F}_S$, for some state-space $S \in \mathcal{S}$. Note that unless $\mathcal{S}$ is a singleton, Σ is not an algebra because it contains distinct $\emptyset^S$ for all $S \in \mathcal{S}$.

2.4 Negation

If $(D^{\uparrow}, S)$ is an event where $D \subseteq S$, the negation $\neg(D^{\uparrow}, S)$ of $(D^{\uparrow}, S)$ is defined by $\neg(D^{\uparrow}, S) := ((S \setminus D)^{\uparrow}, S)$. Note, that by this definition, the negation of a (measurable) event is a (measurable) event. Abusing notation, we write $\neg D^{\uparrow} := (S \setminus D)^{\uparrow}$. Note that by our notational convention, we have $\neg S^{\uparrow} = \emptyset^S$ and $\neg \emptyset^S = S^{\uparrow}$, for each space $S \in \mathcal{S}$. The event $\emptyset^S$ should be interpreted as a "logical contradiction phrased with the expressive power available in S." $\neg D^{\uparrow}$ is typically a proper subset of the complement $\Omega \setminus D^{\uparrow}$. That is, $(S \setminus D)^{\uparrow} \subsetneqq \Omega \setminus D^{\uparrow}$.

Intuitively, there may be states in which the description of an event $D^{\uparrow}$ is both expressible and valid – these are the states in $D^{\uparrow}$; there may be states in which its description is expressible but invalid – these are the states in $\neg D^{\uparrow}$; and there may be states in which neither its description nor its negation are expressible – these are the states in

$$\Omega \setminus \left(D^{\uparrow} \cup \neg D^{\uparrow}\right) = \Omega \setminus S\left(D^{\uparrow}\right)^{\uparrow}.$$

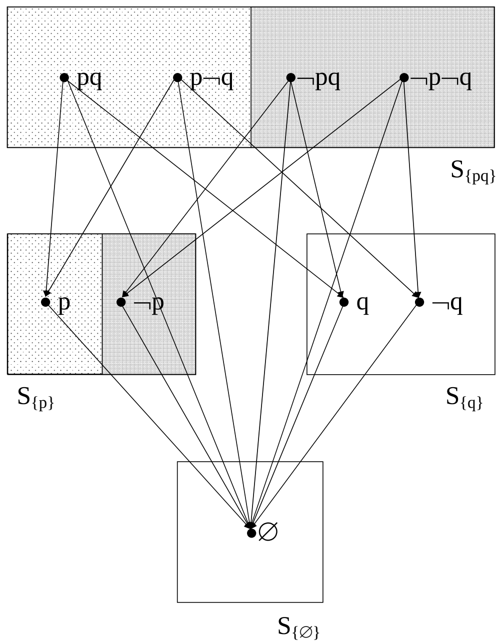
Thus our structure is not a standard state-space model in the sense of Dekel, Lipman, and Rustichini (1998).

2.5 Conjunction and Disjunction

If $\left\{\left(D_{\lambda}^{\uparrow}, S_{\lambda}\right)\right\}_{\lambda \in L}$ is a finite or countable collection of events (with $D_{\lambda} \subseteq S_{\lambda}$, for $\lambda \in L$), their conjunction $\bigwedge_{\lambda \in L}\left(D_{\lambda}^{\uparrow}, S_{\lambda}\right)$ is defined by $\bigwedge_{\lambda \in L}\left(D_{\lambda}^{\uparrow}, S_{\lambda}\right) := \left(\left(\bigcap_{\lambda \in L} D_{\lambda}^{\uparrow}\right), \sup_{\lambda \in L} S_{\lambda}\right)$. Note, that since $\mathcal{S}$ is a *complete* lattice, $\sup_{\lambda \in L} S_{\lambda}$ exists. If $S = \sup_{\lambda \in L} S_{\lambda}$, then we have $\left(\bigcap_{\lambda \in L} D_{\lambda}^{\uparrow}\right) = \left(\bigcap_{\lambda \in L}\left(\left(r_{S_{\lambda}}^{S}\right)^{-1}(D_{\lambda})\right)\right)^{\uparrow}$. Again, abusing notation, we write $\bigwedge_{\lambda \in L} D_{\lambda}^{\uparrow} := \bigcap_{\lambda \in L} D_{\lambda}^{\uparrow}$ (we will therefore use the conjunction symbol $\wedge$ and the intersection symbol $\cap$ interchangeably).

We define the relation $\subseteq$ between events (E, S) and (F, S'), by $(E, S) \subseteq (F, S')$ if

Figure 2: Event Structure



and only if $E \subseteq F$ as sets *and* $S' \preceq S$. If $E \neq \emptyset$, we have that $(E, S) \subseteq (F, S')$ if and only if $E \subseteq F$ as sets. Note however that for $E = \emptyset^S$ we have $(E, S) \subseteq (F, S')$ if and only if $S' \preceq S$. Hence we can write $E \subseteq F$ instead of $(E, S) \subseteq (F, S')$ as long as we keep in mind that in the case of $E = \emptyset^S$ we have $\emptyset^S \subseteq F$ if and only if $S \succeq S(F)$. It follows from these definitions that for events E and F, $E \subseteq F$ is equivalent to $\neg F \subseteq \neg E$ only when E and F have the same base, i.e., $S(E) = S(F)$.

The disjunction of $\left\{D_\lambda^\uparrow\right\}_{\lambda \in L}$ is defined by the de Morgan law $\bigvee_{\lambda \in L} D_\lambda^\uparrow = \neg \left(\bigwedge_{\lambda \in L} \neg \left(D_\lambda^\uparrow\right)\right)$. Typically $\bigvee_{\lambda \in L} D_\lambda^\uparrow \subsetneqq \bigcup_{\lambda \in L} D_\lambda^\uparrow$, and if all D_λ are nonempty we have that $\bigvee_{\lambda \in L} D_\lambda^\uparrow = \bigcup_{\lambda \in L} D_\lambda^\uparrow$ holds if and only if all the $D_\lambda^\uparrow$ have the same base-space. Note, that by these definitions, the conjunction and disjunction of (at most countably many measurable) events is a (measurable) event.

Apart from the measurability conditions, the event-structure outlined so far is analogous to Heifetz, Meier and Schipper (2006, 2008). An example is shown in Figure 2. It depicts a lattice with four spaces and projections. The event that p obtains is indicated by the dotted areas, whereas the grey areas illustrate the event that not p obtains. $S_p \cup S_q$ is for instance not an event in our structure.

2.6 Probability Measures

Here and in what follows, the term 'events' always measurable events in Σ unless otherwise stated.

Let $\Delta(S)$ be the set of probability measures on $(S, \mathcal{F}_S)$. We consider this set itself as a measurable space endowed with the σ-field $\mathcal{F}_{\Delta(S)}$ generated by the sets $\{\mu \in \Delta(S) : \mu(D) \geq p\}$, where $D \in \mathcal{F}_S$ and $p \in [0,1]$.

2.7 Marginals

For a probability measure $\mu \in \Delta(S')$, the marginal $\mu_{|S}$ of μ on $S \preceq S'$ is defined by

$$\mu_{|S}(D) := \mu\left(\left(r_S^{S'}\right)^{-1}(D)\right), \quad D \in \mathcal{F}_S.$$

Let S_μ be the space on which μ is a probability measure. Whenever $S_\mu \succeq S(E)$ then we abuse notation slightly and write

$$\mu(E) = \mu(E \cap S_\mu).$$

If $S(E) \npreceq S_\mu$, then we say that $\mu(E)$ is undefined.

2.8 Types

I is the nonempty set of individuals. For every individual, each state gives rise to a probabilistic belief over states in some space.

Definition 1 *For each individual $i \in I$ there is a type mapping $t_i : \Omega \to \bigcup_{\alpha \in \mathcal{A}} \Delta(S_\alpha)$, which is measurable in the sense that for every $S \in \mathcal{S}$ and $Q \in \mathcal{F}_{\Delta(S)}$ we have $t_i^{-1}(Q) \cap S \in \mathcal{F}_S$.*

We require the type mapping t_i to satisfy the following properties:

(0) Confinement: If $\omega \in S'$ then $t_i(\omega) \in \triangle(S)$ for some $S \preceq S'$.

(1) If $S'' \succeq S' \succeq S$, $\omega \in S''$, and $t_i(\omega) \in \triangle(S)$ then $t_i(\omega_{S'}) = t_i(\omega)$.

(2) If $S'' \succeq S' \succeq S$, $\omega \in S''$, and $t_i(\omega) \in \triangle(S')$ then $t_i(\omega_S) = t_i(\omega)_{|S}$.

(3) If $S'' \succeq S' \succeq S$, $\omega \in S''$, and $t_i(\omega_{S'}) \in \triangle(S)$ then $S_{t_i(\omega)} \succeq S$.

$t_i(\omega)$ represents individual i's belief at state ω. Properties (0) to (3) guarantee the consistent fit of beliefs and awareness at different state-spaces. *Confinement* means that at any given state $\omega \in \Omega$ an individual's belief is concentrated on states that are all described with the same "vocabulary" - the "vocabulary" available to the individual at ω. This "vocabulary" may be less expressive than the "vocabulary" used to describe statements in the state ω."

Properties (1) to (3) compare the types of an individual in a state ω and its projection to ω_S. Property (1) and (2) mean that at the projected state ω_S the individual believes everything she believes at ω given that she is aware of it at ω_S. Property (3) means that at ω an individual can not be unaware of an event that she is aware of at the projected state ω_S.

Define[4]

$$Ben_i(\omega) := \left\{ \omega' \in \Omega : t_i(\omega')_{|S_{t_i(\omega)}} = t_i(\omega) \right\}.$$

This is the set of states at which individual i's type or the marginal thereof coincides with her type at ω. Such sets are events in our structure:

Remark 1 *For any $\omega \in \Omega$, $Ben_i(\omega)$ is an $S_{t_i(\omega)}$-based event, which is not necessarily measurable.*[5]

Assumption 1 *If $Ben_i(\omega) \subseteq E$, for an event E, then $t_i(\omega)(E) = 1$.*

This assumption implies introspection (Property (va)) in Proposition 9 in the appendix. Note, that if $Ben_i(\omega)$ is measurable, then Assumption 1 implies $t_i(\omega)(Ben_i(\omega)) = 1$.

Definition 2 *We denote by $\underline{\Omega} := \left\langle \mathcal{S}, \left(r^{S_\alpha}_{S_\beta}\right)_{S_\beta \preceq S_\alpha}, (t_i)_{i \in I} \right\rangle$ an interactive unawareness belief structure.*

2.9 Awareness and Unawareness

The definition of awareness is analogous to the definition in unawareness knowledge structures (see Remark 6 in Heifetz, Meier and Schipper, 2008).

[4] The name "Ben" is chosen analogously to the "ken" in knowledge structures.

[5] Even in a standard type-space, if the σ-algebra is not countably generated, then the set of states where a player is of a certain type might not be measurable.

Definition 3 *For $i \in I$ and an event E, define the awareness operator*

$$A_i\left(E\right) := \left\{\omega \in \Omega : t_i\left(\omega\right) \in \Delta\left(S\right), S \succeq S\left(E\right)\right\}$$

if there is a state ω such that $t_i(\omega) \in \Delta(S)$ with $S \succeq S(E)$, and by

$$A_i(E) := \emptyset^{S(E)}$$

otherwise.

An individual is aware of an event if and only if his type is concentrated on a space in which the event is "expressible."

Proposition 1 *If E is an event then $A_i(E)$ is an $S\left(E\right)$-based event.*

This proposition shows that the set of states in which an individual is aware of an event is indeed an event in our structure. Moreover, the operator is convenient to work with since the event $A_i(E)$ has the same base-space as the event E.

Unawareness is naturally defined as the negation of awareness:

Definition 4 *For $i \in I$ and an event E, the unawareness operator is defined by*

$$U_i(E) = \neg A_i(E).$$

Note that the definition of our negation and Proposition 1 imply that if E is an event, then $U_i(E)$ is an $S\left(E\right)$-based event.

Note further that Definition 3 and 4 apply also to events that are not necessarily measurable.

2.10 Belief

The p-belief-operator is defined as usual (see for instance Monderer and Samet, 1989):

Definition 5 *For $i \in I$, $p \in [0,1]$ and an event E, the p-belief operator is defined, as usual, by*

$$B_i^p(E) := \{\omega \in \Omega : t_i(\omega)(E) \geq p\},$$

if there is a state ω such that $t_i(\omega)(E) \geq p$, and by

$$B_i^p(E) := \emptyset^{S(E)}$$

otherwise.

Proposition 2 *If E is an event then $B_i^p(E)$ is an $S(E)$-based event.*

This proposition shows that the set of states in which an individual believes an event with probability at least p is an event in our structure that has the same base-space as the event E.

The p-belief operator has the standard properties stated in Proposition 9 in the appendix.

2.11 Properties of Unawareness

Dekel, Lipman and Rustichini (1998) showed that in a standard state-space unawareness must be trivial, even if the belief operator satisfies only very weak properties. In contrast, we show that we obtain all properties suggested in the literature.[6]

Proposition 3 *Let E be an event and $p, q \in [0,1]$. The following properties of awareness and belief obtain: 1. Plausibility: $U_i(E) \subseteq \neg B_i^p(E) \cap \neg B_i^p \neg B_i^p(E)$, 2. Strong Plausibility: $U_i(E) \subseteq \bigcap_{n=1}^{\infty} (\neg B_i^p)^n (E)$, 3. B^pU Introspection: $B_i^p U_i(E) = \emptyset^{S(E)}$ for $p \in (0,1]$ and $B_i^0 U_i(E) = A_i(E)$, 4. AU Introspection: $U_i(E) = U_i U_i(E)$, 5. Weak Necessitation: $A_i(E) = B_i^1\left(S(E)^{\uparrow}\right)$, 6. $B_i^p(E) \subseteq A_i(E)$ and $B_i^0(E) = A_i(E)$, 7. $B_i^p(E) \subseteq A_i B_i^q(E)$, 8. Symmetry: $A_i(E) = A_i(\neg E)$, 9. A Conjunction: $\bigcap_{\lambda \in L} A_i(E_\lambda) = A_i\left(\bigcap_{\lambda \in L} E_\lambda\right)$, 10. AB^p Self Reflection: $A_i B_i^p(E) = A_i(E)$, 11. AA Self Reflection: $A_i A_i(E) = A_i(E)$, and 12. $B_i^p A_i(E) = A_i(E)$.*

Note that properties 3, 4, 5, 8, 9, 11, and 12 hold also for non-measurable events, because even if E is not measurable, by 5. $A_i(E)$ is measurable.

Although we model awareness of events, Property 8 suggests that we model a notion of awareness of issues or questions. Let an issue or question (E.g., "is the stock market crashing?") be such that it can be answered in the affirmative ("The stock market is crashing.") or the negative ("The stock market is not crashing."). By symmetry (Property 8), an individual is aware of an event if and only if she is aware of the its negation. Thus we model the awareness of questions and issues rather than just single events. In fact, by weak necessitation, Property 5, an individual is aware of an event E

[6] These properties are analogous to the properties in unawareness knowledge structures (Heifetz, Meier and Schipper, 2006, 2008). Properties 1 to 5 have been suggested by Dekel, Lipman and Rustichini (1998), and 8 to 11 by Fagin and Halpern (1988), Modica and Rustichini (1999) and Halpern (2001).

if and only if she is aware of any event that can be "expressed" in a space with the same "expressive power" as the base-space of E.

In Proposition 11 in the appendix, we state some multi-person properties of awareness and belief. For instance, we show that if an individual is aware of an event E, then she can also conceive that others are aware of the event E. Moreover, we show that common awareness and mutual awareness coincide. That is, if everybody is aware of an event, then everybody can conceive that everybody is aware of the event, everybody is aware of that, etc.

2.12 Unawareness versus Zero Probability

In this section we discuss how unawareness of an event can be characterized as zero probability belief of the *existence* of the event.

For any event $E \in \Sigma$, consider the event $S(E)^{\uparrow}$. We interpret this event *as the event that E exists*. That is, in any state $\omega \in S(E)^{\uparrow}$ either E or $\neg E$ obtains. This interpretation of $S(E)^{\uparrow}$ is crucial for the interpretation of unawareness as zero probability belief.

Let S_ω be the space $S \in \mathcal{S}$ with $\omega \in S$.

We extend the type mapping t_i to a mapping t_i^Z by for all events $E \in \Sigma$,

$$t_i^Z(\omega)(E) \quad := \quad \begin{cases} t_i(\omega)(E) & \text{if } S_{t_i(\omega)} \succeq S(E) \\ 0 & \text{if } S_{t_i(\omega)} \not\succeq S(E) \text{ and } S_\omega \succeq S(E) \\ \text{undefined} & \text{otherwise.} \end{cases}$$

That is, for every $\omega \in \Omega$ the extended type mapping induces a belief over all events that exist at ω.

We use the extended type mapping to define the zero probability belief operator on events in Σ.

Definition 6 *For $i \in I$ and an event $E \in \Sigma$, the zero-probability operator is defined by*

$$Z_i(E) = \{\omega \in \Omega : t_i^Z(\omega)(E) = 0\}$$

if there is a state $\omega \in \Omega$ such that $t_i^Z(\omega)(E) = 0$, and by

$$Z_i(E) = \emptyset^{S(E)}$$

otherwise.

Note that different from standard state-spaces, we may have $Z_i(S(E)^{\uparrow}) \neq \emptyset$ for some event E. (In this case it must be that $S(E) \succ \inf \mathcal{S}$.) That is, if $Z_i(S(E)^{\uparrow}) \neq \emptyset$ then there is a state in which agent i assigns zero probability belief to the event that the event E exists.

Zero probability belief about the event that the event E exists characterizes unawareness of the event.

Proposition 4 *For any event $E \in \Sigma$, $Z_i(S(E)^{\uparrow}) = U_i(E)$.*

Zero probability so defined in unawareness structures behaves differently from zero probability belief in standard type-spaces because agents can assign not only zero probability to an event but also to the *existence of an event.* In particular we show in Proposition 5 that in our unawareness structures, an agent assigns zero probability to the existence of an event if and only if it assigns zero probability to the event *and* zero probability to the negation of the event. This is impossible in a standard type-space.

Proposition 5 *For any event $E \in \Sigma$, $Z_i(S(E)^{\uparrow}) = Z_i(E) \cap Z_i(\neg E)$.*

In applications one may be tempted to work with zero probability only instead of the notion of unawareness. The difficulty is that zero probability of an event becomes an "overburdened" notion. This is because when an agent assigns zero probability to an event she may do so because she is unaware of the event or because she is aware of the event but assigns zero probability to it. In other words - taking Proposition 4 into account - if an agent assigns zero probability to an event then she may do so because she assigns zero probability to the existence of the event or she is certain of the existence of the event but assigns zero probability to that the event obtains. Having both a notion of unawareness and belief allows us to precisely distinguish between these two cases. This is shown in the following proposition. Note that in the event $B_i^1(\neg E)$ agent i assigns zero probability to the event E (and is aware of the event E).

Proposition 6 *For any event $E \in \Sigma$, $Z_i(E) = U_i(E) \cup (A_i(E) \cap B_i^1(\neg E))$.*

Proposition 6 also implies that the zero probability operator is a map from Σ to Σ.

Corollary 1 *For every event $E \in \Sigma$, $Z_i(E)$ is a $S(E)$-based event.*

To sum up, if instead of unawareness one wants to work with a notion of zero probability, then unawareness of an event corresponds to zero probability of the existence of that event. Since such zero probability statements are impossible to express in standard state-spaces, unawareness structures are useful because the lattice of spaces allows us to model for each event E the event that E exists.

Furthermore, the notion of zero probability is less useful than the notion of unawareness because it does not allow us to distinguish between zero probability of the event E due to zero probability assigned to the existence of E or certainty of its existence but zero probability that the event E obtains. Together with the fact that unawareness is conceptually different from probability zero belief, this makes the zero probability model inferior to the model of unawareness.

2.13 The Connection to Standard Type Spaces

We show how to derive standard type-space from our unawareness structure by "flattening" our lattice of spaces. Moreover, we demonstrate with a simple example that not every standard type-space can be derived from non-trivial unawareness structures.

Definition 7 *$G \subseteq \Omega$ is a* measurable set *if and only if for all $S \in \mathcal{S}$, $G \cap S \in \mathcal{F}_S$.*

Notice that a measurable set is not necessarily an event in our special event structure.

Remark 2 *The collection of measurable sets forms a sigma-algebra on Ω.*

Remark 3 *Let $\mathcal{S}$ be at most countable and G be a measurable set, $p \in [0,1]$ and $i \in I$. Then $\{\omega \in \Omega : t_i(\omega)(G) \geq p\}$ is a measurable set.*

Let $\underline{\Omega}$ be an unawareness belief structure. We define the *flattened type-space* associated with the unawareness belief structure $\underline{\Omega}$ by

$$F(\underline{\Omega}) := \langle \Omega, \mathcal{F}, (t_i^F)_{i \in I} \rangle,$$

where Ω is the union of all state-spaces in the unawareness belief structure $\underline{\Omega}$, $\mathcal{F}$ is the collection of all measurable sets in $\underline{\Omega}$, and $t_i^F : \Omega \longrightarrow \Delta(\Omega, \mathcal{F})$ is defined by

$$t_i^F(\omega)(E) \;:=\; \begin{cases} t_i(\omega)(E \cap S_{t_i(\omega)}) & \text{if } E \cap S_{t_i(\omega)} \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

A *standard type-space* on Y for the player set I is a tuple

$$\underline{Y} := \left\langle Y, \mathcal{F}_Y, (t_i)_{i \in I} \right\rangle,$$

where Y is a nonempty set, $\mathcal{F}_Y$ is a sigma-field on Y, and for $i \in I$, t_i is a $\mathcal{F}_Y - \mathcal{F}_{\Delta(Y)}$ measurable function from Y to $\Delta(Y, \mathcal{F}_Y)$, the space of countable additive probability measures on $(Y, \mathcal{F}_Y)$, such that for all $\omega \in Y$ and $E \in \mathcal{F}_Y : [t_i(\omega)] \subseteq E$ implies $t_i(\omega)(E) = 1$, where $[t_i(\omega)] := \{\omega' \in Y : t_i(\omega') = t_i(\omega)\}$.

Proposition 7 *If $\underline{\Omega}$ is an unawareness belief structure, then $F(\underline{\Omega})$ is a standard type-space. Moreover, it has the following property: For every $p > 0$, measurable set $E \in \mathcal{F}$ and $i \in I$: $\{\omega \in \Omega : t_i(\omega)(E) \geq p\} = \{\omega \in \Omega : t_i^F(\omega)(E) \geq p\}$.*

A flattened unawareness structure is just a standard type-space. To derive such a type-space, one extends a player's type mapping by assigning probability zero to measurable sets for which the player's belief was previously undefined. Of course, once an unawareness structure is flattened, there is no way to analyze reasoning about unawareness anymore since by Dekel, Lipman and Rustichini (1998) unawareness is trivial. Moreover, there is no way to analyze probability zero belief of the existence of an event as in the previous section.

Note that the converse to Proposition 7 is not true. I.e., given a standard type-space, it is not always possible to find some unawareness structure with non-trivial unawareness. This is illustrated in the following counter-example. We conclude that not every standard types-space with zero probability can be used to model unawareness. The precise restrictions required for modeling unawareness are made transparent in unawareness belief structures.

Example 1 Let $Y = \{\omega_1, \omega_2, \omega_3\}$ with $t_i(\omega_1) = t_i(\omega_2) = t_i(\omega_3) = \tau_i$ and $\tau_i(\{\omega_1\}) = \tau_i(\{\omega_2\}) = \frac{1}{2}$ and $\tau_i(\{\omega_3\}) = 0$. If $\Omega = S = Y$, then by Dekel, Lipman and Rustichini (1998) the unawareness structure has trivial unawareness only. Any non-trivial partition of Y into separate spaces yields either no projections or violates properties (0) to (3). $\square$

3 Common Prior, Agreement, and Speculation

In this section, we define a common prior and explore the implications. In Section 1.1, we showed by example that the common prior assumption is too weak to rule out speculative trade under unawareness. With unawareness, we can have common certainty of willingness to trade but strict preference to trade. Yet, we are able to prove a "No-Trade" theorem according to which there can not be common certainty of strict preference to trade under unawareness. In the same vein, we prove a "No-Agreeing-to-Disagree" theorem.

3.1 Common Belief

From now on, we assume that the set of individuals I is at most countable.

We define mutual and common belief as usual (e.g. Monderer and Samet, 1989):

Definition 8 *The mutual p-belief operator on events is defined by*

$$B^p(E) = \bigcap_{i \in I} B_i^p(E).$$

The common certainty operator on events is defined by

$$CB^1(E) = \bigcap_{n=1}^{\infty} \left(B^1\right)^n (E).$$

That is, the mutual p-belief of an event E is the event in which everybody p-believes the event E. Common certainty of E is the event that everybody is certain of the event E, and everybody is certain that everybody is certain of the event E, everybody is certain of that, ... ad infinitum. Common certainty is the generalization of common knowledge to the probabilistic notion of certainty. Note that Proposition 2 and the definition of the conjunction of events imply that $B^p(E)$ and $CB^1(E)$ are $S(E)$-based events, for any measurable event E.

We say that an event E is *common certainty* at $\omega \in \Omega$ if $\omega \in CB^1(E)$.

Propositions 10 and 11 (see appendix) state some properties of belief and awareness in the multiperson context.

3.2 Priors and Common Priors

In a standard type-space S, a *prior* P_i^S of player i is a convex combination of the beliefs of i's types in S (Samet, 1998). That is, for every event $E \in \mathcal{F}_S$,

$$P_i^S(E) = \int_S t_i(\cdot)(E)\, dP_i^S(\cdot). \tag{1}$$

In particular, if S is finite or countable, this equality holds if and only if

$$P_i^S(E) = \sum_{s \in S} t_i(s)(E) P_i^S(\{s\}). \tag{2}$$

In words, to find the probability $P_i^S(E)$ that the prior P_i^S assigns to an event E, one should check the beliefs $t_i(s)(E)$ ascribed by player i to the event E in each state $s \in S$, and then average these beliefs according to the weights $P_i^S(\{s\})$ assigned by the prior P_i^S to the different states $s \in S$.

P^S is a *common prior* on S if P^S is a prior for every player $i \in I$.

Here we generalize these definitions to unawareness structures, as follows.

Definition 9 (Prior) *A prior for player i is a system of probability measures $P_i = \left(P_i^S\right)_{S \in \mathcal{S}} \in \prod_{S \in \mathcal{S}} \Delta(S)$ such that*

1. *The system is projective: If $S' \preceq S$ then the marginal of P_i^S on S' is $P_i^{S'}$. (That is, if $E \in \Sigma$ is an event whose base-space $S(E)$ is lower or equal to S', then $P_i^S(E) = P_i^{S'}(E)$.)*
2. *Each probability measure P_i^S is a convex combination of i's beliefs in S: For every event $E \in \Sigma$ such that $S(E) \preceq S$,*

$$P_i^S(E \cap S \cap A_i(E)) = \int_{S \cap A_i(E)} t_i(\cdot)(E)\, dP_i^S(\cdot). \tag{1u}$$

$P = \left(P^S\right)_{S \in \mathcal{S}} \in \prod_{S \in \mathcal{S}} \Delta(S)$ is a common prior if P is a prior for every player $i \in I$.

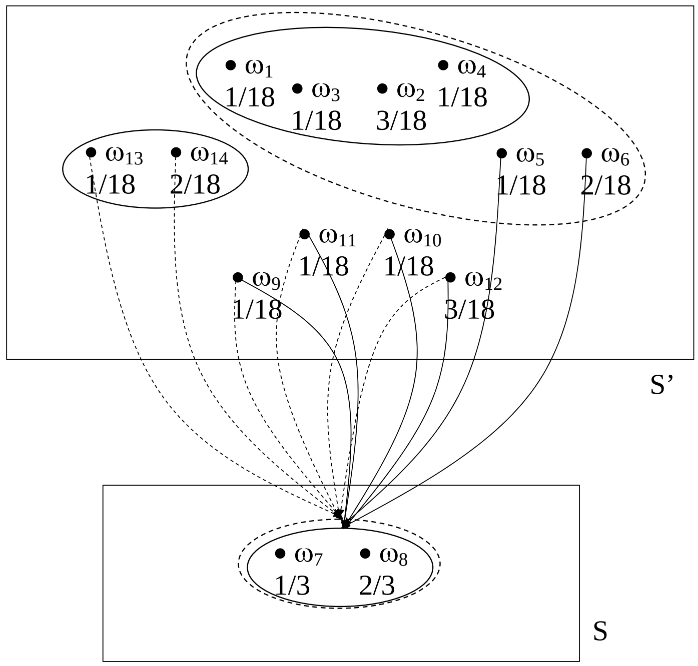
In particular, if S is finite or countable, equality (1u) holds if and only if

$$P_i^S(E \cap S \cap A_i(E)) = \sum_{s \in S \cap A_i(E)} t_i(s)(E) P_i^S(\{s\}). \tag{2u}$$

What is the reason for the difference between (1) and (1u) (or similarly between (2) and (2u))? With unawareness, $t_i(s)(E)$ is well defined only for states $s \in S$ in which player i is aware of E, i.e., the states $s \in S \cap A_i(E)$. This is the cause for the difference in the definition of the domain of integration (or summation) on the right-hand side. Consequently, E (or equivalently $E \cap S$) on the left-hand side of (1) and (2) is replaced by $E \cap S \cap A_i(E)$ in (1u) and (2u).

An example of an unawareness structure with a common prior is given in Figure 3. A discussion of the common prior (and Figure 3) is deferred to Section 4.1.

Figure 3: Illustration of a Common Prior



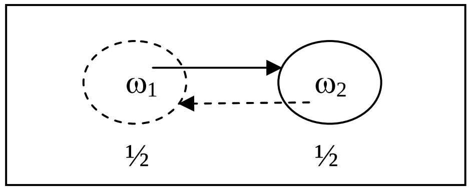
3.3 Speculative Trade

In this section, we investigate whether the common prior assumption implies the absence of speculative trade (e.g. Milgrom and Stokey, 1982). The example in Section 1.1 shows that speculation is possible under unawareness even if we assume that there is a common prior. Despite this counter example to the "No-trade" theorems, we prove below a generalized "No-trade" theorem according to which, if there is a common prior, then there can not be common certainty of *strict* preference to trade. That is, even with unawareness it is not the case that "everything goes". We find this surprising, because unawareness can be interpreted as a special form of "delusion": At a given state of a space, a player's belief may be concentrated in a very different lower state-space.

The following example demonstrates that speculative trade is possible in delusional standard state-space structures with a common prior.

Example 4 (Speculative Trade with Delusion) Consider the information structure in Figure 4. The common prior and the information structure allows the dashed player to have a posterior of $t_{dashed}(\omega_1)(\{\omega_1\}) = t_{dashed}(\omega_2)(\{\omega_1\}) = 1$ and the solid player $t_{solid}(\omega_1)(\{\omega_2\}) = t_{solid}(\omega_2)(\{\omega_2\}) = 1$. So they may happily disagree on the expected

Figure 4: Speculative Trade with Delusion



value of a random variable defined on this standard state-space. $\square$

Denote by $[t_i(\omega)] := \{\omega' \in \Omega : t_i(\omega') = t_i(\omega)\}$.

Definition 10 *A common prior* $P = \left(P^S\right)_{S \in \mathcal{S}} \in \prod_{S \in \mathcal{S}} \Delta(S)$ *is* positive *if and only if for all* $i \in I$ *and* $\omega \in \Omega$*: If* $t_i(\omega) \in \triangle(S')$*, then* $[t_i(\omega)] \cap S' \in \mathcal{F}_{S'}$ *and* $P^S\left(([t_i(\omega)] \cap S')^{\uparrow} \cap S\right) > 0$ *for all* $S \succeq S'$*.*

For every type, a positive common prior puts a positive weight on each "stationary" state where the player has this type. This technical condition serves the same purpose as the assumption in Aumann (1976) that the prior puts strict positive weight on each partition cell in his finite partitional structure. Our condition implies that for each player there can be at most countably many types in each space.

Definition 11 *Let* x_1 *and* x_2 *be real numbers and* v *a random variable on* Ω*. Define the sets* $E_1^{\leq x_1} := \left\{\omega \in \Omega : \int_{S_{t_1(\omega)}} v(\cdot)\, d(t_1(\omega))(\cdot) \leq x_1\right\}$ *and* $E_2^{\geq x_2} := \left\{\omega \in \Omega : \int_{S_{t_2(\omega)}} v(\cdot)\, d(t_2(\omega))(\cdot) \geq x_2\right\}$*. We say that at* ω*, conditional on his information, player 1 (resp. player 2) believes that the expectation of* v *is weakly below* x_1 *(resp. weakly above* x_2*) if and only if* $\omega \in E_1^{\leq x_1}$ *(resp.* $\omega \in E_1^{\geq x_2}$*).*

Note that the sets $E_1^{\leq x_1}$ or $E_2^{\geq x_2}$ may not be events in our unawareness belief structure, because $v(\omega) \neq v(\omega_S)$ is allowed, for $\omega \in S' \succ S$. Yet, we can define p-belief, mutual p-belief and common certainty for measurable subsets of Ω, and show that the properties stated in Propositions 9 and 10 obtain as well. The proofs are analogous and thus omitted.[7]

[7] Contrary to our definition of the negation of an event, in point (ii) of Proposition 9, $\neg E$ is here understood to be the relative complement of E with respect to the union of state-spaces.

Theorem 1 *Let $\underline{\Omega}$ be a finite unawareness belief structure and $P = \left(P^S\right)_{S \in \mathcal{S}} \in \prod_{S \in \mathcal{S}} \Delta(S)$ be a positive common prior. Then there is no state $\tilde{\omega} \in \Omega$ such that there are a random variable $v : \Omega \longrightarrow \mathbb{R}$ and $x_1, x_2 \in \mathbb{R}$, $x_1 < x_2$, with the following property: at $\tilde{\omega}$ it is common certainty that conditional on her information, player 1 believes that the expectation of v is weakly below x_1 and, conditional on his information, player 2 believes that the expectation of v is weakly above x_2.*

The theorem says that if there is a positive common prior, then there can not be common certainty of strict preference to trade. Together with our example of speculative trade under unawareness we conclude that a common prior does not rule out speculation under unawareness but it can never be common certainty that both players expect to strictly gain from speculation. The theorem implies as a corollary that given a positive common prior, *arbitrary small transaction fees rule out speculative trade under unawareness.*

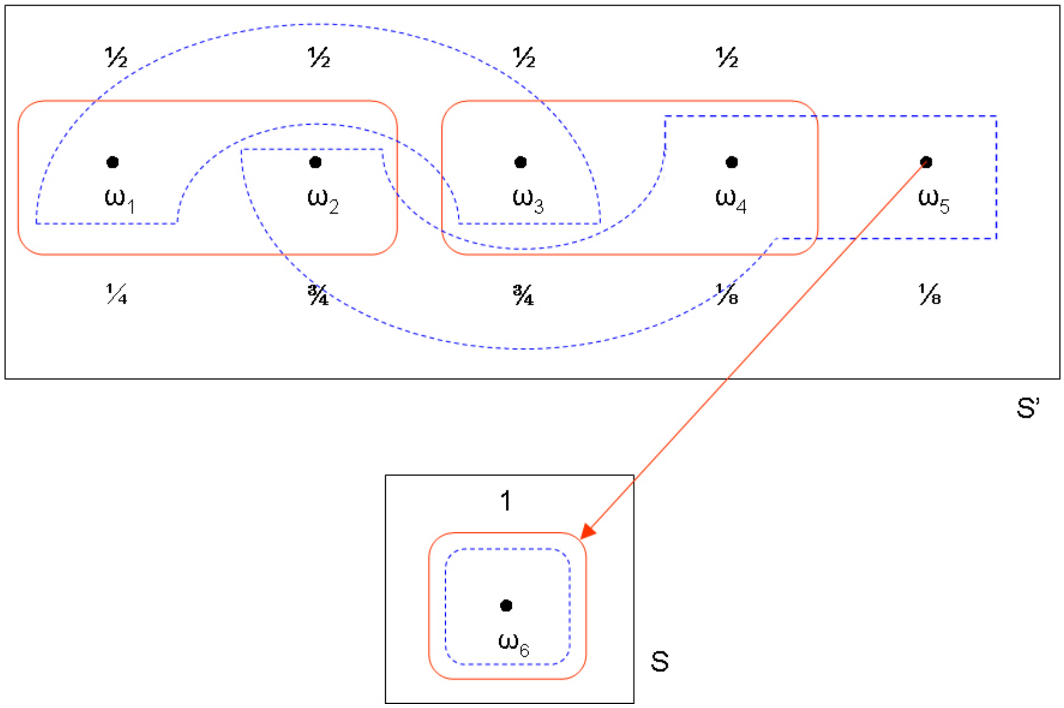
So, with respect to speculative trade, heterogeneous unawareness with a common prior is "intermediate" between common awareness with heterogeneous priors on the one hand, and common awareness with a common prior on the other hand. With heterogeneous priors even in standard state-spaces, common certainty of strict preference to trade is possible.

In Meier and Schipper (2009), we extend the above "No-trade" theorem to infinite unawareness belief structures. To this end we introduce topological unawareness belief structures.

The following example shows that the converse of the "No-trade" theorem does not hold.

Example 5 Consider the information structure with two spaces in Figure 5. There are two players: The information structure of the first (resp. second) player is given by the solid (resp. intermitted) objects. The belief of the first (resp. second) player is given above (resp. below) the states. Since the relative weights differ, there can not be a positive common prior. In fact, there is not even a common prior since equation (2u) of Definition 9 imposed on the priors of both individuals would imply that the common prior assigns probability zero to all states in S'. Note that the only measurable sets that are common certainty among both players are $\Omega = S' \cup S$ and S. Yet, it is not true that in all states in Ω or S player 1's expectation of a random variable differs from player 2's expectation. E.g., at ω_6 both player's expectations of the random variable must agree. Thus, the absence of common certainty of strict preference to trade does not imply the existence of a (positive) common prior.

Figure 5: Information Structure of the Counter-Example



3.4 Agreement

For an event E and $p \in [0,1]$ define the set $[t_i(E) = p] := \{\omega \in \Omega : t_i(\omega)(E) = p\}$, if $\{\omega \in \Omega : t_i(\omega)(E) = p\}$ is nonempty, and otherwise set $[t_i(E) = p] := \emptyset^{S(E)}$.

Lemma 1 *$[t_i(E) = p]$ is a $S(E)$-based event.*

Proof. $[t_i(E) = p] = B_i^p(E) \cap B_i^{1-p}(\neg E)$. Hence the claim follows from Proposition 2.□

The following proposition is a generalization of the standard *"No-Agreeing-to-Disagree" theorem* (Aumann, 1976):

Proposition 8 *Let $\underline{\Omega}$ be an unawareness belief structure, G be an event and $p_i \in [0,1]$, for $i \in I$. Suppose there exists a common prior $P = \left(P^S\right)_{S \in \mathcal{S}} \in \prod_{S \in \mathcal{S}} \Delta(S)$ such that* *for some space $S \succeq S(G)$ we have $P^S(CB^1(\bigcap_{i \in I}[t_i(G) = p_i])) > 0$. Then $p_i = p_j$, for all $i, j \in I$.*[8]

The proposition asserts the following: Suppose individuals have a common prior that is weakly positive in the sense that it assigns strict positive probability to the event that posteriors of G are common certainty. Then common certainty of posteriors for the event G implies that those posteriors must agree across all individuals. So individuals with a common prior can not agree-to-disagree on the posteriors of events which they are all aware of.

Remark 4 *A positive common prior (Definition 10) implies the condition $P^S(CB^1(\bigcap_{i \in I}[t_i(G) = p_i])) > 0$ in Proposition 8 if $CB^1(\bigcap_{i \in I}[t_i(G) = p_i])$ is nonempty and $S \succeq S(G)$.*

4 Discussion

4.1 Common Priors

How could a *prior* be interpreted? Following the discussion of the notion of a prior in standard Bayesian analysis by Savage (1954), Morris (1995) and Samet (1999), we like to distinguish three interpretations: First, a prior is interpreted verbally as a player's subjective belief at a prior stage. Second, the prior is a coherence condition on the player's types. Third, the prior is the long run relative frequency of repeated events observed by the player in the past.

Consider the first interpretation. A prior is a subjective belief at a prior stage before the player received further information which led her to the interim belief $t_i(\omega)$. With unawareness, this interpretation is nonsensical. One would have to imagine that the player had been aware of all relevant aspects of reality at the prior stage, but then became unaware of some of them (while nevertheless having received more information regarding other aspects).

In standard Bayesian analysis, Samet (1999) put forward a second interpretation of a prior as a coherence condition on types: For every event $E \in \Sigma$ and every $p \in [0, 1]$, every type of the player answers affirmatively to the question "Given that tomorrow you will assign to the event E probability at least p, do you assign to E probability at least

[8] In the appendix, we prove a more general version in which we require only a common prior on a space $S \succeq S(G)$ satisfying the condition stated in the proposition.

p now?" This interpretation is conceptually valid also for unawareness belief structures with an important qualification: Every type of the player is asked these questions only for events of which she is aware because otherwise a question by itself may make the type aware of an event of which she was previously unaware. While this qualification is vacuous in standard Bayesian analysis - because of the implicit assumption of full awareness - it implies for unawareness belief structures that each type is "aware" only of the prior restricted to the events that she is aware of. Moreover, every type can only perceive the beliefs of her types of which she is aware. This emphasizes that the prior is derived from types rather than being a primitive of the model.

The third interpretation views the prior as the relative frequency of events observed previously by the individual as history goes to infinity and before receiving information which led to her interim belief $t_i(\omega)$. Again, with unawareness such a interpretation is nonsensical. One would have to imagine that the player had been measuring all events in history, but then became unaware of some of them (while nevertheless having received more information regarding other events). To recapture the validity of the frequentist interpretation, we must assume that every player can observe only events that she is aware of interim. This assumption is quite reasonable since a player can only measure what she is aware of. For instance, meteorologists were unable to measure ozone before they became aware of it. Yet, the applicability of the frequentist interpretation may be limited since we allow also for conditioning on unobservable events (such as types of other players), a caveat that applies not only to unawareness belief structures but to belief structures in general.

A *common prior* is an identical prior among all players. In an unawareness belief structures with a common prior, each type is only "aware" of the common prior among the types (of hers or other players) that she is aware of. Figure 3 illustrates a common prior in an unawareness belief structure. Odd (resp. even) states in the upper space project to the odd (resp. even) state in the lower space. There are two individuals, one indicated by the solid lines and ellipses and another by dashed lines and ellipses. Note that the ratio of probabilities over odd and even states in each "information cell" coincides with the ratio in the "information cell" in the lower space.

The *positivity* condition (Definition 10) requires that for every player and every type, the common prior puts strict positive weight on the set of "stationary" states where the player has this type. It ensures that the common prior indeed imposes consistency on the types. To see this, consider once again Figure 3. Replace the common prior by a prior that assigns $\frac{1}{6}$ to each state ω_9, ω_{10}, ω_{11} and $\frac{3}{6}$ to ω_{12}, and zero to all other states in S'.

The prior probabilities for states in S remain unchanged. This prior is common prior but it does not satisfy the positivity assumption of Definition 10. In particular, this common prior does not constrain any player's types with beliefs on S'. So, for unawareness belief structures the positivity assumption on the common prior ensures that the common prior constrains the beliefs of types not just locally on some space but across the lattice. Essentially, it is in the spirit of common prior assumption according to which different beliefs are only due to differences in information. The positivity condition also implies that for each player there can be at most countably many types in each space. Moreover, in terms of awareness it implies that for every pair of players, i and j, and every event E, if i is certain that j is aware of the event E, then j is indeed aware of the event E.

If an unawareness belief structure has a common prior, then the associated flattened model (see Section 2.13) has a common prior. To see this, note that the common prior always induces a common prior on the smallest space, which implies that there is a common prior in the flattened model. If an unawareness belief structure has a positive common prior, then it does not follow that there is a positive common prior in the flattened model. To see this consider once again Figure 3. A common prior in the associated flattened model must ascribe probability zero to all states in S'. Such common prior clearly violates the positivity assumption of Definition 10. Again, this example demonstrates a difference between unawareness belief structures and standard type-spaces.

What are the implications of the absence of speculation on the priors? For standard type-spaces, the converse to the "No-trade" theorem characterizes the common prior assumption through the absence of speculative trade (Morris, 1994, Bonanno and Nehring, 1999, Feinberg, 2000, Halpern, 2002, Heifetz, 2006). Example 5 shows that we can not characterize positive common priors or even just common priors on unawareness belief structures by the absence of common certainty of strict preference to trade. Does our notion of "No-trade" imply at least the existence of a common prior in the flattened model? First, note that our notion of "No-trade" is slightly different from the literature: For instance, Feinberg (2000) characterizes the common prior by the absence of common certainty of speculation for *some* states. We show that a positive common prior implies the absence of common certainty of speculation for *all* states. Hence, our notion of "No-trade" implies Feinberg's notion of "No-trade".[9] Since Feinberg showed that his notion of "No-trade" implies a common prior for standard type-spaces, the existence of

[9]We opted for our notion of "local" speculation because intuitively one is interested to know whether there are *some* states (as opposed to *all* states) where players speculate. Our notion of "No-trade" coincides with Feinberg's notion on belief closed subsets.

a common prior for the flattened model of an unawareness belief structure follows from his result. Note that the impossibility of the converse to a "No-trade" theorem for unawareness belief structures is not due to the different notion of "No-trade" employed. To see this, consider once again Example 5. At state ω_6 it is not common certainty that players want to speculate. Yet, we noticed already that there is no common prior in this model. Hence, also "No-trade" in the sense of Feinberg does not imply a common prior in unawareness belief structures. To sum up, we show that it is still possible to define the common prior assumption under unawareness. Moreover, our "No-trade" theorem demonstrates that the common prior assumption enhanced by positivity imposes discipline. Yet, contrary to standard type-spaces the common prior assumption is not "provable" by the absence of speculation under unawareness, it just remains (in principle) "falsifiable". The possibility of characterizing a common prior by absence of speculation in the standard type-space versus the impossibility of such characterization in unawareness belief structures illustrates an important difference between unawareness belief structures and standard type-spaces.

4.2 Related Literature

There is a growing literature on unawareness both in economics and computer science. The independent parallel work of Sadzik (2006) is closest to ours. Building to a certain extent on our earlier work, Heifetz, Meier and Schipper (2006), he presents a framework of unawareness with probabilistic beliefs in which the common prior on the upmost space is a primitive. In contrast, we take types as primitives and define a prior on the entire unawareness belief structure as a convex combination of the type's beliefs.

In a companion paper, Heifetz, Meier and Schipper (2009a), we apply unawareness belief structures to develop Bayesian games with unawareness, define solutions, and prove existence. Moreover, we investigate the robustness of equilibria to uncertainty about opponents' unawareness of actions.

Feinberg (2009) discusses games with unawareness by modeling games and many views thereof, each (mutual) view being a finite sequence of player names $i_1, ..., i_n$ with the interpretation that this is how i_1 views how .... how i_n views the game. This differs from our unawareness belief structures in which each state "encapsulates" the views of the players, their views about other players' views etc. in a standard and parsimonious way.

Halpern and Rêgo (2006), Rêgo and Halpern (2007), Li (2006) and Heifetz, Meier and Schipper (2009b) and Feinberg (2009) present models of extensive form games with unawareness and analyze solution concepts for them. Li (2006) is based on Li (2009), in which she presents a set theoretic model with knowledge and non-trivial unawareness. A state-space is a product set where each dimension corresponds to an issue. A decision maker may be unaware of some issues by "living in" a space with less dimensions. Modica (2008) studies the updating of probabilities and argues that new information may change posteriors more if it implies also a higher level of awareness. A dynamic framework for a single decision maker with unawareness is introduced by Grant and Quiggin (2007). Ewerhart (2001) studies the possibility of agreement under a notion of unawareness different from the aforementioned literature. Lastly, Ahn and Ergin (2009) consider explicitly more or less fine descriptions of acts and characterize axiomatically a partition-dependent subjective expected utility representation. Since the set of all partitions of a state-space forms a complete lattice, their approach suggests a decision theoretic foundation of subjective probabilities on our lattice structure.

More recently we learned that Board and Chung (2009) presented a different model of unawareness in which they also study speculative trade under what they term living in "denial" and "paranoia". The precise connection to our results is yet to be explored.

Appendices

A Properties of Belief and Awareness

Proposition 9 *Let E and F be events, $\{E_l\}_{l=1,2,\dots}$ be an at most countable collection of events, and $p, q \in [0,1]$. The following properties of belief obtain:*

(o) $B_i^p(E) \subseteq B_i^q(E)$, *for* $q \leq p$,

(i) Necessitation: $B_i^1(\Omega) = \Omega$,

(ii) Additivity: $B_i^p(E) \subseteq \neg B_i^q(\neg E)$, *for* $p + q > 1$,

(iiia) $B_i^p\left(\bigcap_{l=1}^{\infty} E_l\right) \subseteq \bigcap_{l=1}^{\infty} B_i^p(E_l)$,

(iiib) for any decreasing sequence of events $\{E_l\}_{l=1}^{\infty}$, $B_i^p\left(\bigcap_{l=1}^{\infty} E_l\right) = \bigcap_{l=1}^{\infty} B_i^p(E_l)$,

(iiic) $B_i^1\left(\bigcap_{l=1}^{\infty} E_l\right) = \bigcap_{l=1}^{\infty} B_i^1(E_l)$,

(iv) Monotonicity: $E \subseteq F$ *implies* $B_i^p(E) \subseteq B_i^p(F)$,

(va) Introspection: $B_i^p(E) \subseteq B_i^1 B_i^p(E)$,

(vb) Introspection II: $B_i^p B_i^q(E) \subseteq B_i^q(E)$, *for* $p > 0$.

In our unawareness belief structure, Necessitation means that an individual always is certain of the universal event Ω, i.e. she is certain of "tautologies with the lowest expressive power." (ii) means that if an individual believes an event E with at least probability p, then she can not believe the negation of E with any probability strictly greater than $1 - p$. Property (iii a - c) are variations of conjunction, i.e., if an individual believes a conjunction of events with probability at least p, then she p-believes each of the events. The interpretation of monotonicity is: If an event E implies an event F, then p-believing the event E implies that the individual also p-believes the event F. Property (v) concerns the introspection of belief: If an individual believes the event E with at least probability p then she is certain that she believes the event E with at least probability p. Also, if she believes with positive probability that she p-believes an event, the she actually p-believes this event.

Definition 12 *An event E is evident if for each $i \in I$, $E \subseteq B_i^1(E)$.*

Proposition 10 *For every event $F \in \Sigma$:*

(i) $CB^1(F)$ is evident, that is $CB^1(F) \subseteq B_i^1(CB^1(F))$ for all $i \in I$.

(ii) There exists an evident event E such that $\omega \in E$ and $E \subseteq B_i^1(F)$ for all $i \in I$, if and only if $\omega \in CB^1(F)$.

The proof is analogous to Proposition 3 in Monderer and Samet (1989) for a standard state-space and thus omitted.

Analogously to mutual belief and common belief, we define mutual awareness and common awareness:

Definition 13 *The mutual awareness operator on events is defined by*

$$A(E) = \bigcap_{i \in I} A_i(E),$$

and the common awareness operator on events is defined by

$$CA(E) = \bigcap_{n=1}^{\infty} (A)^n (E).$$

Mutual awareness of an event E is the event that everybody is aware of E. Common awareness of an event E is the event that everybody is aware of E, everybody is aware that everybody is aware of E, everybody is aware of that ... ad infinitum.

Proposition 11 *Let E be an event and $p, q \in [0,1]$. The following multi-person properties obtain:*

1.	$A_i(E) = A_i A_j(E)$	7.	$B^p(E) \subseteq CA(E)$, $B^0(E) = CA(E)$,
2.	$A_i(E) = A_i B_j^p(E)$	8.	$B^p(E) \subseteq A(E)$, $B^0(E) = A(E)$,
3.	$B_i^p(E) \subseteq A_i B_j^q(E)$,	9.	$A(E) = B^1(S(E)^\uparrow)$
4.	$B_i^p(E) \subseteq A_i A_j(E)$,	10.	$CA(E) = B^1(S(E)^\uparrow)$
5.	$CA(E) = A(E)$,	11.	$CB^1(S(E)^\uparrow) \subseteq A(E)$
6.	$CB^1(E) \subseteq CA(E)$,	12.	$CB^1(S(E)^\uparrow) \subseteq CA(E)$

Note that properties 1, 5, 9, 10, 11 and 12 also hold for non-measurable events.

B Proofs

B.1 Proof of Remark 1

Define $D := \{\omega' \in S_{t_i(\omega)} : t_i(\omega') = t_i(\omega)\}$. I.e., $D = Ben_i(\omega) \cap S_{t_i(\omega)}$. We need to show that $D^\uparrow = Ben_i(\omega)$.

Consider first "$\subseteq$": If $\omega' \in D^\uparrow$ then $\omega'_{S_{t_i(\omega)}} \in Ben_i(\omega)$. This is equivalent to $t_i(\omega'_{S_{t_i(\omega)}}) = t_i(\omega) \in \triangle(S_{t_i(\omega)})$. By (3) we have $S_{t_i(\omega')} \succeq S_{t_i(\omega)}$. By (2), $t_i(\omega'_{S_{t_i(\omega)}}) = t_i(\omega')_{|S_{t_i(\omega)}}$. It follows that $t_i(\omega')_{|S_{t_i(\omega)}} = t_i(\omega)$. Thus $\omega' \in Ben_i(\omega)$.

"$\supseteq$": $\omega' \in Ben_i(\omega)$ if and only if $t_i(\omega')_{|S_{t_i(\omega)}} = t_i(\omega)$. Hence for $\omega' \in Ben_i(\omega)$, we have $S_{t_i(\omega')} \succeq S_{t_i(\omega)}$. By (2) $t_i(\omega'_{S_{t_i(\omega)}}) = t_i(\omega')_{|S_{t_i(\omega)}} = t_i(\omega)$. Hence $\omega'_{S_{t_i(\omega)}} \in D$. Thus $\omega' \in D^\uparrow$. □

B.2 Proof of Proposition 1

$A_i(E)$ is an $S(E)$-based event if there exists a subset $D \subseteq S(E)$ s.t. $D^\uparrow = A_i(E)$.

Assume that $A_i(E)$ is non-empty. Define $D := \{\omega \in S(E) : t_i(\omega) \in \Delta(S(E))\}$. By definition of the awareness operator, $D = A_i(E) \cap S(E)$. We show that $D^\uparrow = A_i(E)$.

Let $\omega \in D^{\uparrow}$, that is $\omega \in S'$ for some $S' \succeq S(E)$ and $\omega_{S(E)} \in D$. This is equivalent to $t_i(\omega_{S(E)}) \in \Delta(S(E))$. By 0. follows $S' \succeq S_{t_i(\omega)}$. By 3. we have $S_{t_i(\omega)} \succeq S(E)$. Thus $\omega \in A_i(E)$. (Note that $A_i(E) = \{\omega \in \Omega : S_{t_i(\omega)} \succeq S(E)\}$.)

In the reverse direction, let $\omega \in A_i(E)$, i.e., $t_i(\omega) \in \Delta(S)$ with $S \succeq S(E)$. By 0., $\omega \in S'$ with $S' \succeq S$. Consider $\omega_{S(E)}$. By 2., $t_i(\omega_{S(E)}) = t_i(\omega)_{|S(E)}$. Hence $\omega_{S(E)} \in D$. Thus $\omega \in D^{\uparrow}$.

Finally, if $A_i(E)$ is empty, then by definition of the awareness operator, we have $A_i(E) = \emptyset^{S(E)}$. $\square$

B.3 Proof of Proposition 2

$B_i^p(E)$ is an $S(E)$-based event if there exists a subset $D \subseteq S(E)$ s.t. $D^{\uparrow} = B_i^p(E)$. Assume that $B_i^p(E)$ is non-empty. Define $D := \{\omega \in S(E) : t_i(\omega)(E) \geq p\}$. By definition of the p-belief operator, $D = B_i^p(E) \cap S(E)$. We show that $D^{\uparrow} = B_i^p(E)$.

Let $\omega \in D^{\uparrow}$, that is $\omega \in S'$ for some $S' \succeq S(E)$ and $\omega_{S(E)} \in D$. This is equivalent to $t_i(\omega_{S(E)})(E) \geq p$. By 0. $S_{t_i(\omega_{S(E)})} = S(E)$. By 3. we have $S_{t_i(\omega)} \succeq S(E)$. By 2. it follows that $p \leq t_i(\omega_{S(E)})(E) = t_i(\omega)_{|S(E)}(E)$. Hence $t_i(\omega)(E) \geq p$. Thus $\omega \in B_i^p(E)$.

In the reverse direction, let $\omega \in B_i^p(E)$, i.e., $t_i(\omega)(E) \geq p$. Since $t_i(\omega)(E) \geq p$ it follows that $S_{t_i(\omega)} \succeq S(E)$. Let $\omega \in S'$. By 0. $S' \succeq S_{t_i(\omega)}$. Consider $\omega_{S(E)}$. By 2., $t_i(\omega_{S(E)})(E) = t_i(\omega)(E)_{|S(E)} \geq p$. Hence $\omega_{S(E)} \in D$. Thus $\omega \in D^{\uparrow}$.

Finally, if $B_i^p(E)$ is empty, then by definition of the p-belief operator, we have $B_i^p(E) = \emptyset^{S(E)}$. $\square$

B.4 Proof of Proposition 3

1. This property is equivalent to $B_i^p(E) \cup B_i^p \neg B_i^p(E) \subseteq A_i(E)$. By Property 5. we have $B_i^p(E) \subseteq A_i(E)$. To see that $B_i^p \neg B_i^p(E) \subseteq A_i(E)$, note that $\omega \in B_i^p \neg B_i^p(E)$ if and only if $t_i(\omega)(\neg B_i^p(E)) \geq p$. This implies that $S_{t_i(\omega)} \succeq S(\neg B_i^p(E)) = S(E)$. The last equality follows by Property 8 and Proposition 2. Hence $\omega \in A_i(E)$.

2. The proof is analogous to 1. The is property is equivalent to $\bigcap_{n=1}^{\infty} B_i^p \left(\neg B_i^p\right)^{n-1}(E) \subseteq A_i(E)$. $\omega \in B_i^p \left(\neg B_i^p\right)^{n-1}(E)$ for any $n = 1, 2, ...$ if and only it $t_i(\omega)\left(\left(\neg B_i^p\right)^{n-1}(E)\right) \geq p$ for any $n = 1, 2, ...$ It follows that $S_{t_i(\omega)} \succeq S\left(\left(\neg B_i^p\right)^{n-1}(E)\right)$ for any $n = 1, 2, ...$ By Proposition 2, $S\left(\left(\neg B_i^p\right)^{n-1}(E)\right) = S(E)$ for any $n = 1, 2, ...$ Hence $\omega \in A_i(E)$.

3. First, we show $B_i^p U_i(E) \subseteq A_i(E)$. $\omega \in B_i^p U_i(E)$ if and only if $t_i(\omega)(U_i(E)) \geq p$.

It implies $S_{t_i(\omega)} \succeq S(U_i(E))$. By Proposition 1 $S(U_i(E)) = S(E)$. Hence $S_{t_i(\omega)} \succeq S(E)$ which is equivalent to $\omega \in A_i(E)$.

Second, we show that $B_i^p U_i(E) = \emptyset^{S(E)}$ for $p \in (0, 1]$. Since $B_i^p U_i(E) \subseteq A_i(E)$ we have by monotonicity $B_i^1 B_i^p U_i(E) \subseteq B_i^1 A_i(E)$. By introspection $B_i^p U_i(E) \subseteq B_i^1 B_i^p U_i(E) \subseteq B_i^1 A_i(E)$. By additivity, we have $B_i^p U_i(E) \subseteq \neg B_i^1 A_i(E)$. Hence $B_i^p U_i(E) = \emptyset^{S(E)} = \neg B_i^1 A_i(E) \cap B_i^1 A_i(E)$.

Third, we show that $B_i^0 U_i(E) = A_i(E)$. $\omega \in A_i(E)$ if and only if $\omega \in A_i U_i(E)$ since by AA-self-reflection $A_i(E) = A_i A_i(E)$ and by symmetry $A_i A_i(E) = A_i U_i(E)$. Hence, if $\omega \in A_i(E)$ then $t_i(\omega)(U_i(E))$ is defined. Therefore $\omega \in B_i^0 U_i(E)$, and hence $A_i(E) \subseteq B_i^0 U_i(E)$. Together with the first part of the proof, we conclude $B_i^0 U_i(E) = A_i(E)$.

4. This property is equivalent to $A_i U_i(E) = A_i(E)$. $\omega \in A_i U_i(E)$ if and only if $S_{t_i(\omega)} \succeq S(U_i(E)) = S(A_i(E)) = S(E)$ by Proposition 1. Hence $\omega \in A_i U_i(E)$ if and only if $\omega \in A_i(E)$.

5. $\omega \in A_i(E)$ if and only if $S_{t_i(\omega)} \succeq S(E)$. For any $t_i(\omega)$, we have $S_{t_i(\omega)} \succeq S(E)$ if and only if $1 = t_i(\omega)(S(E)^{\uparrow})$. This is equivalent to $\omega \in B_i^1(S(E)^{\uparrow})$.

6. First, we show $B_i^p(E) \subseteq A_i(E)$. $\omega \in B_i^p(E)$ if and only if $t_i(\omega)(E) \geq p$. This implies that $S_{t_i(\omega)} \succeq S(E)$, which is equivalent to $\omega \in A_i(E)$.

Second, we show for $p = 0$, $A_i(E) \subseteq B_i^0(E)$. $\omega \in A_i(E)$ if and only if $t_i(\omega) \in \Delta(S)$ with $S \succeq S(E)$. Hence $t_i(\omega)(E) \geq 0$, which implies that $\omega \in B_i^0(E)$.

7. $\omega \in B_i^p(E)$ if and only if $t_i(\omega)(E) \geq p$. This implies that $S_{t_i(\omega)} \succeq S(E)$. By Proposition 2 it is equivalent to $S_{t_i(\omega)} \succeq S(B_i^q(E))$, which is equivalent to $\omega \in A_i B_i^q(E)$.

8. By the definition of negation, $S(E) = S(\neg E)$. Hence for $t_i(\omega) \in \triangle(S)$, $S \succeq S(E)$ if and only if $S \succeq S(\neg E)$.

9. $\omega \in \bigcap_{\lambda \in L} A_i(E_\lambda)$ if and only if $S_{t_i(\omega)} \succeq S(E_\lambda)$ for all $\lambda \in L$. This is equivalent to $S_{t_i(\omega)} \succeq \sup_{\lambda \in L} S(E_\lambda) = S\left(\bigcap_{\lambda \in L} E_\lambda\right)$, which is equivalent to $\omega \in A_i\left(\bigcap_{\lambda \in L} E_\lambda\right)$.

10. By Proposition 2, $S(E) = S(B_i^p(E))$. Hence, $\omega \in A_i(E)$ if and only if $\omega \in A_i B_i^p(E)$.

11. By Proposition 1, $S(E) = S(A_i(E))$. Hence $\omega \in A_i(E)$ if and only if $\omega \in A_i A_i(E)$.

12. $\omega \in B_i^p A_i(E)$ if and only if $t_i(\omega)(A_i(E)) \geq p$. This implies $S_{t_i(\omega)} \succeq S(A_i(E))$. By Proposition 1, $S(A_i(E)) = S(E)$. Thus $\omega \in A_i(E)$. To see the converse, by weak necessitation and introspection, $A_i(E) = B_i^1(S(E)^{\uparrow}) \subseteq B_i^1 B_i^1(S(E)^{\uparrow}) = B_i^1 A_i(E)$. By Proposition 9 (o), $B_i^1 A_i(E) \subseteq B_i^p A_i(E)$. $\square$

B.5 Proof of Proposition 4

Note that $U_i(E) = S(E)^{\uparrow} \setminus A_i(E) = \{\omega \in S(E)^{\uparrow} : t_i(\omega) \in \Delta(S) \text{ s.t. } S \not\succeq S(E)\} = \{\omega \in \Omega : S_{t_i(\omega)} \not\succeq S(E) \text{ and } S_\omega \succeq S(E)\}$. The result now follows from the fact that $t_i(\omega)(S(E)^{\uparrow}) = 1$ if $S_{t_i(\omega)} \succeq S(E)$. $\square$

B.6 Proof of Proposition 5

Note that $S(S(E)^{\uparrow}) = S(\neg E) = S(E)$. If $S_{t_i(\omega)} \not\succeq S(E)$ and $S_\omega \succeq S(E)$, then $t_i^Z(\omega)(S(E)^{\uparrow}) = t_i^Z(\omega)(E) = t_i^Z(\omega)(\neg E) = 0$. Hence in this case $\omega \in Z_i(S(E)^{\uparrow})$ and $\omega \in Z_i(E) \cap Z_i(\neg E)$. If $S_{t_i(\omega)} \succeq S(E)$, then $t_i^Z(\omega)(S(E)^{\uparrow}) = t_i(\omega)(S(E)^{\uparrow}) = 1 = t_i(\omega)(E) + t_i(\omega)(\neg E) = t_i^Z(\omega)(E) + t_i^Z(\omega)(\neg E)$. Hence $\omega \notin Z_i(S(E)^{\uparrow})$ and $\omega \notin Z_i(E) \cap Z_i(\neg E)$. $\square$

B.7 Proof of Proposition 6

"$\supseteq$": $U_i(E) = Z_i(S(E)^{\uparrow}) \subseteq Z_i(E)$ where the first equality follows from Proposition 4 and second inclusion follows from Proposition 5. $B_i^1(\neg E) \subseteq A_i(E)$. Clearly, $B_i^1(\neg E) \subseteq Z_i(E)$.

"$\subseteq$": $\omega \in Z_i(E)$ if and only if $t_i^Z(\omega)(E) = 0$. If $t_i^Z(\omega)(\neg E) = 0$, then $\omega \in Z_i(S(E)^{\uparrow})$ by Proposition 5. Thus $\omega \in U_i(E)$ by Proposition 4. If $t_i^Z(\omega)(\neg E) > 0$, then $t_i^Z(\omega)(\neg E) = 1$. Then $\omega \in B_i^1(\neg E) \subseteq A_i(E)$. $\square$

B.8 Proof of Proposition 7

We only have to show:

1. $t_i^F : \Omega \longrightarrow \Delta(\Omega, \mathcal{F})$ is measurable, where $\Delta(\Omega, \mathcal{F})$ is endowed with the sigma-algebra generated by sets $\{\mu \in \Delta(\Omega, \mathcal{F}) : \mu(E) \geq p\}$ for $p \in [0,1]$ and $E \in \mathcal{F}$.

2. For all $\omega \in \Omega$, $i \in I$, and $E \in \mathcal{F}$: If $[t_i^F(\omega)] = \{\omega' \in \Omega : t_i^F(\omega') = t_i^F(\omega)\} \subseteq E$, then $t_i^F(\omega)(E) = 1$.

But both properties follow directly from the respective properties in the unawareness belief structure $\underline{\Omega}$. $\square$

B.9 Proof of Theorem 1

Before we prove the theorem, we state following observations:

Remark 5 *If* $P = \left(P^S\right)_{S\in\mathcal{S}} \in \prod_{S\in\mathcal{S}} \Delta(S)$ *is a positive (common) prior, then also* $P^S \in \Delta(S)$ *is positive (common) prior on* S *for every* $S \in \mathcal{S}$*.*

Remark 6 *If* $\mu_i \in \Delta(S)$ *is a* positive *prior for player* i *on* S *and* $S' \preceq S$*, then the marginal of* μ_i *on* S'*,* $\left(\mu_i^S\right)_{|S'}$ *is a* positive *prior for player* i *on* S'*.*

Lemma 2 *Let* P^S *be a positive common prior on some finite state-space* S *and let* $i \in I$ *and* $\omega \in \Sigma$ *such that* $t_i(\omega) \in \triangle(S)$*. Then we have for all* $\omega' \in [t_i(\omega)] \cap S$ *that* $t_i(\omega)(\{\omega'\}) = \frac{P^S(\{\omega'\})}{P^S([t_i(\omega)]\cap S)}$*.*

Proof of the Lemma. Because $t_i(\omega) = t_i(\omega')$, we have $A_i\left(S^{\uparrow}\right) = A_i\left(\{\omega'\}^{\uparrow}\right) \supseteq [t_i(\omega)]^{\uparrow} \supseteq \{\omega'\}^{\uparrow}$. By the definition of a prior on S, $P^S(\{\omega'\}) = P^S\left(\{\omega'\}^{\uparrow} \cap A_i\left(\{\omega'\}^{\uparrow}\right)\right) = \int\limits_{A_i\left(\{\omega'\}^{\uparrow}\right)\cap S} t_i(\cdot)\left(\{\omega'\}^{\uparrow}\right) dP^S(\cdot)$. Note that if $\omega'' \in S \setminus [t_i(\omega)] \cap S$, then we do have $t_i(\omega'')\left(\{\omega'\}^{\uparrow}\right) = 0$. Hence, since $t_i(\omega) = t_i(\omega'')$, for $\omega'' \in [t_i(\omega)]$, we have
$\int\limits_{A_i\left(\{\omega'\}^{\uparrow}\right)\cap S} t_i(\cdot)\left(\{\omega'\}^{\uparrow}\right) dP^S(\cdot) = \int\limits_{[t_i(\omega)]\cap S} t_i(\cdot)(\{\omega'\})\, dP^S(\cdot) = t_i(\omega)(\{\omega'\})\, P^S([t_i(\omega)] \cap S)$.
Because P^S is positive, it follows that $t_i(\omega)(\{\omega'\}) = \frac{P^S(\{\omega'\})}{P^S([t_i(\omega)]\cap S)}$. □

Proof of the Theorem. The idea of the proof is follows: First, if the set of states in which there is common certainty that the first player's expectation is strictly above α and the second player's expectations is weakly below α is nonempty, there is a minimal state-space such that the common certainty event restricted to this space is nonempty. Second, this restricted common certainty event is a belief closed subset in which beliefs are stationary. Third, this set, together with the restriction of types to this set constitutes a standard state-space to which a standard no-trade argument can be applied.

Note that $E_1^{>\alpha}$ and $E_2^{\leq\alpha}$ may not be events in our unawareness belief structure. The definition of the belief operator as well as Proposition 9 and 10 can be extended to measurable subsets of Ω. The proofs are analogous and thus omitted.

Suppose that $CB^1\left(E_1^{>\alpha} \cap E_2^{\leq\alpha}\right)$ is non-empty. Then fix a $\preceq$-minimal state-space S such that $W := CB^1\left(E_1^{>\alpha} \cap E_2^{\leq\alpha}\right) \cap S \neq \emptyset$. Such a space S exists by the finiteness of Σ.

By Remark 5, since P is a positive common prior, P^S is a positive common prior on S.

Since $W = CB^1\left(E_1^{>\alpha} \cap E_2^{\leq\alpha}\right) \cap S \subseteq S \cap B_i^1\left(CB^1\left(E_1^{>\alpha} \cap E_2^{\leq\alpha}\right)\right)$, the minimality of S implies that for each $\omega \in CB^1\left(E_1^{>\alpha} \cap E_2^{\leq\alpha}\right) \cap S$ we do have $S_{t_i(\omega)} = S$ and $t_i(\omega)(W) = 1$.

By the definition, $t_i(\omega)([t_i(\omega)] \cap S) = 1$, for each $\omega \in CB^1\left(E_1^{>\alpha} \cap E_2^{\leq\alpha}\right) \cap S$. Since $t_i(\omega)(W) = 1$, we have $t_i(\omega)(([t_i(\omega)] \cap S) \setminus W) = 0$.

By Lemma 2, this implies that $P^S(\{\omega'\}) = 0$, for $\omega' \in ([t_i(\omega)] \cap S) \setminus W$ such that $\omega \in CB^1\left(E_1^{>\alpha} \cap E_2^{\leq\alpha}\right) \cap S$. It follows that $P^S(([t_i(\omega)] \cap S) \setminus W) = 0$ and hence, $P^S(([t_i(\omega)] \cap S) \cap W) = P^S([t_i(\omega)] \cap S) - P^S(([t_i(\omega)] \cap S) \setminus W) = P^S([t_i(\omega)] \cap S) > 0$. So, we do have $P^S(W) > 0$.

The fact that $P^S(\{\omega'\}) = 0$, for $\omega' \in ([t_i(\omega)] \cap S) \setminus W$ such that $\omega \in CB^1\left(E_1^{>\alpha} \cap E_2^{\leq\alpha}\right) \cap S = W$ implies the following: For any random variable x, we have $\sum_{\omega' \in [t_i(\overline{\omega})] \cap S} x(\omega') P^S(\{\omega'\}) = \sum_{\omega' \in W \cap [t_i(\overline{\omega})] \cap S} x(\omega') P^S(\{\omega'\})$, if $[t_i(\overline{\omega})] \cap W \neq \emptyset$. And also $\sum_{\omega \in W} x(\omega) P^S(\{\omega\}) = \sum_{[t_i(\overline{\omega})] \cap W \neq \emptyset} \sum_{\omega \in [t_i(\overline{\omega})] \cap S} x(\omega) P^S(\{\omega\})$. This is so, because there is a $\omega \in [t_i(\overline{\omega})] \cap W$ and for this ω, we do have $\omega \in CB^1\left(E_1^{>\alpha} \cap E_2^{\leq\alpha}\right) \cap S$ and $[t_i(\omega)] = [t_i(\overline{\omega})]$ and this implies $P^S(([t_i(\overline{\omega})] \cap S) \setminus W) = 0$.

For $i = 1, 2$ we have

$$
\begin{aligned}
& \sum_{\omega \in W} P^S(\{\omega\}) \sum_{\omega' \in [t_i(\omega)] \cap S} v(\omega') t_i(\omega)(\{\omega'\}) \\
= & \sum_{\omega \in W} P^S(\{\omega\}) \sum_{\omega' \in [t_i(\omega)] \cap S} v(\omega') \frac{P^S(\{\omega'\})}{P^S([t_i(\omega)] \cap S)} \\
= & \sum_{[t_i(\overline{\omega})] \cap W \neq \emptyset} \sum_{\omega \in [t_i(\overline{\omega})] \cap S} P^S(\{\omega\}) \sum_{\omega' \in [t_i(\omega)] \cap S} v(\omega') \frac{P^S(\{\omega'\})}{P^S([t_i(\omega)] \cap S)} \\
= & \sum_{[t_i(\overline{\omega})] \cap W \neq \emptyset} \sum_{\omega \in [t_i(\overline{\omega})] \cap S} P^S(\{\omega\}) \sum_{\omega' \in [t_i(\overline{\omega})] \cap S} v(\omega') \frac{P^S(\{\omega'\})}{P^S([t_i(\omega)] \cap S)} \\
= & \sum_{[t_i(\overline{\omega})] \cap W \neq \emptyset} P^S([t_i(\overline{\omega})] \cap S) \sum_{\omega' \in [t_i(\overline{\omega})] \cap S} v(\omega') \frac{P^S(\{\omega'\})}{P^S([t_i(\overline{\omega})] \cap S)} \\
= & \sum_{[t_i(\overline{\omega})] \cap W \neq \emptyset} \sum_{\omega' \in [t_i(\overline{\omega})] \cap S} v(\omega') P^S(\{\omega'\}) \\
= & \sum_{\omega' \in W} v(\omega') P^S(\{\omega'\}).
\end{aligned}
$$

But by the assumptions, we have $\sum_{\omega \in W} P^S(\{\omega\}) \sum_{\omega' \in [t_1(\omega)] \cap S} v(\omega') t_1(\omega)(\{\omega'\}) > \alpha P^S(W)$ and $\sum_{\omega \in W} P^S(\{\omega\}) \sum_{\omega' \in [t_2(\omega)] \cap S} v(\omega') t_2(\omega)(\{\omega'\}) \leq \alpha P^S(W)$, a contradiction, since $P^S(W) > 0$. $\square$

B.10 Proof of Proposition 8

Before we prove the proposition, we require following auxiliary results:

Remark 7 *For any $\omega \in \Omega$, $t_i(\omega)(E \cap A_i(E)) = t_i(\omega)(E)$ for any event E s.t. $S(E) \preceq S_{t_i(\omega)}$.*

Proof of the Remark: Let E be an event and $t_i(\omega)$ be such that $S(E) \preceq S_{t_i(\omega)}$. Since $E = (E \cap A_i(E)) \cup (E \cap U_i(E))$ and $A_i(E) \cap U_i(E) = \emptyset^{S(E)}$, we have $(E \cap A_i(E)) \cap (E \cap U_i(E)) = \emptyset^{S(E)}$. Since $t_i(\omega)$ is an additive probability measure, $t_i(\omega)(E) = t_i(\omega)(E \cap A_i(E)) + t_i(\omega)(E \cap U_i(E))$. Since $B_i^p U_i(E) = \emptyset^{S(E)}$ for $p \in (0,1]$ (B^pU-Introspection in Proposition 3), we must have $t_i(\omega)(E \cap U_i(E)) = 0$. $\square$

We slightly abuse terminology and call a probability measure $\mu_i \in \Delta(S)$ a *prior for player i on S* if for every event $E \in \Sigma$ with $S(E) \preceq S$ equation (1u) is satisfied, i.e.,

$$\mu_i(E \cap S \cap A_i(E)) = \int_{S \cap A_i(E)} t_i(\cdot)(E) d\mu_i(\cdot). \tag{3}$$

The following lemma says that if there is a prior on a state-space then the marginal on a lower space is a prior as well.

Lemma 3 *If $\mu \in \Delta(S')$ is a prior for player i on S' and $S \preceq S'$, then $(\mu)_{|S}$ (the marginal of μ on S) is a prior for player i on S.*

Proof of the Lemma. Let E be an event with $S(E) \preceq S$ and let μ be individual i's prior probability measure on S' with $S' \succeq S$. We have to show that $\mu\left((r_S^{S'})^{-1}(E \cap S \cap A_i(E))\right) = \int_{S \cap A_i(E)} t_i(\cdot)(E) d\mu(\cdot)$. Since $S(E) \preceq S$, and by Proposition 1 $S(A_i(E)) = S(E)$, it follows that $(r_S^{S'})^{-1}(E \cap S \cap A_i(E)) = E \cap S' \cap A_i(E)$, and therefore $\mu_{|S}(E \cap S \cap A_i(E)) = \mu(E \cap S' \cap A_i(E))$. So it remains to show that $\int_{S \cap A_i(E)} t_i(\cdot)(E \cap A_i(E)) d(\mu_{|S})(\cdot) = \int_{S' \cap A_i(E)} t_i(\cdot)(E \cap A_i(E)) d\mu(\cdot)$.

We first show the following **Claim:** *Let $\omega \in S(E) \preceq S \preceq S'$ such that $\omega \in A_i(E)$. Then $t_i(\omega)(E \cap A_i(E)) = t_i(\omega_S)(E \cap A_i(E))$.*

Proposition 1, $\omega \in A_i(E)$ and $S(E) \preceq S$ imply that $\omega_S \in A_i(E)$. We have that $\omega \in A_i(E)$ implies $t_i(\omega)(E \cap A_i(E)) = t_i(\omega)(E \cap A_i(E) \cap S_{t_i(\omega)})$. By 3 of Definition 1, we have $S_{t_i(\omega_S)} \preceq S_{t_i(\omega)}$. And by 1 of Definition 1 $t_i(\omega_S)(E \cap A_i(E)) = t_i(\omega_S)(E \cap A_i(E) \cap S_{t_i(\omega_S)}) = t_i(\omega_{S_{t_i(\omega_S)}})(E \cap A_i(E) \cap S_{t_i(\omega_S)})$. By 2 of Definition 1, we have $t_i(\omega_{S_{t_i(\omega_S)}})(E \cap A_i(E) \cap S_{t_i(\omega_S)}) = t_i(\omega)((r^{S_{t_i(\omega)}}_{S_{t_i(\omega_S)}})^{-1}(E \cap A_i(E) \cap S_{t_i(\omega_S)})) = t_i(\omega)(E \cap A_i(E) \cap S_{t_i(\omega)}) = t_i(\omega)(E \cap A_i(E))$. Hence the claim is proved.

We have

$$\begin{aligned} \int_{A_i(E)\cap S} t_i(\cdot)(A_i(E) \cap E) d(\mu_{|S})(\cdot) &= \int_{A_i(E)\cap S'} t_i(r^{S'}_S(\cdot))(A_i(E) \cap E) d\mu(\cdot) \\ &= \int_{A_i(E)\cap S'} t_i(\cdot)(A_i(E) \cap E) d\mu(\cdot), \end{aligned}$$

where the first equation follows from the definition of marginal and the second from the above claim. □

We say that $\mu \in \Delta(S)$ is a common prior on S if it is a prior on S for every player $i \in I$.

Remark 8 *Let $\hat{S}$ be the upmost state-space in the lattice $\mathcal{S}$, and let $(P_i^S)_{S\in\mathcal{S}} \in \prod_{S\in\mathcal{S}} \Delta(S)$ be a tuple of probability measures. Then $(P_i^S)_{S\in\mathcal{S}}$ is a prior for player i if and only if $P_i^{\hat{S}}$ is a prior for player i on $\hat{S}$ and P_i^S is the marginal of $P_i^{\hat{S}}$ for every $S \in \mathcal{S}$.*

This remark together with Lemma 3 implies the following:

Remark 9 *A common prior (Definition 9) induces a common prior on S, for any $S \in \mathcal{S}$. The converse is not necessarily true unless S is the upmost state-space of the lattice. Note that it is possible that players have different priors, but at some space S (below the upmost space) the priors on S coincide. Hence, in such a case they have different priors, but a common prior on S (and by Lemma 3 also a common prior on spaces less expressive than S).*

We are now ready to prove Proposition 8. In fact, we prove below a version just requiring the existence of a common prior P^S on S such that $S(G) \preceq S$ and $P^S(CB^1(\bigcap_{i\in I}[t_i(G) = p_i])) > 0$. By Remark 9, this is more general than the statement of Proposition 8.

Proof of Proposition 8. By Proposition 10, $\omega \in CB^1(F)$ if and only if there exists an event E that is evident such that $\omega \in E \subseteq B^1(F)$.

Since for an evident E we have $E \subseteq B_i^1(E) \subseteq A_i(E)$ for all $i \in I$. It follows that $P^S(E \cap A_i(E)) = P^S(E)$ for $S \succeq S(E)$. Set $F = \bigcap_{i \in I}[t_i(G) = p_i]$ and let $E = CB^1(F)$. By Proposition 1, $S(E) = S(G)$. By Lemma 3 and the properties imposed on t_i, we consider w.l.o.g. a common prior $P^{S(G)}$ on $S(G)$.

$$\begin{aligned} P^{S(G)}(E) &= \int_{S(G)\cap A_i(E)} t_i(\cdot)(E) dP^{S(G)}(\cdot) \\ &= \int_{E\cap S(G)\cap A_i(E)} t_i(\cdot)(E) dP^{S(G)}(\cdot) + \int_{(S(G)\cap A_i(E))\setminus E} t_i(\cdot)(E) dP^{S(G)}(\cdot). \end{aligned}$$

$$\int_{E\cap S(G)\cap A_i(E)} t_i(\cdot)(E) dP^{S(G)}(\cdot) = \int_{E\cap S(G)\cap A_i(E)} 1 dP^{S(G)}(\cdot) = P^{S(G)}(E).$$

The second last equation above follows from the fact that E is evident. So, we have $E \subseteq B_i^1(E)$, that is $t_i(\cdot)(E) = 1$, for $\omega \in E$. It follows that

$$\int_{(S(G)\cap A_i(E))\setminus E} t_i(\cdot)(E) dP^{S(G)}(\cdot) = 0. \tag{4}$$

$$\int_{E\cap A_i(E)\cap S(G)} t_i(\cdot)(G) dP^{S(G)}(\cdot) = \int_{E\cap A_i(E)\cap S(G)} p_i dP^{S(G)}(\cdot) = p_i P^{S(G)}(E)$$

If $\omega \in E = CB^1(F)$, then $\omega \in E \subseteq B_i^1(F) \subseteq B_i^1([t_i(G) = p_i])$. Note that $[t_i(G) = p_i] = B_i^{p_i}(G) \cap B_i^{1-p_i}(\neg G)$. Therefore, by monotonicity $B_i^1([t_i(G) = p_i]) \subseteq B_i^1(B_i^{p_i}(G)) \cap B_i^1(B_i^{1-p_i}(\neg G))$. Introspection II implies now that $\omega \in B_i^{p_i}(G) \cap B_i^{1-p_i}(\neg G) = [t_i(G) = p_i]$. So we have $t_i(\omega)(G) = p_i$, for $\omega \in E$.

$$\begin{aligned} \int_{E\cap A_i(E)\cap S(G)} t_i(\cdot)(G) dP^{S(G)}(\cdot) &= \int_{E\cap A_i(E)\cap S(G)} t_i(\cdot)(G\cap E) dP^{S(G)}(\cdot) \\ &= \int_{S(G)\cap A_i(E)} t_i(\cdot)(G\cap E) dP^{S(G)}(\cdot) \\ &\quad - \int_{(S(G)\cap A_i(E))\setminus E} t_i(\cdot)(G\cap E) dP^{S(G)}(\cdot). \end{aligned}$$

Since by the monotonicity of probability measures

$$\int_{(S(G)\cap A_i(E))\setminus E} t_i(\cdot)(G\cap E) dP^{S(G)}(\cdot) \leq \int_{(S(G)\cap A_i(E))\setminus E} t_i(\cdot)(E) dP^{S(G)}(\cdot),$$

we must have by equation (4) and non-negativity of probability measures

$$\int_{(S(G)\cap A_i(E))\setminus E} t_i(\cdot)(G\cap E) dP^{S(G)}(\cdot) = 0.$$

Note that $P^{S(G)}(G \cap E) = \int_{S(G) \cap A_i(E)} t_i(\cdot)(G \cap E) dP^{S(G)}(\cdot)$.

Note further that $P^{S(G)}(E) = P^{S(G)}(E \cap A_i(E))$ for all $i \in N$ since $E = CB^1(F) \subseteq A_i(E)$ for all $i \in N$. Similarly, $P^{S(G)}(G \cap E) = P^{S(G)}(G \cap E \cap A_i(E))$ for all $i \in N$.

Thus

$$p_i P^{S(G)}(E) = P^{S(G)}(G \cap E). \tag{5}$$

Note that by assumption $P^{S(G)}(E) > 0$.

Since equation (5) holds for all $i \in I$, we must have $p_i = p_j$, for all $i, j \in I$. $\square$

B.11 Proof of Remark 4

By Lemma 1 each $[t_i(E) = p_i]$ is an $S(G)$-based event. By the definition of the conjunction of events, $\bigcap_{i \in I}[t_i(G) = p_i]$ is an $S(G)$-based event. As remarked after the definition of the CB^1-operator (page 18), this implies that $CB^1(\bigcap_{i \in I}[t_i(G) = p_i])$ is an $S(G)$-based event. Since by assumption this event is nonempty, its base, that is its intersection with $S(G)$, must be nonempty. Therefore, since by assumption $S \succeq S(G)$, $S \cap CB^1(\bigcap_{i \in I}[t_i(G) = p_i])$ must be nonempty (recall that the $r_S^{S'}$ are surjective, whenever defined). The positivity of P^S implies now that $P^S(CB^1(\bigcap_{i \in I}[t_i(G) = p_i])) = P^S(S \cap CB^1(\bigcap_{i \in I}[t_i(G) = p_i])) > 0$. $\square$

B.12 Proof of Proposition 9

(0) $B_i^p(E) \subseteq B_i^q(E)$ for $p, q \in [0,1]$ with $q \leq p$ is trivial.

(i) $B_i^1(\Omega) \subseteq \Omega$ holds trivially. In the reverse direction, note that $t_i(\omega)(\Omega) = t_i(\omega)(\Omega \cap S_{t_i(\omega)}) = t_i(\omega)(S_{t_i(\omega)}) = 1$ for all $\omega \in \Omega$. Thus $\Omega \subseteq B_i^1(\Omega)$.

(ii) $\omega \in B_i^p(E)$ if and only if $t_i(\omega)(E) \geq p$. Since $t_i(\omega)$ is an additive probability measure, $t_i(\omega)(\neg E) \leq 1 - p$. Hence $\omega \in \neg B_i^q(\neg E)$ for $q > 1 - p$.

(iiia) $\omega \in B_i^p\left(\bigcap_{l=1}^{\infty} E_l\right)$ if and only if $t_i(\omega)\left(\bigcap_{l=1}^{\infty} E_l\right) \geq p$. Monotonicity of the probability measure $t_i(\omega)$ implies $t_i(\omega)(E_l) \geq p$ for all $l = 1, 2, ...$, which is equivalent to $\omega \in \bigcap_{l=1}^{\infty} B_i^p(E_l)$.

(iiib) It is enough to show that any sequence of events $\{E_l\}_{l=1}^{\infty}$ with $E_l \supseteq E_{l+1}$ for $l = 1, 2, ...$ we have $B_i^p\left(\bigcap_{l=1}^{\infty} E_l\right) \supseteq \bigcap_{l=1}^{\infty} B_i^p(E_l)$. $\omega \in \bigcap_{l=1}^{\infty} B_i^p(E_l)$ if and only if $t_i(\omega)(E_l) \geq p$ for $l = 1, 2, ...$ Since $t_i(\omega)$ is a countable additive probability measure, it is continuous from above. That is, if $E_l \supseteq E_{l+1}$ for $l = 1, 2, ...$, we have $\lim_{l \to \infty} t_i(\omega)(E_l) =$

$t_i(\omega)\left(\bigcap_{l=1}^{\infty} E_l\right)$. Since for every $l = 1, 2, ...$, $t_i(\omega)(E_l) \geq p$, we have $p \leq \lim_{l\to\infty} t_i(\omega)(E_l) = t_i(\omega)\left(\bigcap_{l=1}^{\infty} E_l\right)$. Hence $\omega \in B_i^p\left(\bigcap_{l=1}^{\infty} E_l\right)$.

(iiic) It is enough to show that $B_i^1\left(\bigcap_{l=1}^{\infty} E_l\right) \supseteq \bigcap_{l=1}^{\infty} B_i^1(E_l)$. $\omega \in \bigcap_{l=1}^{\infty} B_i^1(E_l)$ if and only if $t_i(\omega)(E_l) = 1$ for $l = 1, 2, ...$ Since $t_i(\omega)$ is a countable additive probability measure, it satisfies Bonferroni's Inequality. I.e., $t_i(\omega)\left(\bigcap_{l=1}^{\infty} E_l\right) \geq 1 - \sum_{l=1}^{\infty} 1 - t_i(\omega)(E_l)$. Since $t_i(\omega)(E_l) = 1$ for all $l = 1, 2, ...$, we have $1 - t_i(\omega)(E_l) = 0$ for all $l = 1, 2, ...$, and hence $\sum_{l=1}^{\infty} 1 - t_i(\omega)(E_l) = 0$. It follows that $t_i(\omega)\left(\bigcap_{l=1}^{\infty} E_l\right) = 1$. We conclude that $\omega \in B_i^1\left(\bigcap_{l=1}^{\infty} E_l\right)$.

(iv) Since $t_i(\omega)$ is a probability measure (satisfying monotonicity) for any $\omega \in \Omega$, $E \subseteq F$ implies that if $t_i(\omega)(E) \geq p$ then $t_i(\omega)(F) \geq p$.

(va) Let $\omega \in B_i^p(E)$. Then $t_i(\omega)(E) \geq p$. It follows that for all $\omega' \in Ben_i(\omega)$ we have $t_i(\omega')(E) \geq p$. Hence $Ben_i(\omega) \subseteq B_i^p(E)$. Thus $t_i(\omega)(B_i^p(E)) = 1$, which implies that $\omega \in B_i^1 B_i^p(E)$.

(vb) Let $\omega \in B_i^p(B_i^q(E))$, for some $p \in (0, 1]$ and assume by contradiction that $\omega \notin B_i^q(E)$. Then, since by Propositions 1 and 2 $\omega \in A_i(E)$, we must have $q > 0$ and $\omega \in B_i^{1-r}(\neg E)$ for some r with $q > r \geq 0$. By (va), we have $\omega \in B_i^1\left(B_i^{1-r}(\neg E)\right)$. Note that $B_i^{1-r}(\neg E)$ and $B_i^q(E)$ are disjoint because of (ii), and hence $B_i^{1-r}(\neg E) \subseteq \neg B_i^q(E)$. Monotonicity implies now that $\omega \in B_i^1\left(\neg B_i^q(E)\right)$, hence, by (ii) $\omega \in \neg B_i^p(B_i^q(E))$ a contradiction to $\omega \in B_i^p(B_i^q(E))$. $\square$

B.13 Proof of Proposition 11

1. By Proposition 1, $S(E) = S(A_j(E))$. Hence $\omega \in A_i(E)$ if and only if $\omega \in A_i A_j(E)$.

2. By Proposition 2, $S(E) = S(B_j^p(E))$. Hence, $\omega \in A_i(E)$ if and only if $\omega \in A_i B_j^p(E)$.

3. $\omega \in B_i^p(E)$ if and only if $t_i(\omega)(E) \geq p$. This implies that $S_{t_i(\omega)} \succeq S(E)$. By Proposition 2, this is equivalent to $S_{t_i(\omega)} \succeq S(B_j^q(E))$, which is equivalent to $\omega \in A_i B_j^q(E)$.

4. The proof is analogous to 3.

5. We show by induction that $A^n(E) = A(E)$, for all $n \geq 1$. We have $\omega \in A(A^n(E))$ if and only if $S_{t_i(\omega)} \succeq S(A^n(E))$, for all $i \in I$, which, by the induction hypothesis, is the case if and only if $S_{t_i(\omega)} \succeq S(A(E))$, for all $i \in I$. By the definition of "$\cap$", it is the case that $S(A(E)) = \sup_{i\in I} S(A_i(E))$. By Proposition 1 we have $S(A_i(E)) = S(E)$ and hence $S(A(E)) = S(E)$. It follows that $S_{t_i(\omega)} \succeq S(A(E))$ if and only if $S_{t_i(\omega)} \succeq S(E)$. But $S_{t_i(\omega)} \succeq S(E)$ if and only if $\omega \in A_i(E)$. Hence we have $A^n(E) = A(E)$, for all $n \geq 1$, and therefore $CA(E) = A(E)$.

6. $\omega \in CB^1(E)$ implies $\omega \in B_i^1(E)$ for all $i \in I$. This is equivalent to $t_i(\omega)(E) = 1$ for all $i \in I$, which implies $S_{t_i(\omega)} \succeq S(E)$ for all $i \in I$. Hence, by 5. we have $\omega \in A(E) = CA(E)$.

7. First, we show that $B^p(E) \subseteq A(E)$. $\omega \in B^p(E)$ if and only if $t_i(\omega)(E) \geq p$ for all $i \in I$. Hence $t_i(\omega) \in \Delta(S)$ with $S \succeq S(E)$, for all $i \in I$. This implies that $\omega \in A_i(E)$, for all $i \in I$. It follows that $\omega \in A(E)$.

Second, we show that $A(E) = B^0(E)$. $\omega \in A(E)$ if and only if $\omega \in A_i(E)$ for all $i \in I$ if and only if (by 6. of Proposition 3) $\omega \in B_i^0(E)$ for all $i \in I$ if and only if $\omega \in B^0(E)$.

8. The proof follows from 7. and 5.

9. By weak necessitation, $A(E) := \bigcap_{i \in I} A_i(E) = \bigcap_{i \in I} B_i^1(S(E)^{\uparrow}) := B^1(S(E)^{\uparrow})$.

10. The proof follows from 9. and 5.

11. By definition of common certainty, $CB^1(S(E)^{\uparrow}) \subseteq B^1(S(E)^{\uparrow})$. By 9., $B^1(S(E)^{\uparrow}) = A(E)$.

12. The proof follows from 11. and 5. $\square$

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