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Lottery versus All-Pay Auction Contests – A Revenue Dominance Theorem

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ECONOMIC PAPERS

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Christian Kanzow
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Alexandra Schwartz

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A Revenue Dominance Theorem

technische universität dortmund

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Jörg Franke, Christian Kanzow, Wolfgang Leininger,
and Alexandra Schwartz[1]

Lottery versus All-Pay Auction Contests – A Revenue Dominance Theorem

Abstract

We allow a contest organizer to bias a contest in a discriminatory way, that is, she can favor specific contestants through the choice of contest success functions in order to maximize total equilibrium effort (resp. revenue). The scope for revenue enhancement through biasing is analyzed and compared for the two predominant contest regimes; i.e. all-pay auctions and lottery contests. Our main result reveals that an appropriately biased all-pay auction revenue-dominates the optimally biased lottery contest for all levels of heterogeneity among contestants. Moreover, such a biased all-pay auction will never make use of the celebrated exclusion principle advanced by Baye et al. (1993).

JEL Classification: C72, D72

Keywords: All-pay auction; lottery contest; bias; revenue

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1 Introduction

The all-pay auction and the lottery contest game are the most frequently used setups to model strategic competition among agents that exert non-refundable effort to influence their respective chances to win a fixed prize. Both types of models have been extensively used in applied analysis, for instance, in the areas of R&D competition, lobbying, sports, rent-seeking, procurement, etc., see Konrad (2009) [11] for a survey. One of the reasons for the popularity might be the analytical tractability, especially if employed under the assumptions that the rules that govern the competition are anonymous and that agents are homogeneous. Recently, there is a growing interest in relaxing these limiting assumptions: The heterogeneity of contestants comes into the focus of analysis and, as a consequence, also the question of how the contest organizer should exploit heterogeneity among contestants by treating different contestants differently. Recent examples that follow this approach are Siegel (2010, 2011), [12, 13], Kirkegaard (2010), [10], Epstein et al. (2011), [5], Szech (2011), [15], and Franke et al. (2011), [7].

Due to the prominence of the rent-seeking interpretation in this literature an important aspect in the strategic analysis is the relation between aggregate equilibrium efforts of the agents (i.e., the revenue of the auction or contest) and the underlying institutional rules and characteristics that govern the specific form of the contest. In Baye et al. (1993), [2], for instance, an analysis of the all-pay auctions with heterogeneous players established the so called exclusion principle, which implies that a revenue-maximizing contest organizer might optimally exclude strong agents from the competition ex-ante. This result is in contrast to the symmetric lottery case considered in Fang (2002), [6], where it is shown that exclusion of strong players is never optimal for the contest organizer. Moreover, the direct comparison between these two contest regimes reveals that neither revenue-dominates the other a priori. The intuition for this result can be attributed to the trade-off between competitive pressure and entry which is differently resolved in the two regimes. Competitive pressure in an all-pay auction is primarily generated by the institution itself. Its highly discriminative, all-pay deterministic winner-takes-it-all nature endogenously restricts entry in equilibrium to (generically) just two contestants. However, competition between those two is so intense, that in (mixed strategy) equilibrium only one has a positive payoff in expectation (while both have a positive probability of winning). In contrast, a lottery contest with its characteristic probabilistic contest rule is much less discriminative as an institution because it does not require to be the highest bidder in order to win. This characteristic is highly conducive for attracting entry; i.e., competitive pressure in a lottery contest is primarily generated by the interaction of many active contestants in equilibrium. Fang (2002), [6] shows that from a revenue maximizing contest designer's point of view it depends on degree and nature of heterogeneity of contestants whether it is better to ignite competitive pressure *ex ante* (through the choice of a very discriminative contest success function like the all-pay auction), which is reduced endogenously *ex post* due to a minimal amount of entry, or to opt for weaker competitive pressure *ex ante* (by choosing a less discriminative contest success function like the lottery contest) which is endogenously reinforced *ex post* due to entry of more contestants.

Importantly, both of these models are based on the assumption that the contest organizer is neutral with respect to the contestants; that is, she chooses among contest regimes, which treat contestants anonymously. This is certainly not the case in many real world contests (just think of the contest rules for a job opening of a professorship), where the contest organizer has control over some variables, which bias the contest systematically (and legally) in favor of certain contestants (see also Epstein et al. (2011), [5], for a detailed discussion of this type of bias in public procurement in Israel). This gives the contest organizer additional power to promote her interests, in particular in the presence of heterogeneous contestants. This situation is analyzed for the case of two contestants in Epstein et al. (2011), [5], where the contest organizer can specify individual weights for each of two contestants. Setting individual weights reflects her potential for discriminating between the two contestants which has consequences for the revenue comparison between all-pay auction and lottery contest: The optimally biased all-pay auction revenue-dominates any biased lottery contest, independently of heterogeneity between the two contestants. However, the restriction to the two-player case is particularly severe for at least two reasons: Firstly, it ignores the basic trade-off with regard to competitive pressure as described above. The ”minimal entry” feature of the all-pay auction is eliminated, likewise the scope of the lottery contests for increased competitive pressure through additional entry. Secondly, the solution theory of the biased lottery contest with only two contestants is a degenerate case of the general n-player solution, see Franke et al. (2011), [7]. More precisely, the optimal weight for a contestant in the two-player case only depends on his own characteristics, whereas with three or more players any optimal individual bias weight depends on the characteristics of *all* contestants.

The objective of this paper is to determine a revenue (or total effort) maximizing contest organizer's choice of contest, when she is faced with n heterogeneous contestants. Her choice set consists of a set of (potentially biased) contest success function, which contains lotteries and all-pay auctions. We can partly rely on Franke et al. (2011), [7], who analyze the optimal choice of the contest organizer, if the choice set is restricted to (biased) lotteries. Her optimal choice from the set of (biased) all-pay auctions and the corresponding revenue (more precisely: a lower bound) is analyzed in this paper. Moreover, a comparison of maximal revenue in the two regimes is provided. Our main result (Theorem 4.3) states that revenue dominance of an appropriately biased all-pay auction over the optimally biased lottery holds for any given set of heterogeneous contestants. This result is far from trivial, but has a clear intuition: The ability of the contest organizer to discriminate between contestants in the all-pay auction is used to make the exclusion principle obsolete. Under the appropriate bias it will always be the two strongest contestants who choose to be active, and they are made to compete with each other in the strongest possible way, i.e., in a playing field that is completely leveled due to the bias. No strong player is excluded a priori by the organizer. As expected, the discriminatory power of the contest organizer in the lottery contest is used to encourage further entry: In any optimally biased lottery contest at least the three strongest contestants are active. However, the playing field among active contestants is not completely leveled in the optimally biased lottery contest because balancing the playing field negatively affects incentives for strong contestants. Moreover, the optimal bias is specified such that not all contestants might be induced to become active. This incompatibility of high entry and high competitive pressure due to a leveled field *given* entry in the lottery contest contributes to its inferiority with respect to the appropriately biased all-pay auction. Our theoretical results therefore provide a new explanation for the often observed phenomenon that only two strong contestants endogenously decide to participate in contests although the potential field of contestants is substantially larger (see the introduction of Fullerton and McAfee (1999), [8], for some real world examples of this phenomenon in research contests). Our revenue dominance theorem demonstrates that the reason for this observation might not be the irrational manipulation of the contest design from the side of the contest organizer (or outright illegal collusion with specific contestants) but instead her motivation to maximize total efforts.

The paper is organized as follows. Section 2 contains the model, Section 3 considers the biased all-pay auction. In Section 4 we explain the optimally biased lottery contest as derived in Franke et al. (2011), [7], and compare it to the result from Section 3. Section 5 concludes.

2 The Model

There are n agents $N = \{1, \ldots, n\}$, that participate either in a contest or in an all-pay auction which implies that they can influence the probability to win a non-divisible prize by exerting non-refundable effort. They are heterogeneous with respect to their valuation of the prize; that is, agent $i \in N$ values the prize at $v_i \in (0, \infty)$ and chooses a strategy (exerts effort) $x_i \in [0, \infty)$ to influence the probability $Pr_i(x_i, x_{-i}) : [0, \infty)^n \to [0, 1]$ of winning the prize, where $\sum_{i \in N} Pr_i(x_i, x_{-i}) = 1$ and $(x_1, \ldots, x_n) = (x_i, x_{-i})$ for all $i = 1, \ldots, n$. Hence, the payoff function of agent i is:

$$\pi_i(x_i, x_{-i}) = Pr_i(x_i, x_{-i})v_i - x_i \text{ for all } i \in N.$$

The formal rule of a contest, which maps an individual's effort into his winning probability as a function of the other contestants' efforts is called a contest success function (CSF). We are going to consider deterministic and probabilistic CSFs; we refer to the former as all-pay auctions and to the latter as lotteries. Technically speaking, lotteries are logit CSFs with linear component functions.

We will assume without loss of generality that agents are ordered with respect to their valuations: $v_1 \geq v_2 \geq \ldots \geq v_n$. The contest organizer has the power to bias the contest outcome with respect to specific agents, see Epstein et al. (2011) [5] for a detailed motivation based on the case of public procurement. This implies that the contest organizer can specify a vector of agent specific weights $\alpha = \{\alpha_1, \ldots, \alpha_n\} \in (0, \infty)^n$ that affect the impact of the agents' effort on the win probability as specified below. In line with Fang (2002), [6], and Epstein et al. (2011), [5], we consider two different classes of contest success functions that govern the probability to win the prize for player i:

- The *biased all-pay auction (BAA)* framework:

$$Pr_i^{BAA}(x_i, x_{-i}) = \begin{cases} 1, & \text{if } \alpha_i x_i > \alpha_j x_j \text{ for all } j \neq i, \\ \frac{1}{k+1}, & \text{if } \alpha_i x_i = \alpha_j x_j \text{ for } k \text{ agents } j \neq i \text{ and } \alpha_i x_i > \alpha_l x_l \text{ for all other agents } l \neq i, \\ 0, & \text{if } \alpha_i x_i < \alpha_j x_j \text{ for some } j \neq i. \end{cases}$$

- The *biased lottery contest game (BLC)* framework:

$$Pr_i^{BLC}(x_i, x_{-i}) = \begin{cases} \frac{\alpha_i x_i}{\sum_{j=1}^n \alpha_j x_j}, & \text{if } \sum_{j=1}^n \alpha_j x_j \neq 0, \\ 0, & \text{if } \sum_{j=1}^n \alpha_j x_j = 0. \end{cases}$$

We are going to evaluate the two regimes with respect to the maximal total (expected) revenue $X^{*,c}$ that they induce in equilibrium: $X^{*,c} = \sum_{j \in N} \mathrm{E}[x_j^{*,c}]$ with $c \in \{BAA, BLC\}$, where $x^{*,c}$ is the (potentially mixed) Nash equilibrium strategy under the respective optimal bias and $\mathrm{E}[x_j^{*,c}]$ is the corresponding expected equilibrium effort of agent j. Alternatively, we can formulate the following three stage game, where the objective of the contest organizer is to maximize total revenue:

1. Stage: The contest organizer chooses the competitive regime *BAA* or *BLC*.

2. Stage: The contest organizer specifies the optimal bias α; i.e., chooses the best CSF.

3. Stage: The agents choose the payoff maximizing strategies.

Note that the previous contributions by Baye et al. (1993), [2], who introduced the exclusion principle, and Fang (2002), [6], who compared the standard (unbiased) all-pay auction and lottery contest, can be viewed as restricting the contest organizer's choice of α_i to 0 or 1 for all $i = 1, \ldots, n$. The former means 'exclusion', the latter 'participation', i.e. 'becoming a finalist' in the language of these authors.

We will derive the subgame-perfect Nash-equilibrium by backward induction; we partially rely on results from [7] and [5].

The characterization and existence proof of the Nash equilibrium in the third stage given a fixed bias α is standard: For asymmetric lottery contest games the methods presented in Stein (2002), [14], as well as Cornes and Hartley (2005), [4], can be used, for the biased all-pay auction a similar argument as in Baye et al. (1993), [2], can be applied. Hence, we directly concentrate on the second stage. In the following section we derive a lower bound for total revenue in the all-pay auction framework under the condition that the bias is specified appropriately. For the lottery contest we rely on the results in Franke et al. (2011), [7], where a closed form expression for total revenue under the optimal bias is provided. Note, that the respective bias α^* for the all-pay auction and the lottery contest framework is not unique (but revenue equivalent within the specific framework) which has to be expected because all contest success functions are homogeneous of degree zero in both frameworks.

3 Revenue Maximization in the All-Pay Auction

The scope of discrimination and the corresponding (lower bound) on total revenue in the all-pay auction is derived as follows. In the next lemma, we are going to show that the biased all-pay auction is strategically equivalent to a standard all-pay auction with transformed valuations. This allows us to use the results from this literature, e.g. Baye et al. (1993 and 1996), [2], [1], and Hillman and Riley (1989), [9]. We derive the maximal revenue and the corresponding bias in closed form if the resulting equilibrium is unique. However, there are also degenerate cases where multiple equilibria exist that are not revenue equivalent. As the contest organizer can induce each possible equilibrium by specifying the appropriate bias, the derived closed form expression for total revenue in the case with a unique equilibrium must be a lower bound for total revenue in all other cases which is sufficient for the comparison with the lottery contest framework in Section 4.

Before we are going to present the equivalence result in the next lemma we introduce the following notation. Denote by $y_i = \alpha_i x_i$ and $\tilde{v}_i = \alpha_i v_i$ for all $i \in N$. In line with Baye et al. (1993), [2], the expected effort from agent i's (potentially mixed) strategy y_i is denoted by $\mathrm{E}[y_i]$ and her expected probability to win by $P_i = \mathrm{E}[Pr_i^{BAA}(y_i, y_{-i})]$.

Lemma 3.1 *The BAA framework is equivalent to a standard unbiased all-pay auction based on transformed valuations $\tilde{v} = \{\tilde{v}_1, \ldots, \tilde{v}_n\}$, where total (expected) equilibrium revenue is equal to:*

$$\tilde{X}^{AA} = \sum_{i=1}^{n} \frac{1}{\alpha_i} \mathrm{E}[y_i^*] \tag{1}$$

with y^* being a solution of the unbiased all-pay auction.

Proof. We consider the following transformation of variables: $y_i = \alpha_i x_i$ for all $i \in N$. Then the contest success function can be formulated as:

$$Pr_i^{BAA}(y_i, y_{-i}) = \begin{cases} 1, & \text{if } y_i > y_j \text{ for all } j \neq i, \\ \frac{1}{k+1}, & \text{if } y_i = y_j \text{ for } k \text{ agents } j \neq i \text{ and } y_i > y_l \text{ for all other agents } l \neq i, \\ 0, & \text{if } y_i < y_j \text{ for some } j \neq i, \end{cases}$$

while the payoff function of agent i can be expressed as:

$$\pi_i(y_i, y_{-i}) = Pr_i^{BAA}(y_i, y_{-i}) v_i - \frac{y_i}{\alpha_i} \text{ for all } i \in N.$$

Multiplying the payoff function of agent $i \in N$ by the constant factor $\alpha_i > 0$ does not affect the equilibrium of the transformed game. Let $\tilde{\pi}_i = \alpha_i \pi_i$ for all $i \in N$. The transformed game is then equivalent to a standard all-pay auction with payoff-function:

$$\tilde{\pi}_i(y_i, y_{-i}) = Pr_i^{BAA}(y_i, y_{-i}) \tilde{v}_i - y_i \text{ for all } i \in N,$$

where total revenue is calculated as: $\tilde{X}^{AA} = \sum_{i=1}^{n} \frac{1}{\alpha_i} \mathrm{E}[y_i^*]$ because $\mathrm{E}[x_i^*] = \frac{1}{\alpha_i} \mathrm{E}[y_i^*]$ for all $i \in N$. □

Note, that the bias weights $(\alpha_1, \ldots, \alpha_n)$, which transform original valuations $(v_1, \ldots, v_N)$ with $v_1 \geq \ldots \geq v_n$ into transformed valuations $(\tilde{v}_1, \ldots, \tilde{v}_n) = (\alpha_1 v_1, \ldots, \alpha_n v_n)$, need not preserve the order of the original valuations. Thus, it might be necessary to permute the contestants in order to reobtain ordered valuations. As the permutation depends on the respective bias, the contest organizer can induce each possible ordering of contestants and each possible constellation of transformed valuations by specifying the appropriate bias. However, we know from Baye et al. (1993 and 1996), [2], [1], and Hillman and Riley (1989), [9], that a unique equilibrium only exists for constellations that satisfy $\tilde{v}_1 \geq \tilde{v}_2 > \tilde{v}_3 \geq \ldots \geq \tilde{v}_n$. For constellations with $\tilde{v}_2 = \tilde{v}_3$ there exist multiple equilibria that might not generate the same total equilibrium effort. Luckily, in our analysis only the special case $\tilde{v}_1 = \tilde{v}_2 > \tilde{v}_3 \geq \ldots \geq \tilde{v}_n$ appears, which is known to have a unique and symmetric Nash equilibrium. We will use this to prove the next theorem, which is the main result of this section and provides a lower bound for the total equilibrium revenue under an optimal bias.

Theorem 3.2 *Let $v_1 \geq v_2 \geq v_3 \geq \ldots \geq v_n$; then specifying an optimal bias $\alpha^* = (\alpha_1^*, \ldots, \alpha_n^*)$ in the BAA framework yields a total equilibrium revenue that satisfies*

$$X^{*,BAA} \geq \frac{v_1 + v_2}{2}. \tag{2}$$

Proof. To prove this result, it suffices to provide one bias α, which yields a corresponding equilibrium revenue $\tilde{X}^{AA} = \frac{v_1+v_2}{2}$. For this reason we consider the special bias $\alpha \in (0, \infty)^n$, where $\alpha_1 = \frac{1}{v_1}$, $\alpha_2 = \frac{1}{v_2}$ and $\alpha_i = \frac{1}{2v_3}$ for all $i > 2$. The corresponding transformed valuations are then $\tilde{v}_1 = \tilde{v}_2 = 1$ and $\tilde{v}_i \leq \frac{1}{2} < 1$ for all $i > 2$. Note that this special bias preserves the ordering of the contestants, i.e. we still have

$$\tilde{v}_1 = \tilde{v}_2 > \tilde{v}_3 \geq \ldots \geq \tilde{v}_n.$$

It is known that the equivalent unbiased all-pay auction from Lemma 3.1 has a unique and symmetric Nash equilibrium y^*, where

$$\mathrm{E}[y_1^*] = \mathrm{E}[y_2^*] = \tfrac{\tilde{v}_1}{2} = \tfrac{1}{2}$$

and $\mathrm{E}[y_i^*] = 0$ for all $i > 2$, see for example Theorem 1 in Baye et al. (1996) [1]. This yields an equilibrium revenue of

$$\tilde{X}^{AA} = \frac{1}{\alpha_1}\mathrm{E}[y_1^*] + \frac{1}{\alpha_2}\mathrm{E}[y_2^*] = \frac{v_1 + v_2}{2}$$

and thus concludes the proof. □

The special bias α in the previous proof is specified in such a way that the playing field among the two contestants with highest valuations is completely balanced; i.e. $\tilde{v}_1 = \alpha_1 v_1 =$

$\alpha_2 v_2 = \tilde{v}_2$, and the remaining contestants are inactive. This levelled contest leads to payoff 0 for both contestants, cf. again Theorem 1 in Baye et al. (1996) [1]. Compare this solution to the solution of the unbiased case (i.e. $\alpha_1 = \alpha_2 = 1$): It is well-known that in equilibrium of the all-pay auction the (expected) payoff to contestant 1 is $v_1 - v_2 \geq 0$, while the payoff to contestant 2 is 0 ; the contest organizer can expect $E[y_1] + E[y_2] = \frac{v_2}{2} + \frac{v_2^2}{2v_1}$ (see e.g. Konrad (2009), [11], p. 26). Hence, the contestants, namely contestant 1, lose the entire rent of $(v_1 - v_2)$, while the contest organizer by biasing the contest in the prescribed way generates additional revenue of $\frac{v_1+v_2}{2} - (\frac{v_2}{2} + \frac{v_2^2}{2v_1}) = \frac{(v_1+v_2)}{2v_1}(v_1 - v_2)$. Since $\frac{v_1+v_2}{2v_1} \leq 1$ the loss in contestants' payoff is larger than the gain in revenue for the organizer who applies this bias. Moreover, the closer v_1 and v_2, the more efficiently can the organizer capture additional revenue: The rate r, at which winner's payoff in the unbiased case can be converted into additional revenue to the organizer in the biased case, is given by $r = \frac{v_1+v_2}{2v_1}$; e.g. if $v_2 = \frac{1}{2}v_1$ then $r = \frac{3}{4}$, whereas if $v_2 = \frac{3}{4}$ then $r = \frac{7}{8}$; in any case $r > \frac{1}{2}$.

We strongly conjecture that the lower bound given in Theorem 3.2 is tight; i.e., it is also an upper bound. This is not difficult to show *under the assumption* that the optimal bias α^* yields transformed valuations with the property that $\tilde{v}_1 \geq \tilde{v}_2 > \tilde{v}_3 \geq \ldots \geq \tilde{v}_n$. In this case the equilibrium in the standard all-pay auction with these transformed valuations is unique. However, if transformed valuations would be such that $\tilde{v}_2 = \tilde{v}_3$ several equilibria, which need not be revenue equivalent, can arise. It appears, that a demonstration that neither of these equilibria can yield higher revenue than $\frac{v_1+v_2}{2}$ requires explicit use of equilibrium strategies, whereas the elegant method provided in Baye et al. (1993) is applicable to all equilibria independently of the supporting equilibrium strategies.

4 Lotteries Versus All-Pay Auctions

The optimal bias for the asymmetric lottery contest has been derived in Franke et al. (2011), [7], under the condition that heterogeneity affects marginal costs to exert effort. However, a simple transformation leads to the framework as presented here, see [7], p. 6 f. We repeat the result in its transformed version in the following proposition to maintain a consistent notation.

Proposition 4.1 *There exists an optimal bias α^{BLC} in the BLC framework that is not unique. However, any optimal bias α^{BLC} leads to:*

$$X^{*,BLC} = \frac{1}{4}\left[\sum_{j\in K^*} v_j - \frac{(k^*-2)^2}{\sum_{j\in K^*}\frac{1}{v_j}}\right], \textit{ where} \tag{3}$$

$$K^* \quad = \quad \left\{ i \in N \;\middle|\; \frac{k^* - 2}{v_i} < \sum_{j \in K^*} \frac{1}{v_j} \right\}. \tag{4}$$

It is shown in [7] that K^*, the set of active contestants, is well-defined and unique and can equivalently be written as

$$K^* = \left\{ i \in N \;\middle|\; \frac{i - 2}{v_i} < \sum_{j=1}^{i} \frac{1}{v_j} \right\}.$$

Based on Theorem 3.2 and Proposition 4.1 we are now almost in a position to solve the first stage by comparing the total revenue under the two regimes; that is, comparing the lower bound of equilibrium revenue in optimally biased all-pay auction in eq. (2) with total revenue in the optimally biased lottery contest in eq. (3). But before we can state the main result of our paper, we need one more auxiliary result. In the following lemma, we consider biased lottery contests, where the agents' valuations are distributed according to $V = \{v_1, \ldots, v_n\}$, and denote by $X^{*,BLC}(V)$ the revenue of the lottery contest as defined in eq. (3) and by $K(V)$ the set of active agents under the optimal bias according to eq. (4). The lemma basically says that maximal possible revenue $X^{*,BLC}(V)$ increases with the valuations $V = \{v_1, \ldots, v_n\}$.

Lemma 4.2 *The function $X^{*,BLC}(V)$ is continuously differentiable on $(0, \infty)^n$ and the partial derivatives are given by*

$$\frac{\partial X^{*,BLC}(V)}{\partial v_i} = \begin{cases} 0 & \textit{if } i \notin K^*(V), \\ \frac{1}{4}\left[1 - \frac{(k(V)-2)^2}{\left(\sum_{j \in K(V)} \frac{1}{v_j}\right)^2} \frac{1}{v_i^2}\right], & \textit{if } i \in K^*(V). \end{cases}$$

In particular, for all $i \in N$ it holds that $\frac{\partial X^{*,BLC}(V)}{\partial v_i} \geq 0$, i.e. $X^{*,BLC}(V)$ is monotonically increasing in v_i.

Proof. Recall that $X^{*,BLC}(V)$ gives the maximal total effort of all active contestants in equilibrium after the contest organizer has chosen the optimal bias α^* given valuations $V = (v_1, \ldots, v_n)$. Hence, $\alpha^* = \alpha^*(V)$; in the same vein, $K^* = K^*(V)$ denotes the set of active contestants in the optimally biased contest given $V = (v_1, \ldots, v_n)$. Denote by $k^*(V)$ the cardinality of $K^*(V)$, $k^*(V) = |K^*(V)|$.

We now define the index set

$$L^*(V) = \left\{ i \in N \;\middle|\; \frac{k^*(V) - 2}{v_i} = \sum_{j \in K^*(V)} \frac{1}{v_j} \right\}$$

and set $l^*(V) = |L^*(V)|$. $L^*(V)$ contains those indices - if any - which belong to contestants who are indifferent between becoming active (with a bid of 0) and staying inactive (recall the definition of $K^*(V)$ from eq. (4)).

So let $V = (v_1, \ldots, v_n)$ be given and consider any ε-neighborhood of V, $U_\varepsilon(V)$, for $\varepsilon > 0$ sufficiently small. It is then true that for any $V' \in U_\varepsilon(V)$

$$K^*(V) \subseteq K^*(V') \subseteq K^*(V) \cup L^*(V)$$

holds; i.e. for all valuations V' sufficiently close to V the set of active contestants in the optimally biased lottery contest for V' consists of all contestants active in the optimally biased lottery contest for V plus - possibly - contestants from $L^*(V)$, who have become active in V'. Intuitively, since the participation condition in eq. (4) for a contestant i is given by an *in*equality, an active contestant in V, who satisfies the inequality, must stay active for sufficiently small changes in V as those cannot lead to a violation of the inequality. For the same reason, inactive contestants, who even violate the condition in $L^*(V)$, must stay inactive for sufficiently small changes in V. Formally, this is proven in [7], Theorem 3.2.

So let $M \subseteq L^*(V)$ and $m = |M|$. An alternative representation of $X^{*,BLC}(V)$ then reads:

$$\begin{aligned}
X^{*,BLC}(V) &= \frac{1}{4}\left[\sum_{j\in K^*(V)} v_j - \frac{(k^*(V)-2)^2}{\sum_{j\in K^*(V)}\frac{1}{v_j}}\right] \\
&= \frac{1}{4}\left[\sum_{j\in K^*(V)} v_j + \frac{m(k^*(V)-2)}{\sum_{j\in K^*(V)}\frac{1}{v_j}} - \frac{(k^*(V)+m-2)(k^*(V)-2)}{\sum_{j\in K^*(V)}\frac{1}{v_j}}\right] \\
&= \frac{1}{4}\left[\sum_{j\in K^*(V)} v_j + \sum_{j\in M} v_j - \frac{(k^*(V)+m-2)^2}{\sum_{j\in K^*(V)}\frac{1}{v_j} + \frac{m}{k^*(V)-2}\sum_{j\in K^*(V)}\frac{1}{v_j}}\right] \\
&= \frac{1}{4}\left[\sum_{j\in K^*(V)\cup M} v_j - \frac{(k^*(V)+m-2)^2}{\sum_{j\in K^*(V)\cup M}\frac{1}{v_j}}\right].
\end{aligned}$$

The second equality results from a trivial split of the last term, the third and fourth equalities result from using the definition of $L^*(V)$ (and hence M).

From the last expression of $X^{*,BLC}(V)$ we immediately see that $X^{*,BLC}(V)$ must be continuous at V: Any sequence $V^j \to V$ can be decomposed into - at most $2^{l^*(V)}$ - subsequences, such that each of these subsequences satisfies $K(V^j) = K^*(V) \cup M$ for all elements V^j of this subsequence with a fixed subset $M \subseteq L^*(V)$. Consequently $X^{*,BLC}(V^j)$ converges to $X^{*,BLC}(V)$.

In order to show continuous differentiability of $X^{*,BLC}(V)$ it suffices to show partial differentiability with respect to all $v_i,\ i = 1, \ldots, n$, and continuity of all the partial derivatives. For all $i \notin K^*(V) \cup L^*(V)$ we obviously have $\frac{\partial}{\partial v_i} X^{*,BLC}(V) = 0$ since $X^{*,BLC}(V)$ does not depend on v_i in a whole neighborhood. So let $i \in K^*(V) \cup L^*(V)$ and consider an arbitrary sequence $v_i^j \to v_i$. If we define $V^j = (v_1, \ldots, v_{i-1}, v_i^j, v_{i+1}, \ldots, v_n)$, then obviously $V^j \to V$. Again, consider any subsequence of V^j such that $K(V^j) = K^*(V) \cup M$ for a fixed $M \subseteq L^*(V)$ and consequently $\left|K(V^j)\right| = k^*(V) + m$ for all j in this subsequence. We then have on this subsequence:

$$
\begin{aligned}
\lim_{j\to\infty} \frac{X^{*,BLC}(V) - X^{*,BLC}(V^j)}{v_i - v_i^j} &= \begin{cases} 0, & \text{for } i \in L^*(V) \setminus M, \\ \frac{1}{4}\left[1 - \frac{(k^*(V)+m-2)^2}{\left(\sum_{l\in K^*(V)\cup M} \frac{1}{v_l}\right)^2} \frac{1}{v_i^2}\right], & \text{for } i \in K^*(V) \cup M \end{cases} \\
&= \begin{cases} 0, & \text{for } i \in L^*(V) \setminus M, \\ \frac{1}{4}\left[1 - \frac{(k^*(V)-2)^2}{\left(\sum_{l\in K^*(V)} \frac{1}{v_l}\right)^2} \frac{1}{v_i^2}\right], & \text{for } i \in K^*(V) \cup M \end{cases} \\
&= \begin{cases} 0 & \text{for } i \in L^*(V) = M \cup L^*(V) \setminus M \\ \frac{1}{4}\left[1 - \frac{(k^*(V)-2)^2}{\left(\sum_{l\in K^*(V)} \frac{1}{v_l}\right)^2} \frac{1}{v_i^2}\right] & \text{for } i \in K^*(V) \end{cases}
\end{aligned}
$$

Here we have again made use of the definition of $L^*(V)$, which contains M.

Obviously, the above limit exists and is independent of the sequence V^j, or the respective subsequences. Hence, $X^{*,BLC}(V)$ is partially differentiable with respect to all $v_i,\ i = 1, \ldots, n$. Continuity of the partial derivatives $\frac{\partial}{\partial v_i} X^{*,BLC}(V)$ derived above can now be shown in the same way as we have shown continuity of $X^{*,BLC}(V)$. The nonnegativity of the partial derivatives is again a direct consequence of the definition of the set $K^*(V)$. □

Now we can finally state the main result of this section. We shall prove that for any $V = (v_1, \ldots, v_n)$ the optimal BLC regime yields less total effort than $\frac{v_1+v_2}{2}$, which was shown to be always achievable through an appropriately biased all-pay auction in Theorem 3.2.

Theorem 4.3 *The optimal BAA regime induces higher total effort in equilibrium than the optimal BLC regime.*

Proof. We prove the theorem by first considering a special case, where the theorem can be verified by a simple calculation, and then reducing all other cases to the first one.

First we consider the case where all agents – possibly except the first one – have the same valuation, that is $v_1 \geq v_2 = v_3 = \ldots = v_n$. If we determine the set K^* of active agents in regime

BLC based on eq. (4), the condition simplifies to $\frac{k^*-2}{v_2} < \frac{1}{v_1} + \frac{k^*-1}{v_2}$. This inequality holds if $0 < \frac{1}{v_1} + \frac{1}{v_2}$, which is satisfied independently of i. Hence, in this case all agents will be active in the regime BLC and, applying Theorem 3.2, the crucial inequality $X^{*,BAA} > X^{*,BLC}$ is thus satisfied if

$$\frac{v_1 + v_2}{2} > \frac{1}{4}\left[v_1 + (n-1)v_2 - \frac{(n-2)^2}{\frac{1}{v_1} + \frac{n-1}{v_2}}\right].$$

After some algebra this inequality can be simplified to:

$$(n-1)v_1^2 + 2v_1v_2 > (n-3)v_2^2$$

The first expression on the left hand side is larger than the expression on the right hand side which implies that the inequality holds.

Now consider arbitrary valuations $v_1 \geq v_2 \geq v_3 \geq \ldots \geq v_n$ and denote the corresponding optimal revenues by $X^{*,BAA}$ and $X^{*,BLC}$. To prove the assertion in this case, we construct auxiliary distributions of valuations by first substituting v_3 with $v_3' = v_2$ and then compare the resulting revenue $X_{V_3}^{*,BLC}$ with $X^{*,BLC}$. Then we continue this procedure with $j = 4, \ldots, n$.

1. Consider the auxiliary distribution V_3 with $v_1 \geq v_2 = v_3' \geq v_4 \geq \ldots \geq v_n$. By Lemma 4.2 we know that $X_{V_3}^{*,BLC} \geq X^{*,BLC}$ as revenue can only increase.

2. Consider the auxiliary distribution V_4 with $v_1 \geq v_2 = v_3' = v_4' \geq v_5 \geq \ldots \geq v_n$. By Lemma 4.2 and step 1 we know that $X_{V_4}^{*,BLC} \geq X_{V_3}^{*,BLC} \geq X^{*,BLC}$.

(n-2). Consider the last step; i.e., the distribution V_n, where $v_1 \geq v_2 = v_3' = \ldots = v_n'$. By Lemma 4.2 and the previous steps we know that $X_{V_n}^{*,BLC} \geq X_{V_{n-1}}^{*,BLC} \geq \ldots \geq X_{V_3}^{*,BLC} \geq X^{*,BLC}$. Note that V_n coincides with a distribution considered at the beginning of the proof. Hence, our previous considerations and Theorem 3.2 can be applied to derive the following chain of inequalities:

$$X^{*,BAA} \geq \frac{v_1 + v_2}{2} > X_{V_n}^{*,BLC} \geq X^{*,BLC},$$

which completes the proof of this theorem.

□

5 Concluding Remarks

We have shown that in the presence of heterogeneous contestants an appropriately biased all-pay auction always revenue-dominates the optimally biased lottery contest. This is in contrast to the comparison of the unbiased versions of these contest models, if there are more than two contestants. The (unbiased) all-pay auction might yield less revenue (total effort), if in particular the exclusion principle applies; i.e., heterogeneity is such that it is revenue-enhancing to exclude the strongest player from participation. The two active but weaker contestants then may expend less effort than all the active players in the lottery contest. In contrast, we show that if the contest organizer has the ability to bias the contest, the exclusion principle of the all-pay auction becomes obsolete. No player is excluded. The contest organizer can always bias the all-pay auction in such a way that the two strongest players will be active and, moreover, compete on equal terms (the strongest player is therefore not excluded but sufficiently weakened in her effectiveness). All other players choose to be inactive. In short, the two strongest contestants are exposed on equal terms to the extreme discriminative all-pay auction. Reducing the discriminativeness of the contest by using a lottery CSF will attract more entry into the contest; i.e., more contestants (at least three) will be active in equilibrium. But having more active contestants in the less discriminative contest does not pay off for the contest organizer: The increase in competitiveness due to a higher number of competitors cannot offset the loss of competitiveness due to a 'softer' contest. Economic policy instruments aimed at facilitating entry do not work for contests, if revenue maximization is the goal of the contest organizer because participation effects are not strong enough to outweigh incentive effects.

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