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Games for Cautious Players: the Equilibrium in Secure Strategies*

Mikhail ISKAKOV[1], Alexey ISKAKOV[2] and Claude d'ASPREMONT[3]

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Abstract

A non-cooperative solution, the Equilibrium in Secure Strategies (EinSS), is defined that extends the Nash equilibrium in pure strategies when it does not exist and is meant to solve games where players are "cautious", *i.e.* looking for secure positions and avoiding threats. This concept abstracts and unifies various *ad hoc* solutions already formulated in various applied economic games that have been discussed extensively in the literature. It complements usefully mixed strategy Nash equilibria that are usually not explicit and difficult to interpret in these games. Like the Nash equilibrium, the EinSS is a static concept, and the basic requirement of excluding at equilibrium some deviations remains. But it also appeals to dynamic intuitions, tolerating at equilibrium the possibility of some deviations, which would be blocked by counter-deviations punishing the deviator. This is in line with the "objection-counter-objection" rationale first introduced in cooperative games. A general existence theorem is provided and then applied to the price-setting game in Hotelling location model, to Tullock's rent-seeking contests and to Bertrand-Edgeworth duopoly. Finally competition in the insurance market game is re-examined and the Rothchild-Stiglitz-Wilson contract shown to be an EinSS even when the Nash equilibrium breaks down.

Keywords: Noncooperative games, Equilibrium existence, Discontinuous games, Equilibrium in secure strategies, Hotelling model, Tullock contest, Insurance market, Bertrand-Edgeworth duopoly

JEL Classification: C72, D03, D43, D72, L12, L13

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1. Introduction

There are well-known economic games where a Nash-Cournot equilibrium does not exist. Examples include Hotelling's game of price competition on the line when the sellers locations are close[1], Rothchild and Stiglitz game of competitive insurance markets with adverse selection[2], Tullock's rent-seeking game with the success function parameter greater than two[3]. This existence problem was highlighted by Dasgupta and Maskin in their seminal paper (1986). They proved existence of mixed strategy Nash equilibria for a family of games with discontinuous payoff functions that covers all mentioned models. However these equilibria are not easy to characterize in most cases. Also they are usually difficult to interpret when applied to specific economic contexts. On the other hand, in some contexts there is an intuitive expectation that a stable position in pure strategies should exist. In support of this hypothesis a variety of *ad hoc* equilibrium concepts have been developed to describe the specific behavior of players in particular models. For instance, Wilson (1977) and Riley (1979) suggested two different pseudo-equilibrium concepts for the insurance market model. Eaton and Lipsey (1978) proposed the 'zero conjectural variation solution' to restore equilibrium existence in Hotelling's model. In d'Aspremont and Gabzewicz (1980) a concept of quasi-monopoly is introduced, that ensures the existence of a pseudo-equilibrium in the Bertrand-Edgeworth duopoly model when one capacity is small compared to the other. A common feature of these and many other concepts is that they describe a particular logic of rational behavior that takes into account the interaction of the players. This raises the key question: is there a general rationale, independent of the specifics of a particular model, behind these intuitively perceived equilibrium positions.

In this paper we try to address this question looking for some natural reasoning of independent players, which follows purely from their strategic thinking. In addition, we also assume that this logic is not associated with the formation of coalitions, agreements or any other preliminary contracts about common rules. The main rationale at the basis of the proposed concept (without excluding the possibilities of other concepts based on other logics)[4], is the assumption that players are looking for "secure" positions, namely positions where they cannot be "threatened" by another player. For that purpose, a *threat* of one player against another will be defined as a unilateral deviation of the first increasing his own payoff and simultaneously decreasing the payoff of the other player. Clearly at a Nash-Cournot equilibrium no player can threaten another: it is a "secure" strategy profile. We shall maintain this property. But this first requirement will be supplemented by another, appealing to some dynamic intuition, but without modeling this intuition explicitly. In this respect we shall follow an idea that was already used in cooperative game theory, that we will call for short the "objection and counter-objection rationale" (Aumann and Maschler,

[1]Hotelling (1929), d'Aspremont (1979).

[2]Rothchild and Stiglitz (1976), Wilson (1977).

[3]Tullock (1967, 1980) and Baye et al. (1994), for discussion and references.

[4]For example, another logic follows the general notion of "rationalizability" and various coarsening of the Nash equilibrium have been studied along this line (see Bernheim, 1984; Pearce, 1984; Basu and Weibull, 1991; Rubinstein and Wolinsky, 1994).

1964) to deal with the vacuity of the Core in cooperative games with side payments and lead to the definition of the Bargaining Set concept. In general terms the concept can be formulated as follows: defining the Core as the set of imputations at which there is no objection from some coalition of players, the Bargaining set is the set of imputations at which there is no "justified objection" from some coalition, the objection being justified in the sense that no other coalition has a counter-objection to it. Translated in our context the idea becomes: defining the Nash-Cournot equilibrium as a (secure) strategy profile at which there is no profitable deviation of some player, an Equilibrium in Secure Strategies (EinSS) is a secure strategy profile at which there is no profitable "secure deviation" of some player. The deviation is secure in the sense that no other player has a counter-deviation, which will be defined as a threat against the deviator making him worse-off than initially. By definition when a Nash Equilibrium exists, it is an Equilibrium in Secure Strategies. Notice that we apply the "objection and counter-objection rationale" while keeping the requirement that an EinSS is a secure strategy profile. Without this requirement we get a much coarser equilibrium concept (named *threatening-proof profile* in Iskakov and Iskakov, 2012a)[5]. In many games the set of such equilibria is very large. As a refinement the EinSS reduces this set drastically.

From the standpoint of applications our research is motivated by the fact that the EinSS concept abstracts and unifies various concepts in applied models in which there is no Nash equilibrium. The first formulation of an EinSS (Iskakov, 2005) has been applied to Hotelling's game (Iskakov and Iskakov, 2012b). For such economic models the EinSS concept offers meaningful solutions, which in many cases are unique. At the same time the proposed concept retains all Nash equilibria, when they exist.

In this paper we reformulate the EinSS concept and prove its existence in the four well-known economic games that we mentioned above. For each game we provide a unique (or, with symmetry, unique up to a permutation of players) EinSS solution and its intuitive interpretation[6]. For the first three applications (including Hotelling's game) existence is derived from a general existence theorem. This theorem establishes the existence of an EinSS in a class of games that may be neither continuous nor quasi-concave. For the last application, the insurance game, a specific proof has to be derived.

Related Literature. The proposed equilibrium concept can be associated with several areas of game theory. First, in applying the objection and counter-objection rationale, it is not only related to the Bargaining Set literature (for more recent references, see Holzman, 2001), but also to the literature on farsighted solution concepts, defining profile stability by checking wether a player (or group of players) can deviate non-myopically, that is, anticipating a possible sequence of deviations by other players. The farsighted solution concepts originally arose as a result of the study of vNM stable set in coalitional games (von Neumann and Morgenstern, 1944) and the non-myopic alternative proposed by Harsanyi (1974) and

[5]See also (Sandomirskaia, 2014; Iskakov and Iskakov, 2016).

[6]In general the set of EinSS is not large. From a theoretical point of view, this follows from the fact that the problem of finding EinSS can be formulated as an optimization problem for players' payoffs on the sets of their secure strategies.

formalized and extended by Chwe (1994)[7]. Although these concepts can formally be used to study also noncooperative games (Suzuki and Muto, 2005; Nakanishi, 2007; Jamroga and Melissen, 2011), it raises questions concerning the tractability of the obtained results. The first two papers assume that in the chain of deviations there could be unilateral deviations that are not profitable for deviators. This assumption does not really fit in a non-cooperative setting. As for the Farsighted Pre-Equilibrium (FPE) proposed by Jamroga and Melissen (2011), it is closer to our concept of secure deviations, but with important differences. At an EinSS, strategic thinking only goes for two deviations ahead, whereas at an FPE it goes for three and more steps ahead, assuming (contrary to what we do) that the initiating deviator has the opportunity to make a last unilateral deviation in order to recover his original payoff. Finally, a substantial difference between the EinSS and all farsighted solution concepts is the requirement that an EinSS be a secure strategy profile. In particular, dropping this assumption may lead to the emergence of multiple equilibria.

As we prove existence of an EinSS in a class of discontinuous games, our work is also related to the literature on existence of Nash equilibrium in discontinuous games. Developing ideas of Dasgupta and Maskin (1986), Reny's (1999) important paper proposes a fairly simple existence criterion covering most of previous existence results. The key concept of Reny is that, at any strategy profile, some player has a deviation to "secure a payoff" with respect to arbitrary local deviations of other players[8]. What we propose is another approach (and another definition of secure deviation) for finding equilibria in discontinuous games. Rather than imposing a condition on the game, we relax the requirements at equilibrium, allowing for applications to games in which a pure Nash equilibrium does not exist.

In the following section, the concept of EinSS is defined and its basic properties are analyzed. Then in section 3, we state and prove an existence theorem for discontinuous games and apply it, successively, to Hotelling's model, Tullock's contest and Bertrand-Edgeworth duopoly model. Section 4 gives an *ad hoc* existence results for the competitive insurance market with adverse selection. Proofs omitted in the main text are given in Appendix.

2. Equilibrium in secure strategies

2.1. Equilibrium concept

Consider a non-cooperative game $G = (S_i, u_i)_{i=1}^N$ of N players in normal form. We use the standard notation s_{-i} for strategies of all players other than i. If $s_i \in S_i$ is a deviation of player i from the profile s into profile s', we use the following notation $s \xrightarrow{i} s'$ with the obvious constraint $s_{-i} = s'_{-i}$. The proposed equilibrium concept is based on the notions of threat and secure deviation.

[7]Harsanyi's definition has been recently modified by Ray and Vohra (2015) in order to respect coalitional sovereignty.

[8]Reny's conditions have been weakened or simplified in a number of ways. See, for example, Bagh and Jofre (2006), Bich (2009), McLennan et al. (2011), Barelli and Meneghel (2013).

Definition 2.1. ***A threat*** *of player i against player j at strategy profile s is a deviation s_i' such that $u_i(s_i', s_{-i}) > u_i(s)$ and $u_j(s_i', s_{-i}) < u_j(s)$. A deviation of player i at strategy profile s is* ***a competitive deviation*** *if it is a threat against some player at s.*

A threat (and a competitive deviation) indicates a situation, in which it is profitable for one player to worsen the situation of another. In that respect, an important property of a strategy is to avoid such threats.

Definition 2.2. *A strategy s_i of player i is* ***a secure strategy*** *at strategy profile s if no player $j \neq i$ has a threat against player i at s. Otherwise it is an insecure strategy at s. A strategy profile s is* ***a secure profile****, if all players' strategies are secure.*

Alternatively, a strategy profile is secure if no player has a competitive deviation. Let us now introduce a complementary notion to competitive deviation.

Definition 2.3. ***A non-competitive deviation*** *of player i at strategy profile s is a deviation s_i' such that $u_i(s_i', s_{-i}) > u_i(s)$ for player i and $u_j(s_i', s_{-i}) \geq u_j(s)$ for all other players $j \neq i$.*

In this way all profitable unilateral deviations of players can be divided into competitive and non-competitive deviations depending on their effect on the other players. Non-competitive deviations do not pose an immediate danger to other players. Thus, from the standpoint of mutual security of players at unilateral deviations a Nash equilibrium can be characterized in the following manner.

Definition 2.4. *A secure strategy profile is* ***a Nash Equilibrium*** *if no player has a non-competitive deviation.*

Following the "objection and counter-objection rationale" (Aumann and Maschler, 1964), and reinforcing the security objective of the players, one can extend the applicability of a non-cooperative equilibrium by enlarging the class of games for which an equilibrium exists. The idea is that a non-competitive deviation by some player may give the opportunity to some other player to threaten the deviator at the new strategy profile. Then there are two possibilities after such a non-competitive deviation. Either the threat can make the deviator worse off than before the deviation, in which case we assume the deviation to be *insecure* for the deviator. This should induce the deviator, as a cautious player, to refrain from deviating. Or the threat makes the deviator as well off than initially and in this case the deviation can be considered as *secure* for the deviator. Formally, we have:

Definition 2.5. ***A secure non-competitive deviation*** *of player i at strategy profile s is a non-competitive deviation s_i' such that $u_i(s_i', s_j', s_{-ij}) \geq u_i(s)$ for any threat s_j' of player $j \neq i$ against player i at profile (s_i', s_{-i}).*

In order to formulate a concept of equilibrium in a game of cautious players, we may assume that such players avoid *insecure* non-competitive deviations. Then it is sufficient in the corresponding definition of equilibrium to only exclude *secure* non-competitive deviations at a non-cooperative equilibrium profile. Together with the requirement that the equilibrium is a secure profile, this leads us to the proposed definition for an Equilibrium in Secure Strategies:

Definition 2.6. *A secure strategy profile is* ***an Equilibrium in Secure Strategies (EinSS)*** *if no player has a secure non-competitive deviation.*

Before looking at economic applications, analyze some of properties of the EinSS concept.

2.2. Basic properties

The first important property of the EinSS follows immediately from its definition:

Proposition 2.1. *Any Nash equilibrium is an Equilibrium in Secure Strategies.*

This means that the existence problem can not be worse for EinSS than for Nash equilibrium. Whenever a Nash equilibrium exists an EinSS also exists. However for some practically important problems without Nash equilibrium (such as Hotelling's model and Bertrand-Edgeworth duopoly which will be considered in this paper) the EinSS exists and provides an interesting interpretation.

Let us now consider a simple matrix game example having no Nash equilibrium in order to illustrate the definition:

	t_1	t_2
s_1	(1,1)	(2,0)
s_2	(2,2)	(0,3)

One can find all threats in the game. First, in the strategy profile (s_2, t_1) there is a threat of t-player against s-player as we move from payoffs $(2,2)$ to payoffs $(0,3)$. Second, in the strategy profile (s_1, t_2) there is a threat of t-player against s-player as we move from payoffs $(2,0)$ to payoffs $(1,1)$. And finally in the profile (s_2, t_2) there is a threat of s-player against t-player as we move from payoffs $(0,3)$ to payoffs $(2,0)$. In all three cases one player can make himself better off and the other player worse off. These threats in the game can be visualized graphically in the following way:

$$\begin{array}{ccc} (1,1) & \leftarrow & (2,0) \\ & & \uparrow \\ (2,2) & \rightarrow & (0,3) \end{array}$$

The only secure profile in the game (which is secure for both players) is the profile (s_1, t_1) with payoffs $(1,1)$. If players were choosing best responses sequentially in the game they would end up in an infinite cycle so that there is no Nash equilibrium in pure strategies.

This situation can change if we take into account the considerations of security. The profiles with payoffs $(2,2)$, $(0,3)$ and $(2,0)$ can not be an equilibrium in secure strategies because they pose threats. The profile (s_1,t_1) is the only secure profile in the game. The t-player can not increase his profit by any deviation from it. There is a profitable deviation for the s-player from this profile into the profile (s_2,t_1) with payoffs $(2,2)$. However it is not a secure deviation since the s-player can lose more after the response deviation of t-player from the profile (s_2,t_1) with payoffs $(2,2)$ into the profile (s_2,t_2) with payoffs $(0,3)$. Therefore, in the profile (s_1,t_1), no player has a secure deviation and this profile is an EinSS. This means that a cautious player would prefer the guaranteed payment of 1 in the (s_1,t_1) to the possibility of gaining 2 in (s_2,t_1) accompanied by a high-risk to get zero in (s_2,t_2).

For some games the reverse of Proposition 2.1 is true. For instance for strictly competitive games (i.e., two-person games where all pairs of strategies are Pareto optimal), we get:

Proposition 2.2. *In a strictly competitive game, any Equilibrium in Secure Strategies is a Nash equilibrium.*

Proof. Consider a strictly competitive game and suppose there is an EinSS which is not a Nash equilibrium. Then there is at least one player who has a profitable deviation. But (by Pareto optimality) the other player will get a strictly lower payoff. Therefore, this is a competitive deviation so that the profile is not secure and can not be EinSS. □

However this property may not hold if the condition of strict competitiveness is weakened. For instance it does not hold for almost strictly competitive games introduced by Aumann (1961) on the basis of the concept of twisted equilibrium. A twisted equilibrium point in the two-player game is a pair of strategies at which neither player can decrease the other player's payoff by a unilateral change in strategy. A game is called almost strictly competitive if (i) the set of payoffs vectors to Nash equilibrium strategy profiles is equal to the set of payoffs vectors to twisted equilibrium strategy profiles and (ii) if the set of Nash equilibrium strategy profiles and the set of twisted equilibrium strategy profiles intersect. Let us modify the previous matrix game in the following way:

	t_1	t_2	t_3
s_1	(1,1)	(2,0)	(-1,1)
s_2	(2,2)	(0,3)	(-1,1)
s_3	(1,-1)	(1,-1)	(0,0)

There is a unique Nash equilibrium profile (s_3,t_3) with payoffs $(0,0)$ which at the same time is a unique twisted equilibrium. So the game is almost strictly competitive. The profile (s_3,t_3) is also an EinSS. But there is another EinSS. This is the profile (s_1,t_1) with payoffs $(1,1)$. It is not a Nash equilibrium. This example shows that cautious players (avoiding threatening or being threatened by others) may enforce (as an EinSS) a strategy profile that is not a Nash equilibrium but may dominate another EinSS which is a Nash equilibrium.

As the following example shows, a cautious player can even play a strictly dominated strategy at an EinSS, but the deviations offered to him at this equilibrium should be non-competitive and non-secure.

	t_1	t_2
s_1	(9,6)	(4,5)
s_2	(10,7)	(8,8)

The strategy s_2 dominates strategy s_1. There is a Nash equilibrium profile with payoffs $(8,8)$, which is an EinSS. But the strategy profile with payoffs $(9,6)$ is another EinSS solution, in which the s-player plays the strictly dominated strategy s_1. If the s-player was using the deviation $(s_1,t_1) \xrightarrow{s} (s_2,t_1)$, both players would gain but the t-player would be able to gain even more by deviating then to the profile (s_2,t_2) where the s-player is worse-off than at the EinSS.

Another way to view the EinSS, is to consider that each player, whenever possible, restrict to secure strategies (strategies such that no other player has a threat against him). In that respect, the definition of an EinSS implicitly implies that each player maximizes his payoff functions over the set of secure strategies. We let $Q_i(s_{-i}) \subseteq S_i$ be the set of secure strategies of player i for given strategies s_{-i} of the other players. We can define a profile in which the "secure strategy" of each player is the best one in the same way as the Nash equilibrium is a profile in which the strategy of each player is a best response.

Definition 2.8. *A secure strategy $s_i^* \in Q_i(s_{-i})$ of player i is* ***a best secure response*** *to strategies s_{-i} of all other players if player i has no more profitable secure strategy in $Q_i(s_{-i})$. A profile s^* is* ***the Best Secure Profile (or BS-profile)*** *if the strategies of all players are best secure responses, i.e. $s_i^* \in Q_i(s_{-i}^*)$ and $u_i(s^*) = \max\limits_{s_i \in Q_i(s_{-i}^*)} u_i(s_i, s_{-i}^*)$ for all i.*

There is a simple relation between EinSS and BS-profiles: the set of BS-profiles is larger than the set of EinSS:

Proposition 2.3. *Any Equilibrium in Secure Strategies is a BS-profile. A BS-profile may not be an Equilibrium in Secure Strategies.*

Proof. An EinSS is a secure profile by definition. And it must be the best secure response for each player since otherwise there is a player who can increase his payoff by a secure non-competitive deviation. Therefore an EinSS is a BS-profile. The reverse is not true. Consider the following matrix game example:

	t_1	t_2	t_3
s_1	(0,0)	(2,2)	(2,2)
s_2	(2,2)	(1,3)	(3,1)
s_3	(2,2)	(3,1)	(1,3)

The profile (s_1,t_1) is the only secure profile in the game and therefore it is a BS-profile. However it is not an EinSS because of secure non-competitive deviations $(s_1,t_1) \xrightarrow{s} (s_2,t_1)$ or $(s_1,t_1) \xrightarrow{s} (s_3,t_1)$. $\square$

Hence a BS-profile is an EinSS if and only if no player has secure non-competitive deviation. Even if the concept of BS-profile is weaker than the concept of EinSS, it will be useful in applications (as we shall see later) as an intermediate step in finding an EinSS.

3. Existence

3.1. Existence result

Recall that a strategy s_i of player i is insecure at strategy profile s, *if there is another player j who has a threat against player i at s.* Pursuing on the BS-profile idea, and as a tool to obtain an existence theorem, we shall associate to every game $G = (S_i, u_i)_{i=1}^N$ a modified game $\widetilde{G} = (S_i, v_i)_{i=1}^N$ with same strategies but payoffs adjusted for insecure strategies by taking into account the worst threat.

Definition 3.1. ***A secure payoff*** *of player i at strategy profile s is the function:*

$$v_i(s) = \begin{cases} \inf\limits_{j \neq i,\ s'_j:\ u_j(s'_j, s_{-j}) > u_j(s)} u_i(s'_j, s_{-j}), & \text{if } s_i \text{ is an insecure strategy for player } i \text{ in } G \\ u_i(s), & \text{if } s_i \text{ is a secure strategy for player } i \text{ in } G \end{cases}$$

The following proposition is an immediate but useful consequence:

Proposition 3.1. *If a secure strategy profile s^* is a strict Nash equilibrium in the modified game $\widetilde{G}$, then it is an EinSS. If s^* is an EinSS of the game G, then s^* is a Nash equilibrium of the modified game $\widetilde{G}$.*

On the basis of this observation, we now introduce a class of games for which the existence of an EinSS can be shown. Roughly speaking this is a class of games such that, whenever a profile s is insecure (in the game G) for some player i, i.e. $s_i \notin Q_i(s_{-i})$, this player can increase his secure payoff v_i by deviating to a profile (s'_i, s_{-i}) which is secure for him, i.e. $s'_i \in Q_i(s_{-i})$. Player i is then said to have *a better secure alternative* at profile s. However this condition will be weakened by requiring its fulfillment only relative to some box $B = \times_{i=1}^N B_i$, where B_i is assumed to be a compact convex subset of S_i.

Definition 3.2. *A player i has **a better secure alternative** in B, if for every $s_{-i} \in B_{-i}$ there exists a non-empty subset $\widetilde{Q}_i(s_{-i}) \subseteq Q_i(s_{-i}) \cap B_i$ such that, for every strategy $s_i \notin \widetilde{Q}_i(s_{-i})$ there exists a strategy $s'_i \in \widetilde{Q}_i(s_{-i})$ such that $u_i(s'_i, s_{-i}) = v_i(s'_i, s_{-i}) > v_i(s_i, s_{-i})$. A game $G = (S_i, u_i)_{i=1}^N$ is said to be a **BSA-game** relative to B if every player has a better secure alternative in B.*

For any $s_{-i} \in B_{-i}$, the set $\widetilde{Q}_i(s_{-i})$ in a BSA-game is always understood to be non-empty. The graph of the multi-valued function $\widetilde{Q}_i(s_{-i})$, $\Gamma(\widetilde{Q}_i) = \{(s_i, s_{-i}) \mid s_i \in \widetilde{Q}_i(s_{-i}),\ s_{-i} \in B_{-i}\}$ is defined as a subset of B.

We shall use Debreu (1952) theorem to prove the existence of an EinSS in BSA games in finite Euclidean spaces. For this class of games we relax the standard conditions for the existence of a (pure) Nash equilibrium by requiring the fulfillment of these conditions only in the corresponding box B. In this case a Nash equilibrium in the set B turns out to be an EinSS in the original game.

Theorem 3.1. *Let G be a BSA-game relative to B such that for all i, the graph $\Gamma(\widetilde{Q}_i)$ is closed, $u_i(s)$ is a continuous function from $\Gamma(\widetilde{Q}_i)$ to $\mathbb{R}$, and $\phi_i(s_{-i}) = \max_{s_i \in \widetilde{Q}_i(s_{-i})} u_i(s_i, s_{-i})$ is continuous. If for every i and $s_{-i} \in B_{-i}$ the set $M_{s_{-i}} = \{s_i \in \widetilde{Q}_i(s_{-i}) \mid u_i(s_i, s_{-i}) = \phi_i(s_{-i})\}$ is contractible, then there exists an EinSS in B.*

Proof. As the basis of the proof we shall use the social equilibrium existence theorem of Debreu (1952). Since the sets $\widetilde{Q}_i(s_{-i})$ assumed to be non-empty for all $s_{-i} \in B_{-i}$ in a BSA-game we can consider them as a multivalued function from B_{-i} to B_i. Next, following Debreu we define a profile s^* as a social equilibrium point, if for all $i = 1, ..., N$: $s_i^* \in \widetilde{Q}_i(s_{-i}^*)$ and $u_i(s^*) = \max_{s_i \in \widetilde{Q}_i(s_{-i}^*)} u_i(s_i, s_{-i}^*)$. Then all conditions of Debreu's Existence Theorem (1952) are satisfied: there exists a social equilibrium point $s^* \in B$. Consider a profitable deviation s_i' of any player i at profile s^*. It is obviously non-competitive, because s^* is a secure profile. If $s_i' \in \widetilde{Q}_i(s_{-i}^*)$ then $u_i(s_i', s_{-i}^*) \leq u_i(s^*)$ and the deviation is not profitable. If $s_i' \notin \widetilde{Q}_i(s_{-i}^*)$ then according to BSA condition there exists $s_i'' \in \widetilde{Q}_i(s_{-i}^*)$ such that $u_i(s_i'', s_{-i}^*) > v_i(s_i', s_{-i}^*)$. Since $u_i(s_i'', s_{-i}^*) \leq u_i(s^*)$ we obtain $v_i(s_i', s_{-i}^*) < u_i(s^*)$. By Definition 3.1 of the function v_i this implies that the non-competitive deviation s_i' is not secure. Therefore in the secure profile s^* no player has a secure non-competitive deviation, i.e. s^* is an EinSS in the original game G. $\square$

The rest of this section will be devoted to well-known economic games which can be proved to be BSA-games and to satisfy the conditions of Theorem 3.1.

3.2. Applications of Theorem 3.1

To apply Theorem 3.1 it is necessary to examine the structure of the set of secure strategies of players. This step seems inevitable when searching for the EinSS. Nevertheless, in the following economic games the set of secure strategies is rather simple. In this case Theorem 3.1 applies very well.

3.2.1. Hotelling's model

In the Hotelling's model (1929) two sellers of a homogeneous product are located on a closed line of length l at respective distances a and b from the ends of this line ($a + b \leq l, a \geq 0, b \geq 0$). Customers are evenly distributed with a unit density along the line and consume a unit of this product per unit of time. A customer will buy from the seller who quotes the lowest delivery price, namely the mill price plus transportation cost, which is assumed linear

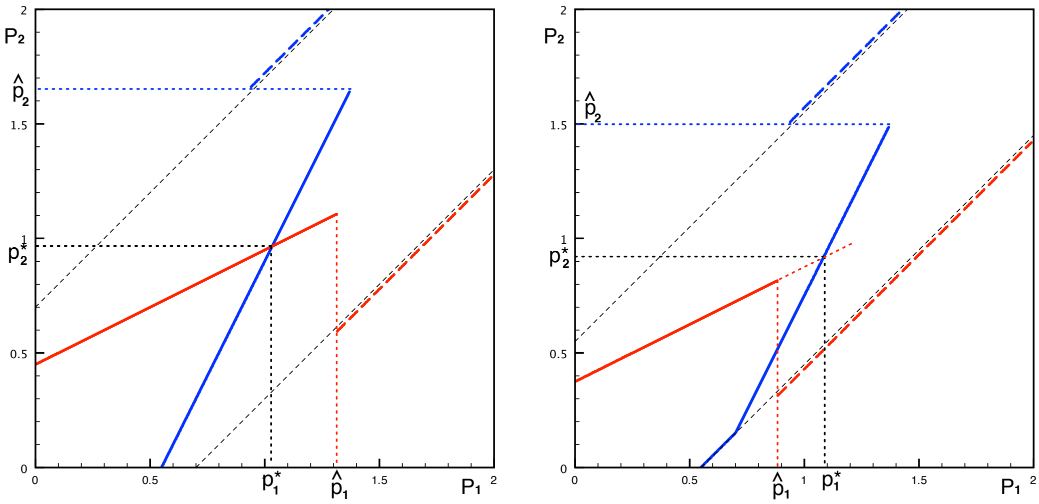

Figure 1: Two cases of the best responses behavior in the price-setting Hotelling's subgame.

with respect to the distance. Let p_1 and p_2 denote, respectively, the mill price of sellers and let $d = l - a - b$ denote the distance between them. In the price-setting subgame, where the two sellers are the players, the payoff function of seller 1 is

$$u_1(p_1, p_2) = \begin{cases} u_1^I = p_1 l, & p_1 < p_2 - d \\ u_1^{II} = p_1(a + \frac{d+p_2-p_1}{2}), & |p_1 - p_2| \leq d \\ 0, & p_1 > p_2 + d \end{cases} \tag{1}$$

The payoff function of seller 2 is symmetric. These functions are two-peak. The best response functions of the two players (drawn in red for player 2 and in blue for player 1 in Fig. 1) are discontinuous respectively at the points $\hat{p}_1$ and $\hat{p}_2$, where there is a jump from one peak to the other of the payoff functions of player 2 and player 1, respectively.

Two cases are possible (see Fig. 1). The best response functions can intersect at one point $(p_1^*, p_2^*) = (l + \frac{a-b}{3}, l + \frac{b-a}{3})$, at which the price-setting game admits a Nash equilibrium. This case occurs when $p_1^* \leq \hat{p}_1$ and $p_2^* \leq \hat{p}_2$. If on the contrary $p_1^* > \hat{p}_1$ or $p_2^* > \hat{p}_2$ then the jump occurs before the two best response functions intersect. In this case there is no Nash equilibrium in the game. These conditions can be written as (d'Aspremont et al.,1979):

$$\left(l + \frac{a-b}{3}\right)^2 < \frac{4}{3}l(a+2b) \quad \text{or} \quad \left(l + \frac{b-a}{3}\right)^2 < \frac{4}{3}l(b+2a) \tag{2}$$

To apply Theorems 3.1 it is necessary to examine the structure of the sets of secure strategies of the two players in the price-setting subgame. In the area $p_i < p_{-i} - d$, where seller i "undercuts" its competitor and take away the entire market, all strategies of i are insecure, because the competitor can reduce its price and get positive profit. In the area $p_i > p_{-i} + d$ seller i gets zero payoff. Therefore, all secure profiles with positive payoffs for the two players belong to the diagonal domain $|p_1 - p_2| \leq d$ of the strategy space. In this area there are two types of threats against a player i. If $u_{-i}^{II}(p_i, p_{-i}) < u_{-i}^{I}(p_i - d)$, the competitor $-i$ can profitably "undercut" the price of seller i and take away the entire market. In the contrary case, the best response of i's competitor is simply $BR_{-i}(p_i) = \underset{|p_i - p'_{-i}| \leq d}{\operatorname{argmax}} u_{-i}^{II}(p_i, p'_{-i})$. But, if $p_{-i} > BR_{-i}(p_i)$, the competitor $-i$ can also profitably reduce the price and capture a market share from player i. Therefore, the set of strategy profiles in the diagonal domain, which are secure for player i is defined by the condition

$$\begin{cases} u_{-i}^{II}(p_i, p_{-i}) \geq u_{-i}^{I}(p_i - d) \\ p_{-i} \leq BR_{-i}(p_i) \end{cases} \tag{3}$$

It remains to choose the box B so that for every (p_1, p_2) in B the inverse $BR_1^{-1}(p_1)$ and $BR_2^{-1}(p_2)$ of the best response functions are defined and continuous, since then the strategy of player $-i$ in the profile $(p_i, BR_i^{-1}(p_i))$ is clearly secure. The best response functions $BR_1(p_2)$ and $BR_2(p_1)$ strictly increase up to the points of their discontinuity, respectively at profiles $(BR_1(\hat{p}_2), \hat{p}_2)$ and $(\hat{p}_1, BR_2(\hat{p}_1))$. Therefore, these points determine the size of B. Finally we get:

$$B = B_1 \times B_2 = [0, p_1^B] \times [0, p_2^B], \text{ with } p_1^B = \min\{\hat{p}_1, BR_1(\hat{p}_2)\},\ p_2^B = \min\{\hat{p}_2, BR_2(\hat{p}_1)\}.$$

Proposition 3.2. *Under condition (2) the price-setting Hotelling's subgame, with payoffs given by (1), satisfies the conditions of Theorem 3.1 in B and admits an EinSS in B.*

See proof in Appendix.

For locations outside condition (2), a Nash equilibrium (hence a EinSS) exists, which coincides with the solution found by Hotelling. One can show in addition that at each location pairs there is a unique EinSS (Iskakov and Iskakov, 2012b). The existence and uniqueness of EinSS in the price-setting subgame allows for the correct solution (i.e. the *subgame perfect* EinSS) of the two-stage location-price Hotelling's game. In the first stage seller 1 chooses its location a and seller 2 its location b, anticipating the equilibrium prices $(p_1^*(a, b), p_2^*(a, b))$ of the second stage subgame. As shown by Iskakov and Iskakov (2012b), the Nash equilibrium in locations gives

$$a^* = 3l + b^* - 6\sqrt{lb^*},\ b^* \geq l/4, \text{ and } b^* = 3l + a^* - 6\sqrt{la^*},\ a^* \geq l/4 \tag{4}$$

There are no other EinSS equilibrium in the first stage game.

We see that the equilibrium locations lie on the boundary between the domain in which the Nash equilibrium in prices found by Hotelling exists, and the domain in which it doesn't because of undercutting, i.e. at this boundary conditions (2) hold as equalities. By symmetry, we get for the (unique) equilibrium locations: $a^* = b^* = l/4$.

To sum, by taking explicitly into account the threat of undercutting, one gets a unique solution at each location pair, the EinSS, which coincides with Nash where it exists, but exists anyway because the sellers secure themselves against this threat. The resulting equilibrium location pair minimizes the differentiation on the domain where the Nash equilbrium in prices exists. The Principle of Minimum Differentiation applies only on this domain.

This is in contrast with the effect of the modified ZCV assumption introduced by Eaton (1972, p. 269), saying "the action of one producer does not completely eliminate the other"[9]. In the EinSS concept sellers consider the possibility of being completely eliminated as a real threat, and therefore, they keep prices sufficiently low to secure themselves against such undercutting. Also, in contrast to the price solution in mixed strategies the EinSS solution is obtained in explicit form and can be interpreted as an equilibrium of bilateral containment by cautious players.

3.2.2. Tullock contest of two players

In the basic formulation of Tullock's rent-seeking contest, N players compete for a prize and each player exerts effort $x_i \in [0,1]$ so as to increase his probability of winning $x_i / \sum_{j=1}^{N} x_j$ (Tullock, 1980). A more generalized form of the game has the expected profits of the players given by $x_i^\alpha / \sum_{j=1}^{N} x_j^\alpha - x_i,\ \alpha > 0$ (see, e.g., Skaperdas, 1996). Here we consider a contest of two players with payoff functions defined as:

$$u_1 = \frac{x_1^\alpha}{x_1^\alpha + x_2^\alpha} - x_1,\ u_2 = \frac{x_2^\alpha}{x_1^\alpha + x_2^\alpha} - x_2,\ \alpha > 0 \tag{5}$$

This game has a unique Nash equilibrium $(\alpha/4, \alpha/4)$ when $\alpha \leq 2$ and there is no Nash equilibrium when $\alpha > 2$. When $\alpha > 1$ the payoff functions u_i have a double peak in x_i. The left peak arises at $x^{1peak} = 0$ and the position of the right peak x^{2peak} is defined by

$$x^{2peak} = \xi^{-1}(x_{-i}) \equiv \begin{cases} \left(\xi^+\right)^{-1}(x_{-i}), & x_{-i} > \frac{\alpha}{4} \\ \left(\xi^-\right)^{-1}(x_{-i}), & x_{-i} \leq \frac{\alpha}{4} \end{cases}, \quad \text{where}$$

$$\xi^+(x_i) \equiv \left(\frac{x_i^{\alpha-1}}{2}\left(\alpha - 2x_i + \sqrt{\alpha^2 - 4\alpha x_i}\right)\right)^{1/\alpha}, \quad \max\left\{0, \frac{\alpha^2-1}{4\alpha}\right\} \leq x_i \leq \alpha/4 \tag{6}$$

$$\xi^-(x_i) \equiv \left(\frac{x_i^{\alpha-1}}{2}\left(\alpha - 2x_i - \sqrt{\alpha^2 - 4\alpha x_i}\right)\right)^{1/\alpha}, \quad 0 \leq x_i \leq \alpha/4$$

When $\alpha > 2$ the best response of player i is

$$BR_i(x_{-i}) = \begin{cases} \xi^{-1}(x_{-i}), & x_i \leq \bar{x}_i \equiv \frac{1}{\alpha}(\alpha-1)^{\frac{\alpha-1}{\alpha}} \\ 0, & x_i \geq \bar{x}_i \end{cases} \tag{7}$$

The player's strategy is secure in the game G if and only if the competitor can not increase his payoff by increasing his effort. Following this criterion consider the set of

[9] This rule is also known as the no-mill-price-undercutting restriction. See Eaton and Lipsey (1989) and the references given in their survey.

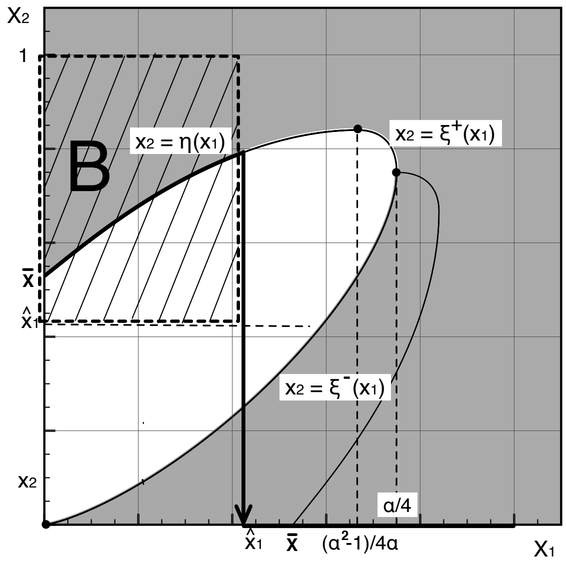

Figure 2: Secure strategies (gray area) and BSR function of player 2.

profiles at which the strategy of player 2 is secure. Let us define $\bar{\bar{x}} = \frac{1}{4\alpha}(\alpha+1)^{\frac{\alpha+1}{\alpha}}(\alpha-1)^{\frac{\alpha-1}{\alpha}}$ and introduce an auxiliary invertible function $\tilde{\eta}$ for $\alpha > 2$ on the interval $0 \le x \le \frac{\alpha^2-1}{4\alpha}$ as $\tilde{\eta}(x) \equiv x_2 : u_1(x, x_2) = u_1(\xi^{-1}(x_2), x_2)$. As we know the payoff function $u_1(x_1, x_2)$ can be two-peak in x_1 depending on x_2. When $0 \le x_2 < \bar{x}$ the right peak is higher than the left one and player 1 has a threat at this profile against player 2 when $x_1 < x^{2peak} = \xi^{-1}(x_2)$. When $\bar{x} < x_2 < \bar{\bar{x}}$ the left peak is higher than the right one and insecure strategies for player 2 lie in the interval $\tilde{\eta}^{-1}(x_2) < x_1 < \xi^{-1}(x_2)$. Finally when $x_2 \ge \bar{\bar{x}}$ the function u_1 decreases monotonically in x_1 and all profiles are secure for player 2. The set of profiles (x_1, x_2) that are secure for player 2 is shaded in gray in Fig. 2 and can be formally written as

$$\{x_2 < \bar{x}, x_1 \ge \xi^{-1}(x_2)\} \cup \{\bar{x} \le x_2 < \bar{\bar{x}},\ x_1 \le \tilde{\eta}^{-1}(x_2) \text{ or } x_1 \ge \xi^{-1}(x_2)\} \cup \{x_2 \ge \bar{\bar{x}}\},$$

or, alternatively, it can be written in the form:

$$\{x_1 \ge \alpha/4\}\ \cup\ \{x_1 < \alpha/4, x_2 \ge \eta(x_1) \text{ or } x_2 \le \xi^-(x_1)\}. \tag{8}$$

$$\text{where } \ \eta(x_1) \equiv \begin{cases} \xi^+(x_1), & \frac{\alpha^2-1}{4\alpha} \le x_1 \le \frac{\alpha}{4} \\ x_2 :\ u_1(x_1, x_2) = u_1\left(\xi^{-1}(x_2), x_2\right), & x_1 \le \frac{\alpha^2-1}{4\alpha} \end{cases} \tag{9}$$

As can be seen from Fig.2 the set of secure strategies of player 2 at a given $x_1 < \alpha/4$ consists of two intervals separated by the right peak $x_2 = x^{2peak}$ of the function u_2. Therefore, the best secure response $BSR_2(x_1)$ of player 2 could be either at $x_2 = 0$, or at $x_2 = \xi^-(x_1)$, or at $x_2 = \eta_2(x_1)$. The accurate estimation (which will be proven in the first step of the proof of Proposition 3.3) shows that

$$BSR_2(x_1) = \begin{cases} \eta_2(x_1),\ x_1 \le \hat{x} \\ 0,\ \ x_1 \ge \hat{x} \end{cases}, \quad \text{where } \hat{x} :\ \hat{x} = x_2\left(\frac{1}{x_2} - 1\right)^{1/\alpha},\ x_2 = \eta_2(\hat{x})$$

This function $BSR_2(x_1)$ is marked in Fig.2 by a thick line. This implies that the connected subsets of secure strategies of player 2, including the more profitable secure strategies, can be chosen for $x_1 < \hat{x}$ as $\widetilde{Q}_2(x_1) = \{\eta_2(x_1) \leq x_2 \leq 1\}$ and for $x_1 > \hat{x}$, as $\widetilde{Q}_2(x_1) = \{0 \leq x_2 \leq \delta\}$, with $\delta > 0$ small enough. When $x_1 = \hat{x}$ any such BSA subset $\widetilde{Q}_2(x_1)$ must include the two points $x_2 = 0$ and $x_2 = \eta_2(x_1)$, and hence the corresponding set $M_{\hat{x}}$ from the Theorem 3.1 would be non-contractible. Thus, if we want to apply Theorem 3.1, the set B should not include strategy profiles at $x_1 = \hat{x}$ and $x_2 = \hat{x}$ (by symmetry). However, the above analysis suggests that one can choose a strategy profile box B as (see Fig. 2):

$$B = B_1 \times B_2 = [0, \hat{x} - \epsilon] \times [\hat{x} + \epsilon, 1] \text{ with an arbitrary small } \epsilon > 0. \tag{10}$$

Proposition 3.3. *When $\alpha > 2$ the two-person Tullock contest (5) satisfies the conditions of Theorem 3.1 in B and admits an EinSS in B.*

See proof in Appendix.

A full solution of the Tullock contest for two players in secure strategies as well as some generalizations for many players were obtained in (Iskakov et al., 2014). In particular, when a contest admits no Nash equilibrium, the following proposition applies:

Proposition 3.4. *When $\alpha > 2$ the two-person Tullock contest (5) admits the following EinSS (which is unique up to a permutation of players):*

$$(0, x^*) \text{ or } (x^*, 0), \quad x^* = \frac{1}{\alpha}(\alpha - 1)^{\frac{\alpha-1}{\alpha}} \tag{11}$$

When there is no Nash equilibrium (i.e. a symmetric competition is unprofitable for players), the EinSS concept allows us to discover a different type of non-cooperative equilibrium in Tullock contest, where one player exerts a high level of effort to keep the prize while the other player refuses to participate. In this situation the first player prefers to fix his secure monopolistic position and maintains a high level of effort to create an entry barrier into the contest for the competitor. The observed EinSS solution is unique up to a permutation of players. The efficiency of equilibrium in the contest is usually characterized by the rent dissipation, which is a ratio of the total effort of all players to the value of the prize. The higher the rent dissipation, the lower the efficiency of equilibrium. When a pure Nash equilibrium does not exist, the rent dissipation in mixed strategy equilibria is equal to one, so the rent is completely dissipated (Baye et al., 1993). However, in the discovered monopolistic EinSS the rent dissipation is significantly less. Thus, in this case the EinSS provides a more efficient solution than the Nash equilibrium in mixed strategies.

3.2.3. Bertrand-Edgeworth duopoly model

In this subsection we consider the Bertrand-Edgeworth model of price-setting duopolists, selling an homogeneous product, under capacity constraints and with identical marginal costs, normalized to zero. We assume a continuous strictly decreasing consumer's demand function $D(p)$. Each firm $i = 1, 2$, has productive capacity S_i and it is assumed that $D(0) \geq S_1 + S_2$. Firms compete in prices (p_1, p_2) and play non-cooperatively. The firm quoting the lower price serves the entire market up to its capacity and the residual demand is met by the other firm. Consumers served at the lower price are randomly selected so that the residual demand to the firm quoting the higher price is a proportion of total demand at that price. If setting the same prices, firms share the market in proportion to their capacities. Formally we define the payoff functions of players to be:

$$\begin{aligned} u_1(p_1,p_2) &= \begin{cases} p_1 \min\{S_1, D(p_1)\}, & p_1 < p_2 \\ p_1 \min\{S_1, \frac{S_1}{S_1+S_2} D(p_1)\}, & p_1 = p_2 \\ p_1 \min\{S_1, \frac{D(p_1)}{D(p_2)} \max\{0, D(p_2) - S_2\}\}, & p_1 > p_2 \end{cases} \\ u_2(p_1,p_2) &= \begin{cases} p_2 \min\{S_2, D(p_2)\}, & p_2 < p_1 \\ p_2 \min\{S_2, \frac{S_2}{S_1+S_2} D(p_2)\}, & p_2 = p_1 \\ p_2 \min\{S_2, \frac{D(p_2)}{D(p_1)} \max\{0, D(p_1) - S_1\}\}, & p_2 > p_1 \end{cases} \end{aligned} \tag{12}$$

It is well known since Edgeworth (1925) that such model may not posses a Nash equilibrium (see e.g. d'Aspremont and Gabszewicz, 1980). We now show that, unless capacities are excessive (in some specific sense), there is an Equilibrium in secure strategies. Denote by p^* the price such that demand equal capacity, i.e. $D(p^*) = S_1 + S_2$. We have:

Proposition 3.5. *Assume that the receipt function $pD(p)$ is strictly concave. If the Bertrand-Edgeworth game with payoff functions (12) satisfies the conditions:*

$$\begin{cases} \arg\max\limits_{p>0}\{p(D(p) - S_1)\} \leq p^* \\ \arg\max\limits_{p>0}\{p(D(p) - S_2)\} \leq p^* \end{cases}, \quad \textit{where} \quad D(p^*) = S_1 + S_2, \tag{13}$$

then it satisfies the conditions of Theorem 3.1 and admits an EinSS.

Proof. Let us take $B = [0, p^*] \times [0, p^*]$ as strategy profile box. If $p \in B$ the payoff functions of each player, $u_1 = p_1 S_1$ and $u_2 = p_2 S_2$, are independent of the other player strategies. All strategy profiles in B are secure. Choose $\tilde{Q}_1(p_2) = [0, p^*]$, $\tilde{Q}_2(p_1) = [0, p^*]$. Clearly, the graph of each function $\tilde{Q}_i$, $\Gamma(\tilde{Q}_i) = B$, is closed, the maximum payoff $\phi_i(p_{-i}) = p^* S_i$ is constant, and each set $M_{p_{-i}}$ consists of a unique point and so contractible. It only remains to prove that G is a BSA-game in B. For that we need only to prove that player 1 with insecure strategy $p_1 > p^*$ always has a BSA in B when $p_2 \leq p^*$. Observe that, at the profile (p_1, p_2), there exists a deviation $(p_1, p_2) \xrightarrow{2} (p_1, p^* + \varepsilon)$, with $\varepsilon > 0$ small enough so that $p^* + \varepsilon < p_1$, $S_2 < D(p^* + \varepsilon)$ and player 2 strictly increases its payoff from $p_2 S_2$ to $(p^* + \varepsilon) S_2$. Then, if $p^* D(p^*) > p_1 D(p_1)$, $v_1(p_1, p_2) \leq u_1(p_1, p^* + \varepsilon) \leq p_1 \frac{D(p_1)}{D(p^*+\varepsilon)} (D(p^* + \varepsilon) - S_2) < p^*(D(p^*) - S_2) = p^* S_1 = u_1(p^*, p_2)$, i.e. the deviation $(p_1, p_2) \xrightarrow{1} (p^*, p_2)$ provides for player 1 a BSA in B_1.

Now suppose $p_1 D(p_1) \geq p^* D(p^*)$. If $p^* D(p^*) = p_1 D(p_1)$ then it implies that $p^* < p_M < p_1$ and that, anyway, $(p_1 - \varepsilon) D(p_1 - \varepsilon) > p^* D(p^*)$, with $\varepsilon > 0$ arbitrarily small. Then, at the profile (p_1,p_2), there exists a deviation $(p_1,p_2) \xrightarrow{2} (p_1,p_1-\varepsilon)$, at which player 2 strictly increases its payoff from p_2S_2 to $u_2(p_1,p_1-\varepsilon) = \min\{(p_1-\varepsilon)D(p_1-\varepsilon),(p_1-\varepsilon)S_2\} > \min\{p^*D(p^*),p^*S_2\} = p^*S_2 \geq p_2S_2$. From the condition (13) we obtain $u_1(p_1,p_1-\varepsilon) = \max\{0,p_1\frac{D(p_1)}{D(p_1-\varepsilon)}(D(p_1-\varepsilon)-S_2)\} < p^*(D(p^*)-S_2) = p^*S_1 = u_1(p^*,p_2)$, i.e. the deviation $(p_1,p_2) \xrightarrow{1} (p^*,p_2)$ provides for player 1 a BSA in B_1. Arguments for player 2 follow by symmetry. Thus G is a BSA game in B. □

Under strict concavity of the receipt function, it is known and easy to see that (p^*,p^*) is the unique Nash Equilibrium if $p^* \geq p_M = \arg\max\limits_{p>0}\{pD(p)\}$. In such a case, firm i, anticipating that its competitor chooses p^*, cannot do better than quoting price p^* and producing at capacity. For any larger price p_i, $p_iD(p_i) < p^*D(p^*)$ and its profit would only be $p_i\frac{D(p_i)}{D(p^*)}(D(p^*)-S_{-i}) < p^*S_i$. The following proposition goes further and provides a full characterization of EinSS in the Bertrand-Edgeworth game.

Proposition 3.6. *Let the receipt function $pD(p)$ be strictly concave and reach its maximum at p_M. Then in the Bertrand-Edgeworth game with payoff functions (12), (p^*,p^*) is an EinSS (with$D(p^*) = S_1+S_2$) if and only if*

$$\begin{cases} \arg\max\limits_{p>0}\{p(D(p)-S_1)\} \leq p^* \\ \arg\max\limits_{p>0}\{p(D(p)-S_2)\} \leq p^* \end{cases} \tag{14}$$

If $p^* \geq p_M$ it is a Nash equilibrium. There are no other EinSS in the game.

Proof. Since the receipt function $pD(p)$ is strictly concave then the function $p(D(p)-S)$ at a given S is also strictly concave in p and reaches the unique maximum at $p>0$. Therefore $\arg\max\limits_{p>0}\{p(D(p)-S)\}$ can be considered as a function of S. See proof in Appendix. □

Corollary. *If function $pD(p)$ is differentiable the condition (14) is equivalent to*

$$\frac{d}{dp}\Big(pD(p)\Big)\Big|_{p=p^*} \leq \min\{S_1,S_2\} \tag{14'}$$

Proof. One can easily check that $\hat{p} = \arg\max\limits_{p>0}\{p(D(p)-S)\}$ if and only $\frac{d}{dp}\Big(pD(p)\Big)\Big|_{p=\hat{p}} = S$. Besides $\frac{d}{dp}\Big(pD(p)\Big)$ is strictly decreasing. Therefore $\hat{p} \leq p^*$ if and only if $\frac{d}{dp}\Big(pD(p)\Big)\Big|_{p=p^*} \leq \frac{d}{dp}\Big(pD(p)\Big)\Big|_{p=\hat{p}} = S$. Hence we get the equivalence of (14) and (14'). □

As an example let us consider the demand function $D(p) = 1-p$. Then $p^* = 1-S_1-S_2$, $\arg\max\limits_{p>0}\{p(D(p)-S)\} = \frac{1-S}{2}$, $p_M = 1/2$ and conditions (14) take the form:

$$S_1+2S_2 \leq 1 \quad \text{and} \quad S_2+2S_1 \leq 1 \tag{15}$$

As mentioned, when the equilibrium price p^* is equal to or exceeds the price p_M, an EinSS coincides with the Nash equilibrium solution. When a Nash equilibrium does not exist (when $1 - S_1 - S_2 < \frac{1}{2}$) and (15) holds, there is a unique EinSS solution with equilibrium prices (p^*, p^*). Then p^* is less than the monopoly price (equal to $max\{1/2, 1 - S_1 - S_2\}$). The corresponding difference in price $S_1 + S_2 - \frac{1}{2}$ can be interpreted as an additional cost to maintain security when duopolists secure themselves against the threat of being undercut in the industry competition. This example also illustrates the fact that, for an EinSS to exist when p^* is less than p_M, one firm should have a productive capacity which is not too large (less than 1/3 in the example).

4. Insurance market model of Rothschild, Stiglitz and Wilson

In this section we shall consider the model of insurance market analyzed by Rothschild and Stiglitz (1976) and Wilson (1977) and show that it always has an equilibrium in secure strategies. The proof will not be an application of Theorem 3.1, but will follow the graphical procedure introduced by Rothchild and Stiglitz.

Two insurance companies sell insurance contracts to consumers which fall into two classes: there are n_H high risk consumers and n_L low risk consumers. High risk consumers have accidents with probability p_H and low risk consumers with probability $p_L < p_H$. All consumers have the same strictly positive initial endowment $w = (w_1, w_2) \in R^2$ representing their income in the two states of nature: w_2 if having an accident and w_1 if not. Preferences of all consumers are represented by the same strictly concave utility function u. Each insurance contract is a vector $c = (c_1, c_2) \in R^2$, where c_1 is the insurance premium and c_2 is the accident benefit net of premium. The endowment of consumer with insurance contract becomes $(w_1 - c_1, w_2 + c_2)$. Consumers of a given risk class j buy at most one insurance contract c (if they prefer it to their initial endowment w) which maximizes their expected utility:

$$V_j(c) = p_j u(w_2 + c_2) + (1 - p_j)u(w_1 - c_1), \quad \text{where} \quad j = H \text{ or } L \tag{16}$$

Each insurance company offers a pair of contracts (c^H, c^L), where without loss of generality one can assume that high risk consumers find c^H at least as desirable as c^L and low risk consumers find c^L at least as desirable as c^H. The expected profit of the company from the contract $c^j = (c^j_1, c^j_2)$ sold to a customer of class j,

$$\pi_j(c^j) = -p_j c^j_2 + (1 - p_j)c^j_1, \quad \text{where} \quad j = H \text{ or } L \tag{17}$$

Suppose that company 1 offers contracts $(c^H(1), c^L(1))$ and company 2 $(c^H(2), c^L(2))$. Then the expected profit of company 1 is

$$U_1 = \sum_{j=H,L} \begin{cases} n_j \pi_j(c^j(1)), & \text{if } \; V_j(c^j(1)) > V_j(c^j(2)) \\ \frac{1}{2} n_j \pi_j(c^j(1)), & \text{if } \; V_j(c^j(1)) = V_j(c^j(2)) \\ 0, & \text{otherwise} \end{cases} \tag{18}$$

And expected profit of company 2 is defined symmetrically.

The detailed interpretation and investigation of this model can be found in Rothschild and Stiglitz (1976) and Wilson (1977). In particular it was shown that, if a pure strategy equilibrium exists, both companies must offer the same contract pair $c^* = (c^{*H}, c^{*L})$ satisfying $w_2 + c_2^{*H} = w_1 - c_1^{*H}$ (i.e. high risks are perfectly insured, and only them), $\pi_H(c^{*H}) = \pi_L(c^{*L}) = 0$ (i.e. customers of each risk class generate zero expected profits for companies) and $V_H(c^{*H}) = V_H(c^{*L})$ (i.e. high risk customers are indifferent between the low risk contract and their own). Following Dasgupta and Maskin (1986) we will call contract pair c^* a "Rothschild-Stiglitz-Wilson" or RSW contract pair. However if there is a sufficiently high proportion of low risk customers one company can deviate from c^* and earn positive profit. It can offer a "pooling" contract c^{**} that both high and low risk customers prefer to c^*. It was shown that there is no Nash equilibrium in this case. We can show however that contract pair c^* is still an EinSS.

Proposition 4.1. *A RSW contract pair c^* is always an Equilibrium in Secure Strategies in the insurance market game.*

See proof in Appendix.

For the described model Wilson (1976) introduced and analyzed an equilibrium concept ("E2 equilibrium") based on the following assumption. Each insurance company believes that after offering its contract, the other company would immediately withdraw any unprofitable contract. Under this assumption an equilibrium in the insurance market game always exists. In general this solution is different from EinSS. When the RSW contract is a Nash equilibrium both the EinSS and the E2 equilibrium coincide with it. When the RSW contract is not a Nash equilibrium, the unique EinSS is the RSW contract, and the E2 equilibrium is the corresponding optimal pooling contract (which is also unique except on the boundary of nonexistence of the Nash equilibrium).

Conclusion

The non-cooperative equilibrium that we have analyzed, the EinSS, is meant to extend the Nash equilibrium concept in order to solve games where the Nash equilibrium does not exist and where it is not unreasonable to introduce, as a behavioral assumption, that players are cautious, namely that players are looking for secure positions and avoid threats. In that respect this concept abstracts and unifies various *ad hoc* solutions already formulated in various applied economic games that have been discussed extensively in the literature. It complements usefully mixed strategy Nash equilibria that are not explicit and difficult to interpret in these games. Like the Nash equilibrium, the EinSS is a static concept, and the basic requirement of excluding at equilibrium competitive deviations remains. But it also appeals to dynamic intuitions, tolerating at equilibrium the possibility of a non-competitive deviation which would be blocked by some counter-deviation punishing the deviator. This is in line with the "objection-counterobjection" logic first introduced in cooperative games.[10]

In the first application, the Hotelling price-setting game, with no Nash equilibrium when sellers are too close to each other but a unique EinSS for all location pairs, the subgame perfect EinSS solution selects one location pair. This is the location pair that minimizes differentiation on the domain where the Nash equilibrium in prices exists. It also minimizes transportation costs. In the second example, the symmetric two-player contest, a pure-strategy Nash-Cournot equilibrium does not exist when the success function parameter is greater than two, but there is an EinSS (unique up to permutations of players) providing a more efficient solution than a Nash equilibrium in mixed strategies. In the no-Nash-equilibrium cases (i.e. symmetric competition being unprofitable), one player exerts a high level of effort while the other player refuses to participate. In the third example, the Bertrand-Edgeworth duopoly with capacity constraints, we have shown that in many cases where the Nash equilibrium does not exists, but with some restriction on firm productive capacities, there is an EinSS with equilibrium prices lower than the monopoly price. The corresponding difference in prices can be interpreted as an additional cost supported by the firms to protect themselves from the threat of price undercutting. Existence of an EinSS in these three applications can be derived from a general existence theorem for discontinuous games (the payoff functions or the best response functions being discontinuous). Finally, we consider Rothchild-Stiglitz-Wilson (RSW) insurance game, in which at a Nash equilibrium both companies must offer the same RSW-contract pair. However, if there is a sufficiently high proportion of low risk customers, a single pooling contract will be preferred by every customer and will therefore upset the RSW-contract as a Nash equilibrium. But a RSW-contract always remains an EinSS.

These four applications are only examples, but they well illustrate the kind of determinate solutions cautious players may reach in some classes of games. To confirm this analysis, it would be interesting to enlarge this set of applications and, as required, the conditions for existence of an EinSS.

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[10]The possibility of introducing an explicit dynamic process converging in some way or another to the EinSS, as it was done for some cooperative concepts, remains a topic for future research.

Appendix A. Omitted proofs

Proof of Proposition 3.2. The sets $\widetilde{Q}_i(p_{-i})$ in the BSA condition that we select are the sets of all secure strategies in the diagonal domain defined by (3). These sets are closed segments, which can be explicitly described as:

$$\begin{aligned} \widetilde{Q}_1(p_2) &= \Big[\max\{p_2 - d, 2p_2 - 2b - d\}, \min\Big\{p_2 + d, \frac{p_2(2b + d - p_2) + 2dl}{2l - p_2}\Big\}\Big] \cap B_1 \\ \widetilde{Q}_2(p_1) &= \Big[\max\{p_1 - d, 2p_1 - 2a - d\}, \min\Big\{p_1 + d, \frac{p_1(2a + d - p_1) + 2dl}{2l - p_1}\Big\}\Big] \cap B_2 \end{aligned} \tag{A.1}$$

We now prove that the game G is a BSA-game in B. First let us prove that player 1 with insecure strategy p_1 always has a BSA in B_1 when $p_2 \le p_2^B$. If $\min\{d, \frac{2a+d}{2}\} \le p_2 \le p_2^B$ we have according to our choice of B: $BR_2^{-1}(p_2) \in \widetilde{Q}_1(p_2) \neq \varnothing$. If $p_2 < \min\{d, \frac{2a+d}{2}\}$ player 1 always has secure strategies with a small enough price and we also have $\widetilde{Q}_1(p_2) \neq \varnothing$. Therefore player 1 always has a deviation in secure strategy in B_1 with a positive payoff. If $v_1(p_1, p_2) = 0$ any such deviation provides a BSA in B_1 for player 1. Consider now the case $v_1(p_1, p_2) > 0$. If $|p_1 - p_2| \le d$ then the strategy p_1 is insecure only if $p_2 > BR_2(p_1)$. Because $BR_2(p_1)$ is increasing at $p_1 \in B_1$ we obtain $p_1 < BR_2^{-1}(p_2) < BR_1(p_2)$ (the second inequality follows from the condition $p_2 \le p_2^B < p_2^*$). Then due to the strict quasiconcavity of $u_1(p)$ in p_1 at $|p_1 - p_2| \le d$ the deviation $p_1 \xrightarrow{1} BR_2^{-1}(p_2)$ is a profitable deviation in secure strategy, which provides a BSA in B_1 for player 1. If $p_1 < p_2 - d$ we have $0 < v_1(p_1, p_2) \le u_1(p_1, 0) \le p_1(a + \frac{d - p_1}{2}) < (p_2 - d)(a + d) = u_1^{II}(p_2 - d, p_2)$. Therefore player 1 can profitably deviate in the profile $(p_2 - d, p_2)$. If it is not secure, player 1 can deviate profitably further (as was proven before) in a secure strategy, which provides a BSA in B_1 for player 1.

Let us check the fulfillment of the other conditions of Theorem 3.1. Clearly, the graphs $\Gamma(\widetilde{Q}_i)$ of functions $\widetilde{Q}_i(p_{-i})$ are closed sets. Payoff functions of players are continuous on these graphs (as well as in the entire diagonal domain). Maximum payoffs of players $\phi_i(p_{-i})$ on the sets $\widetilde{Q}_i(p_{-i})$ are continuous, because the payoff functions are one-peak in the diagonal domain, and the boundaries of segments $\widetilde{Q}_i(p_{-i})$ change continuously with p_{-i}. Finally, each set $M_{p_{-i}}$ consists of a unique point, because of the strict quasiconcavity of payoffs in the diagonal domain. Therefore these sets are contractible. By Theorem 3.1 G admits an EinSS in B. $\square$

Proof of Proposition 3.3. As sets $\widetilde{Q}_i(x_{-i})$ in B for the BSA condition, we choose the closed segments

$$\widetilde{Q}_2(x_1) = \{\eta_2(x_1) \le x_2 \le 1\}, \quad \widetilde{Q}_1(x_2) = \{0 \le x_1 \le \delta\}, \tag{A.2}$$

where $\delta > 0$ is sufficiently small so that $u_1(x_1)$ always decreases on the interval $\widetilde{Q}_1$. Let us prove that the game G is a BSA-game in B.

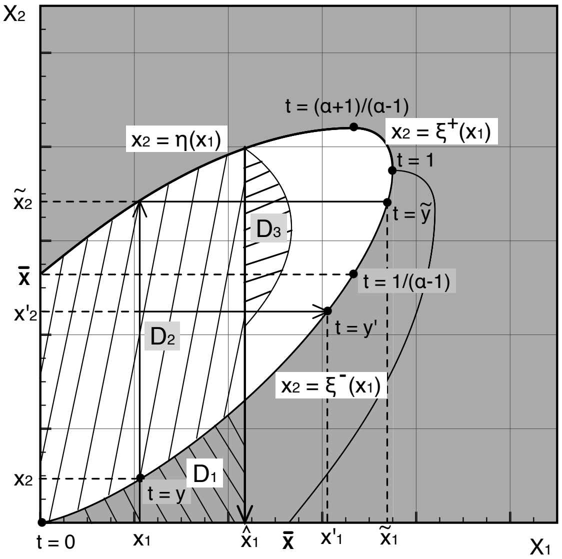

Figure A.3: Location of the considered domains and points on the curve $\xi^{-1}(t)$ for different values of t.

(1). Consider first the domain $D_1 = \{0 < x_1 < \hat{x},\ 0 \le x_2 \le \xi^-(x_1)\}$ with secure strategies of player 2 and prove that it is always profitable for player 2 to deviate from it into $x_2 = \eta(x_1)$ (see Fig. A.3). Choose an arbitrary $x_1 < \hat{x}$. Since $u_2(x_1, x_2)$ is quasiconvex in D_1 with respect to x_2 and $u_2(x_1, 0) = 0 < u_2(x_1, \eta(x_1))$ then it suffices to show that

$$u_2(x_1, \xi^-(x_1)) < u_2(x_1, \eta(x_1)),\ x_1 < \hat{x} \tag{A.3}$$

A. Consider $x_1 \le \frac{\alpha^2-1}{4\alpha}$. Introduce the notations: $x_2 \equiv \xi^-(x_1)$, $\tilde{x}_2 \equiv \eta(x_1)$, $\tilde{x}_1 \equiv \xi^{-1}(\tilde{x}_2)$, $u_2^\xi \equiv u_2(x_1, \xi^-(x_1)) = \frac{x_2^\alpha}{x_1^\alpha + x_2^\alpha} - x_2$, where $x_1 = \xi^{-1}(x_2)$, $u_2^\eta \equiv u_2(x_1, \eta(x_1)) = \frac{\tilde{x}_2^\alpha}{x_1^\alpha + \tilde{x}_2^\alpha} - \tilde{x}_2$. From the definition (9) of $\eta(x_1)$ at $x_1 \le \frac{\alpha^2-1}{4\alpha}$: $\frac{x_1^\alpha}{\tilde{x}_2^\alpha + x_1^\alpha} - x_1 = \frac{\tilde{x}_1^\alpha}{\tilde{x}_2^\alpha + \tilde{x}_1^\alpha} - \tilde{x}_1$. It follows that $\frac{\tilde{x}_2^\alpha}{x_1^\alpha + \tilde{x}_2^\alpha} = 1 + \tilde{x}_1 - x_1 - \frac{\tilde{x}_1^\alpha}{\tilde{x}_2^\alpha + \tilde{x}_1^\alpha}$ and $u_2^\eta = \tilde{x}_1 - x_1 + \frac{\tilde{x}_2^\alpha}{\tilde{x}_2^\alpha + \tilde{x}_1^\alpha} - \tilde{x}_2$, where $\tilde{x}_1 = \xi^{-1}(\tilde{x}_2)$. The curve $x_1 = \xi^{-1}(x_2)$ can be conveniently parametrized using a monotonically increasing along it a variable $t:\ 0 \le t \le \frac{\alpha+1}{\alpha-1}$ so that at $t = y$: $x_1 = \frac{\alpha y}{(1+y)^2}$, $x_2 = y^{1/\alpha} \cdot x_1$, and at $t = \tilde{y}$: $\tilde{x}_1 = \frac{\alpha \tilde{y}}{(1+\tilde{y})^2}$, $\tilde{x}_2 = \tilde{y}^{1/\alpha} \cdot \tilde{x}_1$, where parameters $y, \tilde{y} \in [0, \frac{\alpha+1}{\alpha-1}]$. Using these notations we obtain $u_2^\eta - u_2^\xi = f(\tilde{y}) - f(y)$, where the function

$$f(t) = \frac{t}{t+1} + \frac{\alpha t(1 - t^{1/\alpha})}{(t+1)^2}, \quad 0 \le t \le \frac{\alpha+1}{\alpha-1} \tag{A.4}$$

is quasiconcave and reaches maximum at $t_{max} = 1$, and admissible sets of parameters are such that $\frac{1}{\alpha-1} \le \tilde{y} \le \frac{\alpha+1}{\alpha-1}$, $0 \le y \le \frac{\alpha-1}{\alpha+1}$. Note that $y < \frac{1}{\alpha-1}$ and $f(y) < f(\frac{1}{\alpha-1})$ (if $y \ge \frac{1}{\alpha-1}$, then $x_1(y) \ge \frac{\alpha-1}{\alpha}$, i.e. at $\alpha > 2$: $x_1(y) \ge \frac{1}{\alpha}(\alpha-1)^{\frac{\alpha-1}{\alpha}} \equiv \bar{x}_1$, according to (7) this means that $u_2(x_1, x_2) < 0$ for any x_2 in contradiction with the condition $u_2(x_1, \eta(x_1)) \ge 0$). By direct substitution into (A.4) one can verify that $f\left(\frac{\alpha-1}{\alpha+1}\right) < f\left(\frac{\alpha+1}{\alpha-1}\right)$ for all $\alpha > 2$ and $f(y) \le f(\frac{\alpha-1}{\alpha+1}) < f(\frac{\alpha+1}{\alpha-1})$. Therefore, $f(y) < \min\{f\left(\frac{1}{\alpha-1}\right), f\left(\frac{\alpha+1}{\alpha-1}\right)\}$ and $\tilde{y} \in \left[\frac{1}{\alpha-1}, \frac{\alpha+1}{\alpha-1}\right]$. Because of the qusiconcavity of $f(t)$, it follows now that $f(y) < f(\tilde{y})$, and accordingly, the required inequality (A.3) holds.

B. Consider $x_1 \ge \frac{\alpha^2-1}{4\alpha}$. From the definition (9) of $\eta(x_1)$: $u_2^\eta \equiv u_2(x_1, \eta(x_1)) = u_2(x_1, \xi^+(x_1))$, and $u_2^\xi \equiv u_2(x_1, \xi^-(x_1))$. We use again the variable $y \equiv \frac{1}{2x_1}\left(\alpha - 2x_1 - \sqrt{\alpha^2 - 4\alpha x_1}\right)$. Then according to (5) and (6):

$$u_2^\eta = \frac{y^{-1}}{1+y^{-1}} - x_1 y^{-1/\alpha}, \quad u_2^\xi = \frac{y}{1+y} - x_1 y^{1/\alpha}, \quad x_1 = \frac{\alpha}{(1+y)(1+y^{-1})}, \quad 0 < y < 1.$$

It follows that $u_2^\eta - u_2^\xi = x_1\left((y^{1/\alpha} - y^{-1/\alpha}) - \frac{1}{\alpha}(y - y^{-1})\right)$. When $y = 1$ the function $\Phi \equiv (u_2^\eta - u_2^\xi)/x_1 = 0$. Its derivative $\frac{d\Phi}{dy} = \frac{1}{\alpha y}\left((y^{1/\alpha} + y^{-1/\alpha}) - (y + y^{-1})\right)$. When $\alpha > 2$ $(0 < 1/\alpha < 1)$: $0 < y < y^{1/\alpha} < 1 \Rightarrow y^{1/\alpha} + y^{-1/\alpha} < y + y^{-1}$ and $\frac{d\Phi}{dy} < 0$, $\Phi > 0$, $u_2^\eta > u_2^\xi$. Hence we obtain the inequality (A.3).

(2). Consider the domain $D_2 = \{0 < x_1 < \hat{x},\ \xi^-(x_1) < x_2' < \eta(x_1)\}$ where the strategies of player 2 are insecure. We prove that at any strategy profile in D_2 there is a threat of player 1 against player 2 to deviate profitably $x_1 \xrightarrow{1} x_1'$ into the strategy of its right peak $x_1' = \xi^{-1}(x_2')$, which lies on the curve (6). Besides, $u_2(x_1, \eta(x_1)) > u_2(x_1', x_2')$ (see Fig. A.3).

A. Consider the case $x_1 \le \frac{\alpha^2-1}{4\alpha}$. We use the function $f(t)$ (A.4) and the same notations as in the first part A of the proof. In addition, we denote $x_1' = \frac{\alpha y'}{(1+y')^2}$, $x_2' = y'^{1/\alpha} \cdot x_1'$ for the parameter value $t = y'$. Then,

$$\Delta u_2 \equiv u_2(x_1, \eta(x_1)) - u_2(x_1', x_2') = f(\tilde{y}) + \frac{\alpha y'}{(1+y')^2} - \frac{\alpha y}{(1+y)^2} - f(y')$$

Let us prove that $\Delta u_2 > 0$. Consider first $y' \ge 1$. Note that $\tilde{y} > y'$ (because of $\tilde{x}_2 > x_2'$) and the function $u_2(t) = \frac{t}{t+1} - \frac{\alpha t \cdot t^{1/\alpha}}{(t+1)^2}$ increases at $t \ge 1$. Therefore, $u_2(x_1', x_2') = u_2(y') < u_2(\tilde{y}) = u_2(\tilde{x}_1, \tilde{x}_2) < u_2(x_1, \tilde{x}_2) = u_2(x_1, \eta(x_1))$, i.e. $\Delta u_2 > 0$. Consider now $\frac{1}{\alpha-1} \le y' \le 1$. In this case we have $x_1'(y') \ge \frac{\alpha-1}{\alpha}$, i.e. at $\alpha > 2$: $x_1' \ge \frac{1}{\alpha}(\alpha-1)^{\frac{\alpha-1}{\alpha}} \equiv \bar{x}_1$, and according to (7) this implies that $u_2(x_1', x_2') < 0$. Because of $u_2(x_1, \eta(x_1)) \ge 0$ we obtain $\Delta u_2 > 0$. Finally, consider $y < y' < \frac{1}{\alpha-1}$. In this case $\Delta u_2 \ge u_2(x_1, \eta(x_1)) - \max\limits_{y \le t \le \frac{1}{\alpha-1}} u_2(t)$. The function $u_2(t)$ is quasiconcave on the interval $t \in [0,1]$. Let $t = \hat{y}$ be the position of its maximum. If $\hat{y} \in [0, y]$, then $\max\limits_{y \le t \le \frac{1}{\alpha-1}} u_2(t) = u_2(x_1(y), x_2(y)) = u_2(x_1, \xi^-(x_1))$ and the inequality $\Delta u_2 > 0$ follows immediately from the previously proven condition (A.3). If $\hat{y} \in (y, 1]$ then the following estimate holds: $\hat{y} < \min\{\frac{\alpha-1}{\alpha+1}, \frac{1}{\alpha-1}\}$. It follows from direct verification that $\frac{du_2}{dt}|_{t=\frac{\alpha-1}{\alpha+1}} = (\frac{\alpha+1}{\alpha-1})^{1/\alpha} - 2 < 0$ when $\alpha > 2$, and $\frac{du_2}{dt}|_{t=\frac{1}{\alpha-1}} = (\alpha-1)^{1/\alpha} - (\alpha-1) < 0$ when $\alpha > 2$. Therefore, $\max\limits_{y \le t \le \frac{1}{\alpha-1}} u_2(t) = u_2(x_1(\hat{y}), x_2(\hat{y})) = u_2(\hat{y})$, and

$$\Delta u_2 \ge u_2(x_1, \eta(x_1)) - u_2(\hat{y}) = f(\tilde{y}) + \frac{\alpha \hat{y}}{(1+\hat{y})^2} - \frac{\alpha y}{(1+y)^2} - f(\hat{y}) > f(\tilde{y}) - f(\hat{y})\ .$$

Since $f(t)$ increases at $t \le 1$, then it follows from the above estimate that

$$f(\hat{y}) < \min\left\{ f\left(\frac{1}{\alpha-1}\right), f\left(\frac{\alpha-1}{\alpha+1}\right) \right\} \le \min\left\{ f\left(\frac{1}{\alpha-1}\right), f\left(\frac{\alpha+1}{\alpha-1}\right) \right\}.$$

Because $f(t)$ is quasiconcave and $\tilde{y} \in [\frac{1}{\alpha-1}, \frac{\alpha+1}{\alpha-1}]$, we obtain $f(\tilde{y}) > f(\hat{y})$ and the inequality $\Delta u_2 > 0$ is proved.
B. Consider the case $x_1 \ge \frac{\alpha^2-1}{4\alpha}$. In this case the inequality required for the BSA condition takes the form $u_2(x_1, \xi^+(x_1)) > u_2(x_1', x_2')$. We use previously introduced notations for y, y' and $\hat{y}$. Because of $\hat{y} = \operatorname*{argmax}_{0 \le t \le 1} u_2(t) < \frac{\alpha-1}{\alpha+1}$, it follows that the function $u_2(t) = \frac{t}{t+1} - \frac{\alpha t \cdot t^{1/\alpha}}{(t+1)^2}$ is quasiconcave on the interval $\left[\frac{\alpha-1}{\alpha+1}, \frac{\alpha+1}{\alpha-1}\right]$. Then taking into account that $\frac{\alpha-1}{\alpha+1} \le y < y' < y^{-1} \le \frac{\alpha+1}{\alpha-1}$ we obtain $u_2(y') < \max\{u_2(y), u_2(y^{-1})\}$. In the first part B of the proof it was shown that $u_2(y) < u_2(y^{-1})$ when $y < 1$. Consequently, we obtain $u_2(y') < u_2(y^{-1})$, or using initial notations $u_2(x_1', x_2') < u_2(x_1, \xi^+(x_1))$, q.e.d.

(3). Let us now prove that the BSA condition holds for the set $\widetilde{Q}_1(x_2)$. For the convenience of illustration we shall give the proof for the symmetrical set $\widetilde{Q}_2(x_1) = \{0 \le x_2 \le \delta\}$, $x_1 > \hat{x}$. Consider an arbitrary $x_2' > \delta$. In strategy profiles where $u_2(x_1, x_2') < 0$ the BSA condition is met because $u_2(x_1, x_2') < u_2(x_1, 0) = 0$. Consider the domain $D_3 = \{(x_1, x_2') |\ x_1 > \hat{x},\ x_2' > \delta,\ u_2(x_1, x_2') \ge 0\}$. The above proof of the inequality $u_2(x_1, \xi^-(x_1)) < u_2(x_1, \eta(x_1))$ does not use the condition $x_1 < \hat{x}$ and thus it remains true when $x_1 > \hat{x}$. Therefore, from the condition $u_2(x_1, \eta(x_1)) < 0$ at $x_1 > \hat{x}$, it follows that $u_2((x_1, \xi^-(x_1)) < u_2(x_1, \eta(x_1)) < 0$. Since $u_2(x_1, x_2)$ is quasiconcave in x_2 when $\xi^-(x_1) < x_2 < \eta(x_1)$, the strategy of player 2 must belong to the interval $\xi^-(x_1) < x_2' < \eta(x_1)$ (see Fig. A.3), for all strategy profiles $(x_1, x_2') \in D_3$. This implies that at any strategy profile in D_3 there is a threat of player 1 against player 2 to deviate profitably $x_1 \xrightarrow{1} x_1'$ into the strategy of its right peak $x_1' = \xi^{-1}(x_2')$, which lies on the curve (6). Consider again the parametrization $u_2(t) = \frac{t}{t+1} - \frac{\alpha t \cdot t^{1/\alpha}}{(t+1)^2}$ and define the following values of the parameter t: y, y' and $\tilde{y}$ so that $u_2(x_1, \xi^-(x_1)) = u_2(y)$, $u_2(x_1', x_2') = u_2(y')$ and $u_2(\xi^{-1}(\eta(x_1)), \eta(x_1)) = u_2(\tilde{y})$. Since $u_2(t)$ is quasiconvex on the interval $t \in \left[y, \frac{\alpha+1}{\alpha-1}\right]$ and $y < y' < \tilde{y} \le \frac{\alpha+1}{\alpha-1}$, then $u_2(y') = u_2(x_1', x_2') < \max\{u_2(y), u_2(\tilde{y})\} \le \max\{u_2(x_1, \xi^-(x_1)), u_2(x_1, \eta(x_1))\} < 0 = u_2(x, 0)$, and consequently, the BSA condition holds true also in the domain D_3.

(4). Let us check the fulfillment of the other conditions of Theorem 3.1. Clearly, the graphs $\Gamma\left(\widetilde{Q}_i(x_{-i})\right)$ of the functions $\widetilde{Q}_i(x_{-i})$ are closed sets. The payoff functions of players are continuous on these graphs (as well as for all strategy profiles except for the profile $(0, 0)$). The maximum payoffs $\phi_i(x_{-i})$ on the sets $\widetilde{Q}_i(x_{-i})$ are continuous, and the boundaries of segments $\widetilde{Q}_i(x_{-i})$ change continuously with x_{-i}. Finally, each set $M_{x_{-i}}$ consists of a unique point, because of the strict quasiconcavity of payoffs on $\widetilde{Q}_i(x_{-i})$. Therefore, these sets are contractible. By Theorem 3.1 G admits an EinSS in B. □

Proof of Proposition 3.6. (1). Any EinSS in the game must be a BSR-profile with positive payoffs (since any profile with zero payoffs always possesses a secure deviation into profiles with positive payoffs). First, let us find all secure profiles in the game with positive payoffs. Consider the case $p^* < p_1 < p_2$. If $D(p_1) > S_1$ player 1 can always threaten player 2 by slightly increasing his price p_1. If $D(p_1) \leq S_1$ then according to (12), $u_2(p_1, p_2) = 0$. Symmetrically, if $p^* < p_2 < p_1$ either player 2 can threaten player 1 or $u_1(p_1, p_2) = 0$. If $p^* < p_2 = p_1$, each player can undercut the other. If $p_1 \leq p^* < p_2$ player 1 can threaten player 2 by increasing his price to p^*+0 which exceeds p^* by an arbitrarily small amount. Indeed in this case $D(p_1) > D(p^*+0) \geq S_1$ and $u_1(p_1, p_2) = p_1 S_1 < (p^*+0)S_1 = u_1(p^*+0, p_2)$. On the other hand, $u_2(p^*+0, p_2) = p_2 \frac{D(p_2)}{S_1+S_2} S_2 < p_2 S_2$ and $u_2(p^*+0, p_2) = p_2 D(p_2)\left(1 - \frac{S_1}{D(p^*+0)}\right) < p_2 D(p_2)\left(1 - \frac{S_1}{D(p_1)}\right) \Rightarrow u_2(p^*+0, p_2) < u_2(p_1, p_2)$. Symmetrically, if $p_2 \leq p^* < p_1$ player 2 can threaten player 1. Therefore all secure profiles with positive payoffs must lie in the set $\{(p_1, p_2) : 0 < p_i \leq p^*,\ i = 1, 2\}$. On the other hand if $p_1 \leq p^*$, $u_1(p_1, p_2) = S_1 p_1$ linearly increases in p_1 and does not depend on p_2. Hence there are no threats available to player 1. Symmetrically, if $p_2 \leq p^*$ there are no threats available to player 2. Therefore (p_1, p_2) is a secure profile with positive payoffs in the game (12) if and only if it lies in the set $M = \{(p_1, p_2) : 0 < p_i \leq p^*,\ i = 1, 2\}$.

(2). The payoff functions (12) u_1 and u_2 increase in the set M linearly in p_1 and in p_2 respectively. Therefore there is only one BSR-profile (p^*, p^*) with positive payoffs in the set M (otherwise one player could securely slightly increase his price). According to Proposition 2.3 there are no other EinSS in the game except this profile.

(3). Let us consider profile (p^*, p^*) and prove the conditions (14). Suppose for example that $p^* < \hat{p}(S_2) \equiv \arg\max_{p>0}\{p(D(p) - S_2)\}$. Then player 1 can deviate $p_1^* \to \hat{p}$. His payoff will increase since $p^* < \hat{p} \leq p_M = \arg\max_{p>0} pD(p)$ and $u_1(p_1, p_2)$ is strictly increasing in p_1 if $p_1 \leq p_M$ according to (12). Any retaliatory threat of player 2 according to (12) can not make the payoff of player 1 less than $\min_{p_2} u_1(\hat{p}, p_2) = \min_{p_2<\hat{p}} u_1(\hat{p}, p_2) = u_1(\hat{p}, p_2)|_{p_2=\hat{p}-0} = \hat{p}\min\{S_1, D(\hat{p}) - S_2\}$. The payoff of player 1 in the initial profile does not exceed this value. Indeed $p(D(p) - S_2)$ is strictly increasing at $p < \hat{p}$ and we have $u_1(p^*, p^*) \leq p^*(D(p^*) - S_2) < \hat{p}(D(\hat{p}) - S_2)$ and $u_1(p^*, p^*) = p^* S_1 < \hat{p} S_1$. Therefore the deviation of player 1 into $\hat{p}(S_2)$ is always a secure deviation according to Definition 2.5. Hence profile (p^*, p^*) is not an EinSS. Symmetrically if $p^* < \hat{p}(S_1)$ then player 2 can make a secure deviation into $\hat{p}(S_1)$ and profile (p^*, p^*) is not an EinSS either. The necessity of (14) is proven.

(4). Let us now assume that (14) holds (i.e. $\hat{p}(S_1) \leq p^*$ and $\hat{p}(S_2) \leq p^*$). Consider an arbitrary deviation $p^* \to p_1$ of player 1. If $p_1 < p^*$ it can not be a profitable deviation for player 1. Therefore $p_1 > p^*$. Player 1 increases the payoff if and only if $u_1(p^*, p^*) = p^* S_1 = p^* \frac{D(p^*)-S_2}{D(p^*)} D(p^*) < u_1(p_1, p^*) = p_1 \frac{D(p^*)-S_2}{D(p^*)} D(p_1)$, i.e. $p^* D(p^*) < p_1 D(p_1)$ should hold. Then there is retaliatory threat of player 2 to deviate from profile (p_1, p^*) into a profile $(p_1, p_1 - 0)$, with $p_1 - 0$ arbitrarily close to p_1. From $p^* S_2 < p_1 S_2$ and $p^* D(p^*) < p_1 D(p_1)$ it follows that player 2 increases his payoff by this deviation. The payoff of player 1 in this profile is arbitrarily close to $u_1(p_1, p_1 - 0) = p_1 \min\{S_1, D(p_1) - S_2\}|_{p^*<p_1} =$

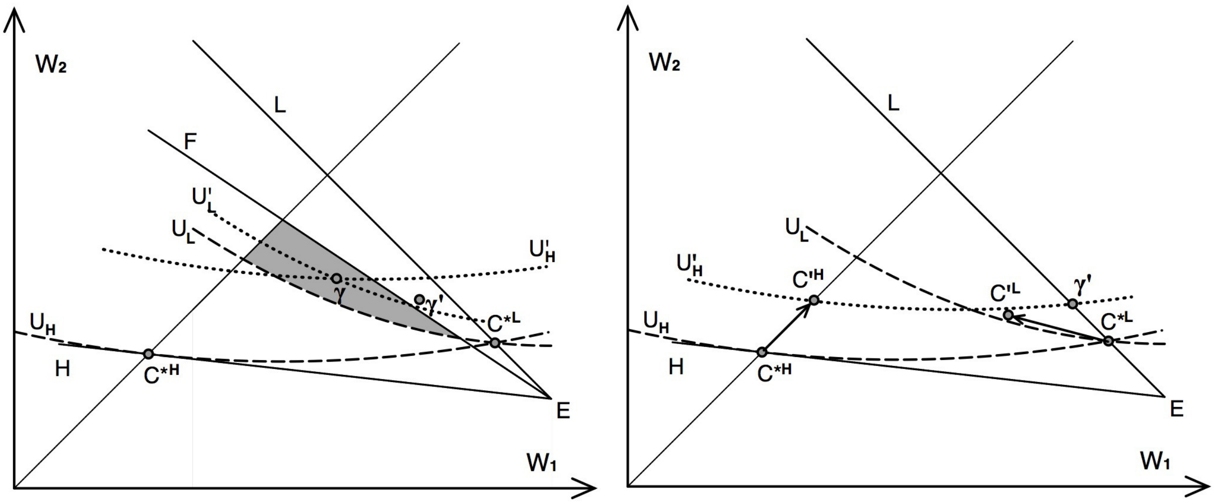

Figure A.4: Deviations from RSW solution (c^{*H}, c^{*L}) are not secure. Both "pooling" deviation γ (on the left) and separating deviation (c'^H, c'^L) (on the right) pose a threat γ' to receive negative payoff.

$p_1(D(p_1) - S_2)$. Since $p(D(p) - S_2)$ is strictly decreasing at $p \geq \hat{p}(S_2)$ and $p_1 > p^* \geq \hat{p}(S_2)$ then $u_1(p^*, p^*) = p^*(D(p^*) - S_2) > p_1(D(p_1) - S_2) = u_1(p_1, p_1 - 0)$. Therefore the deviation of player 1 into profile (p_1, p^*) is not a secure deviation. Symmetrically an arbitrary deviation of player 2 is not a secure deviation either. No player can make secure deviation in the profile (p^*, p^*). By definition it is an EinSS. The sufficiency of (14) is proven. □

Proof of Proposition 4.1. In our proof we will follow the graphical procedure introduced in Rothschild and Stiglitz (1976). In Fig. A.4 the horizontal and vertical axis represent income of customers in the states of no accident and accident respectively. The point E with coordinates $w = (w_1, w_2)$ is the uninsured state of customer. Purchasing the insurance contract $c = (c_1, c_2)$ moves the individual from E to the point $(w_1 - c_1, w_2 + c_2)$. The set of insurance contracts for low-risk customers that break even lies on the line EL. The set of contracts for high-risk customers lies on the line EH respectively. If company offers a "pooling" contract which is the same for both groups (such that $c^H = c^L$) should, in case of equilibrium, lie on the market odds line EF. The pair of contracts (c^{*H}, c^{*L}) in Fig. A.4 represents the RSW solution of the insurance market game. The indifference curves through c^{*H} and c^{*L} for high-risk and low-risk customers U_H and U_L are shown by broken lines.

Let us consider position when both insurance companies offer the RSW contract c^* and obtain zero payoffs. If it is a Nash equilibrium it is also an EinSS according to Proposition 2.1. Consider the case when c^* is not a Nash equilibrium. It is still a secure profile since any change in the insurance policies of one company will not bring losses to the other company. Its payoffs will still remain zero. Suppose one company can deviate by offering a new insurance policy. It is either **(A)** a "pooling" insurance contract or **(B)** a separating insurance contract.

(A) If it is a pooling contract γ it must lie above the low-risk indifference curve U_L through c^{*L} in order to be profitable for both low- and high-risk customers (see Fig.A.4 on the left). A deviating company can make a positive profit only if γ lies below the market odds line EF in the shaded area. Let us draw the indifference curves U'_H and U'_L for low-risk and high-risk customers through γ. Then the second company as a response to γ can offer a pooling contract γ' between curves U'_H and U'_L somewhere to the right from the γ and below low-risk line EL. In this case all low-risk customers would choose γ' and the second company could make a profit. All high-risk customers would stay with γ contract and the first company would lose money. Hence there is a retaliatory threat to deviate into γ' such that the deviating company loses more money than it gains at deviation γ. Therefore offering pooling contract γ is not a secure deviation.

(B) Let us now assume that a deviating company offers a new separating contract (c'^H, c'^L) which is more profitable than (c^{*H}, c^{*L}) (see Fig. A.4 on the right). If it is more profitable for low-risk customers it also must be more profitable for high-risk customers (since in this case they always prefer c'^L to c^{*H}). In order to be more profitable for high-risk customers c'^H must lie above the high-risk indifference curve U_H through c^{*H}. Therefore c'^H also lies above high-risk line EH and makes a loss for a deviating company. Consequently c'^L must lie below low-risk line EL and make a profit for a deviating company. In this case profits from c'^L subsidize the losses of c'^H and (c'^H, c'^L) can be more profitable than the RSW solution (c^{*H}, c^{*L}). Let us draw the indifference curves U'_H through c'^H. Then the second company as a response to (c'^H, c'^L) can offer a pooling contract γ' at the intersection of U'_H with low-risk line EL. In this case all low-risk customers would choose γ' and the second company could make a profit. All high-risk customers would stay with (c'^H, c'^L) contract and the deviating company would lose money. Hence there is a retaliatory threat to deviate into γ' such that the deviating company loses more money than it gains at deviation into (c'^H, c'^L). Therefore offering separating contract (c'^H, c'^L) is not a secure deviation either. No company can make a secure deviation from (c^{*H}, c^{*L}). Therefore it is an EinSS. $\square$

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