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The complexity of computing a (quasi-)perfect equilibrium for an n-player extensive form game of perfect recall

Kousha Etessami
University of Edinburgh
Email: `kousha "at" inf.ed.ac.uk`

Abstract

We study the complexity of computing or approximating refinements of Nash equilibrium for finite n-player extensive form games of perfect recall (EFGPR), $n \geq 3$. Our results apply to a number of well-studied refinements, including sequential equilibrium, extensive-form perfect equilibrium, and quasi-perfect equilibrium.

Informally, we show that, for all these refinements, approximating such a refined equilibrium for an n-player EFGPR is not any harder than (i.e., can be efficiently reduced to) approximating a Nash equilibrium for a 3-player normal form game. More specifically, we show that approximating such a refined equilibrium for a given EFGPR, within given desired precision, is FIXP_a-complete. We also study corresponding notions of "almost" equilibrium for these refinements, and we show that computing one is PPAD-complete. (In all these cases our main results show containment FIXP_a and containment in PPAD. Hardness follows from earlier results for simpler games.)

For 2-player EFGPRs, analogous complexity results follow from the algorithms of Koller, Megiddo, and von Stengel (1996), von Stengel, van den Elzen, and Talman (2002), and Miltersen and Sørensen (2010). For n-player EFGPRs, an analogous result for Nash and subgame-perfect equilibrium was given by Daskalakis, Fabrikant, and Papadimitriou (2006). No analogous results were known for more refined notions of equilibrium for EFGPRs with 3 or more players.

1 Introduction

Extensive form games are the fundamental mathematical model of games that transpire as a sequence of moves by players over time. A finite extensive form game is described by a finite tree, where each internal node belongs to one of the players (or to chance), and where each leaf indicates a payoff to every player. A "play" of the game traces a path in this tree from the root to a leaf, with each player choosing the child to move to at nodes belonging to it (the child being chosen randomly at chance nodes, or when players decide to randomize their moves). In general, an extensive form game may be of *imperfect information*, meaning roughly that players may need to make moves without having full knowledge of the current "state" (i.e., current node of the game tree). A basic sanity condition for imperfect information games, called *perfect recall*, requires (roughly) that every player in the game should remember all of its own prior moves and information sets. This condition was already put forward by Kuhn ([21]), who showed that games with perfect recall have nice properties and avoid certain pathologies of general extensive form games. Subsequently, Selten [43], in his seminal work on perfect equilibria, argued that non-cooperative extensive form games that lack perfect recall should be rejected as misspecified models. The assumption of perfect recall has indeed become standard practice in much of the large literature on extensive form games. Henceforth, we use the abbreviations: EFGPR for "extensive form game of perfect recall", EFG for "extensive form game", and NFG for "normal form game".

Selten's work made clear that Nash equilibrium, and even subgame-perfect equilibrium, is not an adequately refined solution concept for extensive form games. In particular, there are Nash and subgame-perfect equilibria of EFGPRs that involve "non-credible threats", rendering them implausible. Motivated by this, Selten defined a more refined notion of perfect equilibrium, based on *"trembling hand"* perfection, and showed that any EFGPR has at least one perfect equilibrium. (Selten was awarded a Nobel prize in economics, together with Nash and Harsanyi, largely for his work on refinement of equilibria.) Subsequent work, e.g., by Kreps and Wilson on sequential equilibria [20], and by many others, has reaffirmed the imperative for considering refinements of equilibrium, especially for extensive form games. By now EFGPRs, and equilibrium refinements for them, are treated in most standard textbooks on game theory (see, e.g., [33, 30, 23, 46]).

This paper studies the complexity of computing or approximating an equilibrium for a given EFGPR, with $n \geq 3$ players. We study various important refinements of NE, including: sequential equilibrium (SE), extensive form trembling-hand perfect equilibrium (PE), and quasi-perfect equilibrium (QPE). All of these notions refine subgame-perfect equilibrium (SGPE). Of these, PE and QPE are the most refined notions.[1] Quasi-perfect equilibrium (QPE), defined by van Damme [45], is incompatible with PE, meaning that a PE need not be a QPE and a QPE need not be a PE. Like PE, QPE also refines NE, SGPE, and SE. Furthermore, QPE also refines "normal-form perfect equilibrium" (NF-PE) for EFGPRs, which differs from, and is incompatible with (extensive form) PE for EFGPRs. For the benefit of readers confused by all the different mentioned notions of equilibrium for EFGPRs, Figure 1 of Section 2 summarizes the refinement relationships that exist (and don't exist) between them, by depicting the Hasse diagram of their refinement partial order.

Informally, we show that for all these notions of equilibrium, approximating an equilibrium for given n-player EFGPR within a given desired precision $\delta > 0$ (or computing an "δ-almost equilibrium" for given $\delta > 0$) is no harder than approximating a (δ-almost) NE for a given 3-player *normal form game*. NFGs are trivially encodable as EFGPRs without blowup in size. Thus our results extend the celebrated complexity results for computing/approximating an NE for NFGs to the much more general setting of EFGPRs, and "perfection comes at no extra cost in complexity". Before stating our results more precisely, we must first discuss prior related work.

For readers unfamiliar with computational complexity theory, and with notions such as complexity classes, polynomial time, NP, reductions, and hardness: these are very rich subjects. Although we attempt to be self-contained wherever possible, we can not review all the relevant background on computational complexity in this paper. We recommend as references, firstly, good textbooks on complexity theory such as [2, 34], as well as texts on algorithmic game theory [32, 39] which in particular contain chapters on the complexity of equilibrium computation. Finally, we recommend specific papers that provide background on the the total search complexity classes PPAD and FIXP, such as [35, 7, 11] and [13].

Related work. Equilibrium computation, and its connection to fixed point computation, has been studied for decades, both for normal form and extensive form games. Papadimitriou [35] defined the search problem complexity class PPAD in order to capture the complexity of problems related

[1] However, unlike PE and QPE, an SE consists not just of a suitable behavior profile, but also a *system of beliefs*. We'll see later in what sense PE (and QPE) "refines" SE ([20]). Our complexity results for SE are also for computing its associated belief system.

to computing an equilibrium.[2] It follows from the correctness of the Lemke-Howson algorithm that computing an NE for 2-player NFGs is in PPAD. It similarly follows from Scarf's algorithm that given an n-player NFG (for any n), and given $\epsilon > 0$, computing a "ϵ-NE" (which we call "ϵ-almost-NE" in this paper, to avoid confusion with other notions[3]) is in PPAD; this is a strategy profile where no player can improve its own payoff by more than ϵ by unilaterally deviating from its strategy. In a celebrated series of result in 2006, Chen and Deng [7], and Daskalakis et. al. [11], showed that both of these problems are PPAD-complete. For games with 3 (or more) players, specified by an integer payoff table, all the NEs may have irrational numbers ([31]). Thus, we can not compute an NE *exactly* for them (at least not in the Turing model of computation). With Yannakakis in [13], we showed that for games with 3 (or more) players, an ϵ-NE may in fact be nowhere near any actual NE, unless $\epsilon > 0$ is so small that its *binary* encoding size is exponential in the size of the game; thus, finding an ϵ-NE may tell us nothing about the location of any actual NE. In [13] we considered the complexity of computing an actual NE to within a desired number of bits of precision, i.e., computing a strategy profile that has ℓ_∞-distance at most $\delta > 0$ to some NE, for a given δ (given in binary). We showed that this problem is complete for a natural complexity class which we called FIXP_a.[4] Informally, FIXP_a is the class of discrete search problems that can be reduced to approximating, within desired ℓ_∞-distance $\delta > 0$, a (any) Brouwer fixed point of a continuous function given by an algebraic circuit using gates $\{+, -, *, /, \max, \min\}$. (We will later formally define FIXP_a, as well as its *real-valued* progenitor FIXP, and the piecewise-linear fragment linear-FIXP (= PPAD).) Very recently, in a paper with Hansen, Miltersen, and Sørensen [12], building on [13], we have shown that for NFGs with $n \geq 3$ players, approximating a "trembling-hand perfect equilibrium" (PE) within desired precision is also FIXP_a-complete. Since PEs refine NEs, we only had to show containment in FIXP_a. Interestingly, it was shown previously in [15] that given a 3-player NFG, deciding whether a *given* strategy profile is a PE is NP-hard (unlike for NEs, for which this is easily in P-time).

Research on the computation of equilibria for EFGs, with and without perfect recall, also has a long and rich history. Of course for perfect information games computing a NE or SGPE is easily in P-time using Kuhn's classic "backward induction" ([21]). On the other hand, for imperfect information games *without* perfect recall, it was pointed out by Koller and Megiddo [18] (and by others, e.g., [5]) that even for 1-player games computing or approximating a (any) NE is NP-hard (it can easily encode SAT[5]). By contrast, for 1-player EFGPRs an equilibrium (i.e., an optimal strategy) can be computed easily in P-time by dynamic programming, as shown by Wilson [49].

Of course, one way to compute an equilibrium for an EFGPR (or EFG) is to first convert it to an NFG, and then apply any algorithm applicable to NFGs. The problem with this approach is that, even for EFGPRs, a *standard* conversion from extensive to normal form incurs exponential blowup.[6] Thus, even a P-time algorithm for NFGs requires exponential time if applied naively in this way to EFGPRs. In the other direction, an NFG can trivially be encoded as an "equivalent"

[2] It is well-known that already for 2-player NFGs, computing a *specific* NE, e.g., that optimizes total payoff or other objectives, is NP-hard [14, 8]. So, in this paper, whenever we speak of a problem of computing (or approximating) "an" equilibrium, possibly of a *refined* kind, we are not more specific than that: *any* equilibrium *of that kind* will do.

[3] We do so to avoid confusion when we combine "ϵ-almost" with other notions, particularly Myerson's ϵ-PEs ([29]).

[4] We also showed in [13] that approximating an actual NE, even within ℓ_∞-distance any *fixed constant* $\delta < 1/2$ for 3-player NFGs, is *"hard"*: even placing this in NP would place PosSLP in NP, and thereby resolve long standing open problems in arithmetic vs. Turing complexity.

[5] SAT is a prototypical NP-complete problem: deciding satisfiability of a given propositional boolean formula.

[6] Even notions of *reduced* normal form in general incur exponential blowup for EFGPRs. We will not elaborate on reduced norm form, but roughly it means redundant strategies of the EFGPR are not considered in the normal form.

EFGPR which is not much bigger, so that any equilibrium computation problem for NFGs is P-time reducible to an analogous problem for EFGPRs.

In a series of important works in the 1990s, Koller, Megiddo, and von Stengel [18, 47, 19] obtained equilibrium algorithms for 2-player EFGPRs with complexity bounds that essentially match those of 2-player NFGs. In particular, Koller and Megiddo [18] showed that for 2-player zero-sum EFGPRs an NE (i.e., a minimax profile) in behavior strategies can be computed in P-time using linear programming. Furthermore, by using the *sequence form* ([38, 47]) of EFGPRs, Koller, Megiddo, and von Stengel ([19]) showed that one can apply variants of Lemke's algorithm to certain LCPs associated with 2-player EFGPRs to compute an (exact) NE in behavior strategies. A consequence of their result (when combined with Chen and Deng's PPAD-hardness result for 2-player NFGs [7]) is that computing an NE for 2-player EFGPRs is PPAD-complete. Later, von Stengel, van den Elzen, and Talman [48], using the sequence form, gave a similar Lemke-like algorithm for computing a "normal form perfect equilibrium" (NF-PE) [7] for 2-player EFGPRs. More recently, Miltersen and Sørensen have used the sequence form to give related Lemke-like algorithms for computing both a SE [27] and a QPE [28] for 2-player EFGPRs. As pointed out by Miltersen and Sørensen in [28], van Damme's existence proof for a QPE in any EFGPR, given in [45], is somewhat roundabout: it uses the existence of a *proper* equilibrium in a NFG ([29]), and it uses a relationship established in [45] between proper equilibrium in NFGs and QPEs of any EFGPR that has that NFG as its standard normal form. Miltersen and Sørensen state in [28] that "As far as we know, no very simple and direct proof of existence [of QPE] is known." They note that their results furnish a different proof of existence of QPE for 2-player EFGPRs. One of the consequences of our results is a simple and direct proof, via application of Brouwer's fixed point theorem (and Bolzano-Weierstrass), of the existence of a QPE in any n-player EFGPR. In a similar way, our results furnish a direct existence proof for all the notions of equilibrium for EFGPRs that we study.

More closely related to our complexity results for n-player EFGPRs, with $n \geq 3$, von Stengel in [47] used the sequence form of EFGPRs to describe an interesting nonlinear program, associated with a given n-player EFGPR, such that the optimal solutions to the nonlinear program are the NEs of the EFGPR, where the encoding size of the nonlinear program is polynomial in the size of the EFGPR. One can use von Stengel's nonlinear programming formulation, together with results on decision procedures for the theory of reals [37, 3], to show that approximating an NE for a given n-player EFGPR, to within given ℓ_∞-distance $\delta > 0$, is in PSPACE.

Even more closely related to our results is a result by Daskalakis, Fabrikant, and Papadimitriou in [10]. Specifically, Theorem 4 of [10] states that the problem of computing a [ϵ-]Nash equilibrium and a [ϵ-almost] subgame-perfect equilibrium, for an extensive form game [of perfect recall] is polynomial time reducible to computing a [ϵ-]Nash equilibrium for a 2-player normal form game. The statement of Theorem 4 in [10] does not make a distinction between computing an actual Nash equilibrium (within desired precision $\epsilon > 0$), versus computing an ϵ-NE. Indeed, [10] appeared prior to the publication of the paper [13] where the distinction between the complexity of these two problems was highlighted, and where the complexity class FIXP and FIXP_a were defined. The proof of Theorem 4 in [10] can be used ([9]) to establish a reduction from the problem of computing an exact Nash or subgame perfect equilibrium (within given desired precision $\epsilon > 0$) for a given

[7] A *normal-form perfect equilibrium* (NF-PE), is a (behavior) profile that induces a (mixed profile) PE of the *standard* NFG associated with the EFGPR. This is *not* equivalent to extensive-form PE (see [46], Chapter 6). In fact, unlike extensive-form PE, a NF-PE need not be subgame-perfect. Our results apply to both PE and NF-PE.

n-player EFGPR, to the problem of computing a Nash equilibrium (within given desired precision $\epsilon > 0$) for a 3-player normal form game. In [10] a brief proof sketch for Theorem 4 is provided, which builds on the earlier PPAD-completeness results in [11, 7] and goes via reductions to graphical games. However, the sketched proof provided in [10] contains an error ([9]): it assumes that any behavior strategy profile (even when not fully mixed) necessarily defines a distribution on the nodes of every information set, but this need not be the case, in particular because some information sets may be reached with probability 0. Thus, the distributions on information sets described in the proof sketch in [10] are in general ill-defined. The authors of [10] have communicated ([9]) a fix for this error to the author of this paper. The fix involves defining the probability distribution on a given information set using the most recent common single-node ancestor of all vertices in that information set. The authors of [10] will make their fixed proof available in some future expanded version of [10]. We will not elaborate further on their fix, since our results make no use of any of the results in [10]. In particular, we make no use of graphical games. Instead we directly provide algebraically-defined functions whose fixed points give ϵ-perfect equilibria of the given EFGPR. Our results imply essentially the same complexity results for computing Nash and subgame-perfect equilibrium as those implied by Theorem 4 of [10], as well as for computing various other important refinements of equilibrium.[8]

Our results. We consider the complexity of various equilibrium computation problems for which an input instance consists of $\langle \mathcal{G}, \delta \rangle$, where $\mathcal{G}$ is an n-player EFGPR (for any n: n can be part of the input), and where the rational "error" parameter $\delta > 0$ is *given in binary representation.* Our main results are the following:

1. Computing a behavior (strategy) profile, b, such that there exists a PE (or NE, or SGPE) b^* of $\mathcal{G}$, with $\|b - b^*\|_\infty < \delta$, is FIXP_a-complete. (Theorem 10, Part 1.)
2. Computing a behavior profile, b, such that there exists a QPE (or NF-PE), b^* of $\mathcal{G}$, with $\|b - b^*\|_\infty < \delta$, is FIXP_a-complete. (Theorem 10, Part 2.)
3. Computing an *assessment*, (b, μ), such that there exist an SE, (b^*, μ^*) of $\mathcal{G}$, with $\|(b, \mu) - (b^*, \mu^*)\|_\infty < \delta$, is FIXP_a-complete. (Theorem 10, Part 3.)
   An assessment (b, μ) consists of both a behavior profile b, as well as an associated *system of beliefs*, μ. (We shall define all this formally later.)
4. Given, additionally, $\epsilon > 0$ (again, in binary representation) as input, computing a *δ-almost ϵ-perfect equilibrium* (δ-almost-ϵ-PE) of $\mathcal{G}$ is PPAD-complete. (Theorem 19, Part 1.)

   A δ-almost-ϵ-PE is a relaxation of Myerson's notion of ϵ-PE ([29]) applied to EFGPRs. Roughly (we provide fully formal definitions later), it is a fully mixed behavior profile, b, such that for any player i, and for any action a played by player i in b with probability $> \epsilon$, in some information set j, it must be the case that the action a is "δ-almost local best response", meaning that the utility for player i if it switches unilaterally to pure action a in the information set j (and retains its behavior strategy in b in all other information sets), will be at most δ less than the maximum utility it could obtain by unilaterally switching its "local" distribution on actions within the information set j (while retaining its behavior strategy in b in all other information sets).

[8] Although it is worth pointing out that, by contrast, our results *do not* imply that computing an exact PE, SE, or QPE, is in FIXP, only that computing a (δ-almost) approximation of these is in FIXP_a (and PPAD respectively).

We show that a δ-almost-ϵ-PE suitably "refines" *δ-almost subgame-perfect equilibrium* (δ-almost-SGPE). A δ-almost-SGPE of $\mathcal{G}$ is a behavior profile, b, where no player can improve its own payoff *in any subgame of* $\mathcal{G}$ by more than δ, by unilaterally changing its strategy in that subgame.

Thus, as a consequence we also obtain (cf. [10]) that computing a δ-almost-NE and a δ-almost-SGPE is PPAD-complete (Theorem 19, Part 3.).

5. Likewise, we define a notion of δ-almost-ϵ-QPE, which is a relaxation of the notion of ϵ-QPE, defined by van Damme in [45], and we show that computing a δ-almost-ϵ-QPE of $\mathcal{G}$, given $\mathcal{G}$, and given $\delta > 0$ and $\epsilon > 0$, is PPAD-complete. (Theorem 19, Part 2.)

In all the above results, the "hardness" result follows immediately (already for 3-player games) from the prior known hardness results for NFGs ([13, 7, 11]). The new results are the upper bounds, all of which are new, except for the consequence that computing a δ-almost-SGPE and δ-almost-NE is contained in PPAD (this, as we explained before, follows from a prior result in [10] and its fixed but unpublished proof [9]).

Outline of proof ideas. By contrast to the prior work on algorithms for 2-player EFGPRs, our results make no explicit use of the *sequence form* for EFGPRs. Also, by contrast to [10] we make no use of reductions to graphical games. Instead, we combine older insights, including Kuhn and Selten's original *agent normal form* for EFGPRs, and Myerson's alternative definition of PE using ϵ-PEs (both for normal and extensive form), with recently developed fixed point functions for equilibria of n-player normal form games, $n \geq 3$, developed in [13] and [12].

More specifically, a key to our results is this: in Section 3, we adapt a construction in [12] of a fixed point function for "ϵ-PEs" of a given NFG (which itself is an adaptation of a fixed point function for NEs of NFGs given in [13]) to show that to any n-player EFGPR, $\mathcal{G}$, we can associate a continuous function $F^{\epsilon}_{\mathcal{G}}(x)$, defined by a "small" algebraic circuit over $\{+, *, \max\}$ (whose encoding size is polynomial in that of $\mathcal{G}$), where ϵ in an input parameter to the circuit, and such that, for any fixed $\epsilon > 0$, the function $F^{\epsilon}_{\mathcal{G}}(x)$ maps the space of behavior strategy profiles of $\mathcal{G}$ to itself, such that the Brouwer fixed points of $F^{\epsilon}_{\mathcal{G}}(x)$ constitute ϵ-PEs of $\mathcal{G}$. This proves that computing an ϵ-PE, given $\langle \mathcal{G}, \epsilon \rangle$, is in FIXP, *even when $\epsilon > 0$ is given succinctly by an algebraic circuit.*

Also, we similarly define another continuous function, $H^{\epsilon}_{\mathcal{G}}(x)$ using a "small" algebraic circuit, such that, for any fixed $\epsilon > 0$ the function $H^{\epsilon}_{\mathcal{G}}(x)$ maps the space of behavior profiles to itself, and such that every fixed point of $H^{\epsilon}_{\mathcal{G}}(x)$ is a ϵ-QPE.

The reason why we can construct the functions $F^{\epsilon}_{\mathcal{G}}(x)$ and $H^{\epsilon}_{\mathcal{G}}(x)$ with a "small" (poly-sized) algebraic circuit is related to properties of the *agent normal form* of EFGPRs, and to the fact that the "realization probabilities" and the expected payoff functions for EFGPRs can be expressed as "small" (multilinear) polynomials. In particular, a simple but important fact ([43],[29]; see Proposition 4 in this paper) is that an EFGPR has exactly the same (ϵ-)PEs as its agent normal form. (It *does not* necessarily have the same NEs.) Even though we can not construct the agent normal form explicitly (because it is exponentially large), it turns out that we do not need to: by combining these various facts, we can nevertheless construct a "small" algebraic circuit for $F^{\epsilon}_{\mathcal{G}}(x)$, by adapting the analogous construction from [12].

With the functions $F^{\epsilon}_{\mathcal{G}}(x)$ (and $H^{\epsilon}_{\mathcal{G}}(x)$) in hand, in Section 4 we then use (similar to [12]) algebraic circuits to construct a "very very small" $\epsilon^* > 0$ (but whose encoding size, when expressed as a circuit, still remains polynomial) for which we can prove, using results from real algebraic geometry ([37, 3]), that every fixed point of $F_{\mathcal{G}}^{\epsilon^*}(x)$ is δ-close (in ℓ_∞) to an actual PE. Likewise, we show that every fixed point of $H_{\mathcal{G}}^{\epsilon^*}(x)$ is δ-close to a QPE. This allows us to show containment in FIXP_a for approximating a PE, and for approximating a QPE. We furthermore show how to extend the function $F_{\mathcal{G}}^{\epsilon}(x)$ to define another "small" algebraic function $G_{\mathcal{G}}^{\epsilon}(x, z)$ that serves the same purpose for *sequential equilibrium* (SE), and in particular that additionally includes a corresponding *system of beliefs* inside its fixed points. This shows containment in FIXP_a for approximating an SE.

Finally, in Section 5, we observe some properties of the functions $F_{\mathcal{G}}^{\epsilon}(x)$ (they are "polynomially continuous" and "polynomially computable"), which when combined with results in [13] imply that computing a "δ-almost fixed point" of $F_{\mathcal{G}}^{\epsilon}(x)$, given $\mathcal{G}$ and given $\delta > 0$ and $\epsilon > 0$, is in PPAD. We then show that a "δ-almost fixed point" of $F_{\mathcal{G}}^{\epsilon}(x)$ is a (3δ)-almost-$(\delta + \epsilon)$-PE of $\mathcal{G}$. We also show that a "δ-almost fixed point" of $H_{\mathcal{G}}^{\epsilon}(x)$ is a (3δ)-almost-$(\delta + \epsilon)$-QPE. Lastly, we show that a δ'-almost-ϵ'-PE, for "polynomially small" δ' and ϵ', is a δ-almost-SGPE of $\mathcal{G}$. These results allow us to show containment in PPAD for the "δ-almost" equilibrium notions that we study.

This last part, for establishing PPAD-completeness for "δ-almost" equilibria, is technically one of the more involved parts of our proofs. Also, our proof of FIXP_a-completeness for computing a QPE involves a novel fixed point characterization. By comparison to these, our proof of FIXP_a-completeness for PE is technically easier, *given* the prior results in [12, 13], and given long existing results in the literature on EFGPRs which we exploit.

Potential computational applications. We believe our results could potentially provide a "reasonably practical" method for computing δ-almost relaxations of equilibrium refinements for n-player EFGPRs, including δ-almost ϵ-perfect and δ-almost ϵ-quasi-perfect equilibrium, as well as less refined notions of δ-almost equilibrium like SGPE and Nash (for which see also the result of [10]), by applying classic discrete path following algorithms for "almost" fixed point computation, such as variants of Scarf's algorithm [40, 41], on the "small" algebraic fixed point functions we associate with EFGPRs. We believe this is a promising approach for "almost equilibrium" computation for EFGPRs that should be implemented and explored experimentally. We note that the well-known software package GAMBIT ([24]), which provides a variety of state-of-the-art algorithms for solving various classes of games, does not currently provide any algorithm for computing or approximating an equilibrium (of any kind) for a general n-player EFGPR, for any $n \geq 3$. Indeed, a survey on equilibrium computation from 1996 ([25]), by McKelvey and McLennan who helped to develop GAMBIT, discusses the algorithms by Koller et. al. ([18, 47, 19]) for 2-player EFGPRs, but does not discuss any general algorithms for n-player EFGPRs, beyond first converting to (reduced) normal form, and using heuristics like iterated elimination of dominated strategies. We believe our results can potentially be used to remedy this gap in the availability of "practical" software for (refined) equilibrium computation for n-player EFGPRs. Of course, since we prove that computing δ-almost ϵ-PE is PPAD-complete (= linear-FIXP-complete), it follows from [7, 11, 13] that these problems are all ultimately reducible to computing a Nash Equilibrium in a 2-player normal form game. Thus one could simply aim to apply available implementations of algorithms for computing an NE for 2-player NFGs. However, this is a rather indirect approach, since it goes through reductions that result in relatively large (albeit polynomial) blowups. We believe that instead it is better to work directly with the fixed point equations we use to capture these refined equilibria for EFGPRs, and apply classic discrete path following algorithms for "almost" fixed point computation, such as variants of Scarf's algorithm to these.

2 Definitions and Background

Dear Reader: EFGPRs, and refinements of equilibrium for them, are treated in nearly every modern textbook on game theory (see, e.g., [23, 33, 30, 46]). Nevertheless, for us to discuss our problems rigorously, we can not just point you to a book or paper with relevant definitions. We must fix (a considerable amount of) notation and terminology, and we must describe various essential background results. This is especially because we will be addressing various subtle refinements of equilibrium, and corresponding notions (in some cases, new) of "approximate" and "almost" equilibrium, where slight differences in definitions can have major consequences, particularly for computational complexity. We also have to define the relevant complexity classes like FIXP, FIXP_a, and PPAD. So, we proceed to carefully fix notation and definitions, and to describe the needed background results. Readers familiar with EFGPRs, or with other parts of the background, can skip ahead to subsequent sections that contain the new results, and return to this section as needed, using it as a "reference". (Although some things are likely to become harder to follow that way.)

For a finite set X, we let $\Delta(X)$ denote the set of probability distributions on X, i.e., the set of functions $f : X \to [0, 1]$ such that $\sum_{x \in X} f(x) = 1$. For $f \in \Delta(X)$, we let $\texttt{support}(f) = \{x \in X \mid f(x) > 0\}$ denote its *support set*. For a positive integer k, we let $[k] = \{1, \ldots, k\}$.

Extensive Form Games. Intuitively, a finite game tree is just a rooted, labeled, finite tree. We will find it convenient to view such a tree as a finite, prefix-closed, set of strings over a finite alphabet of "actions". Formally, let Σ be a finite set called the *action alphabet*. We shall use the symbols $a, a', a_1, a_2, \ldots$, to denote letters in the alphabet Σ. For a string $u \in \Sigma^*$, we use $|u|$ to denote the length of u. A *tree*, $T = (V, E)$ over action alphabet Σ, consists of a finite set $V \subseteq \Sigma^*$ of *nodes* (or *vertices*), where furthermore V is prefix-closed, meaning that if $w \in V$ and $w = ua$, where $a \in \Sigma$, then $u \in V$. Note that by definition the empty string ϵ is in V. We refer to ϵ as the *root* of the tree. The directed edge relation $E \subseteq V \times V$, of the tree T (which points "away from" the root) is defined by: $E = \{(u, w) \in V \times V \mid \exists a \in \Sigma : w = ua\}$. For two nodes $u, w \in V$, if $(u, w) \in E$, we say that w is a *child* of u, and that u is the (unique) *parent* of w. For $u \in V$, we let $\texttt{Ch}(u) = \{w \in V \mid (u, w) \in E\}$ denote the set of children of u. Let $\sqsubseteq$ denote the reflexive transitive closure of E. Thus, $u \sqsubseteq w$ is just the prefix relation on the set V. We use $u \sqsubset w$ to denote the strict prefix relation: $(u \sqsubseteq w \wedge u \neq w)$. When $u \sqsubset w$, we say that u is an *ancestor* of w, and that w is a *descendant* of u. For each node $u \in V$, we define $\texttt{Act}(u) = \{a \in \Sigma \mid ua \in V\}$ to be the set of actions *available* at node u. A *leaf* is a node $u \in V$ with no children, i.e., where $\texttt{Ch}(u) = \emptyset$. Let $\mathbb{L} = \{u \in V \mid \texttt{Ch}(u) = \emptyset\}$ denote the set of leaves of the tree T. A non-leaf node is called an *internal node*; let $\mathbb{W} = V \setminus \mathbb{L}$ denote the set of internal nodes. A *path* ψ in the tree T is a non-empty sequence $\psi = u_0, u_1, u_2, \ldots, u_m$ of nodes, where for all $0 \leq i < m$, $(u_i, u_{i+1}) \in E$. The path ψ is called a *play* if $u_0 = \epsilon$, and it is called a *complete play* if additionally u_m is a leaf. In other words, a (complete) play is just a path that starts at the root (and ends at a leaf). Note that a node $u \in V$ is a string in Σ^* that encodes all the information needed to reconstruct the unique path in T from the root to u.

A *Finite Game in Extensive Form* (EFG), $\mathcal{G} = (N, \Sigma, T, P, I, p, r)$, is a tuple consisting of:

1. <u>*Players*</u>: A set $N = [n] = \{1, \ldots, n\}$ of *players*.
2. <u>*Action alphabet*</u>: a finite set Σ, called the *action alphabet*. Let $k_{\mathcal{G}} = |\Sigma|$ denote the size of Σ.
3. <u>*Game Tree*</u>: A finite tree $T = (V, E)$ over the action alphabet Σ, called the *game tree*.

4. *Player partition*: A partition $P = (P_0, P_1, \ldots, P_n)$ of the set $\mathbb{W}$ of internal nodes, i.e., $P_i \subseteq \mathbb{W}$, $\bigcup_{k=0}^{n} P_k = \mathbb{W}$, and $P_i \cap P_j = \emptyset$, for all $i \neq j$, $i, j \in \{0, \ldots, n\}$.

   For $i = 1, \ldots, n$, the nodes in P_i are the internal nodes "belonging" to player i: these are the nodes where player i has to choose the next move. The set P_0 consists of the internal nodes belonging to *chance* (or *nature*). The next move at a node $u \in P_0$ is chosen randomly, according to a provided distribution, p_u, given in item (6.) below.
   We define the *player map*, $\mathscr{P} : V \to \mathbb{N}$, by: for all $i \in \{0, \ldots, n\}$ and $u \in P_i$, $\mathscr{P}(u) := i$.

5. *Information set partition*: A tuple $I = (I_1, \ldots, I_n)$, such that for each $i \in [n] = \{1, \ldots, n\}$, $I_i = (I_{i,1}, \ldots, I_{i,d_i})$ is a partition of the set P_i of vertices belonging to player i, where each *information set* $I_{i,j} \subseteq P_i$ is non-empty & $\bigcup_{j=1}^{d_i} I_{i,j} = P_i$, $I_{i,j} \cap I_{i,k} = \emptyset$ for all $j \neq k$, $j, k \in [d_i]$.

   It is furthermore assumed that, for every information set $I_{i,j}$, and for any two nodes $u, v \in I_{i,j}$, $\texttt{Act}(u) = \texttt{Act}(v)$. In other words, the same set of actions is available to player i at every node in $I_{i,j}$. Let $\mathcal{A}_{i,j} := \texttt{Act}(u)$, where $u \in I_{i,j}$. By assumption, $\mathcal{A}_{i,j}$ is well-defined.

   We define the map $\mathcal{I}(\cdot)$, which maps a node u to the index of the information set to which u belongs. Thus, if $u \in I_{i,j}$, then $\mathcal{I}(u) := j$. For convenience, we extend the map $\mathcal{I}(\cdot)$ to chance nodes $u \in P_0$ as follows: for all $u \in P_0$, we define $\mathcal{I}(u) := u$.

   The extensive form game, $\mathcal{G}$, is said to have *perfect information* if all information sets $I_{i,j}$ are singleton sets, for all $i \in [n]$, $j \in [d_i]$. Otherwise, it is called a game of *imperfect information*.

6. *Probability distributions for chance nodes*: A tuple of probability distributions $p = (p_u)_{u \in P_0}$, one for each chance node $u \in P_0$, where $p_u : \texttt{Act}(u) \to (0, 1] \cap \mathbb{Q}$ is a positive, rational[9], probability distribution on actions available at u. So, $p_u(a) > 0$ and $p_u(a) \in \mathbb{Q}$ for all $a \in \texttt{Act}(u)$, and $\sum_{a \in \texttt{Act}(u)} p_u(a) = 1$. Let $p_{0,\min}^{\mathcal{G}} := \min_{u \in P_0, a \in \texttt{Act}(u)} p_u(a)$.

7. *Payoff functions*: An n-tuple $r = (r_1, \ldots, r_n)$ of payoff functions. For each player i, the payoff function $r_i : \mathbb{L} \mapsto \mathbb{N}_{>0}$, maps each leaf $u \in \mathbb{L}$ of the tree T to a positive integer payoff for player i.[10] Let $M_{\mathcal{G}} := \max_{i \in [n], u \in \mathbb{L}} r_i(u)$ denote the largest possible (positive integer) payoff.

We denote the bit encoding size of an EFG, $\mathcal{G}$, by $|\mathcal{G}|$, where we assume binary encoding for the integer payoff values at the leaves of $\mathcal{G}$, as well as the rational probabilities of actions at chance nodes (with numerator and denominator given in binary).[11] For a rational number $q \in \mathbb{Q}$, we use $\texttt{size}(q)$ to denote its bit encoding size. Similarly, for a rational vector $\mathtt{v} \in \mathbb{Q}^m$, we use $\texttt{size}(\mathtt{v}) := \sum_{i=1}^{m} \texttt{size}(\mathtt{v}_i)$ to denote its encoding size.

For a game $\mathcal{G}$ with tree $T = (V, E)$, let $\mathtt{h}^{\mathcal{G}} := \max\{|u| \mid u \in V\}$ denote the *height* of T. For $u \in V$, we define the *subtree* rooted at u, $T_u = (V_u, E_u, u)$, by: $V_u = \{w \in V \mid u \sqsubseteq w\}$, and $E_u = \{(u, w) \in E \mid u, w \in V_u\}$. We let $\mathtt{h}_u^{\mathcal{G}} := \max\{|w| - |u| \mid w \in V_u\}$ denote the *height* of T_u.

[9] We restrict the distributions p_u to have rational probabilities for computational purposes.

[10] We restrict to positive integer payoffs, rather than real payoffs, for computational purposes. One can of course also consider rational payoff functions $r_i : \mathbb{L} \mapsto \mathbb{Q}$. However, as is well-known, restricting to positive integer payoffs is w.l.o.g. for computational purposes: we can always "clear denominators" by multiplying by their LCM, and then add a large enough positive value to the resulting integers to get positive payoffs. This does not increase by much the encoding size of $\mathcal{G}$, and the resulting game can be shown to be "suitably isomorphic" to the original for all our purposes, including equilibrium approximation within desired precision, and δ-almost equilibrium computation.

[11] We assume natural representations for the various pieces of $\mathcal{G}$, including the tree T, player partition, information partition, payoff functions, and the probability distributions at chance nodes (with rational probabilities encoded in binary) . The details of the natural encoding are irrelevant for our purposes, so we do not spell them out.

(Note that $\mathtt{h}^{\mathcal{G}} = \mathtt{h}^{\mathcal{G}}_{\epsilon}$.) Consider an EFG, $\mathcal{G} = (N, \Sigma, T, P, I, p, r)$. For a node u of the game tree T, if the subtree T_u satisfies the property that for every node $w \in V_u$, the information set $I_{\mathscr{P}(w),\mathcal{I}(w)}$ is a subset of V_u, then the subtree T_u naturally defines a *subgame*, $\mathcal{G}_u = (N', \Sigma, T', P', I', p', r')$, which is rooted at the node u instead of at ϵ, and where the player partition, information set partition, payoff functions, and probability function for chance nodes, are all inherited directly from $\mathcal{G}$ by restricting them to the subtree T_u in the obvious way.

Note that a node $u \in V$ is a string in Σ^* which also encodes the unique *history* of actions, starting at the root, which lead to that node in T. For any node $u \in V$, with $|u| = k$, $u = a_1 a_2 \ldots a_k$, and for any $m \in \{0, 1, \ldots, k\}$, let $u[m] = a_1 \ldots a_m$ denote the length m prefix of u. For a node u, with $|u| = k$, we define the *information-action history at* u, denoted $Y(u)$, to be the following sequence of k triples:

$$Y(u) = \langle\ (\mathscr{P}(u[m]), \mathcal{I}(u[m]), a_{m+1})\ \mid\ m = 0, \ldots k-1\ \rangle$$

For each player $i \in [n]$, we define the *visible history for player* i *at* u, denoted $Y_i(u)$, to be the subsequence of $Y(u)$ obtained by retaining only those triples (i', j', a') in the sequence $Y(u)$ for which $i' = i$, and deleting all other triples. In other words, $Y_i(u)$ records the sequence of information sets belonging to player i encountered along the path from the root ϵ to u (not including u), and the actions player i chose at each of those information sets, prior to reaching u.

An EFG, $\mathcal{G}$, is said to have ***perfect recall*** if the following condition holds: for any two nodes $u, v \in V$, if $\mathscr{P}(u) = \mathscr{P}(v) = i \in [n]$ and $\mathcal{I}(u) = \mathcal{I}(v)$, then $Y_i(u) = Y_i(v)$. In other words, during play, players remember their own prior sequence of actions as well as the information sets they were in when they took those prior actions. So, it can not be the case that two nodes u and v are in the same information set for some player i, and yet the visible history for player i at u is different from the visible history for player i at v. Note that perfect recall implies there do not exist nodes $u \neq v$ belonging to the same information set such that u is an ancestor of v. Otherwise, since $Y_i(u)$ is a strict prefix of $Y_i(v)$, we would have $Y_i(u) \neq Y_i(v)$, violating perfect recall. For a game $\mathcal{G}$ of perfect recall, let us define the visible history associated with an information set $I_{i,j}$ as follow: Let $Y_{i,j} := Y_i(u)$, where $u \in I_{i,j}$. Note that by perfect recall $Y_{i,j}$ is well-defined.

Assumption: *Throughout this paper, extensive form games are assumed to have perfect recall.*

As mentioned, this assumption is standard practice in much of the literature on extensive form games. As mentioned, we use EFGPR to refer to an EFG with perfect recall.

Strategies. For an extensive form game, $\mathcal{G}$, where the information sets for player i are indexed by the set $[d_i] = \{1, \ldots, d_i\}$, a *pure strategy*, s_i, for player $i \in [n]$, is a function $s_i : [d_i] \to \Sigma$ that assigns an available action to each information set belonging to player i, so for all $j \in [d_i]$, $s_i(j) \in \mathcal{A}_{i,j}$. In other words, when using pure strategy s_i, player i chooses the available action $s_i(j)$ at every node in the information set $I_{i,j}$. Let S_i denote the set of pure strategies for player i. Let $S = S_1 \times S_2 \times \ldots \times S_n$ denote the set of *profiles* of pure strategies.

A *mixed strategy* for player i, $\sigma_i \in \Delta(S_i)$, is a probability distribution on pure strategies S_i (note: for a finite game $\mathcal{G}$, S_i is a finite set). For a pure strategy $c \in S_i$, we shall use π^c_i to denote this pure strategy as an element of $\Delta(S_i)$; so $\pi^c_i(c) = 1$, and π^c_i assigns probability 0 to other pure strategies. We let $M_i = \Delta(S_i)$ denote the set of mixed strategies for player i. Let $M = M_1 \times M_2 \times \ldots \times M_n$ denote the set of *profiles* of mixed strategies. Let $M^{>0}$ denote the set of *fully mixed* profiles of mixed strategies, that is, $M^{>0} := \{\sigma = (\sigma_1, \ldots, \sigma_n) \in M \mid \sigma_i(c) > 0,\ \text{for all } i \in [n] \text{and } c \in S_i\}$.

A *behavior strategy*, b_i, for player i, is a d_i-tuple $b_i = (b_{i,1}, b_{i,2}, \ldots, b_{i,d_i})$ of probability distributions, such that for each $j \in [d_i]$, $b_{i,j} \in \Delta(\mathcal{A}_{i,j})$ is a probability distribution on the set of actions $\mathcal{A}_{i,j}$ available in information set $I_{i,j}$. In other words, for all $a \in \mathcal{A}_{i,j}$, $0 \leq b_{i,j}(a) \leq 1$, and $(\sum_{a\in\mathcal{A}_{i,j}} b_{i,j}(a)) = 1$. We shall find it convenient to sometimes write $b_{i,j,a}$ instead of $b_{i,j}(a)$, and to view $b_{i,j}$ as a vector of probabilities, $b_{i,j} = (b_{i,j,a})_{a\in\mathcal{A}_{i,j}}$. Let $B_{i,j} := \Delta(\mathcal{A}_{i,j})$. We call $b_{i,j} \in B_{i,j}$ a *local strategy* at information set $I_{i,j}$. For an action $a \in \mathcal{A}_{i,j}$, we shall use $\pi^a_{i,j}$ to denote the *pure* local strategy in $B_{i,j}$, that assigns probability 1 to the action a. Let $B_i = B_{i,1} \times \ldots \times B_{i,d_i}$ denote the set of behavior strategies for player i. Let $B = B_1 \times B_2 \times \ldots \times B_n$ denote the set of profiles of behavior strategies. Let $B^{>0}$ denote the set of *fully mixed* behavior profiles, that is $B^{>0} := \{b = (b_1,\ldots,b_n) \in B \mid b_{i,j}(a) > 0, \text{ for all } i \in [n],\ j \in [d_i], \text{ and } a \in \mathcal{A}_{i,j}\}$.

For a behavior strategy $b_i = (b_{i,1},\ldots,b_{i,d_i}) \in B_i$, for $j \in [d_i]$ and a local strategy $b'_{i,j} \in B_{i,j}$, we use $(b_i \mid b'_{i,j})$ to denote the revised behavior strategy $(b_{i,1},\ldots,b_{i,j-1}, b'_{i,j}, b_{i,j+1},\ldots,b_{i,j})$. In other words, $(b_i \mid b'_{i,j}) \in B_i$ consists of the same local strategies as b_i, except at information set $I_{i,j}$ the local strategy is switched from $b_{i,j}$ to $b'_{i,j}$. Likewise, for a behavior profile $b \in B$, and a behavior strategy $b'_i \in B_i$, we let $(b \mid b'_i) = (b_1,\ldots,b_{i-1}, b'_i, b_{i+1},\ldots,b_n)$. In other words, $(b \mid b'_i) \in B$ consists of the same behavior strategies as b, except for player i the behavior strategy is switched form b_i to b'_i. Lastly, for a behavior profile $b = (b_1,\ldots,b_n) \in B$ and a local strategy $b'_{i,j} \in B_{i,j}$, we define the shorthand notation $(b \mid b'_{i,j}) := (b \mid (b_i \mid b'_{i,j}))$.

We also define a more general set of strategies, generalizing both B_i and M_i, called *mixed-behavior strategies*, MB_i. A mixed-behavior strategy $\sigma_i \in MB_i$ is a probability distribution over a finite subset of behavior strategies in B_i. Clearly, $S_i \subseteq B_i \subseteq MB_i$ and $S_i \subseteq M_i \subseteq MB_i$. We let $MB = MB_1 \times \ldots \times MB_n$ denote the set of profiles of mixed-behavior strategies.

Once we fix a strategy profile, $\sigma = (\sigma_1,\ldots,\sigma_n) \in MB$ for the players, this determines a *realization probability* function, $\mathbb{P}_\sigma(u)$, that assigns to every node $u \in V$ the probability of reaching u starting from the root, when players use their respective strategies in the profile σ. Then the *expected payoff*, $U_i(\sigma)$, to player i under the strategy profile σ is:

$$U_i(\sigma) = \sum_{z\in\mathbb{L}} \mathbb{P}_\sigma(z) \cdot r_i(z) \tag{1}$$

For any profile σ, and a strategy σ'_i for player i, we use $(\sigma \mid \sigma'_i)$ to denote the revised profile $(\sigma_1,\ldots,\sigma_{i-1},\sigma'_i,\sigma_{i+1},\ldots,\sigma_n)$, where everyone's strategy remains the same, except player i's strategy switches to σ'_i. We call two strategies σ'_i and σ''_i for player i *realization equivalent*, denoted by $\sigma'_i \approx \sigma''_i$, if for all $u \in V$ and for all strategy profiles $\sigma \in MB$, $\mathbb{P}_{(\sigma|\sigma'_i)}(u) = \mathbb{P}_{(\sigma|\sigma''_i)}(u)$. Note that if $\sigma'_i \approx \sigma''_i$, then $U_i(\sigma \mid \sigma'_i) = U_i(\sigma \mid \sigma''_i)$ for all $\sigma \in MB$. For games of perfect recall, we have:

Proposition 1 ([21], [43]) *For every EFGPR, $\mathcal{G}$, every mixed-behavior strategy $\sigma_i \in MB_i$ is realization equivalent to a behavior strategy $b_i \in B_i$, i.e., such that $\sigma_i \approx b_i$.*

Thus, w.l.o.g., we can confine our attention to behavior strategies in B_i for all EFGPRs.

Note that also for every behavior strategy $b_i \in B_i$ there exists a realization equivalent mixed strategy, $\sigma^{b_i}_i \in M_i$. Here's how. Define $\chi(x,y)$ by: $\chi(x,y) := 1$ if $x = y$, and otherwise $\chi(x,y) := 0$. We define the mixed strategy $\sigma^{b_i}_i$ as follows. For every $c \in S_i$:

$$\sigma^{b_i}_i(c) := \prod_{\{\ (j,a)\ \mid\ j\in[d_i]\ \&\ a\in\mathcal{A}_{i,j}\ \}} \chi(c(j),a) \cdot b_{i,j}(a).$$

The mixed strategy $\sigma^{b_i}_i$ is realization equivalent to behavior strategy b_i.[12] For a behavior profile

[12]Of course, in general, the support size of $\sigma^{b_i}_i$ can be exponential in the dimension of the vector b_i, so it is not in general efficient to work explicitly with $\sigma^{b_i}_i$ instead of b_i.

$b \in B$, we will use the notation $\sigma[b] := (\sigma_1^{b_1}, \ldots, \sigma_n^{b_n}) \in M$ to denote the (realization equivalent) mixed profile *induced* by b.

For a EFGPR, $\mathcal{G}$, for any node $u \in V$, and any behavior profile $b \in B$, we can define the realization probability $\mathbb{P}_b(u)$ as a multi-variate polynomial $F_u(x)$ (in fact, a multilinear monomial) whose "variables" x correspond to the coordinates of a behavior strategy profile in B, and such that for all $b \in B$, $F_u(b) = \mathbb{P}_b(u)$. Specifically, for all nodes $u \in V$, where[13] $|u| = k$ and $u = a_1a_2\ldots a_k$, we associate the variable $x_{i,j,a}$ with the probability $b_{i,j,a} = b_{i,j}(a)$ in a behavior profile b, and $F_u(x)$ is given by[14]:

$$F_u(x) \equiv \left(\prod_{\{m\in\{0,\ldots,k-1\} \mid u[m]\in P_0\}} p_{u[m]}(a_{m+1}) \right) \cdot \prod_{\{m\in\{0,\ldots,k-1\} \mid u[m]\in \mathbb{W}\setminus P_0\}} x_{\mathscr{P}(u[m]),\mathcal{I}(u[m]),a_{m+1}}$$

Note that, for any $u \in V$, the total degree of $F_u(x)$ is at most $\mathtt{h}^{\mathcal{G}}$, where (recall) $\mathtt{h}^{\mathcal{G}}$ is the height of the game tree. More generally, for a subset $V' \subseteq V$ of nodes, let $\mathtt{Top}(V') := \{u \in V' \mid \neg\exists v \in V' : v \sqsubset u\}$. (Note: for any information set $I_{i,j}$, $\mathtt{Top}(I_{i,j}) = I_{i,j}$.) We define the *realization probability*, $\mathbb{P}_b(V')$, of (*some* node in) $V' \subseteq V$, under (behavior) profile b, as follows: $\mathbb{P}_b(V') \doteq \sum_{u\in\mathtt{Top}(V')} \mathbb{P}_b(u)$. Thus we can also define the multilinear polynomial: $F_{V'}(x) \equiv \sum_{u\in\mathtt{Top}(V')} F_u(x)$, such that for all $b \in B$, $F_{V'}(b) = \mathbb{P}_b(V')$.

Also, using equation (1), we have that the expected payoff function is given by the polynomial:

$$U_i(x) \equiv \sum_{z\in\mathbb{L}} F_z(x) \cdot r_i(z) \tag{2}$$

Thus, restating all this, we have:

Proposition 2 *Given a EFGPR, $\mathcal{G}$, and given any subset $V' \subseteq V$ of nodes of the game tree, there is a multi-variate multilinear polynomial $F_{V'}(x)$ in the vector of variables x, with total degree bounded by $\mathtt{h}^{\mathcal{G}}$, such that for all $b \in B$, $F_{V'}(b) = \mathbb{P}_b(V')$ defines the realization probability of V' under behavior profile b in $\mathcal{G}$. Moreover, there is a multilinear polynomial $U_i(x)$, with total degree bounded by $\mathtt{h}^{\mathcal{G}}$, such that for all $b \in B$, $U_i(b)$ is the expected payoff of player i under behavior profile b in $\mathcal{G}$, and moreover, the polynomials $F_{V'}(x)$ and $U_i(x)$ can be expressed (as a weighted sum of multilinear monomials) with an encoding size that is polynomial in $|\mathcal{G}|$.*

For a fixed $b_i \in B_i$, we shall use the notation $U_k(x \mid b_i)$ to denote the polynomial obtained by fixing the values of the variables x_i, by assigning to them their corresponding values in b_i, in the polynomial $U_k(x)$. Likewise, for a fixed local strategy $b_{i,j} \in B_{i,j}$, we shall use $U_k(x \mid b_{i,j})$ to denote the polynomial obtained by fixing the variables $x_{i,j}$ by assigning to them their corresponding values in $b_{i,j}$ in the polynomial $U_k(x)$.

Information Set Forest. We shall need the concept of the *information set forest* associated with each player in a EFGPR. These forest essentially captures, for each player, the possible sequential orders in which that player may encounter its own information sets during a play of the game.[15] Specifically, for a EFGPR, $\mathcal{G}$, for each player $i \in [n]$, we define a directed, edge-labeled, graph,

[13]Here, recall that a node $u \in V$ is defined by the sequence $a_1a_2\ldots a_k$ of actions in the game tree that reaches it.

[14]Recall that P_0 denotes the set of internal nodes of the game tree that belong to "chance", and that

[15]We need this concept in several proofs, in particular associated with our results for QPEs, where we use dynamic programming, working "bottom up", by induction on the height of the information set forest, in order to compute the optimal utility for a player under certain restricted unilateral deviations from a given behavior profile b.

$\mathcal{F}_i = (V^{\mathcal{F}_i}, E^{\mathcal{F}_i})$, whose nodes are $V^{\mathcal{F}_i} = [d_i]$, i.e., the (indices of) information sets belonging to player i, and whose Σ-labeled directed edges, $E^{\mathcal{F}_i} \subseteq V^{\mathcal{F}_i} \times \Sigma \times V^{\mathcal{F}_i}$, are defined as follows: $(j, a, j') \in E^{\mathcal{F}_i}$ if and only if the last triple in the (non-empty) sequence $Y_{i,j'}$ is (i, j, a). It follows immediately from this definition that $\mathcal{F}_i$ is a directed (edge-labeled) forest, for all i. The source nodes (roots) of the forest $\mathcal{F}_i$ are those information sets which are the first belonging to player i to be encountered along some complete play of the game $\mathcal{G}$. The sink nodes (leaves) of this forest are the last information set for player i encountered along some complete play. The action a labeling the edge $(j, a, j') \in E^{\mathcal{F}_i}$ is the action that player i must take at information set $I_{i,j}$ in order to enable the possibility of reaching information set $I_{i,j'}$ (but whether or not this happens with positive probability can depend on the strategies of other players). We henceforth refer to $\mathcal{F}_i$ as the *information set forest* associated with player i. We shall say that a node $j' \in V^{\mathcal{F}_i}$ is a *descendant* of a node j in $\mathcal{F}_i$ if there is a path in $\mathcal{F}_i$ from j to j' (in other words, if j' is in the subtree rooted at j).

We let $\mathtt{h}^{\mathcal{F}_i}$ denote the *height* of the forest $\mathcal{F}_i$, i.e., the length of the longest path in $\mathcal{F}_i$. For $j \in [d_i]$, we let $\mathtt{h}_j^{\mathcal{F}_i}$ denote the height of information set j in the forest $\mathcal{F}_i$, i.e., the length of the longest path from vertex j to a leaf of the forest $\mathcal{F}_i$. For a node $u \in P_i$ of the game tree T, we will sometimes abuse notation and use $\mathtt{h}_u^{\mathcal{F}_i}$ instead of $\mathtt{h}_{\mathcal{I}(u)}^{\mathcal{F}_i}$. Note that $\mathtt{h}^{\mathcal{F}_i} \leq \mathtt{h}^{\mathcal{G}}$, for all $i \in [n]$.

For a behavior strategy $b_i \in B_i$ for player i, for any information set $j \in [d_i]$, and for any (other) profile $b'_i \in B_i$, we use the notation $(b_i |_{(i,j)} b'_i)$ to denote a new behavior strategy $b''_i := (b_i |_{(i,j)} b'_i) \in B_i$ which is defined as follows. For every information set $j' \in [d_i]$, the local strategy $b''_{i,j'}$ is defined as follows: if j' is a descendant of j in the information forest $\mathcal{F}_i$, or if j' is equal to j, then $b''_{i,j'} := b'_{i,j'}$. Otherwise, $b''_{i,j'} := b_{i,j'}$. We also use the notation $(b |_{(i,j)} b'_i) := (b \mid (b_i |_{(i,j)} b'_i))$ to denote a behavior profile which is identical to b except that player i's behavior strategy b_i is replaced by $(b_i |_{(i,j)} b'_i)$. In other words, $(b |_{(i,j)} b'_i)$ is the profile which is identical to b for all players other than player i, and where for player i, the local strategy at information set j' agrees with b'_i if the information set $I_{i,j'}$ is reachable from $I_{i,j}$, and otherwise it agrees with b_i.

We shall also use $\mathcal{F}_i$ in another way to alter behavior strategies of player i. For the information set forest $\mathcal{F}_i$ of player i, and for integer m such that $0 \leq m \leq \mathtt{h}^{\mathcal{F}_i}$, let $\mathcal{F}_i^m$ denote the sub-forest of $\mathcal{F}_i$ induced by all vertices j in $\mathcal{F}_i$ that have height $\mathtt{h}_j^{\mathcal{F}_i} \leq m$. Let $\mathcal{V}_i^m$ denote the vertices of $\mathcal{F}_i^m$.

For a behavior strategy $b_i \in B_i$ for player i, for $0 \leq m \leq h^{\mathcal{F}_i}$, and for any other behavior strategy, $b'_i \in B_i$, we use $(b_i |_m b'_i)$ to denote the behavior strategy that is given by local strategy $b'_{i,j}$ for every $j \in \mathcal{V}_i^m$, and by the original local strategy $b_{i,j}$, for all other $j \in [d_i] \setminus \mathcal{V}_i^m$. We also use the notation $(b |_m b'_i) := (b \mid (b_i |_m b'_i))$ to describe a profile that is identical to b, except that behavior strategy b_i for player i is replaced by $(b_i |_m b'_i)$.

Recall $U_k(x)$ is the polynomial representing the expected payoff function to player k under a behavior profile x. For fixed $b_i \in B_i$, we will use the notation $U_k(x |_{(i,j)} b_i)$ to denote the polynomial obtained from $U_k(x)$ as follows: for any $j' \in [d_i]$, if information set $I_{i,j'}$ is reachable from information set $I_{i,j}$, then the associated variables $x_{i,j'}$ are fixed to their values in the local strategy $b_{i,j'}$. Likewise, for $0 \leq m \leq h^{\mathcal{F}_i}$, $U_k(x |_m b_i)$ denotes the polynomial obtained from $U_k(x)$ as follows: for every $j' \in \mathcal{V}_i^m$, the variables $x_{i,j'}$ are fixed to their values in $b_{i,j'}$.

Normal Form. A finite *normal form game* (NFG), $\Gamma = (N, (S_i)_{i=1}^n, (u_i)_{i=1}^n)$, consists of a finite set $N = \{1, \ldots, n\}$ of players, a finite set S_i of pure strategies for each player i, and a payoff function $u_i : S \to \mathbb{N}_+$ for each player[16] i, where $S = S_1 \times \ldots \times S_n$. For every finite n-player EFG(PR), $\mathcal{G}$, there is an associated *standard normal form* game, $\mathcal{N}(\mathcal{G}) = (N, (S_i)_{i=1}^n, (u_i)_{i=1}^n)$, where the set

[16] Again, we restrict w.l.o.g. to positive integer payoffs, for computational purposes.

of pure strategies S_i for player i in $\mathcal{N}(\mathcal{G})$ is the set of pure strategies for player i in $\mathcal{G}$, and where the payoff function, $u_i(\cdot)$, for each player i is defined by $u_i(s) := U_i(s)$ for all $s \in S$, where $U_i(s)$ is the expected payoff in $\mathcal{G}$ to player i under pure profile s. For NFGs we use the same notations (σ_i, σ, $U_i(\sigma)$, etc.) for mixed strategies, mixed profiles, and their expected payoffs, etc., as we do for EFGPRs. Note that the encoding size $|\mathcal{N}(\mathcal{G})|$ of the NFG $\mathcal{N}(\mathcal{G})$ is in general exponential in $|\mathcal{G}|$, because already when there are two actions available at each information set, the number of strategies $|S_i|$ of player i is 2^{d_i}, where d_i is the number of information sets belonging to player i.

In the other direction, we can easily convert any NFG $\Gamma = (N, (S_i)_{i=1}^n, (u_i)_{i=1}^n)$ to an "equivalent" EFGPR, $\mathcal{E}(\Gamma)$, which is not much bigger in terms of encoding size than Γ. Specifically, let the action alphabet Σ of $\mathcal{E}(\Gamma)$ be the disjoint union of pure strategies of Γ, $\Sigma = \dot{\bigcup}_{i=1}^n S_i$, and let the nodes V of the game tree of $\mathcal{E}(\Gamma)$ be $V := \{s_1 s_2 \ldots s_k \mid k \leq n \text{ and, for all } j \in [k]: \ s_j \in S_j\}$. The player partition is given as follows: $P_0 = \emptyset$ and for all $i \in [n]$: $P_i := \{u \in V \mid |u| = i-1\}$. There is only one information set for each player $i \in [n]$: namely $I_{i,1} := P_i$. Finally, the leaves are the nodes $\mathbb{L} := \{u \in V \mid |u| = n\}$, and the payoff functions r_i are defined as follows, for all $i \in [n]$: for any leaf $s_1 s_2 \ldots s_n \in \mathbb{L}$, $r_i(s_1 s_2 \ldots s_n) := u_i(s_1, s_2, \ldots, s_n)$. Note that $\mathcal{E}(\Gamma)$ clearly has perfect recall since "there is nothing to remember": for any player $i \in [n]$ and any nodes $u, v \in P_i$, the visible histories $Y_i(u)$ and $Y_i(v)$ are both the empty sequences, and thus equal, because there is no ancestor of u or v belonging to P_i. The encoding size of $\mathcal{E}(\Gamma)$ is certainly polynomial in the encoding size of Γ (and with judicious encoding of the various parts of $\mathcal{E}(\Gamma)$ it could be made essentially linear). It is not hard to see that the games Γ and $\mathcal{E}(\Gamma)$ are essentially "equivalent" in every respect that matters to us (including for computational purposes). Note, in particular, that there is a one-to-one correspondence, which respects payoffs, between the mixed strategies of Γ and the behavior strategies of $\mathcal{E}(\Gamma)$.

Equilibrium. For a NFG, $\Gamma = (N, (S_i)_{i=1}^n, (u_i)_{i=1}^n)$, a mixed strategy σ_i' for player i is called a *best response* to a mixed profile $\sigma = (\sigma_1, \ldots, \sigma_n)$ if $U_i(\sigma \mid \sigma_i') \geq U_i(\sigma \mid \sigma_i'')$ for all mixed strategies σ_i''. Note that σ_i' is a best response to σ if and only if, for every pure strategy $c \in \texttt{support}(\sigma_i')$, and for every strategy $c' \in S_i$, $U_i(\sigma \mid \pi_i^c) \geq U_i(\sigma \mid \pi_i^{c'})$. A mixed profile σ is called a *Nash equilibrium* (NE) for Γ if σ_i is a best response to σ for all i. Nash [31] showed every (finite) NFG has an NE. It follows that the standard normal form game $\mathcal{N}(\mathcal{G})$ associated with an EFGPR, $\mathcal{G}$, has a mixed NE, $\sigma^* \in M$, which by definition is also a mixed Nash equilibrium of $\mathcal{G}$. We can say more. In light of Proposition 1, a behavior strategy $b_i' \in B_i$ for player i is called a *best response* to a behavior profile $b \in B$ if for all $b_i'' \in B_i$, $U_i(b \mid b_i') \geq U_i(b \mid b_i'')$. A profile $b = (b_1, \ldots, b_n) \in B$ is call a *Nash equilibrium (NE)* in behavior strategies if for all players i, b_i is a best response to b. Combining Proposition 1 and Nash's theorem applied to the standard normal form $\mathcal{N}(\mathcal{G})$, it follows that a NE in behavior strategies exists for any EFGPR, $\mathcal{G}$.

A profile $b \in B$ is called a *subgame-perfect equilibrium* (SGPE) if b induces a Nash equilibrium on every subgame $\mathcal{G}_u$ of $\mathcal{G}$. In other words, for every subgame $\mathcal{G}_u$, if we confine the behavior profile b to the subtree T_u rooted at u, it induces a Nash equilibrium b^u for the subgame $\mathcal{G}_u$. Again, a SGPE in behavior strategies exists for any EFGPR [42], and of course subgame-perfection is a *refinement* of NE: the SGPEs form a subset of the NEs.

We now discuss several notions of "approximate" and "almost" equilibrium for normal form and extensive form games. The well known notion of a "ϵ-NE" for a NFG is a profile where, informally, no player can improve its own payoff by more than ϵ by switching its strategy unilaterally. This of course can be defined analogously for EFGs and EFGPRs. However, to avoid confusion in terminology between this notion and the very different notion (introduced by Myerson [29]) of

ϵ-perfect equilibrium (ϵ-PE), which we define shortly, we will use the different terminology "δ-almost-NE" to refer to what would usually be called a "δ-NE" in the literature.

Formally, for $\delta > 0$, we call a behavior strategy $b'_i \in B_i$ for player i a *δ-almost best response* to a profile $b \in B$ if for all $b''_i \in B_i$, $U_i(b \mid b'_i) \geq U_i(b \mid b''_i) - \delta$. We call a profile $b = (b_1, \ldots, b_n) \in B$ a *δ-almost Nash equilibrium* (δ-almost-NE), if for all players i, b_i is a δ-almost best response to b. For $\delta > 0$, we define a *δ-almost subgame-perfect equilibrium* (δ-almost-SGPE), to be a profile $b \in B$ which induces a δ-almost-NE, b^u, on every subgame $\mathcal{G}_u$ of $\mathcal{G}$. Note that "δ-almost-SGPE" is a refinement of "δ-almost-NE".

As mentioned, Selten [43] pointed out that SGPE has inadequacies as a refinement of NE. For this reason, Selten defined a more refined notion of perfect equilibrium, based on *"trembling hand"* perfection. Two distinct notions emerge from this: *normal form perfect equilibrium* (NF-PE) and *extensive form perfect equilibrium* (PE). We shall find it very useful to provide Myerson's [29] alternative definitions for these notions, going via the notion of "ϵ-perfect equilibrium". Myerson originally defined ϵ-PE for NFGs, but his definition adapts readily to EFGPRs (see, e.g., [46, 45]). Although Myerson's definition of PE via ϵ-PEs (adapted to EFGPRs) differs from the original definition of (extensive form) PE given by Selten [43], it is equivalent; see, e.g. [29, 46, 45]. (The key reason for the equivalence was already pointed out by Selten himself in ([43], Lemma 7 & 8), as we shall highlight later.)

For an NFG[17], $\Gamma = (N, (S_i)_{i=1}^n, (u_i)_{i=1}^n)$, and for $\epsilon > 0$, a mixed profile $\sigma \in M$ is called a *ϵ-perfect equilibrium* (ϵ-PE) of Γ if it is both (a): *fully mixed* meaning $\sigma \in M^{>0}$, and (b): for every player i and pure strategy $c \in S_i$, if $\sigma_i(c) > \epsilon$, then the pure strategy π_i^c is a best response for player i to σ, in other words, $U_i(\sigma \mid \pi_i^c) \geq U_i(\sigma \mid \pi_i^{c'})$ for all $c' \in S_i$. Likewise, we call σ a *δ-almost ϵ-perfect equilibrium* (δ-almost-ϵ-PE) of Γ if (a) holds and, instead of condition (b), σ satisfies the following condition (b'): for every player i and pure strategy $c \in S_i$, if $\sigma_i(c) > \epsilon$, then the pure strategy π_i^c is a δ-almost best response for player i to σ, in other words, $U_i(\sigma \mid \pi_i^c) \geq U_i(\sigma \mid \pi_i^{c'}) - \delta$, for all $c' \in S_i$.

We call a mixed profile σ^*, a (trembling hand) *perfect equilibrium* (PE) of Γ if it is a *limit point* of a sequence of ϵ-PEs of Γ (with $\epsilon \to 0$). In other words, σ^* is a PE iff there is a sequence $\epsilon_k > 0$, $k \in \mathbb{N}$, such that $\lim_{k\to\infty} \epsilon_k = 0$, and such that for all $k \in \mathbb{N}$ there is an ϵ_k-PE, σ^{ϵ_k} of Γ, with $\lim_{k\to\infty} \sigma^{\epsilon_k} = \sigma^*$. Every NFG, Γ, has a PE, and every PE is both a NE and a SGPE ([43]).

For a EFGPR, $\mathcal{G}$, a local strategy $b'_{i,j} \in B_{i,j}$ is called a *local best response* to a profile $b \in B$ if for all local strategies $b''_{i,j} \in B_{i,j}$, $U_i(b \mid b'_{i,j}) \geq U_i(b \mid b''_{i,j})$. It is not hard to show that $b'_{i,j}$ is a local best response iff $U_i(b \mid b'_{i,j}) \geq U_i(b \mid \pi_{i,j}^a)$ for all $a \in \mathcal{A}_{i,j}$. For $\delta > 0$, a local strategy $b'_{i,j} \in B_{i,j}$ is called a *δ-almost local best response* to a profile $b \in B$ if for all $b''_{i,j} \in B_{i,j}$, $U_i(b \mid b'_{i,j}) \geq U_i(b \mid b''_{i,j}) - \delta$. Again, $b'_{i,j}$ is a δ-almost local best response to b if and only if for all actions $a \in \mathcal{A}_{i,j}$, $U_i(b \mid b'_{i,j}) \geq U_i(b \mid \pi_{i,j}^a) - \delta$.

For an EFGPR, $\mathcal{G}$, and for $\epsilon > 0$, a behavior profile $b \in B$ is called a *ϵ-perfect equilibrium* (ϵ-PE), if it is (a): *fully mixed*, meaning $b \in B^{>0}$, and (b): for all i, j, and all $a \in \mathcal{A}_{i,j}$, if $b_{i,j}(a) > \epsilon$, then $\pi_{i,j}^a$ is a local best response to b. It other words, if a local strategy $b_{i,j}$ places probability greater than ϵ on action a, then unilaterally switching the local strategy $b_{i,j}$ to pure action a is a local best response to b.

For $\delta > 0$, and $\epsilon > 0$, a behavior profile $b \in B$ is called a *δ-almost ϵ-perfect equilibrium* (δ-almost-ϵ-PE) of $\mathcal{G}$, if it is (a.): fully mixed, $b \in B^{>0}$, and (b.): for all i, j, and all $a \in \mathcal{A}_{i,j}$ if $b_{i,j}(a) > \epsilon$, then $\pi_{i,j}^a$ is a δ-almost local best response to b.

[17]For example, but not necessarily, for the standard normal form $\mathcal{N}(\mathcal{G})$ of an extensive form game $\mathcal{G}$.

We call a behavior profile $b^* \in B$ a *extensive form perfect equilibrium* (PE) of $\mathcal{G}$ if it is a limit point of ϵ-PEs of $\mathcal{G}$ (where $\epsilon \to 0$). Selten [43] showed that every EFGPR, $\mathcal{G}$, has a PE, and that every PE is also a SGPE of $\mathcal{G}$ (so, PE refines both SGPE and NE).[18]

A different refinement of equilibrium for a EFGPR, $\mathcal{G}$, is a *normal form perfect equilibrium* (NF-PE). This is, by definition, a behavior profile $b \in B$ such that the (realization equivalent) mixed profile $\sigma[b]$ induced by b is a PE of the standard normal form game, $\mathcal{N}(G)$. We note that even a pure PE of an EFGPR, $\mathcal{G}$, is not necessarily a NF-PE (i.e., does not necessarily induce a PE of $\mathcal{N}(G)$)), and nor is a pure NF-PE (i.e., a pure PE of $\mathcal{N}(G)$) necessarily a PE of $\mathcal{G}$ (see [46], Chapter 6). So, for EFGPRs, the two notions of PE and NF-PE are incompatible. In fact, a NF-PE of $\mathcal{G}$ is not necessarily even a SGPE (there are examples where it is not), and note that Selten's purpose for defining PE was to refine subgame-perfect equilibrium. So, it is not unreasonable to argue that PE is the more relevant notion for EFGPRs. Our results apply to approximating both a PE and a NF-PE for EFGPRs. (By contrast, the results of [48] apply only to computing NF-PE for 2-player EFGPRs.)

We next define *quasi-perfect equilibrium* (QPE), and the associated notions: ϵ-QPE. Let us first give a informal idea of what a QPE is, and how it differs from a PE. Intuitively, an ϵ-QPE only allows a player, i, to play an action with probability $> \epsilon$ if that action *when combined with optimal actions chosen in all descendant information sets belonging to player* i, amounts to a best response action in that information set, and a QPE is just a limit point of a sequence of ϵ-QPEs, as ϵ's get smaller and smaller. This differs from the notion of ϵ-PE and PE, in which we do *not* allow player i to deviate optimally in descendant information sets belonging to player i, when determining whether a given action amount to a "local best response". It was argued by van Damme [46] that the QPE definition captures a natural, and in some ways better, notion of a "local" best response. Later the superiority of QPE was further advocated by Mertens [26], who argued based on the desirability of dominant strategy equilibria: Mertens observed that there exist EFGPRs with a dominant strategy equilibrium (where every player simply plays a dominant strategy) which is a QPE but not a PE, whereas all dominant strategy equilibria in EFGPRs are necessarily QPEs. (We elaborate on the implications of Mertens' examples later in this section.)

We now formally define ϵ-QPE and QPE. For an EFGPR, $\mathcal{G}$, and for $\epsilon > 0$, a behavior profile $b \in B$ is called a *ϵ-quasi-perfect equilibrium* (ϵ-QPE), if it is (a.): *fully mixed*, $b \in B^{>0}$, and (b.): for all players i, all $j \in [d_i]$, and all actions $a, a' \in \mathcal{A}_{i,j}$, if $(\max_{b'_i \in B_i} U_i(b \mid_{(i,j)} (b'_i \mid \pi^{a}_{i,j}))) < (\max_{b'_i \in B_i} U_i(b \mid_{(i,j)} (b'_i \mid \pi^{a'}_{i,j})))$ then $b_{i,j}(a) \leq \epsilon$.

(We shall delay the analogous definition of "δ-almost ϵ-quasi-perfect equilibrium" until Section 5, because it will require further definitions.)

We call a behavior profile $b^* \in B$ a *quasi-perfect equilibrium* (QPE) of $\mathcal{G}$ if it is a limit point of ϵ-QPEs of $\mathcal{G}$ (where $\epsilon \to 0$). It was shown by van Damme [45] that every EFGPR has at least one QPE. Furthermore, as noted by van Damme in [45], QPE refines NF-PE. (We will highlight this again in Proposition 3 below.)

Finally, we define the notion of *sequential equilibrium* due to Kreps and Wilson [20]. We need the notion of a *system of beliefs*. For a EFGPR, $\mathcal{G}$, with game tree $T = (V, E)$, a *system of beliefs* (or *belief system*) is a map $\mu : (\mathbb{W} \setminus P_0) \to [0, 1]$ such that that for all players $i \in [n]$ and all $j \in [d_i]$, we

[18] Please note that we have overloaded the "(ϵ-)PE" terminology to apply to both (ϵ-)PE for NFGs and *extensive form* (ϵ-)PE for EFGPRs. The reason for this overloading will become clear when we discuss *agent normal form*. We remark that it is easier to see why (extensive form) PE refines SGPE via Selten's original definition of PE (via *perturbed* games). But Myerson's definition, via ϵ-PEs, has particular advantages for our purposes, as we'll see.

have $\sum_{u \in I_{i,j}} \mu(u) = 1$. Let $\mathfrak{B}$ denote the set of all belief systems (associated with the game $\mathcal{G}$). An *assessment* is a pair $(b, \mu) \in B \times \mathfrak{B}$, where b is a behavior strategy profile, and μ is a belief system. Intuitively, in assessment (b, μ), for a node $u \in I_{i,j}$, the belief $\mu(u)$ represents the probability that player i assigns to the play hitting node u assuming profile b is played, if player i finds out that the play has hit information set $I_{i,j}$. For any node $u \in I_{i,j}$, let $\mathbb{P}_b(u \mid I_{i,j}) = \mathbb{P}_b(u)/\mathbb{P}_b(I_{i,j})$ denote the conditional realization probability of reaching node u, under profile b, conditioned on reaching (i.e., realizing) information set $I_{i,j}$. This is well-defined whenever $\mathbb{P}_b(I_{i,j}) > 0$.

We will call a belief system μ *suitable for behavior profile* b if for all information sets $I_{i,j}$ such that $\mathbb{P}_b(I_{i,j}) > 0$, for all nodes $u \in I_{i,j}$, $\mu(u) = \mathbb{P}_b(u \mid I_{i,j})$. Note that if b is a fully mixed profile then there is a *unique* belief system suitable for b, because $\mathbb{P}_b(I_{i,j}) > 0$ for all information sets $I_{i,j}$. Accordingly, when b is a fully mixed behavior profile, we denote the unique belief system suitable for b by μ^b, and we say that μ^b is *the belief system generated by* b. Note that given an EFGPR, $\mathcal{G}$, and given a fully mixed (rational) profile $b \in B^{>0}$, we can easily compute the belief system μ^b generated by b in time polynomial in $|\mathcal{G}| + \texttt{size}(b)$, because the conditional probability $\mu^b(u) = \mathbb{P}_b(u \mid I_{i,j}) = \mathbb{P}_b(u)/\mathbb{P}_b(I_{i,j})$ is easy to compute given $\mathcal{G}$, b, and u. (By Proposition 2 the numerator and denominator are defined by multilinear polynomials, whose value can be easily evaluated at b, given $\mathcal{G}$ and b, in time polynomial in $|\mathcal{G}| + \texttt{size}(b)$.)

For any node $u \in V$, and for any leaf $z \in \mathbb{L}$, let $\mathbb{P}^u_b(z)$ denote the probability that leaf z is reached if the game is started at node u and the profile b is played. For any information set $I_{i,j}$, define the probability distribution $\mathbb{P}^{i,j}_{b,\mu}(z)$ on leaves by: $\mathbb{P}^{i,j}_{b,\mu}(z) := \sum_{u \in I_{i,j}} \mu(u) \cdot \mathbb{P}^u_b(z)$, for all $z \in \mathbb{L}$. Then the *expected payoff with respect to assessment* (b, μ)*, starting in information set* $I_{i,j}$, is defined by $U^{\mu,j}_i(b) = \sum_{z \in \mathbb{L}} \mathbb{P}^{i,j}_{b,\mu}(z) \cdot r_i(z)$. A behavior strategy b'_i for player i is called a *best reply at information set* $I_{i,j}$ *against assessment* (b, μ) if $U^{\mu,j}_i(b \mid b'_i) = \max_{b''_i \in B_i} U^{\mu,j}_i(b \mid b''_i)$. We say that profile b is a *sequential best reply against assessment* (b, μ) if for all players i, and all information sets $I_{i,j}$, b_i is a best reply at information set $I_{i,j}$ against assessment (b, μ). An assessment (b, μ) is called a *sequential equilibrium* (SE) of $\mathcal{G}$ if: there exists a sequence $\langle (b^k, \mu^{b^k}) \mid k \in \mathbb{N} \rangle$ of assessments, such that for all $k \in \mathbb{N}$, b^k is fully mixed and μ^{b^k} is the belief system generated by b^k, and $\lim_{k \to \infty} (b^k, \mu^{b^k}) = (b, \mu)$ (this conditioned is usually called *consistency* of (b, μ)), and furthermore b is a sequential best reply against (b, μ). Kreps and Wilson ([20]) showed the following facts about sequential equilibrium (the facts relating QPE to SE and NF-PE were shown later by van Damme [45]):

Proposition 3 ([20]; [45]) *For any EFGPR, $\mathcal{G}$:*

1. *([20]) An SE, (b', μ'), exists for $\mathcal{G}$.*
2. *([20]) For every SE, (b', μ'), of $\mathcal{G}$, the behavior profile b' is a SGPE of $\mathcal{G}$.*
3. *([20]) For every PE, b^*, of $\mathcal{G}$, there is a system of beliefs μ^* such that (b^*, μ^*) is a SE. In this sense, we say "every PE is a sequential equilibrium".*[19]
   In fact, for every PE, b^*, of $\mathcal{G}$, if $\langle (b^k, \mu^{b^k}) \rangle_{k \in \mathbb{N}}$ denotes any sequence where, for all $k \in \mathbb{N}$, b^k is a fully mixed behavior profile which is a $(1/k)$-PE for $\mathcal{G}$, and μ^{b^k} is the belief system generated by b^k, and where $\lim_{k \to \infty} b^k = b^*$ and $\lim_{k \to \infty} \mu^k = \mu^*$, then (b^*, μ^*) is a SE of $\mathcal{G}$.
4. *([45]) For every QPE, b^*, of $\mathcal{G}$, there is a system of beliefs μ^* such that (b^*, μ^*) is a SE. In this sense, we again say "every QPE is a sequential equilibrium".*[20]

[19]The converse is false: there are EFGPRs with an SE, (b', μ'), such that b' is far from any PE. See, e.g., [20, 46].
[20]The converse is again false: there are EFGPRs with an SE, (b', μ'), such that b' is far from any QPE. See [45].

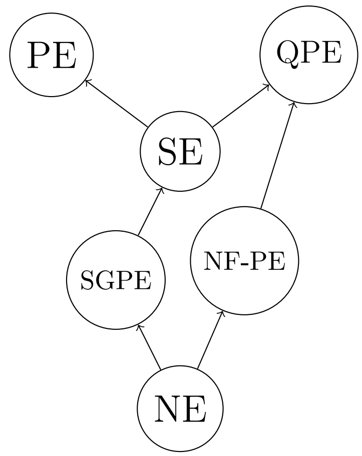

Figure 1: Hasse diagram of the mentioned equilibrium refinements for EFGPRs.

In fact, for every QPE, b^*, of $\mathcal{G}$, if $\langle(b^k, \mu^{b^k})\rangle_{k\in\mathbb{N}}$ denotes any sequence where, for all $k \in \mathbb{N}$, b^k is a fully mixed behavior profile which is a $(1/k)$-QPE for $\mathcal{G}$, μ^{b^k} is the belief system generated by b^k, and where $\lim_{k\to\infty} b^k = b^*$ and $\lim_{k\to\infty} \mu^k = \mu^*$, then (b^*, μ^*) is a SE of $\mathcal{G}$.

5. *([45]) Every QPE, b^*, of $\mathcal{G}$ is a NF-PE.*
   (Recall: for b^* is a NF-PE of $\mathcal{G}$ means that the mixed profile $\sigma[b^*] = (\sigma_1^{b_1^*}, \ldots, \sigma_n^{b_n^*})$ induced by b^* is a PE of the standard NFG, $\mathcal{N}(\mathcal{G})$.)

Figure 1 summarizes the mentioned refinement relationships between the various equilibrium notions that we have defined for EFGPRs: it depicts the Hasse diagram of the refinement partial order. In the diagram, a directed edge $X \to Y$ means that equilibrium notion Y refines notion X, i.e., that every Y-equilibrium is also a X-equilibrium. Moreover, whenever there is no directed path in this Hasse diagram from a node X to a node Y, that means there exist known examples of EFGPRs where a Y-equilibrium is *not* an X-equilibrium. (So, this is a partial order not because we lack knowledge of an underlying richer (total) order: no other refinement relationships exist for general EFGPRs, other than those implied by this Hasse diagram.)

It is noteworthy that there *can not exist* some more refined equilibrium notion that refines *both* PE and QPE, *and* exists in every EFGPR. In particular, Mertens [26] has given a simple example of a 2-player EFGPR whose set of PEs is *disjoint* from its set of NF-PEs (and whose NF-PEs consist of just one dominant strategy equilibrium). Thus, since QPE refines NF-PE, the set of PEs of Mertens' EFGPR is also disjoint from its set of QPEs. Mertens argues, partly based on this example, that QPE is preferable to PE as a refinement for EFGPRs: a dominant strategy equilibrium, when it exists, is generally prized, and it is always a QPE, but it is not necessarily a PE as shown by Mertens's example. Mertens's example shows we can not hope for some (as yet unknown) "most refined" notion of equilibrium for EFGPRs, which always exists, and which refines all the refinements we have mentioned. It is worth mentioning however that the results of [6] and [36] combined show that if a EFGPR is suitably *"generic"*[21], then its set of PEs, QPEs, and SEs

[21] Here "generic" means the EFGPR has some "structure" Ψ (which excludes the payoff information) and has a vector of payoff functions $r = (r_1, \ldots, r_n) \in \mathbb{R}^m$ such that $r \notin R[\Psi]$; where $R[\Psi] \subseteq \mathbb{R}^m$ is a certain (semi-algebraic)

are all the same. However, many natural games that we might encounter may not be "generic" in this sense, as illustrated by the various simple and natural examples of games provided in, e.g., [46, 26, 45, 23], where PE, SE, and QPE do not coincide.

Agent Normal Form. Kuhn [21] and Selten [43] considered an alternative way to associate a normal form game with a given EFGPR, $\mathcal{G}$, which they called the *agent normal form.* The agent normal form game, $\mathcal{AN}(\mathcal{G})$, is defined as follows. $\mathcal{AN}(\mathcal{G})$ has a player, called an *agent*, associated with each information set $I_{i,j}$ of the EFGPR, $\mathcal{G}$. Thus if $\mathcal{G}$ has n players and player i has d_i information sets, then the total number of agents in $\mathcal{AN}(\mathcal{G})$ is $d = \sum_{i=1}^{n} d_i$, which is the total number of information sets in $\mathcal{G}$. We refer to each agent in $\mathcal{AN}(\mathcal{G})$ by its index: (i, j). The set of pure strategies for agent (i, j) in $\mathcal{AN}(\mathcal{G})$ is given by the set $\mathcal{A}_{i,j}$ of actions available to player i of $\mathcal{G}$ in the information set $I_{i,j}$. Thus, note that the set of mixed strategies for agent (i, j) in $\mathcal{AN}(\mathcal{G})$ is in one-to-one correspondence with the set of local strategies $B_{i,j}$ for player i at information set $I_{i,j}$ in the EFGPR, $\mathcal{G}$. Thus also, the set of profiles of mixed strategies in $\mathcal{AN}(\mathcal{G})$ is in one-to-one correspondence with the set B of behavior strategy profiles in $\mathcal{G}$. Moreover, the set of pure strategy profiles of the agents in $\mathcal{AN}(\mathcal{G})$ is in one-to-one correspondence with the set of pure strategy profiles S in $\mathcal{G}$. Thus, hereafter, we use S interchangeably, to denote both the sets of pure profiles for $\mathcal{G}$ and for $\mathcal{AN}(\mathcal{G})$, and we also use B interchangeably, to denote both the set of *behavior* profiles of $\mathcal{G}$ and the set of *mixed* profiles of $\mathcal{AN}(\mathcal{G})$.

We define the payoff functions, $u_{(i,j)}(s)$, of $\mathcal{AN}(\mathcal{G})$ as follows: given a pure profile $s \in S$ for the d agents, the payoff to agent (i, j) is given by $u_{(i,j)}(s) := U_i(s)$. In other words, the payoff for every agent (i, j) in $\mathcal{AN}(\mathcal{G})$ under profile s is the expected payoff of player i in $\mathcal{G}$ under the same profile s. Thus, the goal of all the agents (i, j) who are "acting on behalf of" player i, is aligned exactly with the goal of player i. It follows that also the *expected* payoff, $U_{(i,j)}(b)$, to agent (i, j) under any mixed profile $b \in B$ in $\mathcal{AN}(\mathcal{G})$ is equal to the expected payoff $U_i(b)$ of player i under the same (behavior) profile $b \in B$ of $\mathcal{G}$.

A simple but important fact, that follows immediately from the definitions we have given for $(\epsilon$-)PEs, is that the set of $(\epsilon$-)PEs of $\mathcal{G}$ is equal to the set of $(\epsilon$-)PEs of $\mathcal{AN}(\mathcal{G})$.[22]

Proposition 4 (cf. [43] Lemma 7, & [29]; see also [46]) *For a EFGPR, $\mathcal{G}$, and $\epsilon > 0$, a behavior profile $b \in B$ is a ϵ-PE of $\mathcal{G}$ if and only if b is a mixed ϵ-PE of $\mathcal{AN}(\mathcal{G})$ (this is true by definition). Thus, a profile $b \in B$ is a PE of $\mathcal{G}$ iff b is a PE of $\mathcal{AN}(\mathcal{G})$.*

Note, firstly, that it is *not* true in general that the set of Nash equilibria of $\mathcal{G}$ and $\mathcal{AN}(\mathcal{G})$ are the same. There are simple (even 1-player) examples showing this. This is because even though a profile $b \in B$ might consist entirely of "local best responses" in $\mathcal{G}$, some information sets may be reached with probability 0 under profile b, and therefore "local best responses" together do not necessarily constitute a "global" best response in $\mathcal{G}$.

Note also that, as mentioned already, no such relationship holds in general between the PEs of $\mathcal{G}$ and the PEs of its standard normal form $\mathcal{N}(\mathcal{G})$, in either direction.

Proposition 4 holds by definition because we have used Myerson's [29] alternative definition of PEs, via ϵ-PEs. We remark that the reason why Myerson's definition is equivalent to Selten's original definition (which we will not give formally) was shown already by Selten himself. Namely, Selten defined a PE as a limit point of NEs of a sequence of *perturbed* games (with positive "perturbations" going to zero). In a *perturbed* EFGPR, there is a minimum positive probability specified

"forbidden" set of *dimension* strictly less than m.

[22] This is why we overload the "$(\epsilon$-)PE" terminology for the corresponding notions of both NFGs and EFGPRs.

for each action available in each information set, and that action must be played with at least that probability in any behavior strategy. Selten ([43], Lemma 7) showed that for perturbed EFGPRs, a behavior strategy that consists entirely of "local best responses" is also necessarily a "global" best response. As explained already, this *does not* hold in general when the game is not perturbed.

We shall need the following "almost" variant of Proposition 4, which also follows immediately from our definitions.

Proposition 5 *For all $\delta > 0$ and $\epsilon > 0$, for any EFGPR, $\mathcal{G}$, a (behavior) strategy profile $b \in B$ is a δ-almost-ϵ-PE of $\mathcal{G}$ iff b is a (mixed) δ-almost-ϵ-PE of $\mathcal{AN}(\mathcal{G})$.*

Note that if the agent normal form $\mathcal{AN}(\mathcal{G})$ is represented in the usual way, by providing its table of payoffs for all possible pure strategy profiles of all the agents, then just as was the case for standard normal form, the encoding size $|\mathcal{AN}(\mathcal{G})|$ is also exponential in $|\mathcal{G}|$, because the number $|S|$ of pure profiles of $\mathcal{AN}(\mathcal{G})$ is exponential in $|\mathcal{G}|$. Nevertheless, we shall find $\mathcal{AN}(\mathcal{G})$ very useful for our computational purposes.

The complexity classes FIXP, FIXP$_a$, and linear-FIXP(= PPAD)

We shall now define the search problem complexity classes FIXP, FIXP$_a$, and PPAD, which we shall use to characterize the complexity of computing an equilibrium (of various kinds) for a EFGPR.

A $\{+, -, *, /, \max, \min\}$-*circuit* has inputs consisting of variable $x_1, x_2, \ldots, x_n$, as well as rational constants, and has a finite number of (binary) computation gates taken from $\{+, -, *, /, \max, \min\}$, with a subset of the computation gates labeled $\{o_1, o_2, \ldots, o_m\}$ and called output gates.[23] The class of $\{+, \max\}$-*circuits* are the restricted class of $\{+, -, *, /, \max, \min\}$-circuits, where the only allowed gates are $\{+, \max\}$ in addition to gates for *multiplication by a rational constant*.

When a circuit in this paper is a general $\{+, -, *, /, \max, \min\}$-circuit, we shall often just refer to it simply as "circuit", when it is clear from the context. We shall also refer to $\{+, \max\}$-circuits as *piecewise-linear* circuits. A circuit (of either kind) computes a continuous function from $\mathbb{R}^n \to \mathbb{R}^m$ (and $\mathbb{Q}^n \to \mathbb{Q}^m$) in the natural way. Abusing notation slightly, we shall often identify the circuit with the function it computes.

By a (total) *multi-valued function*, f, with domain A and co-domain B, we mean a function that maps each $a \in A$ to a non-empty subset $f(a) \subseteq B$. We use $f : A \twoheadrightarrow B$ to denote such a function. Intuitively, when considering a multi-valued function as a computational problem, we are interested in producing just one of the elements of $f(a)$ on input a, so we refer to $f(a)$ as the set of *allowed outputs*.

A multi-valued function $f : \{0,1\}^* \twoheadrightarrow \mathbb{R}^*$ is said to be in FIXP if there is a polynomial time computable map, r, that maps each instance $I \in \{0,1\}^*$ of f to $r(I) = \langle 1^{k^I}, 1^{d^I}, P^I, C^I, \phi^I, a^I, b^I \rangle$, where

- k^I and d^I are positive integers.
- P^I is a convex polytope in $\mathbb{R}^{k^I}$, given as a set of linear inequalities with rational coefficients.
- C^I is a circuit, with k^I inputs and k^I outputs, which maps P^I to itself.

[23]The set of gates $\{+, -, *, /, \max, \min\}$ is of course redundant, e.g., using rational constants the gates $\{-, \min\}$ can be simulated by the other gates.

- $\phi^I : [d^I] \rightarrow [k^I]$ is a finite function, given by its table.
- $a^I, b^I \in \mathbb{Q}^{d^I}$.
- $f(I) = \{(a^I_i y_{\phi^I(i)} + b^I_i)_{i=1}^{d^I} \mid y \in P^I \wedge C^I(y) = y\}$. Note that $f(I) \neq \emptyset$, by Brouwer's fixed point theorem.

The above is one of many equivalent characterizations of FIXP [13]. In particular, it was shown in [13] that the gates $\{+, *, \max\}$ together with rational constants suffice for functions computed by the corresponding circuits to characterize FIXP, and furthermore adding other gates such as k'th-root gates for any fixed k does not increase the power of FIXP.

A multi-valued function $f : \{0,1\}^* \twoheadrightarrow \mathbb{R}^*$ is said to be in linear-FIXP if it satisfies the same definition as for FIXP, except that the circuit C^I must be a $\{+, \max\}$-circuit (recall: with multiplication by rational constants allowed).

Informally, FIXP are those real vector multi-valued functions, with discrete inputs, that can be cast as Brouwer fixed point computations for algebraically defined functions, and linear-FIXP is the restriction of those to functions that are piecewise-linear. A multi-valued function $f : \{0,1\}^* \twoheadrightarrow \mathbb{R}^*$ is said to be FIXP-*complete* (respectively, linear-FIXP-*complete*) if:

1. $f \in$ FIXP (respectively, $f \in$ linear-FIXP), and
2. [f is FIXP-*hard* (respectively, f is linear-FIXP-*hard*)]: for all $g \in$ FIXP (respectively, $g \in$ linear-FIXP), there is a polynomial time computable map, mapping instances I of g to $\langle y^I, 1^{k^I}, \phi^I, a^I, b^I \rangle$, where y^I is an instance of f, where $f(y^I) \subseteq \mathbb{R}^{k^I}$, $\phi^I : [d^I] \rightarrow [k^I]$ is a function (given by its table), $d^I \geq 1$, and a^I and b^I are d^I-tuples with rational entries, so that $g(I) \supseteq \{(a^I_i z_{\phi^I(i)} + b^I_i)_{i=1}^{d^I} \mid z \in f(y^I)\}$. In other words, for any allowed output z of f on input y^I, the vector $(a^I_i z_{\phi^I(i)} + b^I_i)_{i=1}^{d^I}$ is an allowed output of g on input I.

In [13] it was shown that the multi-valued function which maps normal forms games, with $n \geq 3$ players, to their Nash equilibria is FIXP-complete.[24]

Since the output of a FIXP function consists of real-valued vectors, and since there exist circuits whose fixed points are all irrational, a FIXP function is not directly computable by a Turing machine, and the class is therefore not directly comparable with standard complexity classes of *discrete* total search problems (such as PPAD, PLS, or TFNP).

Even though we phrased linear-FIXP as a class of real-valued search problems, it can also be viewed as class of *discrete* search problems, because the nature of the functions defined by $\{+, \max\}$-circuits (with multiplication by rational constants), over a convex polytope domain P^I, implies that they always have at least one *rational*-valued fixed point, with encoding size polynomial in that of the circuit.[25] In fact, it was shown in [13] that linear-FIXP = PPAD. (So, linear-FIXP can serve as our definition of PPAD in this paper. We will not need the original definition.)

It was shown by Chen and Deng [7] that the multi-valued function that maps 2-player NFGs to their NEs is PPAD-complete, and by Daskalakis *et al.* [11] that the multi-valued function that maps NFGs (with any number of players), and a given rational $\epsilon > 0$, to their ϵ-NEs is PPAD-complete.

[24] To view the Nash equilibrium problem as a total multi-valued function, $f_{\text{Nash}} : \{0,1\}^* \twoheadrightarrow \mathbb{R}^*$, we can view all strings in $\{0,1\}^*$ as encoding some game, by viewing "ill-formed" input strings as encoding a fixed trivial game.

[25] Technically, to view linear-FIXP as a *discrete* search problem class, comparable to PPAD, etc., we likewise close (discrete) linear-FIXP under polynomial time (search problem) reductions.

We now define the discrete class FIXP_a, also from [13]. A multi-valued function $f : \{0,1\}^* \twoheadrightarrow \{0,1\}^*$ (a.k.a. a totally defined discrete search problem) is said to be in FIXP_a if there is a function $f' \in \mathsf{FIXP}$, and polynomial time computable maps $\delta : \{0,1\}^* \to \mathbb{Q}_+$ and $g : \{0,1\}^* \to \{0,1\}^*$, such that for all instances I,

$$f(I) \supseteq \{\, g(\langle I, y \rangle) \mid y \in \mathbb{Q}^* \ \wedge\ \exists y' \in f'(I) : \ \|y - y'\|_\infty \le \delta(I) \,\}.$$

Informally, FIXP_a are those totally defined discrete search problems that reduce to approximating exact Brouwer fixed points. A multi-valued function $f : \{0,1\}^* \twoheadrightarrow \{0,1\}^*$ is said to be FIXP_a-*complete* if:

1. $f \in \mathsf{FIXP}_a$, and
2. [f is FIXP_a-*hard*]: For all $g \in \mathsf{FIXP}_a$, there are polynomial time computable maps $r_1, r_2 : \{0,1\}^* \to \{0,1\}^*$, such that $g(I) \supseteq \{\, r_2(\langle I, z \rangle) \mid z \in f(r_1(I)) \,\}$.

In [13] it was shown that the multi-valued function that maps pairs $\langle \Gamma, \delta \rangle$, where Γ is a NFG and $\delta > 0$, to the set of rational δ-approximations (in ℓ_∞-distance) of Nash equilibria of Γ, is FIXP_a-complete.

3 Computing a (extensive form) ϵ-PE, and a ϵ-QPE, is in FIXP

Given a EFGPR, $\mathcal{G}$, we now construct an algebraically defined function, $F^\epsilon_{\mathcal{G}}(x)$, whose Brouwer fixed points (for each fixed $\epsilon > 0$), constitute ϵ-PEs of $\mathcal{G}$. We likewise construct a function, $H^\epsilon_{\mathcal{G}}(x)$ whose Brouwer fixed points (for each fixed $\epsilon > 0$), constitute ϵ-QPEs of $\mathcal{G}$. The functions $F^\epsilon_{\mathcal{G}}(x)$ and $H^\epsilon_{\mathcal{G}}(x)$ are both defined using an algebraic $\{+, *, \max\}$-circuit whose encoding size is polynomial in $|\mathcal{G}|$, and where $\epsilon > 0$ is an input of the algebraic circuit. Our construction of $F^\epsilon_{\mathcal{G}}(x)$ essentially amounts to the same construction as given for ϵ-PEs of *normal form games* in [12], except when it is applied to the *agent normal form*, $\mathcal{AN}(\mathcal{G})$. Of course the problem is that we can not afford to actually construct $\mathcal{AN}(\mathcal{G})$, because it is exponentially large. However, it turns out we do not need to construct $\mathcal{AN}(\mathcal{G})$ in order to construct $F^\epsilon_{\mathcal{AN}(\mathcal{G})}(x)$. We instead exploit the fact (Proposition 2) that the expected payoff functions $U_{(i,j)}(x) := U_i(x)$ for agents (i,j) in $\mathcal{AN}(\mathcal{G})$ are expressible as polynomials whose encoding size is polynomial in $|\mathcal{G}|$. This allows us to construct $F^\epsilon_{\mathcal{G}}(x) = F^\epsilon_{\mathcal{AN}(\mathcal{G})}(x)$ with encoding size polynomial in $|\mathcal{G}|$, avoiding the explicit construction of $\mathcal{AN}(\mathcal{G})$.

Our construction of the function $H^\epsilon_{\mathcal{G}}(x)$ for ϵ-QPEs is based on some similar ideas, but is more involved, and does not make direct use of the relationship with $\mathcal{AN}(\mathcal{G})$.

Given a n-player EFGPR, $\mathcal{G}$, the space B of behavior strategy profiles for $\mathcal{G}$ is clearly a compact convex polytope in euclidean space, $\mathbb{R}^m$, where m is the dimension of the vectors $b \in B$ that denote behavior profiles. Moreover, B can clearly be expressed efficiently using a system of less than $3m$ linear inequalities (which define B to be the set of vectors $b \in \mathbb{R}^m$ in which each local strategy $b_{i,j}$ forms a probability distribution on $\mathcal{A}_{i,j}$). For $\epsilon > 0$, let $B^\epsilon \subseteq B$ denote the polytope of behavior profiles defined by:

$$B^\epsilon = \{b \in B \mid b_{i,j}(a) \ge \epsilon, \text{ for all } i \in [n],\, j \in [d_i] \text{ and } a \in \mathcal{A}_{i,j}\}.$$

Theorem 6 *For any EFGPR, $\mathcal{G}$:*

1. *There is a function, $F^{\epsilon}_{\mathcal{G}}(x) : B \to B^{\epsilon}$, given by a $\{+, *, \max\}$-circuit computable in polynomial time from $\mathcal{G}$, with the circuit having both x and $\epsilon > 0$ as its inputs, such that for all fixed $0 < \epsilon < 1/m$ (where m is the dimension of vectors $b \in B$), every Brouwer fixed point of the function $F^{\epsilon}_{\mathcal{G}}(x)$ is a ϵ-PE of $\mathcal{G}$. In particular, the problem of computing an extensive form ϵ-perfect equilibrium for a given EFGPR is in* FIXP.

2. *There is a function, $H^{\epsilon}_{\mathcal{G}}(x) : B \to B^{\epsilon}$, given by a $\{+, *, \max\}$-circuit computable in polynomial time from $\mathcal{G}$, with the circuit having both x and $\epsilon > 0$ as its inputs, such that for all fixed $0 < \epsilon < 1/m$ (where m is the dimension of vectors $b \in B$), every Brouwer fixed point of the function $H^{\epsilon}_{\mathcal{G}}(x)$ is a ϵ-QPE of $\mathcal{G}$. In particular, the problem of computing a ϵ-QPE for a given EFGPR is in* FIXP.

As mentioned, the proof we give below of Part (1.) of Theorem 6 is very similar to the proof of the analogous result for ϵ-PEs of NFGs given in [12], which itself builds on a fixed point characterization of Nash equilibria from [13]. By Proposition 4, to prove Theorem 6 it suffices to find ϵ-PEs of the agent normal form $\mathcal{AN}(\mathcal{G})$, because these are the same as ϵ-PEs of $\mathcal{G}$. We can not "construct" $\mathcal{AN}(\mathcal{G})$, because it has size exponential in $\mathcal{G}$, but we do not need to. We now give the detailed proof for both parts. Although the proof of Part (1.) is very similar to the analogous proof in [12], the proof of Part (2.) also involves additional constructions and does not appeal to the relationship with $\mathcal{AN}(\mathcal{G})$. To facilitate our proof of Part (2.), we need some definitions, and an alternative characterization of ϵ-QPE.

Note that for any fully mixed profile $b \in B^{>0}$, for any player i, $j \in [d_i]$, and any node $u \in I_{i,j}$, the conditional probability $\mathbb{P}_b(u|I_{i,j}) = \frac{\mathbb{P}(u)}{\mathbb{P}_b(I_{i,j})}$ is well-defined, because $\mathbb{P}_b(I_{i,i}) > 0$. Furthermore, importantly, given that $\mathbb{P}_b(I_{i,j}) > 0$, $\mathbb{P}_b(u|I_{i,j})$ is otherwise "independent" of b_i. It only depends on the behavior strategies b_{-i} of players other than i, because, by perfect recall, for all nodes $u \in I_{i,j}$ the visible history for player i is the same: $Y_{i,j}$. For $b \in B^{>0}$, for $i \in [n]$, and for $j \in [d_i]$, we use $U^j_i(b)$ to denote the *conditional expected payoff* to player i, conditioned on reaching information set $I_{i,j}$, under profile b. Again, this conditional expectation is well-defined, since $b \in B^{>0}$. Furthermore, again, except for the fact that $\mathbb{P}_b(I_{i,j}) > 0$, the conditional expectation $U^j_i(b)$ is independent of those local strategy $b_{i,j'}$ in b_i for information sets $I_{i,j'}$ such that the node $j' \in V^{\mathcal{F}_i}$ of the information set forest $\mathcal{F}_i$ is not in the subtree of $\mathcal{F}_i$ rooted at node $j \in V^{\mathcal{F}_i}$. It only depends on those local strategies $b_{i,j''}$ where $j'' \in V^{\mathcal{F}_i}$ is a node in the subtree of $\mathcal{F}_i$ rooted at j. For $i \in [n]$, $j \in [d_i]$ and $a \in \mathcal{A}_{i,j}$, and for $b \in B^{>0}$, we define

$$\mathsf{K}^{j,a}_i(b) := \max_{b'_i \in B_i} U^j_i(b \mid_{(i,j)} (b'_i \mid \pi^a_{i,j})).$$

Thus $\mathsf{K}^{j,a}_i(b)$ denotes the maximum conditional expected payoff to player i, conditioned on reaching information set $I_{i,j}$ using b, where player i switches to action $a \in \mathcal{A}_{i,j}$ at $I_{i,j}$, and chooses the rest of its strategy b'_i (below information set $I_{i,j}$ in $\mathcal{F}_i$) so as to maximize $U^j_i(b \mid_{(i,j)} (b'_i \mid \pi^a_{i,j}))$. Note that, since $b \in B^{>0}$, $\mathsf{K}^{j,a}_i(b)$ is both well defined and "independent" of b_i: it only matters that $\mathbb{P}_b(I_{i,j}) > 0$. Now, observe that, for any $b \in B^{>0}$, for any $i \in [n]$, $j \in [d_i]$, and for any $a, a' \in \mathcal{A}_{i,j}$, we have:

$$(\ \mathsf{K}^{j,a}_i(b) < \mathsf{K}^{j,a'}_i(b)\ ) \quad \Longleftrightarrow \quad (\ (\max_{b'_i \in B_i} U_i(b \mid_{(i,j)} (b'_i \mid \pi^a_{i,j}))) < (\max_{b'_i \in B_i} U_i(b \mid_{(i,j)} (b'_i \mid \pi^{a'}_{i,j}))) \qquad (3)$$

This equivalence holds because the profiles $(b \mid_{(i,j)} (b'_i \mid \pi^a_{i,j})))$ and $(b \mid_{(i,j)} (b'_i \mid \pi^{a'}_{i,j})))$ differ only within player i's local strategies within b_i at information sets j' in the subtree of $\mathcal{F}_i$ rooted at $j \in V^{\mathcal{F}_i}$. Thus, since $\mathbb{P}_b(I_{i,j}) > 0$, the strict inequality on the left of (3) holds if and only if the strict inequality on the right of (3) holds. Thus, an alternative definition for a profile b to be a *ϵ-quasi-perfect equilibrium* (ϵ-QPE), is this: (a.) $b \in B^{>0}$, and (b.) for all $i \in [n]$, $j \in [d_i]$, and $a, a' \in \mathcal{A}_{i,j}$, if $\mathsf{K}^{j,a}_i(b) < \mathsf{K}^{j,a'}_i(b)$, then $b_{i,j}(a) \leq \epsilon$. We will exploit this alternative definition.[26]

Consider a EFGPR, $\mathcal{G}$, and let $b \in B$ have dimension m as vectors in Euclidean space. Suppose we are given $0 < \epsilon < 1/m$. For a vector x of variables corresponding to the coordinates of a behavior strategy $b \in B$, we let $v(x)$ be a m-vector such that for all $i \in [n]$, $j \in [d_i]$, and $a \in \mathcal{A}_{i,j}$ $v(x)_{i,j,a} = U_i(x \mid \pi^a_{i,j}) = U_{(i,j)}(x \mid \pi^a_{i,j})$. In other words, for all behavior profiles $b \in B$, $v(b)_{i,j,a}$ is the expected payoff to agent (i,j) in the agent normal form game $\mathcal{AN}(\mathcal{G})$, if all agents play according to b, except that agent (i,j) switches to pure strategy $\pi^a_{i,j}$. Note that by Proposition 2, $v(x)_{i,j,a}$ can be expressed as a polynomial in the variables x whose encoding size is polynomial in $|\mathcal{G}|$.

Likewise, let us define $v'(x)_{i,j,a} := \mathsf{K}^{j,a}_i(x)$. We shall show, in Lemma 7 below, that the function $\mathsf{K}^{j,a}_i(x)$, defined over $B^{>0}$, can indeed be expressed as a $\{+,-,*,/,\max,\min\}$-formula in the variables x, where the encoding size of the formula is polynomial in $|\mathcal{G}|$.

Lemma 7 *Given a EFGPR, $\mathcal{G}$, for all players $i \in [n]$, all information sets $j \in [d_i]$, and all actions $a \in \mathcal{A}_{i,j}$, there is a $\{+,-,*,/,\max\}$-formula $v'(x)_{i,j,a}$ (i.e., a $\{+,-,*,/,\max,\min\}$-circuit with no re-use of subcircuits), such that the encoding size of $v'(x)_{i,j,a}$ is polynomial in $|\mathcal{G}|$, and each $v'(x)_{i,j,a}$ can be constructed from $\mathcal{G}$ in P-time, and such that for all fully mixed $b \in B^{>0}$, $v'(b)_{i,j,a} = \mathsf{K}^{j,a}_i(b)$.*

Proof. The basic idea of the proof is that, given $b \in B^{>0}$, one can compute $\mathsf{K}^{j,a}_i(b)$ using dynamic programming, by working "bottom up" on the information set forest $\mathcal{F}_i$ for player i. Then the key observation is that this dynamic program can actually be described by a $\{+,-,*,/,\max\}$-formula which has encoding size only polynomial in $\mathcal{G}$.

We next describe the dynamic program, and the resulting formula, in detail. (We will later need to use facts about the detailed structure of the formula.) Consider the information set forest $\mathcal{F}_i$ for player i. Let $\mathbb{L}_{\mathcal{F}_i}$ denote the set of leaves of $\mathcal{F}_i$. Let $\mathbb{W}_{\mathcal{F}_i}$ denote the set of internal nodes of $\mathcal{F}_i$. For a node $j \in [d_i] = V^{\mathcal{F}_i}$, and for $a \in \mathcal{A}_{i,j}$, let us denote the set of a-children of j in $\mathcal{F}_i$ by: $\mathtt{Ch}^a_{\mathcal{F}_i}(j) = \{j' \in V^{\mathcal{F}_i} \mid (j,a,j') \in E^{\mathcal{F}_i}\}$. For an internal node $u \in \mathbb{W}$, and for $a \in \mathtt{Act}(u)$, let $\vec{\mathbb{L}}^{u,a} = \{z \in \mathbb{L} \mid ua \sqsubseteq z \ \&\ \forall m \text{ such that } ua \sqsubseteq z[m],\ z[m] \notin P_{\mathscr{P}(u)}\}$. In other words, $\vec{\mathbb{L}}^{u,a}$ denotes the set of leaves z of the game tree T that are in the subtree rooted at ua, and such that there is no node on the path from ua to z which belongs to the same player $\mathscr{P}(u)$ that u belongs to.

For $u, v \in V$, let $\mathbb{P}_b(v \mid u)$ denote the probability that, using profile b, conditioned on reaching node u, the play eventually thereafter hits node v. For $i \in [n]$ and $j, j' \in [d_i]$, let $\mathbb{P}_b(I_{i,j'} \mid I_{i,j})$ denote conditioned probability of reaching information set $I_{i,j'}$, conditioned on reaching $I_{i,j}$, when using profile b.

We can define $v'(x)_{i,j,a} := \mathsf{K}^{j,a}_i(x)$ inductively in a "bottom up" fashion based on the forest $\mathcal{F}_i$, based on the *height*, $\mathtt{h}^{\mathcal{F}_i}_j$, of the subtree rooted at node $j \in V^{\mathcal{F}_i} = [d_i]$ of $\mathcal{F}_i$. Recall that

[26] Indeed, this is one of the equivalent characterizations of ϵ-QPE that was originally given by van Damme in [45]. We used a different definition for clarity, and for compatibility with the way we defined ϵ-PE. In fact, similarly van Damme [45] used a similar equivalent characterization of ϵ-PE for an EFGPR, defined as follows: (a.) $b \in B^{>0}$, and (b.), for all $i \in [n]$, $j \in [d_i]$, and $a, a' \in \mathcal{A}_{i,j}$, if $U^j_i(b \mid \pi^a_{i,j}) < U^j_i(b \mid \pi^{a'}_{i,j})$ then $b_{i,j}(a) \leq \epsilon$. Again, it is clear that this is equivalent to the definition we have given for ϵ-PE.

$\mathbb{P}_x(u \mid I_{i,j}) = \frac{\mathbb{P}_x(u)}{\mathbb{P}_x(I_{i,j})}$, is defined for all $x \in B^{>0}$, and by Proposition 2 both the numerator and denominator are given by polynomials in x with "small" encoding size (polynomial in $|\mathcal{G}|$). Note that likewise, for $a \in \mathcal{A}_{i,j}$, $\mathbb{P}_{(x|\pi^a_{i,j})}(v \mid u)$ is easily defined by a weighted monomial over the variables x whose encoding size is polynomial in $|\mathcal{G}|$. Furthermore if the node $j' \in V^{\mathcal{F}_i}$ is a child of the node $j \in V^{\mathcal{F}_i}$ in the forest $\mathcal{F}_i$, then

$$\mathbb{P}_{(x|\pi^a_{i,j})}(I_{i,j'} \mid I_{i,j}) = \sum_{u \in I_{i,j}} \mathbb{P}_x(u \mid I_{i,j}) \cdot \sum_{v \in I_{i,j'}} \mathbb{P}_{(x|\pi^a_{i,j})}(v \mid u).$$

Thus $\mathbb{P}_{(x|\pi^a_{i,j})}(I_{i,j'} \mid I_{i,j})$ is also described by a formula over the variables x with encoding size polynomial in $|\mathcal{G}|$. We can now describe a dynamic program for computing $\mathsf{K}^{j,a}_i(x)$, for all $i \in [n]$, $j \in [d_i]$, and $a \in \mathcal{A}_{i,j}$:

$$\mathsf{K}^{j,a}_i(x) := \begin{cases} \sum_{u \in I_{i,j}} \mathbb{P}_x(u \mid I_{i,j}) \cdot \sum_{z \in \vec{\mathbb{L}}^{u,a}} \mathbb{P}_{(x|\pi^a_{i,j})}(z \mid u) \cdot r_i(z) \ , & \text{if } j \in \mathbb{L}_{\mathcal{F}_i} \\ (\ \sum_{j' \in \mathtt{Ch}^a_{\mathcal{F}_i}(j)} \mathbb{P}_{(x|\pi^a_{i,j})}(I_{i,j'} \mid I_{i,j}) \cdot (\max_{a' \in \mathcal{A}_{i,j'}} \mathsf{K}^{j',a'}_i(x))\ )+ & \\ \quad \sum_{u \in I_{i,j}} \mathbb{P}_x(u \mid I_{i,j}) \cdot \sum_{z \in \vec{\mathbb{L}}^{u,a}} \mathbb{P}_{(x|\pi^a_{i,j})}(z \mid u) \cdot r_i(z) \ , & \text{if } j \in \mathbb{W}_{\mathcal{F}_i} \end{cases} \tag{4}$$

It is clear that (4) defines a dynamic program for computing $\mathsf{K}^{j,a}_i(b)$, given $b \in B^{>0}$, and at the same time (4) defines a $\{+, -, *, /, \max\}$-formula with variables x, which when evaluated at $b \in B^{>0}$ yields $\mathsf{K}^{j,a}_i(b)$. Furthermore, the encoding size of the formula given for $\mathsf{K}^{j,a}_i(x)$ is polynomial in $|\mathcal{G}|$. This can be seen by noting, firstly, that all the constituent parts of the inductively defined formula for $\mathsf{K}^{j,a}_i(x)$ are given by formulas with encoding size polynomial in $|\mathcal{G}|$, and furthermore since the inductive definition works "bottom up" on the forest $\mathcal{F}_i$, there is no re-use of subformulas in this inductive definition, i.e., it indeed defines a formula, not a circuit, and the size of the formula is polynomial in $|\mathcal{G}| \times |V^{\mathcal{F}_i}| \leq |\mathcal{G}|^2$. (Later, in Section 5, for "almost" approximation of a QPE, we will also use the fact that the only use of division gates in this formula is in cases where the denominator evaluates to $\mathbb{P}_b(I_{i,j})$ for some information set $I_{i,j}$.) ■

Let $h(x) = x + v(x)$, and let $h'(x) = x + v'(x)$. For each agent (i, j), and for fixed $x \in B$, consider the function $f_{i,j,x}(t) = \sum_{a \in \mathcal{A}_{i,j}} \max(h_{i,j,a}(x) - t, \epsilon)$. Likewise, for $x \in B^{>0}$, consider the function $f'_{i,j,x}(t) = \sum_{a \in \mathcal{A}_{i,j}} \max(h'_{i,j,a}(x) - t, \epsilon)$. Clearly, both $f_{i,j,x}(t)$ and $f'_{i,j,x}(t)$ are continuous, piecewise linear function of t. The functions are strictly decreasing as t ranges from $-\infty$, where $f_{i,j,x}(t) = +\infty$ (respectively, $f'_{i,j,x}(t) = +\infty$), up to $\max_{a \in \mathcal{A}_{i,j}} h_{i,j,a}(x) - \epsilon$ (respectively, $\max_{a \in \mathcal{A}_{i,j}} h'_{i,j,a}(x) - \epsilon$), where $f_{i,j,x}(t) = |\mathcal{A}_{i,j}| \cdot \epsilon$ (respectively, $f'_{i,j,x}(t) = |\mathcal{A}_{i,j}| \cdot \epsilon$). Since we have $|\mathcal{A}_{i,j}| \cdot \epsilon \leq m \cdot \epsilon < 1$, there is a unique value of t, which depends on x, call it $t_{i,j}(x)$ (call it, $t'_{i,j}(x)$, respectively) , where $f_{i,j,x}(t_{i,j}(x)) = 1$ (where $f'_{i,j,x}(t'_{i,j}(x)) = 1$).

The functions $F^\epsilon_{\mathcal{G}} : B \to B^\epsilon$ and $H^\epsilon_{\mathcal{G}} : B \to B^\epsilon$ are defined as follows. First we define $F^\epsilon_{\mathcal{G}}$:

$$F^\epsilon_{\mathcal{G}}(x)_{i,j,a} = \max(h_{i,j,a}(x) - t_{i,j}(x), \epsilon) \tag{5}$$

for every $i = 1, \ldots, n$, and $j \in [d_i]$, and $a \in \mathcal{A}_{i,j}$.

To define $H^\epsilon_{\mathcal{G}} : B \to B^\epsilon$, care is needed since $v'(x)_{i,j,a}$ is only defined for $x \in B^{>0}$. To address this, we use an auxiliary normalizing function. For $\epsilon > 0$, $\mathfrak{D}^\epsilon : B \to B^{>0}$, defined as follows:

$$\mathfrak{D}^\epsilon(x)_{i,j,a} = \frac{\max(x_{i,j,a}, \epsilon)}{\sum_{a' \in \mathcal{A}_{i,j}} \max(x_{i,j,a'}, \epsilon)}$$

$\mathfrak{D}^\epsilon$ clearly does map B to $B^{>0}$. Furthermore, importantly, note that for all $b' \in B^\epsilon$, $\mathfrak{D}^\epsilon(b') = b'$. We only use $\mathfrak{D}^\epsilon$ as a tool to ensure the function H^ϵ_G is defined for all $b \in B$. The range, and thus the fixed points, of H^ϵ_G lies within B^ϵ, and on B^ϵ the function $\mathfrak{D}^\epsilon(x)$ is the trivial identity function. We define $H^\epsilon_G : B \to B^\epsilon$ as follows:

$$H^\epsilon_{\mathcal{G}}(x)_{i,j,a} = \max(h'_{i,j,a}(\mathfrak{D}^\epsilon(x)) - t'_{i,j}(\mathfrak{D}^\epsilon(x)), \epsilon) \tag{6}$$

for every $i = 1, \dots, n$, and $j \in [d_i]$, and $a \in \mathcal{A}_{i,j}$.

From our choice of $t_{i,j}(x)$ and $t'_{i,j}(\mathfrak{D}^\epsilon(x))$, it follows that $\sum_{a \in \mathcal{A}_{i,j}} F^\epsilon_{\mathcal{G}}(x)_{i,j,a} = 1$ and also that $\sum_{a \in \mathcal{A}_{i,j}} H^\epsilon_{\mathcal{G}}(x)_{i,j,a} = 1$, for all $i \in [n]$ and $j \in [d_i]$. Thus, for any behavior profile, $x \in B$, we have $F^\epsilon_{\mathcal{G}}(x) \in B^\epsilon$ and $H^\epsilon_{\mathcal{G}}(x) \in B^\epsilon$. So both $F^\epsilon_{\mathcal{G}}$ and $H^\epsilon_{\mathcal{G}}$ indeed map B to B^ϵ, and since they are clearly also continuous maps, by Brouwer's theorem, they both have a fixed point in B^ϵ.[27]

Lemma 8 *For $0 < \epsilon < 1/m$:*

1. *Every fixed point of the function $F^\epsilon_{\mathcal{G}} : B \to B^\epsilon$ is an ϵ-PE of $\mathcal{AN}(\mathcal{G})$, and thus also of $\mathcal{G}$.*
2. *Every fixed point of the function $H^\epsilon_{\mathcal{G}} : B \to B^\epsilon$ is a ϵ-QPE of $\mathcal{G}$.*

Proof. The proof is essentially the same in both cases:

1. If x is a fixed point of $F^\epsilon_{\mathcal{G}}$, then $x \in B^\epsilon$ and $x_{i,j,a} = \max(x_{i,j,a} + v(x)_{i,j,a} - t_{i,j}(x), \epsilon)$ for all (i,j,a). Recall that $v(x)_{i,j,a} = U_i(x \mid \pi^a_{i,j}) = U_{i,j}(x \mid \pi^a_{i,j})$ is the expected payoff for agent (i,j) under profile $(x \mid \pi^a_{i,j})$.

   Note that the equation $x_{i,j,a} = \max(x_{i,j,a} + U_i(x \mid \pi^a_{i,j}) - t_{i,j}(x), \epsilon)$ implies that $U_i(x \mid \pi^a_{i,j}) = t_{i,j}(x)$ for all i,j,a such that $x_{i,j,a} > \epsilon$, and that $U_i(x \mid \pi^a_{i,j}) \leq t_{i,j}(x)$ for all i,j,a such that $x_{i,j,a} = \epsilon$. Consequently, by definition, x constitutes an ϵ-PE.

2. If x is a fixed point of $H^\epsilon_{\mathcal{G}}$, then $x \in B^\epsilon$, and thus $\mathfrak{D}^\epsilon(x) = x$. Thus, we have will $x_{i,j,a} = \max(x_{i,j,a} + v'(x)_{i,j,a} - t'_{i,j}(x), \epsilon)$ for all (i,j,a), where $v'(x)_{i,j,a} = \mathsf{K}^{j,a}_i(x)$.

   Note, again, that the equation $x_{i,j,a} = \max(x_{i,j,a} + \mathsf{K}^{j,a}_i(x) - t'_{i,j}(x), \epsilon)$ implies that $\mathsf{K}^{j,a}_i(x) = t'_{i,j}(x)$ for all i,j,a such that $x_{i,j,a} > \epsilon$, and that $\mathsf{K}^{j,a}_i(x) \leq t'_{i,j}(x)$ for all i,j,a such that $x_{i,j,a} = \epsilon$. Consequently, by definition, x constitutes an ϵ-QPE.

■

The following Lemma shows that we can implement the functions $F^\epsilon_{\mathcal{G}}(x)$ and $H^\epsilon_{\mathcal{G}}(x)$ by a circuit which has x and ϵ as inputs, by using sorting networks.

Lemma 9 *Given $\mathcal{G}$, we can construct in polynomial time a $\{+, *, \max\}$-circuit that computes the function $F^\epsilon_{\mathcal{G}}(x)$, where x and $\epsilon > 0$ are inputs to the circuit. Likewise, we can construct in P-time a $\{+, *, /, \max\}$-circuit that computes the function $H^\epsilon_{\mathcal{G}}(x)$, where x and $\epsilon > 0$ are inputs to the circuit.*

[27] The reason we specify the domain of these functions as B instead of B^ϵ is technical. To place the approximation problems for PE and QPE in FIXP_a, we shall need make $\epsilon > 0$ *very very* small, and we do so by using a polynomial sized algebraic circuit to define it. However, we shall also need the function domains to be definable by linear inequalities having encoding size only polynomial in $|\mathcal{G}|$. Both can be achieved by retaining the domain B.

Proof. We define the circuits for both $F_{\mathcal{G}}^{\epsilon}(x)$ and $H_{\mathcal{G}}^{\epsilon}(x)$ together, since they are defined very similarly.

Given a vector $x \in B$, and $\epsilon > 0$ as inputs, the respective circuits first compute $y = h(x) = x + v(x)$, and $y' = \mathfrak{D}^{\epsilon}(x) + v'(\mathfrak{D}^{\epsilon}(x))$. It follows from the definition of $v(x)$, $\mathfrak{D}^{\epsilon}(x)$, and $v'(x)$, and from Lemma 7, that both y and y' can be computed by a circuit using $\{+, *, /, \max\}$-gates which has size polynomial in $|\mathcal{G}|$. For each agent (i, j), let $y_{i,j}$ be the corresponding subvector of y induced by the (local) strategy of agent (i, j). Likewise, let $y'_{i,j}$ be the corresponding subvector of y'. Sort the vector $y_{i,j}$ (the vector $y'_{i,j}$) in decreasing order, and let $z_{i,j}$ (respectively, $z'_{i,j}$) be the resulting sorted vector, i.e. the components of $z_{i,j} = (z_{i,j,a_1}, \ldots, z_{i,j,a_{|\mathcal{A}_{i,j}|}})$ are the same as the components of $y_{i,j}$, but they are sorted (likewise for $z'_{i,j} = (z'_{i,j,a'_1}, \ldots, z'_{i,j,a'_{|\mathcal{A}_{i,j}|}}))$. In other words, we are assuming for convenience that $\mathcal{A}_{i,j} = \{a_1, \ldots, a_{|\mathcal{A}_{i,j}|}\}$ and that $z_{i,j,a_1} \geq z_{i,j,a_2} \geq \ldots \geq z_{i,j,a_{|\mathcal{A}_{i,j}|}}$, and likewise that $\mathcal{A}_{i,j} = \{a'_1, \ldots, a'_{|\mathcal{A}_{i,j}|}\}$ and that $z'_{i,j,a'_1} \geq z'_{i,j,a'_2} \geq \ldots \geq z'_{i,j,a'_{|\mathcal{A}_{i,j}|}}$, To obtain the sorted lists $z_{i,j}$ and $z'_{i,j}$, the respective circuits use a polynomial sized sorting network, for each (i, j) (see e.g. Knuth [17] for background on sorting networks). For each comparator gate of the sorting network we use a max and a min gate.

Using this, for each agent (i, j), we compute $t_{i,j}(x)$ and $t'_{i,j}(\mathfrak{D}^{\epsilon}(x))$ as the following expressions:

$$t_{i,j}(x) := \max\{(1/l) \cdot ((\sum_{k=1}^{l} z_{i,j,a_k}) + (|\mathcal{A}_{i,j}| - l) \cdot \epsilon - 1) \mid l = 1, \cdots, |\mathcal{A}_{i,j}|\} \tag{7}$$

$$t'_{i,j}(\mathfrak{D}^{\epsilon}(x)) := \max\{(1/l) \cdot ((\sum_{k=1}^{l} z'_{i,j,a'_k}) + (|\mathcal{A}_{i,j}| - l) \cdot \epsilon - 1) \mid l = 1, \cdots, |\mathcal{A}_{i,j}|\} \tag{8}$$

We will show below that this expression does indeed give the correct value of $t_{i,j}(x)$. The proof for $t'_{i,j}(\mathfrak{D}^{\epsilon}(x))$ is virtually identical, so we omit it.

We output $F_{\mathcal{G}}^{\epsilon}(x)_{i,j,a} = \max(y_{i,j,a} - t_{i,j}(x), \epsilon)$, and $H_{\mathcal{G}}^{\epsilon}(x)_{i,j,a} = \max(y'_{i,j,a} - t'_{i,j}(\mathfrak{D}^{\epsilon}(x)), \epsilon)$, for each $i = 1, \ldots, n$, $j \in [d_i]$, and $a \in \mathcal{A}_{i,j}$.

We now have to establish that $t_{i,j}(x)$, defined above, is the correct value. (Again, we forgo the proof for $t'_{i,j}(\mathfrak{D}^{\epsilon}(x))$, which is virtually identical.) Consider the function $f_{i,j,x}(t) = \sum_{a \in \mathcal{A}_{i,j}} \max(z_{i,j,a} - t, \epsilon)$ as t decreases from $z_{i,j,a_1} - \epsilon$ where the function value is at its minimum of $|\mathcal{A}_{i,j}| \cdot \epsilon$, down until the function reaches the value 1. In the first interval from $z_{i,j,a_1} - \epsilon$ to $z_{i,j,a_2} - \epsilon$ the function is $f_{i,j,x}(t) = z_{i,j,a_1} - t + (|\mathcal{A}_{i,j}| - 1) \cdot \epsilon$; in the second interval from $z_{i,j,a_2} - \epsilon$ to $z_{i,j,a_3} - \epsilon$ it is $f_{i,j,x}(t) = z_{i,j,a_1} + z_{i,j,a_2} - 2t + (|\mathcal{A}_{i,j}| - 2) \cdot \epsilon$, and so forth. In general, in the l-th interval, $f_{i,j,x}(t) = \sum_{k=1}^{l}(z_{i,j,a_k} - t) + (|\mathcal{A}_{i,j}| - l) \cdot \epsilon = \sum_{k=1}^{l} z_{i,j,a_k} - lt + (|\mathcal{A}_{i,j}| - l) \cdot \epsilon$. If the function reaches the value 1 in the l'th interval, then clearly $t_{i,j}(x) = ((\sum_{k=1}^{l} z_{i,j,a_k}) + (|\mathcal{A}_{i,j}| - l) \cdot \epsilon - 1)/l$.

In that case, furthermore for $k' < l$, we have $\sum_{k=1}^{k'}(z_{i,j,a_k} - t_i) + (|\mathcal{A}_{i,j}| - k') \cdot \epsilon \leq \sum_{k=1}^{l}(z_{i,j,a_k} - t_{i,j}(x)) + (|\mathcal{A}_{i,j}| - l) \cdot \epsilon = 1$, because in that case we know $(z_{i,j,a_k} - t_{i,j}(x)) \geq \epsilon$ for every $a \in \{1, \ldots, l\}$. Therefore, in this case $((\sum_{k=1}^{k'} z_{i,j,a_k}) + (|\mathcal{A}_{i,j}| - k') \cdot \epsilon - 1)/k' \leq t_{i,j}(x)$. On the other hand, if $l < |\mathcal{A}_{i,j}|$, then for $k' > l$ we have $t_i \geq z_{i,j,a_{k'}} - \epsilon$, i.e., $z_{i,j,a_{k'}} - t_i \leq \epsilon$, and thus for all $k' > l$, $k' \leq |\mathcal{A}_{i,j}|$, we have $\sum_{k=1}^{k'}(z_{i,j,a_k} - t_{i,j}(x)) + (|\mathcal{A}_{i,j}| - k') \cdot \epsilon \leq \sum_{k=1}^{l}(z_{i,j,a_k} - t_{i,j}(x)) + (|\mathcal{A}_{i,j}| - l) \cdot \epsilon = 1$. Thus again $((\sum_{k=1}^{k'} z_{i,j,a_k}) + (|\mathcal{A}_{i,j}| - k') \cdot \epsilon - 1)/k' \leq t_{i,j}(x)$. Therefore, $t_{i,j}(x) = \max\{(1/l) \cdot ((\sum_{k=1}^{l} z_{i,j,a_k}) + (|\mathcal{A}_{i,j}| - l) \cdot \epsilon - 1) | l = 1, \cdots, |\mathcal{A}_{i,j}|\}$. ■

Lemma 8 and Lemma 9 together immediately imply Theorem 6.

4 Approximating an SE, PE, and QPE is FIXP_a-complete

In this section we exploit the algebraically defined function $F^{\epsilon}_{\mathcal{G}}(x)$ and $H^{\epsilon}_{\mathcal{G}}(x)$ for a EFGPR, $\mathcal{G}$, with input parameter $\epsilon > 0$, devised in the previous section for ϵ-PEs and ϵ-QPEs, and we construct a "small enough" $\epsilon^* > 0$ (using an algebraic circuit, given $\delta > 0$) such that any fixed point of $F^{\epsilon^*}_{\mathcal{G}}(x)$ is a ϵ^*-PE which is also δ-close to an actual PE of $\mathcal{G}$ (in ℓ_∞ distance), and likewise any fixed point of $H^{\epsilon^*}_{\mathcal{G}}(x)$ is a ϵ^*-QPE which is also δ-close to an actual QPE. In this way, we show that approximating a PE, and a QPE, to within given desired precision, $\delta > 0$, for a given EFGPR is FIXP_a-complete. Since PE constitutes a refinement of NE and of SGPE, this of course immediately implies that approximating a NE or SGPE is also FIXP_a-complete (cf. [10]). Likewise, since QPE constitutes a refinement of NF-PE, this also implies that approximating a NF-PE is FIXP_a-complete.

For SEs, we then also show that for any such ϵ^*-PE, b'', if $\mu^{b''}$ is the unique belief system generated by b'' then $(b'', \mu^{b''})$ is δ-close to an actual SE of $\mathcal{G}$ (again in ℓ_∞). Furthermore, using $F^{\epsilon^*}_{\mathcal{G}}(x)$, we define an auxiliary fixed point function $G^{\epsilon^*}_{\mathcal{G}}(x, z)$ with domain $B \times \mathfrak{B}$, such that the Brouwer fixed points of $G^{\epsilon^*}_{\mathcal{G}}$ are pairs $(b'', \mu^{b''})$, where b'' is a ϵ^*-PE and $\mu^{b''}$ is the belief system that it generates. In this way, we show that approximating a SE (including its belief system) to within given desired precision $\delta > 0$, for a given EFGPR, is also FIXP_a-complete.

Theorem 10 *Given as input a EFGPR, $\mathcal{G}$, and a rational $\delta > 0$:*

1. *The problem of computing a vector $b' \in B$ such that there is a PE (or NE or SGPE), b^*, of $\mathcal{G}$, with $\|b' - b^*\|_\infty < \delta$, is FIXP_a-complete.*

2. *The problem of computing a vector $b' \in B$ such that there is a QPE (or NF-PE), b^*, of $\mathcal{G}$, with $\|b' - b^*\|_\infty < \delta$, is FIXP_a-complete.*

3. *The problem of computing a vector $b' \in B$ and a belief system μ' such that there is a SE, (b^*, μ^*) of $\mathcal{G}$, with $\|(b', \mu') - (b^*, \mu^*)\|_\infty < \delta$, is FIXP_a-complete.*

Note that FIXP_a-hardness for these problems follows from the fact that we can encode any NFG, Γ, as an EFGPR, $\mathcal{E}(\Gamma)$, with not much larger encoding size, and from the fact that approximating a NE within desired precision for n-player NFGs is FIXP_a-hard, as shown in [13]. The FIXP_a-hardness of approximating a SGPE, PE, QPE, NF-PE, and SE, then follows because we know that these constitute refinements of NE. Thus, we only need to prove containment in FIXP_a. Our proofs follow closely some of the proofs in [12] used for characterizing the complexity of approximating a PE for NFGs. Although very similar, our proof differs in some details (especially for sequential equilibrium). So, both for clarity and in order to be self-contained, we provide detailed proofs.

Before we prove Theorem 10, we need some Lemmas. The following is a special case of a general paradigm noted by Anderson [1].

Lemma 11 *For any fixed EFGPR, $\mathcal{G}$, and any $\delta > 0$, there is an $\epsilon > 0$, so that any ϵ-(Q)PE, b', of $\mathcal{G}$ has ℓ_∞-distance at most δ from some (Q)PE of $\mathcal{G}$, and furthermore, if $\mu^{b'}$ denotes the belief system generated by b', then $(b', \mu^{b'})$ has ℓ_∞-distance at most δ from some SE of $\mathcal{G}$.*

Proof. Assume to the contrary that there is a EFGPR, $\mathcal{G}$, and a $\delta > 0$ so that for all $\epsilon > 0$, there is an ϵ-(Q)PE, b^ϵ of $\mathcal{G}$ so that there is no (Q)PE in the δ-neighborhood (with respect to ℓ_∞) of b^ϵ or that there is no SE in the δ-neighborhood (with respect to ℓ_∞) of $(b^\epsilon, \mu^{b^\epsilon})$, where μ^{b^ϵ} is the belief system generated by b^ϵ.

Consider the sequence of assessments $(b^{1/n}, \mu^{b^{1/n}})_{n\in\mathbb{N}}$. Since this is a sequence in a compact space (namely, the direct product of the space of behavior profiles and the space of belief systems), it has a limit point (b^*, μ^*). But then b^* is a (Q)PE of $\mathcal{G}$, by definition, since each $b^{1/n}$ is a $1/n$-(Q)PE. But this contradicts the statement that there is no (Q)PE in a δ-neighborhood of any of the behavior profiles $b^{1/n}$. Furthermore, it follows from Proposition 3 (Part 3.) that (b^*, μ^*) is a SE. But this contradicts the statement that there is no SE in a δ-neighborhood of any of the assessments $(b^{1/n}, \mu^{b^{1/n}})$. ∎

A priori, we have no bound on ϵ, but we can use results in real algebraic geometry [3, 4] to obtain a specific bound. We first do this for PE and SE:

Lemma 12 *There is a constant c, so that for all integers $n, m, k, M \in \mathbb{N}$ and $\delta \in \mathbb{Q}_+$, the following holds. Let $\epsilon \leq \min(\delta, 1/(M^{\mathtt{h}^{\mathcal{G}}+1}))^{m^{cm^3}}$. For any n-player EFGPR, $\mathcal{G}$, with a combined total of m pure local strategies for all players in the game, with game tree T having height $\mathtt{h}^{\mathcal{G}}$, and with M a positive integer which is at least as large as any (by assumption, necessarily positive) integer payoff of $\mathcal{G}$ and such that $p_u(a) > 1/M$, for every $u \in P_0$ and every $a \in \mathtt{Act}(u)$. Then any ϵ-PE, b^ϵ, of $\mathcal{G}$ has ℓ_∞-distance at most δ from some PE of $\mathcal{G}$, and furthermore if μ^{b^ϵ} is the belief system generated by b^ϵ, then $(b^\epsilon, \mu^{b^\epsilon})$ has ℓ_∞-distance at most δ from some SE of $\mathcal{G}$.*

Proof. The proof involves constructing formulas in the first order theory of real numbers, which formalize the statement of Lemma 11, with δ being "hardwired" as a constant and ϵ being the only free variable. Then, we apply *quantifier elimination* to these formulas. This leads to a quantifier free statement to which we can apply standard theorems bounding the size of an instantiation of the free variable ϵ making the formula true. We shall apply and refer to theorems in the monograph of Basu, Pollack and Roy [3, 4]. Note that we specifically refer to theorems and page numbers of the online edition [4]; these are in general different from the printed edition [3].

First-order formula for an extensive form ϵ-perfect equilibrium and for the belief system it generates: Let EPS-PE-BS(x, z, ϵ) be the quantifier-free first-order formula, with free variables $x \in \mathbb{R}^m$, $z \in \mathbb{R}^{|\mathbb{W}\setminus P_0|}$, and $\epsilon \in \mathbb{R}$, defined by the conjunction of the following formulas, which together express the fact that x is a behavior profile that is an extensive form ϵ-PE of the given EFGPR, $\mathcal{G}$, and that z is the (unique) belief system generated by x:

$$x_{i,j,a} > 0, \quad \text{for } i \in [n], j \in [d_i], \text{ and } a \in \mathcal{A}_{i,j} \;\;,$$

$$\sum_{a\in\mathcal{A}_{i,j}} x_{i,j,a} = 1, \quad \text{for } i \in [n] \text{ and } j \in [d_i] \;\;,$$

$$\Big(U_i(x \mid \pi^a_{i,j}) \geq U_i(x \mid \pi^{a'}_{i,j})\Big) \vee (x_{i,j,a} \leq \epsilon)\,, \quad \text{for } i \in [n],\, j \in [d_i],\, \text{ and } a, a' \in \mathcal{A}_{i,j} \;\;,$$

$$z_u \cdot \mathbb{P}_x(I_{i,j}) = \mathbb{P}_x(u), \quad \text{for all } u \in V \text{ where } u \in I_{i,j} \text{ for } i \in [n] \text{ and } j \in [d_i].$$

Note that by Proposition 2, $\mathbb{P}_x(I_{i,j})$ and $\mathbb{P}_x(u)$ are expressible as multilinear polynomials in the variables x (whose encoding size is polynomial in $|\mathcal{G}|$).

First-order formula for perfect equilibrium and sequential equilibrium: Let PE-SE(x, z) denote the following first-order formula with free variables $x \in \mathbb{R}^m$, and $z \in \mathbb{R}^{|\mathbb{W}\setminus P_0|}$, expressing that x is a behavior profile that is a PE of $\mathcal{G}$, and that z is a belief system such that (x, z) is a SE of $\mathcal{G}$:

$$\forall \epsilon > 0\, \exists x' \in \mathbb{R}^m\, \exists z' \in \mathbb{R}^{|\mathbb{W}\setminus P_0|} : \text{EPS-PE-BS}(x', z', \epsilon) \wedge \|x - x'\|^2 < \epsilon \wedge \|z - z'\|^2 < \epsilon \ .$$

First-order formula for "almost implies near" statement: Given a *fixed* $\delta > 0$ let PE-SE-bound$_\delta(\epsilon)$ denote the following first-order formula with free variable $\epsilon \in \mathbb{R}$, denoting that any ϵ-perfect equilibrium, x, of $\mathcal{G}$ is δ-close to a PE (in ℓ_2-distance, and therefore also in ℓ_∞-distance), and likewise that if z is the belief system generated by x, then (x, z) is δ-close to a SE:

$$\forall x \in \mathbb{R}^m\, \forall z \in \mathbb{R}^{|\mathbb{W}\setminus P_0|}\ \ \exists x^* \in \mathbb{R}^m\, \exists z^* \in \mathbb{R}^{|\mathbb{W}\setminus P_0|} :$$
$$(\epsilon > 0) \wedge \big(\neg\, \text{EPS-PE-BS}(x, z, \epsilon) \vee \big(\text{PE-SE}(x^*, z^*) \wedge \|x - x^*\|^2 < \delta^2 \wedge \|z - z^*\|^2 < \delta^2\big)\big) \ .$$

Suppose $\delta^2 = 2^{-k}$ and that $M = 2^\tau$ is a positive integer that satisfies the conditions in the statement of the Lemma. Then for this formula we have

- The total degree of all involved polynomials is at most $\max(2, m)$.
- The bitsize of coefficients is at most $\max(k, \tau \cdot (\mathtt{h}^{\mathcal{G}} + 1))$.
- The number of free variables is 1.
- Since $|\mathbb{W} \setminus P_0| \le m$, converting to prenex normal form, the formula has 4 blocks of quantifiers, of sizes at most $2m$, $2m$, 1, $2m$, respectively.

We now apply quantifier elimination [4, Algorithm 14.6, page 555] to the formula PE-SE-bound$_\delta(\epsilon)$, converting it into an equivalent quantifier free formula PE-SE-bound$'_\delta(\epsilon)$ with a single free variable ϵ. This is simply a Boolean formula whose atoms are sign conditions on various polynomials in ϵ. The bounds given by [4] in association with Algorithm 14.6 imply that for this formula:

- The degree of all involved polynomials (which are univariate polynomials in ϵ) is: $$\max(2, m)^{O(m^3)} = m^{O(m^3)}.$$
- The bitsize of all coefficients is at most: $$\max(k, \tau \cdot (\mathtt{h}^{\mathcal{G}} + 1)) \max(2, m)^{O(m^3)} = \max(k, \tau \cdot (\mathtt{h}^{\mathcal{G}} + 1)) m^{O(m^3)}.$$

By Lemma 11, we know that there exists an $\epsilon > 0$ so that the formula PE-SE-bound$'_\delta(\epsilon)$ is true. We now apply Theorem 13.14 of [4, Page 521] to the set of polynomials that are atoms of PE-SE-bound$'_\delta(\epsilon)$ and conclude that PE-SE-bound$'_\delta(\epsilon^*)$ is true for some $\epsilon^* \ge 2^{-\max(k, \tau\cdot(\mathtt{h}^{\mathcal{G}}+1)) m^{\Omega(m^3)}}$. By the semantics of the formula PE-SE-bound$_\delta(\epsilon)$, we also have that PE-SE-bound$_\delta(\epsilon')$ is true for all $\epsilon' \le \epsilon^*$, and the statement of the lemma follows. ■

Proof of Theorem 10, parts (1.) and (3.). We shall combine the proofs of parts (1.) and (3.) of the Theorem together. To do so, we shall first define an auxiliary fixed point function $G^\epsilon_{\mathcal{G}}(x, z)$ defined in terms of $F^\epsilon_{\mathcal{G}}(x)$, such that the Brouwer fixed points of $G^\epsilon_{\mathcal{G}}$ are pairs $(b'', \mu^{b''})$, where b'' is a ϵ-PE and $\mu^{b''}$ is the belief system that it generates. Specifically, we define $G^\epsilon_{\mathcal{G}} : B \times \mathfrak{B} \to B^\epsilon \times \mathfrak{B}$ as follows: For all $(b, z) \in B \times \mathfrak{B}$, $G^\epsilon_{\mathcal{G}}(b, z) := (b', z')$ where $b'_{i,j,a} := F^\epsilon_{\mathcal{G}}(b)$, for all $i \in [n]$, $j \in [d_i]$ and $a \in \mathcal{A}_{i,j}$; and furthermore where $z'_u := \frac{\mathbb{P}_{b'}(u)}{\mathbb{P}_{b'}(I_{i_u,j_u})}$ for all $u \in \mathbb{W} \setminus P_0$, and where $u \in I_{i_u,j_u}$. Note in particular that, for all $u \in \mathbb{W} \setminus P_0$, we can express z'_u as a (efficiently algebraically encodable) rational function of b because, recalling from Proposition 2 that for all $V' \subseteq V$, there is a efficiently encodable polynomial $F_{V'}(x)$ such that for all $b \in B$ $F_{V'}(b) = \mathbb{P}_b(V')$ represents the realization probability of V', we have $z'_u := \frac{\mathbb{P}_{b'}(u)}{\mathbb{P}_{b'}(I_{i_u,j_u})} = \frac{F_u(F^\epsilon_\mathcal{G}(b))}{F_{I_{i_u,j_u}}(F^\epsilon_\mathcal{G}(b))}$.

Thus $G^\epsilon_\mathcal{G} : B \times \mathfrak{B} \to B^\epsilon \times \mathfrak{B}$ is a continuous map, and notably $G^\epsilon_\mathcal{G}$ is defined in the entire compact domain $B \times \mathfrak{B}$, because $b' := F^\epsilon_\mathcal{G}(b) \in B^\epsilon$ and thus the ratio $\frac{\mathbb{P}_{b'}(u)}{\mathbb{P}_{b'}(I_{i_u,j_u})}$ is always well defined (we never divide by 0, because all nodes have positive realization probability under a profile $b' \in B^\epsilon$, for all $\epsilon > 0$). Moreover, by definition of $G^\epsilon_\mathcal{G}$, for all $\epsilon > 0$, for any Brouwer fixed point $(b'', \mu'') \in B^\epsilon \times \mathfrak{B}$ of $G^\epsilon_\mathcal{G}$, b'' must be a ϵ-PE of $\mathcal{G}$ and μ'' must be the unique belief system $\mu^{b''}$ generated by b''.

We now prove that computing a PE to within desired precision is FIXP_a-complete, and that computing a SE to within desired precision is FIXP_a-complete. Let $\mathcal{G}$ be the n-player EFGPR given as input. Let m be the combined total number of pure strategies for all players. Let M' be the minimum positive integer such that $p_u(a) > 1/M'$, for every $u \in P_0$ and every $a \in \texttt{Act}(u)$. Let $M \in \mathbb{N}$ be a positive integer which is the maximum of M' and any (by assumption, necessarily positive) integer payoff of $\mathcal{G}$. By the definition of FIXP_a, our task is the following. Given a parameter $\delta > 0$, we must construct a polytope P, a circuit $C : P \to P$, and a number δ', so that a δ'-approximation to a fixed point of C can be efficiently transformed into δ-approximation of a PE of $\mathcal{G}$, and a δ'-approximation of a fixed point of C can also be efficiently transformed into a δ-approximation of a SE of $\mathcal{G}$. In fact, we shall let $\delta' = \delta/2$ and ensure that δ'-approximations to fixed points of C yield both a δ-approximation of a PE and a δ-approximation of a SE of $\mathcal{G}$. The polytope P is simply the polytope $B \times \mathfrak{B}$, i.e., the cartesian product of the space of behavior profiles of $\mathcal{G}$ and the space of belief systems; clearly we can output the inequalities defining this polytope in polynomial time. The circuit C is the following: We construct the circuit for the function $G^\epsilon_\mathcal{G}$ above. Then, we construct a circuit for the number $\epsilon^* = \min(\delta/2, M^{-h^\mathcal{G}})^{2^{\lceil cm^3 \lg m \rceil}} \leq \min(\delta/2, M^{-h^\mathcal{G}})^{m^{cm^3}}$, where c is the constant of Lemma 12: The circuit simply repeatedly squares the number $\min(\delta/2, M^{-h^\mathcal{G}})$ (which is a rational constant that can be computed in P-time given the input $\mathcal{G}$) and thereby consists of exactly $\lceil cm^3 \lg m \rceil$ multiplication gates, i.e., a polynomially bounded number. We then plug in the circuit for ϵ^* for the parameter ϵ in the circuit for G^ϵ_Γ, obtaining the circuit C, which is obviously a circuit for $G^{\epsilon^*}_\Gamma$. Now, by the above, any fixed point (b'', μ'') of C on P is an ϵ^*-PE of $\mathcal{G}$. Therefore, by Lemma 12, in any fixed point (b'', μ'') of C, we know that b'' is both a ϵ^*-PE and a $\delta/2$-approximation (in ℓ_∞-distance) to a PE b^* of $\mathcal{G}$, and furthermore that μ'' is the unique belief system generated by b'', and that μ'' is a $\delta/2$-approximation (in ℓ_∞-distance) of a belief system μ^* such that (b^*, μ^*) is a SE of $\mathcal{G}$. Finally, by the triangle inequality, any $\delta' = \delta/2$-approximation (b', μ') to a fixed point (b'', μ'') of C on P is a $\delta/2 + \delta/2 = \delta$ approximation (in ℓ_∞) of some pair (b^*, μ^*), such that b^* is a PE of $\mathcal{G}$ and (b^*, μ^*) is a SE of $\mathcal{G}$. We have thus established Theorem 10, parts (1.) and (3.). ■

Next, we want to prove something analogous to Lemma 12, but for QPEs. In order to do so, we first need the following:

Proposition 13 *For any EFGPR, $\mathcal{G}$, with $i \in [n]$, $j \in [d_i]$, and any $a, a' \in \mathcal{A}_{i,j}$, the inequality $\mathsf{K}^{j,a}_i(x) < \mathsf{K}^{j,a'}_i(x)$ can be expressed as formula, $\Phi^{i,j,a,a'}_\mathcal{G}(x) \equiv \exists y \Psi^{i,j,a,a'}_\mathcal{G}(y, x)$, in the existential theory of reals, where $\Psi^{i,j,a,a'}_\mathcal{G}(y, x)$ is quantifier free, where the total degree of all polynomials involved in $\Psi^{i,j,a,a'}_\mathcal{G}(y, x)$ is 2, where the encoding size of $\Phi^{i,j,a}_\mathcal{G}(x)$ is polynomial in $|\mathcal{G}|$, and such* *that for all $b \in B^{>0}$, $\Phi_{\mathcal{G}}^{i,j,a,a'}(b)$ holds true iff $\mathsf{K}_i^{j,a}(b) < \mathsf{K}_i^{j,a'}(b)$.*

Proof. Note that $\mathsf{K}_i^{j,a}(x) < \mathsf{K}_i^{j,a}(x)$ is an inequality between two $\{+,-,*,/,\max\}$-formulas (over the variables x) of encoding size polynomial in $|\mathcal{G}|$. We will show that any such inequality, over any subset of Euclidean space where the formula is always well-defined (i.e., involves no division by 0), can be expressed by an existential theory of reals formula whose encoding size is polynomial in the original inequality (and thus polynomial in $|\mathcal{G}|$).

Specifically, suppose x is an m-vector of variables. By induction on the depth of any $\{+,-,*,/,\max\}$-formula, $\zeta(x)$, which is well-defined over the domain $B^{>0}$ (i.e., which involves no sub formula that performs a division by 0, when x is anywhere in that domain), we prove that there is a existential theory of reals formula $\Psi_\zeta(y_0, y, x)$, of size linear in the size of ζ, with auxiliary variable y_0 and a vector of auxiliary variables y, such that for all $x \in B^{>0}$, $\{y_0 \in \mathbb{R} \mid \exists y \Psi_\zeta(y_0, y, x)\} = \{\zeta(x)\}$. In other words, for the values x in the domain $B^{>0}$, the formula $\exists y \Psi_\zeta(y_0, y, x)$ "expresses" a unique value, $y_0 \in \mathbb{R}$, which is the same value as $\zeta(x)$.

The base case, when $\zeta(x)$ is a variable from x, or a rational constant, is trivial.

Inductively, suppose $\zeta(x) := \zeta_1(x) \odot \zeta_2(x)$, where $\odot \in \{+,-,*,/,\max\}$. By the inductive hypothesis, there is a formula $\exists y \Psi_{\zeta_1}(y_0, y, x)$ using which y_0 expresses $\zeta_1(x)$, and which has size linear in that of ζ_1, and likewise there is a formula $\exists y' \Psi_{\zeta_2}(y_0', y', x)$ using which y_0' expresses $\zeta_2(x)$, and which has size linear in that of ζ_2.

We construct a new formula $\exists y_0, y_0', y, y' \Psi_{zeta}(y_0'', y_0, y_0', y, y', x)$ as follows. If $\odot \in \{+,*,-\}$, then $\Psi_\zeta(y_0'', y_0, y_0', y, y', x) := (y_0'' = y_0 \odot y_0' \wedge \Psi_{\zeta_1}(y_0, y, x) \wedge \Psi_{\zeta_2}(y_0', y', x))$.

If $\odot \doteq /$, then $\Psi_\zeta(y_0'', y_0, y_0', y, y', x) := (y_0'' * y_0' = y_0 \wedge \Psi_{\zeta_1}(y_0, y, x) \wedge \Psi_{\zeta_2}(y_0', y', x))$.

If $\odot \doteq \max$, then $\Psi_\zeta(y_0'', y_0, y_0', y, y', x) := (y_0'' \geq y_0 \wedge y_0'' \geq y_0 \wedge (y_0'' \leq y_0 \vee y_0'' \leq y_0') \wedge \Psi_{\zeta_1}(y_0, y, x) \wedge \Psi_{\zeta_2}(y_0', y', x))$. (The case with $\odot \doteq \min$ is entirely similar and symmetric to the max case.)

Note that, by induction, the new formula $\exists y_0, y_0', y, y' \Psi_\zeta(y_0'', y_0, y_0', y, y', x)$ again has encoding size linear in the encoding size of $\zeta(x)$, and furthermore note that the total degree of all polynomials in $\Psi_\zeta(y_0'', y_0, y_0', y, y', x)$ remains 2.

Finally, for x in the domain $B^{>0}$, let $\mathsf{K}_i^{j,a}(x)$ be expressed by $\exists y \Psi_{\mathsf{K}_i^{j,a}}(y_0, y, x)$, and let $\mathsf{K}_i^{j,a'}(x)$ be expressed by $\exists y' \Psi_{\mathsf{K}_i^{j,a'}}(y_0', y', x)$. We can express the inequality $\mathsf{K}_i^{j,a}(x) < \mathsf{K}_i^{j,a'}(x)$ using the following existential theory of reals formula:

$$\Phi_{\mathcal{G}}^{i,j,a,a'}(x) := \exists y_0, y_0', y, y' \;(\; y_0 < y_0' \wedge \Psi_{\mathsf{K}_i^{j,a}}(y_0, y, x) \wedge \Psi_{\mathsf{K}_i^{j,a'}}(y_0', y', x) \;).$$

■

Lemma 14 *There is a polynomial $q(\cdot)$, such that, for any EFGPR, $\mathcal{G}$, and any $\delta = 2^{-k} > 0$, where k is a positive integer, for any $\epsilon \leq \frac{1}{2^{2^{q(|\mathcal{G}|+k)}}}$, any ϵ-QPE of $\mathcal{G}$ is δ-close (in ℓ_∞) to a QPE.*

Proof. The proof is entirely analogous to that of Lemma 12. We spell out the details for completeness.

First-order formula for ϵ-quasi-perfect equilibrium: Let EPS-QPE(x, ϵ) be the first-order formula (a *universal* formula in the theory of reals), with free variables $x \in \mathbb{R}^m$ and $\epsilon \in \mathbb{R}$, defined by the conjunction of the following formulas, which together express the fact that $x \in B^{>0}$ is a behavior profile that is an extensive form ϵ-QPE of the given EFGPR, $\mathcal{G}$:

$$x_{i,j,a} > 0, \quad \text{for } i \in [n], j \in [d_i], \text{ and } a \in \mathcal{A}_{i,j} \ ,$$
$$\sum_{a \in \mathcal{A}_{i,j}} x_{i,j,a} = 1, \quad \text{for } i \in [n] \text{ and } j \in [d_i] \quad ,$$
$$(\neg \Phi_i^{i,j,a,a'}(x)) \vee (x_{i,j,a} \leq \epsilon)\,, \quad \text{for } i \in [n],\ j \in [d_i],\ \text{and } a, a' \in \mathcal{A}_{i,j} \ .$$

Note that by Proposition 13, $\Phi_i^{i,j,a,a'}(x)$ is expressible as a existential formula in the theory of reals, whose size is polynomial in $|\mathcal{G}|$. Thus, the conjunction EPS-QPE(x, ϵ) of all of the above formulas is expressible as a universal formula in the theory of reals.

First-order formula for quasi-perfect equilibrium: Let QPE(x) denote the following first-order formula with free variables $x \in \mathbb{R}^m$, expressing that x is a behavior profile that is a QPE of $\mathcal{G}$:

$$\forall \epsilon > 0\ \exists x' \in \mathbb{R}^m : \text{EPS-QPE}(x', \epsilon) \wedge \|x - x'\|^2 < \epsilon \ .$$

First-order formula for "almost implies near" statement: Given a fixed $\delta > 0$, let QPE-bound$_\delta(\epsilon)$ denote the following first-order formula with free variable $\epsilon \in \mathbb{R}$, denoting that any ϵ-quasi-perfect equilibrium, x, of $\mathcal{G}$ is δ-close to a QPE:

$$\forall x \in \mathbb{R}^m\ \exists x^* \in \mathbb{R}^m :$$
$$(\epsilon > 0) \wedge \big(\neg\, \text{EPS-QPE}(x, \epsilon) \vee \big(\text{QPE}(x^*) \wedge \|x - x^*\|^2 < \delta^2\big)\big) \ .$$

Suppose $\delta^2 = 2^{-k}$, for some positive integer k, and let $q'(\cdot)$ be some fixed polynomial such that $\tau = q'(|\mathcal{G}|) + k$ is at least the maximum encoding size of any coefficient in any of the polynomials involved in QPE-bound$_\delta(\epsilon)$. (We know that such an explicit polynomial $q'(\cdot)$ exists, given the polynomial bounds as a function of $\mathcal{G}$ on the encoding size of the various parts of the formula QPE-bound$_\delta(\epsilon)$.)

- The total degree of all involved polynomials is at most 2.
- The bitsize of coefficients is at most τ.
- The number of free variables is 1.
- Converting to prenex normal form, the formula has 5 blocks of quantifiers, of sizes at most m, m, 1, m, and $q''(|\mathcal{G}|)$, for some fixed polynomial $q''(\cdot)$, respectively.

We now apply quantifier elimination [4, Algorithm 14.6, page 555] to the formula QPE-bound$_\delta(\epsilon)$, converting it into an equivalent quantifier free formula QPE-bound$'_\delta(\epsilon)$ with a single free variable ϵ. This yields Boolean formula whose atoms are sign conditions on various polynomials in ϵ. Since $m \leq |\mathcal{G}|$, the bounds given by [4] in association with Algorithm 14.6 imply that, for some fixed polynomial $q'''(\cdot)$, we have that in this formula:

- The degree of all involved polynomials (which are univariate polynomials in ϵ) is at most $2^{q'''(|\mathcal{G}|+k)}$.

- The bitsize of all coefficients is at most: $2^{q'''(|\mathcal{G}|+k)}$.

By Lemma 11, we know that there exists an $\epsilon > 0$ so that the formula QPE-bound$'_\delta(\epsilon)$ is true. We now apply Theorem 13.14 of [4, Page 521] to the set of polynomials that are atoms of QPE-bound$'_\delta(\epsilon)$ and conclude that QPE-bound$'_\delta(\epsilon^*)$ is true for some $\epsilon^* \geq 2^{-2^{q'''(|\mathcal{G}|+k)^2}}$. By the semantics of the formula QPE-bound$_\delta(\epsilon)$, we also have that QPE-bound$_\delta(\epsilon')$ is true for all positive $\epsilon' \leq \epsilon^*$, and the statement of the lemma follows. ■

Proof of Theorem 10, part (2.) The proof is completely analogous to the proof of parts (1.) and (3.). We use the algebraically defined functions $H^\epsilon_{\mathcal{G}} : B \to B^\epsilon$, which are parametrized by an input variable ϵ. We "instantiate" ϵ with $\epsilon^* = 2^{-2^{q'''(|\mathcal{G}|+k)^2}}$, where $k = \lceil -\log((\delta/2)^2) \rceil$. We know we can define ϵ^* using an algebraic circuit having encoding size $q'''(|\mathcal{G}| + k)^2$, by repeatedly squaring the rational number $(1/2)$, a total of $q'''(|\mathcal{G}|+k)^2$ times. We thus can construct an $\{+, -, *, /, \max, \min\}$-circuit $C(x)$, having encoding size polynomial in $|\mathcal{G}|$ and $\texttt{size}(\delta)$, which defines the function $H^{\epsilon^*}_{\mathcal{G}} : B \to B^{\epsilon^*}$ on the domain B, and such that every fixed point of $H^{\epsilon^*}_{\mathcal{G}}$ is a ϵ^*-QPE of $\mathcal{G}$, which by Lemma 14 is also $(\delta/2)$-close (in ℓ_∞) to an actual QPE. Thus, applying the triangle inequality, if we approximate a fixed point of $H^{\epsilon^*}_{\mathcal{G}}$ within ℓ_∞ distance $(\delta/2)$, we will have approximated a QPE of $\mathcal{G}$ within ℓ_∞ distance δ. This shows that δ-approximating a QPE, given $\mathcal{G}$ and given $\delta > 0$, is in FIXP_a. ■

5 Computing a δ-almost-ϵ-PE & δ-almost-ϵ-QPE is PPAD-complete.

In this section we again exploit the functions $F^\epsilon_{\mathcal{G}}(x)$ and $H^\epsilon_{\mathcal{G}}(x)$, for a EFGPR, $\mathcal{G}$, devised in Section 3 for ϵ-PEs and ϵ-QPEs. This time we do so in order to show that computing a δ-almost-ϵ-PE, given $\mathcal{G}$, and given $\delta > 0$ and $\epsilon > 0$ (both in binary), is PPAD-complete. We also show that the notion of δ-almost-ϵ-PE suitably "refines" δ-almost-SGPE (and thus also δ-almost-NE), and that as a consequence computing a δ-almost-SGPE (or a δ-almost-NE), given $\mathcal{G}$ and given $\delta > 0$ (in binary), is PPAD-complete ([10]). Furthermore, we also show computing a δ-almost-ϵ-QPE, given $\mathcal{G}$, and given $\delta > 0$ and $\epsilon > 0$ is PPAD-complete. Roughly speaking, we will establish these results by showing that (a): a δ-almost approximate fixed point of the function $F^\epsilon_{\mathcal{G}}(x)$ and the function $H^\epsilon_{\mathcal{G}}(x)$, provides a δ'-almost-ϵ'-PE, and respectively a δ'-almost-ϵ'-QPE, for suitable δ' and ϵ' that are linearly related to δ and ϵ; and that (b): computing a δ-almost approximate fixed point of the function $F^\epsilon_{\mathcal{G}}(x)$ and $H^\epsilon_{\mathcal{G}}(x)$ is in PPAD (and PPAD-complete).

We have not yet actually defined the "almost" relaxation for QPE, which we call δ-almost-ϵ-QPE. We do so now. For this, please recall the notation $\mathsf{K}^{j,a}_i(b)$ defined in section 3, which is the maximum conditional expected payoff to player i conditioned on reaching information set $I_{i,j}$, there playing pure action a, and thereafter (in information sets below $I_{i,j}$ in $\mathcal{F}_i$) player i playing so as to maximize this conditional expected payoff. For $\delta \geq 0$, a behavior profile $b \in B$ is called a *δ-almost ϵ-quasi-perfect equilibrium* (δ-almost-ϵ-QPE) of $\mathcal{G}$, if it is (a): *fully mixed*, $b \in B^{>0}$, and (b): for all players i, all $j \in [d_i]$, and all actions $a, a' \in \mathcal{A}_{i,j}$, if $\mathsf{K}^{j,a}_i(b) < \mathsf{K}^{j,a'}_i(b) - \delta$ then $b_{i,j}(a) \leq \epsilon$. Note that when $\delta = 0$ this definition is equivalent to ϵ-QPE (this is because for a *fully mixed* profile b, $\mathsf{K}^{j,a}_i(b) < \mathsf{K}^{j,a'}_i(b)$ holds *if and only if* $\max_{b'_i \in B_i} U_i(b \mid_{(i,j)} (b'_i \mid \pi^a_{i,j})) < \max_{b''_i \in B_i} U_i(b \mid_{(i,j)} (b''_i \mid \pi^{a'}_{i,j}))$.[28] Thus, our definition is a reasonable "almost" relaxation of ϵ-QPE.

[28]In fact, as noted earlier, van Damme [45] defines QPE using the strict inequalities $\mathsf{K}^{j,a}_i(b) < \mathsf{K}^{j,a'}_i(b)$ instead of

We will make crucial use of some results and definitions from [13], which we now recall. Note that the circuit defining $F^\epsilon_{\mathcal{G}}(x)$ associates a function $F^\epsilon_{\mathcal{G}} : B^\epsilon \to B^\epsilon$ with each given pair $\langle \mathcal{G}, \epsilon\rangle$, where the rational value $\epsilon > 0$ is given in binary as part of the input.[29] Thus $|\mathcal{G}| + \texttt{size}(\epsilon)$ is the encoding size of the input from which the algebraic circuit for $F^\epsilon_{\mathcal{G}}(x)$ is generated.

Following [13], we call the family of functions $\langle F^\epsilon_{\mathcal{G}}(x)\rangle_{\{\langle \mathcal{G},\epsilon\rangle\}}$, associated with input pairs $\langle \mathcal{G}, \epsilon\rangle$, *polynomially continuous* in their domain B^ϵ, if there is a polynomial $q(z)$ such that for all input pairs $\langle \mathcal{G}, \epsilon\rangle$, for every rational $\epsilon_1 > 0$, there is a rational $\delta_1 > 0$, such that $\texttt{size}(\delta_1) \leq q(|\mathcal{G}| + \texttt{size}(\epsilon) + \texttt{size}(\epsilon_1))$ and such that for all $b, b' \in B^\epsilon$:

$$\|b - b'\|_\infty < \delta_1 \implies \|F^\epsilon_{\mathcal{G}}(b) - F^\epsilon_{\mathcal{G}}(b')\|_\infty < \epsilon_1.$$

Again following [13], we call the family of functions $\langle F^\epsilon_{\mathcal{G}}(x)\rangle_{\{\langle \mathcal{G},\epsilon\rangle\}}$ associated with input instances $\langle \mathcal{G}, \epsilon\rangle$, *polynomially computable* if (a): the domain B^ϵ of the functions $F^\epsilon_{\mathcal{G}} : B^\epsilon \to B^\epsilon$ is a convex polytope described by a set of linear inequalities with rational coefficients that can be computed from the input $\langle \mathcal{G}, \epsilon\rangle$ in polynomial time (note that this is clearly always the case for B^ϵ, because $\epsilon > 0$ is part of the input), and (b): there is a polynomial $q(z)$ such that there is an algorithm that given $\langle \mathcal{G}, \epsilon\rangle$, and given a rational vector $b \in B^\epsilon$, computes $F^\epsilon_{\mathcal{G}}(b)$ (which is of course also a rational vector) in time $q(|\mathcal{G}| + \texttt{size}(\epsilon) + \texttt{size}(b))$. We need the following Lemma:

Lemma 15 *The family of functions $\langle F^\epsilon_{\mathcal{G}}(x)\rangle_{\{\langle \mathcal{G},\epsilon\rangle\}}$ for EFGPRs defined in Section 4 (equation (5)) is both (a.) polynomially computable and (b.) polynomially continuous.*

Proof.
(a.): First, we observe that the family of functions $\langle F^\epsilon_{\mathcal{G}}(x)\rangle_{\{\langle \mathcal{G},\epsilon\rangle\}}$ for EFGPRs is polynomially computable. This follows easily from the definition of $F^\epsilon_{\mathcal{G}}(x)$ given Section 4 and in equations (5) and (7). Specifically, given a rational vector $b \in B^\epsilon$, to compute $F^\epsilon_{\mathcal{G}}(b)$, we must first compute a vector $y := h(b) := b + v(b)$, where $v(b)_{i,j,a} := U_{i,j}(b \mid \pi^a_{i,j}) = U_i(b \mid \pi^a_{i,j})$. Note that, given a rational vector $b \in B^\epsilon$, each value $y_{i,j,a} = h(b)_{i,j,a} = b_{i,j,a} + U_i(b \mid \pi^a_{i,j})$ is clearly computable in P-time, because $U_i(x \mid \pi^a_{i,j})$ is given by a polynomial in x whose encoding size, as a sum of multilinear monomials, is polynomial in $|\mathcal{G}| + \texttt{size}(\epsilon)$. Note also that the encoding size of the resulting rational vector y is clearly polynomial in $|\mathcal{G}| + \texttt{size}(\epsilon) + \texttt{size}(b)$. Next, having computed the vector y, we must sort each subvector $y_{i,j}$, associated with agent (i,j), into a non-increasing sequence: $z_{i,j} = (z_{i,j,a_1}, z_{i,j,a_2}, \ldots, z_{i,j,a_{|\mathcal{A}_{i,j}|}})$. We can clearly do so in P-time. Next, for each agent (i,j), we can clearly compute $t_{i,j}(b)$ in P-time using the simple $\{\max, +\}$ formula over the sorted vector of inputs $z_{i,j}$ given in equation (7). Finally, having computed $t_{i,j}(b)$ and $y = h(b)$ in P-time, we have from equation (5) that $F^\epsilon_{\mathcal{G}}(b)_{i,j,a} = \max(h_{i,j,a}(b) - t_{i,j}(b), \epsilon)$. Thus we can compute $F^\epsilon_{\mathcal{G}}(b)$ in time polynomial in $|\mathcal{G}| + \texttt{size}(\epsilon) + \texttt{size}(b)$, given $\mathcal{G}$, $\epsilon > 0$, and any rational vector $b \in B^\epsilon$.
(b.): Next, we want to show that the function family $\langle F^\epsilon_{\mathcal{G}}(x)\rangle_{\{\langle \mathcal{G},\epsilon\rangle\}}$ for EFGPRs is polynomially continuous. We will in fact show that in the domain B^ϵ the function $F^\epsilon_{\mathcal{G}}(x)$ is Lipschitz continuous with Lipschitz constant $2^{q(|\mathcal{G}|+\texttt{size}(\epsilon))}$ (with respect to the ℓ_∞ norm), for some polynomial $q(\cdot)$. In other words, for all $b, b' \in B^\epsilon$, we have:

$$\|F^\epsilon_{\mathcal{G}}(b) - F^\epsilon_{\mathcal{G}}(b')\|_\infty \leq 2^{q(|\mathcal{G}|+\texttt{size}(\epsilon))} \cdot \|b - b'\|_\infty. \tag{9}$$

$\max_{b'_i \in B_i} U_i(b \mid_{(i,j)} (b'_i \mid \pi^a_{i,j})) < \max_{b''_i \in B_i} U_i(b \mid_{(i,j)} (b''_i \mid \pi^{a'}_{i,j}))$.

[29] In this section it will be more convenient to view the domain of the function $F^\epsilon_{\mathcal{G}}$ as B^ϵ, rather than B, because $\epsilon > 0$ will be explicitly given.

Of course, it immediate follows from (9) is that the family of functions $\langle F^{\epsilon}_{\mathcal{G}}(x)\rangle_{\{\langle \mathcal{G},\epsilon\rangle\}}$ is polynomially continuous: in the definition of polynomially continuity, take $\delta_1 := \frac{1}{2^{q(|\mathcal{G}|+\mathtt{size}(\epsilon))}} \cdot \epsilon_1$, it then follows from (9) that for all $b, b' \in B^{\epsilon}$, $\|b - b'\|_\infty < \delta_1 \implies \|F^{\epsilon}_{\mathcal{G}}(b) - F^{\epsilon}_{\mathcal{G}}(b')\|_\infty < \epsilon_1$. Furthermore, clearly $\mathtt{size}(\delta_1) \leq q^*(|\mathcal{G}| + \mathtt{size}(\epsilon) + \mathtt{size}(\epsilon_1))$, for some fixed polynomial $q^*(\cdot)$. So, we only need to establish (9).

Consider any $b, b' \in B^{\epsilon}$. First, let us bound $\|h(b) - h(b')\|_\infty$. Recall that $h_{i,j,a}(x) = x_{i,j,a} + U_i(x \mid \pi^a_{i,j})$. Moreover, we know by Proposition 2 that $U_i(x \mid \pi^a_{i,j})$ is given by an explicit polynomial (a weighted sum of multilinear monomials) in the variables x, with degree bounded by the height $h^{\mathcal{G}}$ of the game tree T, and with encoding size polynomial in $|\mathcal{G}|$.

First, consider any monomial $f(x) = \alpha \cdot x_{i_1} \dots x_{i_k}$. Note that in the domain $B^{\epsilon} \subseteq [0,1]^d$ (for a suitable dimension d), the monomial $f(x)$ is Lipschitz continuous with Lipschitz constant $|\alpha| k$ (with respect to the ℓ_∞ norm). To see this simple fact, note that for $b, b' \in B^{\epsilon}$, we have $|f(b) - f(b')| \leq |\alpha| |b_{i_1} \dots b_{i_k} - b'_{i_1} \dots b'_{i_k}|$. Furthermore, by induction on $k \geq 1$, we have that for $b, b' \in [0,1]^k$, $|b_1 \dots b_k - b'_1 \dots b'_k| \leq k\|b - b'\|_\infty$. The base case, $k = 1$, is trivial. For the inductive case, we have:

$$
\begin{aligned}
|b_1 \dots b_k - b'_1 \dots b'_k| &= |b_1 \dots b_k - b_1 b'_2 \dots b'_k + b_1 b'_2 \dots b'_k - b'_1 \dots b'_k| \\
&\leq |b_1 \dots b_k - b_1 b'_2 \dots b'_k| + |b_1 b'_2 \dots b'_k - b'_1 \dots b'_k| \\
&= |b_1| \cdot |b_2 \dots b_k - b'_2 \dots b'_k| + |b'_2 \dots b'_k| \cdot |b_1 - b'_1| \\
&\leq |b_1| \cdot (k-1)\|b - b'\|_\infty + |b_1 - b'_1| \cdot |b'_2 \dots b'_k| \qquad \text{(by inductive hypothesis)} \\
&\leq (k-1)\|b - b'\|_\infty + |b_1 - b'_1| \qquad \text{(because } |b_1| \in [0,1] \text{ and } |b'_2 \dots b'_k| \in [0,1]) \\
&\leq k\|b - b'\|_\infty.
\end{aligned}
$$

Now suppose that the polynomial $h_{i,j,a}(x) = x_{i,j,a} + U_i(x \mid \pi^a_{i,j})$ is the sum of $M_{i,j,a}$ weighted monomials, and that the maximum absolute value of a coefficient of any of the monomials is $A^{\max}_{i,j,a}$. Then by the above, for any $b, b' \in B^{\epsilon}$, we have $|h_{i,j,a}(b) - h_{i,j,a}(b')| \leq M_{i,j,a} \cdot A^{\max}_{i,j,a} \|b - b'\|_\infty$. Let $M^{\max} = \max_{i,j,a} M_{i,j,a}$, and let $A^{\max} = \max_{i,j,a} A^{\max}_{i,j,a}$. Then we have $\|h(b) - h(b')\|_\infty \leq M^{\max} \cdot A^{\max} \cdot \|b - b'\|_\infty$. Thus, clearly $h(x)$ is Lipschitz continuous in domain B^{ϵ}, with Lipschitz constant $M^{\max} \cdot A^{\max}$, which is clearly upper bounded by $2^{q(|\mathcal{G}|)}$ for some polynomial $q(\cdot)$.

Next, note that the sort function has Lipschitz constant 1, with respect to ℓ_∞. In other words, if $\mathtt{sort}(y)$ is a function that takes a vector $y \in \mathbb{R}^k$ as input, and yields its (non-increasing) sort, $\mathtt{sort}(y) \in \mathbb{R}^k$, then for all $y, y' \in \mathbb{R}^k$, $\|\mathtt{sort}(y) - \mathtt{sort}(y')\|_\infty \leq \|y - y'\|_\infty$.

For completeness, we provide a proof of this easy fact. Suppose for contradiction that $|\mathtt{sort}(y)_{i^*} - \mathtt{sort}(y')_{i^*}| = \|\mathtt{sort}(y) - \mathtt{sort}(y')\|_\infty > \|y - y'\|_\infty$, for some index $i^* \in [k]$. Define the permutations π and π' of $[k]$, such that for all $i \in [k]$, $\mathtt{sort}(y)_i = y_{\pi(i)}$ and $\mathtt{sort}(y')_i = y_{\pi'(i)}$. Suppose, wlog, that $y_{\pi(i^*)} = \mathtt{sort}(y)_{i^*} < \mathtt{sort}(y')_{i^*} = y'_{\pi'(i^*)}$. Since $|\{\pi(1), \dots, \pi(i^*)\}| = i^* > i^* - 1 = |\{\pi'(1), \dots, \pi'(i^* - 1)\}|$, there must exist an $r \in \{1, \dots, i^*\}$ such that $\pi(r) \in \{\pi'(i^*), \pi'(i^* + 1), \dots, \pi'(k)\}$. In other words, $y_{\pi(r)} \leq y_{\pi(i^*)} = \mathtt{sort}(y)_{i^*} < \mathtt{sort}(y')_{i^*} = y'_{\pi'(i^*)} \leq y'_{\pi(r)}$. Thus $\|\mathtt{sort}(y) - \mathtt{sort}(y')\|_\infty = |\mathtt{sort}(y)_{i^*} - \mathtt{sort}(y')_{i^*}| \leq |y'_{\pi(r)} - y_{\pi(r)}| \leq \|y' - y\|_\infty$.

Note also that the composition $f_1(f_2(x))$ of Lipschitz continuous functions $f_1(y)$ and $f_2(x)$, where $f_1(y)$ has Lipschitz constant β_1 and $f_2(x)$ has Lipschitz constant β_2 (both with respect to the ℓ_∞ norm), is Lipschitz continuous with constant $\beta_1 \cdot \beta_2$ (again with respect to ℓ_∞).

Now, consider $t_{i,j}(x)$ as defined by equation (7). The expression defining $t_{i,j}(x)$ is a maximum over linear (affine) expressions (using ϵ as a constant) with at most $|\mathcal{A}_{i,j}|$ terms over the sorted vector of variables $z_{i,j}$. Since the max function has Lipschitz constant 1 (it is just a component of the sort function), it follows that for all $b, b' \in B^{\epsilon}$, we have $\|t_{i,j}(b) - t_{i,j}(b')\|_\infty \leq 2^{q'(|\mathcal{G}|+\texttt{size}(\epsilon))} \cdot \|b - b'\|_\infty$ for some polynomial $q'(\cdot)$.

Finally, since we have $F^{\epsilon}_{\mathcal{G}}(x)_{i,j,a} = \max(h_{i,j,a}(x) - t_{i,j}(x), \epsilon)$, and since max has Lipschitz constant 1, and since the sum of two Lipschitz functions with Lipschitz constant β_1 and β_2 is a Lipschitz function with Lipschitz constant $\leq \beta_1 + \beta_2$, we are done: there is a polynomial $q(\cdot)$ such that for all $b, b' \in B^{\epsilon}$,

$$\|F^{\epsilon}_{\mathcal{G}}(b) - F^{\epsilon}_{\mathcal{G}}(b')\|_\infty \leq 2^{q(|\mathcal{G}|+\texttt{size}(\epsilon))} \cdot \|b - b'\|_\infty. \quad \blacksquare$$

In fact, let us remark that Lemma 15 is a special case of a more general fact, namely that function families defined by $\{+, *, \max, \texttt{sort}\}$-*formulas* whose encoding size is polynomial in the input instance, over a bounded domain such as B^{ϵ}, are necessarily polynomially computable and polynomially continuous. The proof of the next lemma will argue this more explicitly.

Lemma 16 *The family of functions $\langle H^{\epsilon}_{\mathcal{G}}(x)\rangle_{\{\langle \mathcal{G},\epsilon\rangle\}}$ for EFGPRs defined in Section 4 (equation (6)) is both (a.) polynomially computable and (b.) polynomially continuous.*

Proof. (a.): First, we again observe that the family of functions $\langle H^{\epsilon}_{\mathcal{G}}(x)\rangle_{\{\langle \mathcal{G},\epsilon\rangle\}}$ for EFGPRs is polynomially computable over the corresponding domain B^{ϵ}. This again follows easily from the definition of $H^{\epsilon}_{\mathcal{G}}(x)$ given in Section 4, in equations (6) and in the dynamic program (4) defining $\mathsf{K}^{j,a}_i(x)$. Specifically, given a rational vector $b \in B^{\epsilon}$, to compute $H^{\epsilon}_{\mathcal{G}}(b)$, noting that $\mathfrak{D}^{\epsilon}(b) = b$, we must first compute a vector $y' := h'(b) := b + v'(b)$, where $v'(b)_{i,j,a} := \mathsf{K}^{j,a}_i(b)$. We know from the dynamic program given in (4) that given $\mathcal{G}$ and $b \in B^{\epsilon}$, we can compute $v'(b)_{i,j,a}$ in time polynomial in $|\mathcal{G}| + \texttt{size}(\epsilon) + \texttt{size}(b)$, for all i, j, and a. In particular, it is important to emphasize that $\texttt{size}(\mathsf{K}^{j,a}_i(b))$ remains polynomial in $|\mathcal{G}| + \texttt{size}(\epsilon) + \texttt{size}(b)$, and so do the sizes of all the intermediate rational numbers computed by subformulas of $\mathsf{K}^{j,a}_i(b)$. This is not only because the formula has only polynomial size, but also because, importantly, the special kind of $\{+, -, /, \max, \min, \texttt{sort}\}$-formula defining $K^{j,a}_i(b)$, given in (4), has the property that the only occurrences of *division* in the formula occur when the denominator of the division operation evaluates to $\mathbb{P}_b(I_{i,j})$ for some information set $I_{i,j}$. But the probability $\mathbb{P}_b(I_{i,j})$, for any $b \in B^{\epsilon}$ is at least $\epsilon^{h^{\mathcal{G}}}$. Note that $\texttt{size}(\epsilon^{h^{\mathcal{G}}}) \leq h^{\mathcal{G}} \cdot \texttt{size}(\epsilon)$. This ensures that the rational values arising as the result of such division gates in the formula for $K^{j,a}_i(b)$ always have an encoding size that is polynomial in $|\mathcal{G}| + \texttt{size}(\epsilon) + \texttt{size}(b)$. It follows, by an easy induction on the size m of a subformula, that the encoding size of the value computed by a subformula of size m has encoding size polynomial in $m \cdot (|\mathcal{G}| + \texttt{size}(\epsilon) + \texttt{size}(b))$. Since m itself is bounded by a polynomial in $|\mathcal{G}| + \texttt{size}(\epsilon) + \texttt{size}(b)$, this means all values computed in the formula have encoding size bounded by a polynomial in $|\mathcal{G}| + \texttt{size}(\epsilon) + \texttt{size}(b)$. We can thus also compute $h'(b)_{i,j,a}$ in time polynomial in $|\mathcal{G}| + \texttt{size}(\epsilon) + \texttt{size}(b)$. Likewise, computing $t'_{i,j}(b)$ is easily done in time polynomial in $|\mathcal{G}| + \texttt{size}(\epsilon) + \texttt{size}(b)$, using sorting. Thus $H^{\epsilon}_{\mathcal{G}}(b)$ can be computed in time polynomial in $|\mathcal{G}| + \texttt{size}(\epsilon) + \texttt{size}(b)$. Thus we can compute $H^{\epsilon}_{\mathcal{G}}(b)$ in time polynomial in $|\mathcal{G}| + \texttt{size}(\epsilon) + \texttt{size}(b)$, given $\mathcal{G}$, $\epsilon > 0$, and any rational vector $b \in B^{\epsilon}$.

(b.) We now argue that the family of functions $\langle H^{\epsilon}_{\mathcal{G}}(x)\rangle_{\{\langle \mathcal{G},\epsilon\rangle\}}$ is polynomially continuous over the domain B^{ϵ}. We will again actually show that the functions $H^{\epsilon}_{\mathcal{G}}(x)$ are Lipschitz continuous, with a Lipschitz constant of the form $2^{q(|\mathcal{G}|+\texttt{size}(\epsilon))}$, for some polynomial $q(\cdot)$, over domain B^{ϵ}. Just as in Lemma 15, this implies polynomial continuity.

The proof is again similar to the case $F^{\epsilon}_{\mathcal{G}}(x)$. We noted already, after the proof of Lemma 15, that an adaptation of that proof shows that *any* such function that can be defined by a $\{+, *, \max, \texttt{sort}\}$-*formula* and has encoding size polynomial in $|\mathcal{G}| + \texttt{size}(\epsilon)$ is polynomially continuous over the domain B^{ϵ}. We will establish a more direct version of this fact here. $H^{\epsilon}_{\mathcal{G}}(x)$ is defined by a $\{+, *, /, \max, \texttt{sort}\}$-formula, meaning it also involves division. However, in the case of $H^{\epsilon}_{\mathcal{G}}(x)$ we furthermore have the fact that the only use of division is inside subformulas which compute $\mathbb{P}_b(u \mid I_{i,j}) = \frac{\mathbb{P}_b(u)}{\mathbb{P}_b(I_{i,j})}$, for some information set $I_{i,j}$ and some node $u \in I_{i,j}$. Furthermore, we also see easily by inspection of $H^{\epsilon}_{\mathcal{G}}(x)$ that, for all $b \in B^{\epsilon}$, and for *every* subformula $f_1(x)$ of the formula for $H^{\epsilon}_{\mathcal{G}}(x)$, we have $\max_{b \in B^{\epsilon}} |f_1(b)| \leq 2^{q''(|\mathcal{G}|+\texttt{size}(\epsilon))}$ for some fixed polynomial $q''(\cdot)$ which is also independent of the subformula. We will use both of these facts.

Now, for any two subformulas $f_1(x)$ and $f_2(x)$ of $H^{\epsilon}_{\mathcal{G}}(x)$, suppose $f_1(x)$ ($f_2(x)$) has Lipschitz constant β_1 (β_2), with respect to the ℓ_∞ norm, i.e., that for $k \in \{1, 2\}$, if for all $b, b' \in B^{\epsilon}$ we have $|f_k(b) - f_k(b')| < \beta_k \|b - b'\|_\infty$, then:

1. $f_1(x) \cdot f_2(x)$ has Lipschitz constant at most $2^{q''(|\mathcal{G}|+\texttt{size}(\epsilon))} \cdot (\beta_1 + \beta_2)$. To see this, note that for all $b, b' \in B^{\epsilon}$ we have:

$$\begin{aligned} |f_1(b) \cdot f_2(b) - f_1(b') \cdot f_2(b')| &= |f_1(b) \cdot (f_2(b) - f_2(b')) + f_2(b')(f_1(b) - f_1(b'))| \\ &\leq |f_1(b)| \cdot |f_2(b) - f_2(b')| + |f_2(b)| \cdot |f_1(b) - f_1(b')| \\ &\leq 2^{q''(|\mathcal{G}|+\texttt{size}(\epsilon))} (\beta_1 + \beta_2) \cdot \|b - b'\|_\infty \end{aligned}$$

2. $f_1(x) + f_2(x)$ has Lipschitz constant at most $\beta_1 + \beta_2$. (This is obvious.)

3. $\max(f_1(x), f_2(x))$ has Lipschitz constant at most $\max(\beta_1, \beta_2)$. This follows immediately from the more general fact (established in the proof of Lemma 15) that the $\texttt{sort}$ function has Lipschitz constant 1 (under the ℓ_∞ norm), since $\texttt{sort}(y)_1 = \max_i y_i$. More directly (and repeating some the same arguments), we have:
$|\max(f_1(b), f_2(b)) - \max(f_1(b'), f_2(b'))| \leq \max(|f_1(b) - f_1(b')|, |f_2(b) - f_2(b')|) \leq \max(\beta_1, \beta_2) \cdot \|b - b'\|_\infty$. To see why the first inequality holds, assume w.l.o.g. that $\max(f_1(b), f_2(b)) \geq \max(f_1(b'), f_2(b'))$, and that $f_1(b) \geq f_2(b)$. Then, if $f_1(b') \geq f_2(b')$ we have $|\max(f_1(b), f_2(b)) - \max(f_1(b'), f_2(b'))| = |f_1(b) - f_1(b')|$. Otherwise, if $f_1(b') < f_2(b')$, then $|\max(f_1(b), f_2(b)) - \max(f_1(b'), f_2(b'))| = |f_1(b) - f_2(b')| < |f_1(b) - f_1(b')|$, since $f_1(b) \geq f_2(b')$. Thus, w.l.o.g., in all cases, $|\max(f_1(b), f_2(b)) - \max(f_1(b'), f_2(b'))| \leq \max(|f_1(b) - f_1(b')|, |f_2(b) - f_2(b')|)$.

Next, observe that for x in the domain B^{ϵ} the functions $\frac{\mathbb{P}_x(u)}{\mathbb{P}_x(I_{i,j})}$ have Lipschitz constant $2^{q'(|\mathcal{G}|+\texttt{size}(\epsilon))}$, for some fixed polynomial $q'(\cdot)$. This holds because for all i, j and u, and for all $b, b' \in B^{\epsilon}$, we have:

$$
\begin{aligned}
|\frac{\mathbb{P}_b(u)}{\mathbb{P}_b(I_{i,j})} - \frac{\mathbb{P}_{b'}(u)}{\mathbb{P}_{b'}(I_{i,j})}| &= |\frac{\mathbb{P}_{b'}(I_{i,j}) \cdot \mathbb{P}_b(u) - \mathbb{P}_b(I_{i,j}) \cdot \mathbb{P}_{b'}(u)}{\mathbb{P}_b(I_{i,j}) \cdot \mathbb{P}_{b'}(I_{i,j})}| \\
&\leq \frac{1}{|\mathbb{P}_b(I_{i,j}) \cdot \mathbb{P}_{b'}(I_{i,j})|} \cdot |\mathbb{P}_{b'}(I_{i,j}) \cdot \mathbb{P}_b(u) - \mathbb{P}_b(I_{i,j}) \cdot \mathbb{P}_{b'}(u)| \\
&\leq \frac{1}{\epsilon^{2 \cdot h^{\mathcal{G}}}} \cdot |\mathbb{P}_{b'}(I_{i,j}) \cdot \mathbb{P}_b(u) - \mathbb{P}_b(I_{i,j}) \cdot \mathbb{P}_{b'}(u)| \qquad \text{(because for all } b'' \in B^{\epsilon},\ \mathbb{P}_{b''}(I_{i,j}) \geq \epsilon^{h^{\mathcal{G}}}) \\
&\leq 2^{q''(|\mathcal{G}|+\mathtt{size}(\epsilon))} \cdot |\mathbb{P}_{b'}(I_{i,j}) \cdot \mathbb{P}_b(u) - \mathbb{P}_b(I_{i,j}) \cdot \mathbb{P}_{b'}(u)| \\
&= 2^{q''(|\mathcal{G}|+\mathtt{size}(\epsilon))} |\mathbb{P}_{b'}(u)(\mathbb{P}_b(I_{i,j}) - \mathbb{P}_{b'}(I_{i,j})) + \mathbb{P}_{b'}(I_{i,j}))(\mathbb{P}_{b'}(u) - \mathbb{P}_b(u))| \\
&\leq 2^{q''(|\mathcal{G}|+\mathtt{size}(\epsilon))} (|(\mathbb{P}_b(I_{i,j}) - \mathbb{P}_{b'}(I_{i,j}))| + |\mathbb{P}_{b'}(u) - \mathbb{P}_b(u))|) \\
&\leq 2^{q''(|\mathcal{G}|+\mathtt{size}(\epsilon))+q'(|\mathcal{G}|+\mathtt{size}(\epsilon))+1} \|b - b'\|_{\infty}
\end{aligned}
$$

Thus, by induction on the size s of any subformula of $f(x)$ of $H^{\epsilon}_{\mathcal{G}}(x)$, which is either a $\{+, *, \max, \mathtt{sort}\}$-formula or of the form $\frac{\mathbb{P}_x(u)}{\mathbb{P}_x(I_{i,j})}$, we have that for all $b, b' \in B^{\epsilon}$, $|f(b) - f(b')| \leq 2^{(q'(|\mathcal{G}|+\mathtt{size}(\epsilon))+1) \cdot s} \leq 2^{q(|\mathcal{G}|+\mathtt{size}(\epsilon))}$, for some fixed polynomial $q(\cdot)$. Thus, $H^{\epsilon}_{\mathcal{G}}(x)$ is polynomially continuous over the domain B^{ϵ}. ∎

We now define a search problem called the *almost fixed point approximation problem*, called the *weak (fixed point) approximation problem* in [13], specialized to the case of the fixed point functions $F^{\epsilon}_{\mathcal{G}} : B^{\epsilon} \to B^{\epsilon}$. Namely, given as input $\langle \mathcal{G}, \epsilon \rangle$, and a rational $\delta_1 > 0$, compute a rational vector $b' \in B^{\epsilon}$, such that $\|F^{\epsilon}_{\mathcal{G}}(b') - b'\|_{\infty} < \delta_1$. We shall make crucial use of the following fact, which was established in [13] by employing Scarf's [40] algorithm, and Kuhn's [22] related algorithm, for weak (i.e., almost) fixed point approximation:

Proposition 17 ([13], Prop. 2.2 (part 2.)) *If the family of fixed point functions $\langle F^{\epsilon}_{\mathcal{G}}(x) \rangle_{\{\langle \mathcal{G}, \epsilon \rangle\}}$, associated with input instances $\langle \mathcal{G}, \epsilon \rangle$, is polynomially continuous and polynomially computable, then the almost (weak) fixed point approximation problem for $F^{\epsilon}_{\mathcal{G}}(x)$, given input $\langle \mathcal{G}, \epsilon \rangle$, is in* PPAD.

The following Lemma is the key to this section:

Lemma 18 *For any EFGPR, $\mathcal{G}$, and $\epsilon > 0$:*

1. *For any $\delta > 0$, if $b \in B^{\epsilon}$ satisfies $\|b - F^{\epsilon}_{\mathcal{G}}(b)\|_{\infty} < \delta$, then b is a $(3 \cdot \delta)$-almost-$(\delta + \epsilon)$-PE of $\mathcal{G}$.*
2. *For any $\delta > 0$, if $b \in B^{\epsilon}$ satisfies $\|b - H^{\epsilon}_{\mathcal{G}}(b)\|_{\infty} < \delta$, then b is a $(3 \cdot \delta)$-almost-$(\delta + \epsilon)$-PE of $\mathcal{G}$.*
3. *For any $\delta > 0$, let $\epsilon(\mathcal{G}, \delta) := \frac{p^{\mathcal{G}}_{0,\min}}{12 \cdot (\mathtt{h}^{\mathcal{G}}+1) \cdot M_{\mathcal{G}} \cdot |\mathcal{G}|} \cdot \delta$.*
   If $b \in B^{\epsilon(\mathcal{G},\delta)}$ is a $\frac{1}{(\mathtt{h}^{\mathcal{G}}+1)} \cdot \epsilon(\mathcal{G}, \delta)^{(\mathtt{h}^{\mathcal{G}}+1)}$-almost-$(2 \cdot \epsilon(\mathcal{G}, \delta))$-PE, then b is a δ-almost-SGPE.

Proof.
(1.) Suppose that for $b \in B^{\epsilon}$, we have $\|F^{\epsilon}_{\mathcal{G}}(b) - b\|_{\infty} \leq \delta$.
Then $|b_{i,j,a} - \max(b_{i,j,a} + v(b)_{i,j,a} - t_{i,j}(b), \epsilon)| \leq \delta$ for all (i, j, a).
Recall that $v(b)_{i,j,a} = U_i(b \mid \pi^a_{i,j}) = U_{i,j}(b \mid \pi^a_{i,j})$.
Now note that $|b_{i,j,a} - \max(b_{i,j,a} + U_i(b \mid \pi^a_{i,j}) - t_{i,j}(b), \epsilon)| \leq \delta$ implies the following, by case splitting based on the value of $b_{i,j,a}$:

1. If $b_{i,j,a} > \epsilon+\delta$, then $|b_{i,j,a}-(b_{i,j,a}+U_i(b \mid \pi^a_{i,j})-t_{i,j}(b))| \leq \delta$, and thus $|U_i(b \mid \pi^a_{i,j})-t_{i,j}(b))| \leq \delta$. Thus, in this case $t_{i,j}(b) + \delta \geq U_i(b \mid \pi^a_{i,j}) \geq t_{i,j}(b) - \delta$.

2. If $\epsilon \leq b_{i,j,a} \leq \epsilon+\delta$, then $b_{i,j,a}+U_i(b \mid \pi^a_{i,j})-t_{i,j}(b) \leq \epsilon+2\cdot\delta$, and thus $U_i(b \mid \pi^a_{i,j})-t_{i,j}(b) \leq 2\cdot\delta$, and so $U_i(b \mid \pi^a_{i,j}) \leq t_{i,j}(b) + 2 \cdot \delta$.

Thus, for all (i, j, a), we have $U_i(b \mid \pi^a_{i,j}) \leq t_{i,j}(b) + 2 \cdot \delta$, and for all (i, j, a) where $b_{i,j,a} > \epsilon + \delta$, we have $U_i(b \mid \pi^a_{i,j}) \geq t_{i,j}(b) - \delta$. Thus, if $b_{i,j,a'} > \epsilon + \delta$, then $(\max_a U_i(b \mid \pi^a_{i,j})) - U_i(b \mid \pi^{a'}_{i,j}) \leq 3\delta$. In other words, b is a $(3 \cdot \delta)$-almost-$(\epsilon + \delta)$-PE. This completes the proof of Part (1.). of Lemma 18.

(2.): the proof of part (2.) is actually identical to the proof of part (1.), except that instead of $v(b)_{i,j,a} = U_i(b \mid \pi^a_{i,j})$, we have to use $v'(b)_{i,j,a} = \mathsf{K}^{j,a}_i(b)$, and instead of $t_{i,j}(b)$ we have $t'_{i,j}(b)$. If we systematically replace occurrences of $U_i(b \mid \pi^a_{i,j})$ by $\mathsf{K}^{j,a}_i(b)$ in the proof, and likewise replace $U_i(b \mid \pi^{a'}_{i,j})$ by $\mathsf{K}^{j,a'}_i(b)$, and replace $t_{i,j}(b)$ by $t'_{i,j}(b)$, then the proof remains unchanged. Note, in particular, that for $b \in B^\epsilon$, we have $b = \mathfrak{D}^\epsilon(b)$, and thus we can ignore the applications of $\mathfrak{D}^\epsilon(x)$ in the definition of $H^\epsilon_{\mathcal{G}}$, because here we are explicitly given $\epsilon > 0$ and we can view the function as $H^\epsilon_{\mathcal{G}} : B^\epsilon \to B^\epsilon$.

(3.):

Recall that (w.l.o.g.) the payoff functions $r_i : \mathbb{L} \to \mathbb{N}_{>0}$ are positive integer-valued for every player in $\mathcal{G}$, and that $M_{\mathcal{G}}$ denotes the maximum such value. Also recall that $\mathtt{h}^{\mathcal{G}}$ denotes the *height* of the game tree $T = (V, E)$ of $\mathcal{G}$, and that for any node $u \in V$, $\mathtt{h}^{\mathcal{G}}_u$ denotes the height of the subtree rooted at u.

Note that for any profile $b \in B^{\epsilon'}$ for any $\epsilon' > 0$, for any player i, any information set $j \in [d_i]$, and for any node $u \in I_{i,j}$, the conditional probability $\mathbb{P}_b(u \mid I_{i,j})$ of the play reaching node u conditioned on the event of reaching information set $I_{i,j}$, under profile b, is well defined. Furthermore, importantly, again note that the conditional probability $\mathbb{P}_b(u \mid I_{i,j})$ is *independent of* b_i. It only depends on the behavior strategies of players other than i, because, by perfect recall, for all nodes $u \in I_{i,j}$ the visible history for player i at node u is the same: it is $Y_{i,j}$.

For $i \in [n]$, and for $j \in [d_{i'}]$, we use $U^j_i(b)$ to denote the conditional expected payoff to player i, conditioned on the event of reaching information set $I_{i,j}$.

We are now ready to prove (3.). By assumption, $b \in B^{\epsilon(\mathcal{G},\delta)}$, and b is a

$$\frac{1}{(\mathtt{h}^{\mathcal{G}}+1)} \cdot \left(\frac{p^{\mathcal{G}}_{0,\min}}{12\cdot(\mathtt{h}^{\mathcal{G}}+1)\cdot M_{\mathcal{G}}\cdot|\mathcal{G}|} \cdot \delta\right)^{(\mathtt{h}^{\mathcal{G}}+1)}\text{-almost-}\left(\frac{p^{\mathcal{G}}_{0,\min}}{6\cdot(\mathtt{h}^{\mathcal{G}}+1)\cdot M_{\mathcal{G}}\cdot|\mathcal{G}|} \cdot \delta\right)\text{-PE.}$$

We will show that any such b is also a δ-almost-SGPE of $\mathcal{G}$. Consider b from the point of view of a single player i. We need to show that behavior strategy b_i is a δ-almost best response to b, i.e., that $U_i(b) \geq U_i(b \mid \pi^c_i) - \delta$, for any pure strategy $c \in S_i$. Recall that a pure strategy $c : [d_i] \to \Sigma$ for player i maps information sets $j \in [d_i]$ to available actions $c(j) \in \mathcal{A}_{i,j}$.

Claim 1 *For every player i, every $j \in [d_i]$, and every action $a \in \mathcal{A}_{i,j}$ such that $b_{i,j,a} > (\frac{p^{\mathcal{G}}_{0,\min}}{6\cdot(\mathtt{h}^{\mathcal{G}}+1)\cdot M_{\mathcal{G}}\cdot|\mathcal{G}|} \cdot \delta)$, we have for any $a' \in \mathcal{A}_{i,j}$:*

$$U^j_i(b \mid \pi^a_{i,j}) \geq U^j_i(b \mid \pi^{a'}_{i,j}) - \frac{1}{3 \cdot (\mathtt{h}^{\mathcal{G}} + 1)} \cdot \delta$$

Proof. Since $b \in B^{\epsilon(\mathcal{G},\delta)}$ is a $\frac{1}{(\mathtt{h}^{\mathcal{G}}+1)}\left(\frac{p^{\mathcal{G}}_{0,\min}}{12\cdot(\mathtt{h}^{\mathcal{G}}+1)\cdot M_{\mathcal{G}}\cdot|\mathcal{G}|}\cdot\delta\right)^{(\mathtt{h}^{\mathcal{G}}+1)}$-almost-$(\frac{p^{\mathcal{G}}_{0,\min}}{6\cdot(\mathtt{h}^{\mathcal{G}}+1)\cdot M_{\mathcal{G}}\cdot|\mathcal{G}|}\cdot\delta)$-PE, for any $a \in \mathcal{A}_{i,j}$ such that $b_{i,j,a} > (\frac{p^{\mathcal{G}}_{0,\min}}{6\cdot(\mathtt{h}^{\mathcal{G}}+1)\cdot M_{\mathcal{G}}\cdot|\mathcal{G}|}\cdot\delta)$, and any $\pi^{a'}_{i,j}$, we know that:

$$U_i(b \mid \pi^a_{i,j}) \geq U_i(b \mid \pi^{a'}_{i,j}) - \frac{1}{(\mathtt{h}^{\mathcal{G}}+1)}\left(\frac{p^{\mathcal{G}}_{0,\min}}{12\cdot(\mathtt{h}^{\mathcal{G}}+1)\cdot M_{\mathcal{G}}\cdot|\mathcal{G}|}\cdot\delta\right)^{(\mathtt{h}^{\mathcal{G}}+1)} \tag{10}$$

Note that, for any $b' \in B^{\epsilon(\mathcal{G},\delta)}$, we have

$$\mathbb{P}_{b'}(I_{i,j}) \geq \epsilon(\mathcal{G},\delta)^{\mathtt{h}^{\mathcal{G}}} = \left(\frac{p^{\mathcal{G}}_{0,\min}}{6\cdot(\mathtt{h}^{\mathcal{G}}+1)\cdot M_{\mathcal{G}}\cdot|\mathcal{G}|}\cdot\delta\right)^{\mathtt{h}^{\mathcal{G}}}$$

This follows because $\epsilon(\mathcal{G},\delta) \leq p^{\mathcal{G}}_{0,\min}$, and thus under profile $b' \in B^{\epsilon(\mathcal{G},\delta)}$, every "edge" of the game tree will have probability at least $\epsilon(\mathcal{G},\delta)$. Thus already for every node $u \in I_{i,j}$, $\mathbb{P}_{b'}(u) \geq \epsilon(\mathcal{G},\delta)^{\mathtt{h}^{\mathcal{G}}}$, and so $\mathbb{P}_{b'}(I_{i,j}) \geq \mathbb{P}_{b'}(u) \geq \epsilon(\mathcal{G},\delta)^{\mathtt{h}^{\mathcal{G}}}$.

Now note that, for any profile $b' \in B^{\epsilon(\mathcal{G},\delta)}$, the expected payoff $U_i(b')$ can be expressed as a sum $U_i(b') = U^j_i(b')\mathbb{P}_{b'}(I_{i,j}) + U^{\neg j}_i(b')\mathbb{P}_{b'}(\neg I_{i,j})$, where $U^{\neg j}_i(b')$ denotes the expected payoff to player i conditioned on *not* reaching information set $I_{i,j}$, and $\mathbb{P}_{b'}(\neg I_{i,j}) \doteq (1-\mathbb{P}_{b'}(I_{i,j}))$ denotes the probability of not reaching information set $I_{i,j}$.

Note that, if in any such profile b' we change only the local strategy $b'_{i,j}$ to a new strategy $b''_{i,j}$ then this does not effect the probabilities $\mathbb{P}_{b'}(I_{i,j})$ and $\mathbb{P}_{b'}(\neg I_{i,j})$, nor does it effect the conditional expectation $U^{\neg j}_i(b')$. In other words, for any behavior profile $b' \in B^{\epsilon(\mathcal{G},\delta)}$ and any local strategy $b''_{i,j} \in B_{i,j}$, we have:

$$U_i(b' \mid b''_{i,j}) = U^j_i(b' \mid b''_{i,j})\cdot\mathbb{P}_{b'}(I_{i,j}) + U^{\neg j}_i(b')\cdot\mathbb{P}_{b'}(\neg I_{i,j}) \tag{11}$$

Now suppose, *for contradiction*, that for some $\pi^{a'}_{i,j}$, we have:

$$U^j_i(b \mid \pi^a_{i,j}) < U^j_i(b \mid \pi^{a'}_{i,j}) - \frac{1}{3(\mathtt{h}^{\mathcal{G}}+1)}\cdot\delta.$$

But then, by applying equation (11) with $b' := (b \mid \pi^{a'}_{i,j})$ and $b''_{i,j} := \pi^a_{i,j}$, we have:

$$\begin{aligned}
U_i(b \mid \pi^a_{i,j}) &= U^j_i(b \mid \pi^a_{i,j})\cdot\mathbb{P}_{(b|\pi^{a'}_{i,j})}(I_{i,j}) + U^{\neg j}_i(b \mid \pi^{a'}_{i,j})\cdot\mathbb{P}_{(b|\pi^{a'}_{i,j})}(\neg I_{i,j})\\
&< (U^j_i(b \mid \pi^{a'}_{i,j}) - \frac{1}{3(\mathtt{h}^{\mathcal{G}}+1)}\cdot\delta)\cdot\mathbb{P}_{(b|\pi^{a'}_{i,j})}(I_{i,j}) + U^{\neg j}_i(b \mid \pi^{a'}_{i,j})\cdot\mathbb{P}_{(b|\pi^{a'}_{i,j})}(\neg I_{i,j})\\
&\leq U_i(b \mid \pi^{a'}_{i,j}) - \frac{1}{3(\mathtt{h}^{\mathcal{G}}+1)}\cdot\delta\cdot\mathbb{P}_{(b|\pi^{a'}_{i,j})}(I_{i,j})\\
&\leq U_i(b \mid \pi^{a'}_{i,j}) - \frac{1}{3(\mathtt{h}^{\mathcal{G}}+1)}\cdot\delta\cdot\left(\frac{p^{\mathcal{G}}_{0,\min}}{6\cdot(\mathtt{h}^{\mathcal{G}}+1)\cdot M_{\mathcal{G}}\cdot|\mathcal{G}|}\cdot\delta\right)^{\mathtt{h}^{\mathcal{G}}}
\end{aligned}$$

Thus[30], $U_i(b \mid \pi^a_{i,j}) < U_i(b \mid \pi^{a'}_{i,j}) - \frac{1}{(\mathtt{h}^{\mathcal{G}}+1)}\left(\frac{p^{\mathcal{G}}_{0,\min}}{12\cdot(\mathtt{h}^{\mathcal{G}}+1)\cdot M_{\mathcal{G}}\cdot|\mathcal{G}|}\cdot\delta\right)^{\mathtt{h}^{\mathcal{G}}+1}$. But this contradicts inequality (10). Thus, we must have $U^j_i(b \mid \pi^a_{i,j}) \geq U^j_i(b \mid \pi^{a'}_{i,j}) - \frac{1}{3(\mathtt{h}^{\mathcal{G}}+1)}\cdot\delta$. ■

[30] Noting that $\frac{p^{\mathcal{G}}_{0,\min}}{12\cdot(\mathtt{h}^{\mathcal{G}}+1)\cdot M_{\mathcal{G}}\cdot|\mathcal{G}|}\cdot\delta < \frac{\delta}{3}$.

Again, let $b \in B^{\epsilon(\mathcal{G},\delta)}$ be a $\frac{1}{(\mathtt{h}^{\mathcal{G}}+1)}\left(\frac{p^{\mathcal{G}}_{0,\min}}{12\cdot(\mathtt{h}^{\mathcal{G}}+1)\cdot M_{\mathcal{G}}\cdot|\mathcal{G}|}\cdot\delta\right)^{(\mathtt{h}^{\mathcal{G}}+1)}$-almost-$(\frac{p^{\mathcal{G}}_{0,\min}}{6\cdot(\mathtt{h}^{\mathcal{G}}+1)\cdot M_{\mathcal{G}}\cdot|\mathcal{G}|}\cdot\delta)$-PE.

Claim 2 *For every player i, for every integer m where $0 \le m \le \mathtt{h}^{\mathcal{F}_i}$, for every information set $I_{i,j}$ such that $\mathtt{h}^{\mathcal{F}_i}_j = m$, and for every pure strategy $\pi^c_i \in B_i$ for player i:*

$$U^j_i(b) \ge U^j_i(b \,|_m\, \pi^c_i) - \frac{m+1}{(\mathtt{h}^{\mathcal{F}_i}+1)}\cdot\delta$$

Proof. The proof is by induction on m, using Claim 1, starting with base case $m = 0$.
Base case: For $m = 0$ consider an information set $I_{i,j}$ such that $\mathtt{h}^{\mathcal{F}_i}_j \ge 0$. This means that j is a leaf node in the directed information set forest $\mathcal{F}_i$. So, for any pure strategy π^c_i, suppose the local pure strategy (i.e., local action) chosen at $I_{i,j}$ within the pure strategy π^c_i is $a' \in \mathcal{A}_{i,j}$. Note that we then have $U^j_i(b \,|_m\, \pi^c_i) = U^j_i(b \mid \pi^{a'}_{i,j})$. Thus, we have to show that $U^j_i(b) \ge U^j_i(b \mid \pi^{a'}_{i,j}) - \frac{1}{(\mathtt{h}^{\mathcal{F}_i}+1)}\cdot\delta$.

For the local strategy $b_{i,j}$, and for $\eta \ge 0$, let $b^{>\eta}_{i,j} = \sum_{\{a\in\mathcal{A}_{i,j} \mid b_{i,j,a}>\eta\}} b_{i,j,a}$. Likewise, let $b^{\le\eta}_{i,j} = \sum_{\{a\in\mathcal{A}_{i,j} \mid b_{i,j,a}\le\eta\}} b_{i,j,a}$. For $\triangledown \in \{>, \le\}$, for $\epsilon > 0$, for a behavior profile $b \in B^\epsilon$, and for $\eta \ge 0$, let $U^{j,\triangledown\eta}_i(b)$ denote the conditional expected payoff to player i, under profile b, conditioned on the event that the play both reaches information set $I_{i,j}$, and thereupon plays some action in the set $\{a \in \mathcal{A}_{i,j} \mid b_{i,j,a} \triangledown \eta\}$. Note that, for the profile $b \in B^{\epsilon(\mathcal{G},\delta)}$, the conditional expected payoff $U^j_i(b)$ can be written as:

$$U^j_i(b) = U^{j,>\epsilon(\mathcal{G},\delta)}_i(b)\cdot b^{>\epsilon(\mathcal{G},\delta)}_{i,j} + U^{j,\le\epsilon(\mathcal{G},\delta)}_i(b)\cdot b^{\le\epsilon(\mathcal{G},\delta)}_{i,j} \tag{12}$$

But then, for any $a' \in \mathcal{A}_{i,j}$, we have

$$\begin{aligned}
U^j_i(b) &= U^{j,>\epsilon(\mathcal{G},\delta)}_i(b)\cdot b^{>\epsilon(\mathcal{G},\delta)}_{i,j} + U^{j,\le\epsilon(\mathcal{G},\delta)}_i(b)\cdot b^{\le\epsilon(\mathcal{G},\delta)}_{i,j}\\
&\ge U^{j,>\epsilon(\mathcal{G},\delta)}_i(b)\cdot b^{>\epsilon(\mathcal{G},\delta)}_{i,j}\\
&\ge (U^j_i(b \mid \pi^{a'}_{i,j}) - \frac{1}{3\cdot(\mathtt{h}^{\mathcal{G}}+1)}\cdot\delta)\cdot b^{>\epsilon(\mathcal{G},\delta)}_{i,j} \qquad \text{(by Claim 1)}\\
&\ge (U^j_i(b \mid \pi^{a'}_{i,j}) - \frac{1}{3\cdot(\mathtt{h}^{\mathcal{G}}+1)}\cdot\delta)\cdot(1-|\mathcal{A}_{i,j}|\cdot\epsilon(\mathcal{G},\delta))\\
&= (U^j_i(b \mid \pi^{a'}_{i,j}) - \frac{1}{3\cdot(\mathtt{h}^{\mathcal{G}}+1)}\cdot\delta)\cdot(1-|\mathcal{A}_{i,j}|\cdot\frac{p^{\mathcal{G}}_{0,\min}}{12\cdot(\mathtt{h}^{\mathcal{G}}+1)\cdot M_{\mathcal{G}}\cdot|\mathcal{G}|}\cdot\delta)\\
&\ge U^j_i(b \mid \pi^{a'}_{i,j}) - \frac{1}{3\cdot(\mathtt{h}^{\mathcal{G}}+1)}\cdot\delta - U^j_i(b \mid \pi^{a'}_{i,j})\cdot(|\mathcal{A}_{i,j}|\cdot\frac{p^{\mathcal{G}}_{0,\min}}{12\cdot(\mathtt{h}^{\mathcal{G}}+1)\cdot M_{\mathcal{G}}\cdot|\mathcal{G}|}\cdot\delta)\\
&\ge U^j_i(b \mid \pi^{a'}_{i,j}) - \frac{1}{3\cdot(\mathtt{h}^{\mathcal{G}}+1)}\cdot\delta - \frac{1}{3\cdot(\mathtt{h}^{\mathcal{G}}+1)}\cdot\delta\\
&\qquad \text{(because } U^j_i(b \mid \pi^{a'}_{i,j}) \le M_{\mathcal{G}} \text{ and } |\mathcal{A}_{i,j}| \le |\mathcal{G}| \text{ and } p^{\mathcal{G}}_{0,\min} \le 1)\\
&\ge U^j_i(b \mid \pi^{a'}_{i,j}) - \frac{1}{(\mathtt{h}^{\mathcal{F}_i}+1)}\cdot\delta \qquad \text{(because } \mathtt{h}^{\mathcal{F}_i} \le \mathtt{h}^{\mathcal{G}}\text{, for all } i\text{, and } \tfrac{2}{3} \le 1.)
\end{aligned}$$

Thus $U^j_i(b) \ge U^j_i(b \mid \pi^{a'}_{i,j}) - \frac{1}{(\mathtt{h}^{\mathcal{F}_i}+1)}\cdot\delta$, which completes the proof of the base case.[31]

[31] Let us remark that we could have opted for a proof that renders the base case trivial, and "swallows" it into the inductive case, but we felt this would have come at the expense of clarity.

Inductive case: Assume the claim is true for $m-1$ such that $0 \le m-1 < \mathtt{h}^{\mathcal{F}_i}$. We want to show it holds for m. Again, consider any pure strategy π_i^c for player i, and suppose that $\pi_i^c(j) = a'$. In other words, in information set $I_{i,j}$, the action chosen by π_i^c is a'.

Let $J^i(j,a') = \{j' \in [d_i] \mid (j,a',j') \in E^{\mathcal{F}_i}\}$ denote the set of children j' of j in the forest $\mathcal{F}_i$, such that the edge from j to j' is labeled by a'. (In other words, $J^i(j,a')$ denotes the information sets belonging to player i that could possibly be the next information set for that player which is reached, after reaching information set j.) For $j' \in J^i(j,a')$, let $\mathcal{P}^i_{(b|\pi^{a'}_{i,j})}(j' \mid j)$ denote the conditional probability of reaching information set $I_{i,j'}$, conditioned on event of reaching information set $I_{i,j}$ and thereupon taking action $a' \in \mathcal{A}_{i,j}$, under profile b. Furthermore, let $\mathcal{P}^i_{(b|\pi^{a'}_{i,j})}(\neg J^i(j,a') \mid j)$ denote the conditional probability of not reaching any information set in $J^i(j,a')$, conditioned on the event of reaching $I_{i,j}$ and thereupon taking action a'. Finally, let $U_i^{j,\neg J^i(j,a')}(b \mid \pi^{a'}_{i,j})$ denote the conditional expected payoff (under profile b), conditioned on reaching $I_{i,j}$ and thereupon playing a', but thereafter *not* reaching any information set in $J^i(j,a')$. Note that for all $b \in B^{\epsilon(\mathcal{G},\delta)}$, and every $a' \in \mathcal{A}_{i,j}$, we have:

$$(\sum_{j' \in J^i(j,a')} \mathcal{P}^i_{(b|\pi^{a'}_{i,j})}(j' \mid j)) + \mathcal{P}^i_{(b|\pi^{a'}_{i,j})}(\neg J^i(j,a') \mid j) = 1.$$

Note furthermore that:

$$U_i^j(b \mid \pi^{a'}_{i,j}) = \left(\sum_{j' \in J^i(j,a')} U_i^{j'}(b \mid \pi^{a'}_{i,j}) \cdot \mathcal{P}^i_{(b|\pi^{a'}_{i,j})}(j' \mid j) \right) + U_i^{j,\neg J^i(j,a')}(b \mid \pi^{a'}_{i,j}) \cdot \mathcal{P}^i_{(b|\pi^{a'}_{i,j})}(\neg J^i(j,a') \mid j). \quad (13)$$

We now use equation (13), the inductive hypothesis, and equation (12), in order to establish that for any pure strategy π_i^c for player i, we have $U_i^j(b) \ge U_i^j(b \mid_m \pi_i^c) - \frac{m+1}{(\mathtt{h}^{\mathcal{F}_i}+1)} \cdot \delta$.

Suppose that the pure strategy π_i^c has $\pi_i^c(j) = a'$. Observe that in this case:

$$(b \mid_m \pi_i^c) = ((b \mid \pi^{a'}_{i,j}) \mid_{m-1} \pi_i^c) = ((b \mid_{m-1} \pi_i^c) \mid \pi^{a'}_{i,j}) \quad (14)$$

Also observe that:

$$U_i^{j,\neg J^i(j,a')}(b \mid \pi^{a'}_{i,j}) = U_i^{j,\neg J^i(j,a')}((b \mid \pi^{a'}_{i,j}) \mid_{m-1} \pi_i^c) \quad (15)$$

because this conditional expectation does not change when we change the strategy b_i in local strategies (at $J^i(j,a')$ and below) which we have conditioned on not reaching. We thus have:

$$
\begin{aligned}
U_i^j(b) &= U_i^{j,>\epsilon(\mathcal{G},\delta)}(b)\cdot b_{i,j}^{>\epsilon(\mathcal{G},\delta)} + U_i^{j,\leq\epsilon(\mathcal{G},\delta)}(b)\cdot b_{i,j}^{\leq\epsilon(\mathcal{G},\delta)} \quad (by (12))\\
&\geq U_i^{j,>\epsilon(\mathcal{G},\delta)}(b)\cdot b_{i,j}^{>\epsilon(\mathcal{G},\delta)}\\
&\geq (U_i^j(b\mid \pi_{i,j}^{a'}) - \frac{1}{3\cdot(\mathtt{h}^{\mathcal{G}}+1)}\cdot\delta)\cdot b_{i,j}^{>\epsilon(\mathcal{G},\delta)} \qquad \text{(by Claim 1)}\\
&\geq (U_i^j(b\mid \pi_{i,j}^{a'}) - \frac{1}{3\cdot(\mathtt{h}^{\mathcal{G}}+1)}\cdot\delta)\cdot(1-|\mathcal{A}_{i,j}|\cdot\epsilon(\mathcal{G},\delta))\\
&= (U_i^j(b\mid \pi_{i,j}^{a'}) - \frac{1}{3\cdot(\mathtt{h}^{\mathcal{G}}+1)}\cdot\delta)\cdot(1-|\mathcal{A}_{i,j}|\cdot\frac{p_{0,\min}^{\mathcal{G}}}{12\cdot(\mathtt{h}^{\mathcal{G}}+1)\cdot M_{\mathcal{G}}\cdot|\mathcal{G}|}\cdot\delta)\\
&\geq U_i^j(b\mid \pi_{i,j}^{a'}) - \frac{1}{3\cdot(\mathtt{h}^{\mathcal{G}}+1)}\cdot\delta - U_i^j(b\mid \pi_{i,j}^{a'})\cdot(|\mathcal{A}_{i,j}|\cdot\frac{p_{0,\min}^{\mathcal{G}}}{12\cdot(\mathtt{h}^{\mathcal{G}}+1)\cdot M_{\mathcal{G}}\cdot|\mathcal{G}|}\cdot\delta)\\
&\geq U_i^j(b\mid \pi_{i,j}^{a'}) - \frac{1}{3\cdot(\mathtt{h}^{\mathcal{G}}+1)}\cdot\delta - \frac{1}{3\cdot(\mathtt{h}^{\mathcal{G}}+1)}\cdot\delta\\
&\qquad\qquad \text{(because } U_i^j(b\mid \pi_{i,j}^{a'})\leq M_{\mathcal{G}} \text{ and } |\mathcal{A}_{i,j}|\leq|\mathcal{G}| \text{ and } p_{0,\min}^{\mathcal{G}}\leq 1)\\
&\geq U_i^j(b\mid \pi_{i,j}^{a'}) - \frac{1}{(\mathtt{h}^{\mathcal{F}_i}+1)}\cdot\delta \qquad \text{(because } \mathtt{h}^{\mathcal{F}_i}\leq\mathtt{h}^{\mathcal{G}}\text{, for all } i)\\
&= (\sum_{j'\in J^i(j,a')} U_i^{j'}(b\mid \pi_{i,j}^{a'})\cdot\mathcal{P}^i_{(b\mid\pi_{i,j}^{a'})}(j'\mid j))\\
&\quad + U_i^{j,\neg J^i(j,a')}(b\mid \pi_{i,j}^{a'})\cdot\mathcal{P}^i_{(b\mid\pi_{i,j}^{a'})}(\neg J^i(j,a')\mid j) - \frac{1}{(\mathtt{h}^{\mathcal{F}_i}+1)}\cdot\delta \qquad \text{(by equality (13))}\\
&\geq (\sum_{j'\in J^i(j,a')} (U_i^{j'}((b\mid \pi_{i,j}^{a'})\mid_{m-1}\pi_i^c) - \frac{m}{(\mathtt{h}^{\mathcal{F}_i}+1)}\cdot\delta)\cdot\mathcal{P}^i_{(b\mid\pi_{i,j}^{a'})}(j'\mid j))\\
&\quad + U_i^{j,\neg J^i(j,a')}((b\mid \pi_{i,j}^{a'})\mid_{m-1}\pi_i^c) - \frac{m}{(\mathtt{h}^{\mathcal{F}_i}+1)}\cdot\delta)\cdot\mathcal{P}^i_{(b\mid\pi_{i,j}^{a'})}(\neg J^i(j,a')\mid j) - \frac{1}{(\mathtt{h}^{\mathcal{F}_i}+1)}\cdot\delta\\
&\quad \text{(by inductive hypothesis, and by (15))}\\
&= U_i^j(b\mid_m \pi_i^c) - \frac{m}{(\mathtt{h}^{\mathcal{F}_i}+1)}\cdot\delta - \frac{1}{(\mathtt{h}^{\mathcal{F}_i}+1)}\cdot\delta \qquad \text{(by (14) and (13))}\\
&= U_i^j(b\mid_m \pi_i^c) - \frac{m+1}{(\mathtt{h}^{\mathcal{F}_i}+1)}\cdot\delta
\end{aligned}
$$

Thus $U_i^j(b)\geq U_i^j(b\mid_m \pi_i^c) - \frac{m+1}{(\mathtt{h}^{\mathcal{F}_i}+1)}\cdot\delta$. This completes the proof of Claim 2. ■

Part (2.) of Lemma 18 now follows readily from Claim 2. To see this, let $J^{\mathcal{F}_i}$ denote the set of root vertices in the information set forest $\mathcal{F}_i$. Let $\mathcal{P}_b^i(\neg J^{\mathcal{F}_i})$ denote the probability, under profile b, of not reaching any information set in $J^{\mathcal{F}_i}$. Finally, let $U_i^{\neg J^{\mathcal{F}_i}}(b)$ denote the conditional expected payoff to player i, under profile b, conditioned on the event of not reaching any of the information sets in $J^{\mathcal{F}_i}$, and if this event has probability zero, then by definition we let $U_i^{\neg J^{\mathcal{F}_i}}(b) := 0$.

Then, for any pure strategy π_i^c for player i, we have:

$$
\begin{aligned}
U_i(b) &= \Big(\sum_{j' \in J^{\mathcal{F}_i}} U_i^{j'}(b) \cdot \mathbb{P}_b(I_{i,j'})\Big) \quad + \quad U_i^{\neg J^{\mathcal{F}_i}}(b) \cdot \mathcal{P}_b^i(\neg J^{\mathcal{F}_i}) \\
&\geq \Big(\sum_{j' \in J^{\mathcal{F}_i}} (U_i^{j'}(b \mid \pi_i^c) - \delta) \cdot \mathbb{P}_b(I_{i,j'})\Big) \quad + \quad (U_i^{\neg J^{\mathcal{F}_i}}(b \mid \pi_i^c) - \delta) \cdot \mathcal{P}_b^i(\neg J^{\mathcal{F}_i}) \\
&\qquad\qquad \text{(by applying Claim 2, and since } U_i^{\neg J^{\mathcal{F}_i}}(b) = U_i^{\neg J^{\mathcal{F}_i}}(b \mid \pi_i^c)) \\
&= U_i(b \mid \pi_i^c) - \delta.
\end{aligned}
$$

Thus $U_i(b) \geq U_i(b \mid \pi_i^c) - \delta$, which completes the proof of Part (2.) of Lemma 18. ■

Applying Lemma 18, Proposition 17, and Lemma 15, we obtain the main results of this section:

Theorem 19

1. *The problem of computing, given a EFGPR, $\mathcal{G}$, and given rationals $\delta > 0$ and $\epsilon > 0$ (in binary), a δ-almost-ϵ-PE of $\mathcal{G}$, is* PPAD*-complete.*

   Likewise, the problem of computing, given a EFGPR, $\mathcal{G}$, and given rationals $\delta > 0$ and $\epsilon > 0$ (in binary), a δ-almost-ϵ-QPE of $\mathcal{G}$, is PPAD*-complete.*

2. *(cf. [10]) The problem of computing, given a EFGPR, $\mathcal{G}$, and given a rational $\delta > 0$ (in binary), a δ-almost-SGPE of $\mathcal{G}$ is* PPAD*-complete.*

Proof. First, we establish containment in PPAD for all the problems:

1. The fact that computing a δ-almost-ϵ-PE, and computing a δ-almost-ϵ-QPE for a given EFGPR, $\mathcal{G}$, and given $\delta > 0$ and $\epsilon > 0$, is in PPAD follows immediately from Lemma 18, Parts (1.) and (2.), Proposition 17, and Lemma 15.

   Specifically, by Lemma 18, Parts (1.), for $0 < \delta < 1$ and $0 < \epsilon < 1$, a profile $b \in B^{\epsilon/2}$, such that $\|b - F_{\mathcal{G}}^{\epsilon/2}(b)\|_\infty < \frac{\epsilon \cdot \delta}{3}$, is also a δ-almost-ϵ-PE. Likewise, profile $b \in B^{\epsilon/2}$, such that $\|b - H_{\mathcal{G}}^{\epsilon/2}(b)\|_\infty < \frac{\epsilon \cdot \delta}{3}$, is a δ-almost-ϵ-QPE.

   But by Proposition 17 and Lemma 15, since the functions $F_{\mathcal{G}}^{\epsilon}(b)$ and $H_{\mathcal{G}}^{\epsilon}(b)$ are polynomially computable and polynomially continuous (with respect to the input $\langle G, \epsilon \rangle$), the problem of computing such a profile b is in PPAD.

2. For $\delta > 0$, let $\epsilon(\mathcal{G}, \delta) := \frac{p_{0,\min}^{\mathcal{G}}}{12 \cdot (\mathtt{h}^{\mathcal{G}}+1) \cdot M_{\mathcal{G}} \cdot |\mathcal{G}|} \cdot \delta$. Let $\delta' = \frac{1}{3 \cdot (\mathtt{h}^{\mathcal{G}}+1)} \cdot \epsilon(\mathcal{G}, \delta)^{(\mathtt{h}^{\mathcal{G}}+1)}$.

   Since $\delta' < \epsilon(\mathcal{G}, \delta)$, we have $(\delta' + \epsilon(\mathcal{G}, \delta)) \leq (2 \cdot \epsilon(\mathcal{G}, \delta))$. It thus follows from Lemma 18, Part 1., that if $b \in B^{\epsilon(\mathcal{G},\delta)}$ satisfies $\|b - F_{\mathcal{G}}^{\epsilon(\mathcal{G},\delta)}(b)\|_\infty < \delta'$, then
   b is a $\frac{1}{(\mathtt{h}^{\mathcal{G}}+1)} \cdot \epsilon(\mathcal{G}, \delta)^{(\mathtt{h}^{\mathcal{G}}+1)}$-almost-$(2 \cdot \epsilon(\mathcal{G}, \delta))$-PE. But then Lemma 18, Part 2., implies that b is also a δ-almost subgame perfect equilibrium of $\mathcal{G}$.

   Thus, the problem computing a δ-almost-SGPE of $\mathcal{G}$ is P-time reducible to the problem of computing a $b \in B^{\epsilon(\mathcal{G},\delta)}$ such that $\|b - F_{\mathcal{G}}^{\epsilon(\mathcal{G},\delta)}(b)\|_\infty < \delta'$. But since both $\epsilon(\mathcal{G}, \delta) > 0$ and $\delta' > 0$ are rational numbers both of whose encoding size (in binary) is polynomial in the encoding size of the input $\langle \mathcal{G}, \delta \rangle$, by Proposition 17, computing a δ-almost-SGPE is in PPAD.

Finally, to see that both problems are PPAD-hard, recall that Daskalakis, Goldberg, and Papadimitriou [11] established that computing a δ-almost NE (a.k.a., a δ-NE, in the terminology they used), given a n-player *normal form* game, Γ, and given $\delta > 0$, is PPAD-hard. Now recall that a n-player NFG, Γ, is trivially encodable as a n-player EFGPR, $\mathcal{E}(\Gamma)$, and note that a δ-almost-SGPE of $\mathcal{E}(\Gamma)$ is also a δ-almost-NE of Γ. ■

A simple corollary of Theorem 19 is that computing an δ-almost-ϵ-PE for a NFG is also PPAD-complete.

Corollary 20 *The problem of computing, given a NFG, Γ, and given rationals $\delta > 0$ and $\epsilon > 0$ (in binary), a δ-almost-ϵ-PE of Γ, is PPAD-complete.*

Proof. This follows by applying Theorem 19 (Part 1.) to the "equivalent" EFGPR, $\mathcal{E}(\Gamma)$, which we can easily construct from Γ, and from the fact that $\mathcal{E}(\Gamma)$ has exactly the same δ-almost-ϵ-PEs (in behavior strategies) as Γ does (in mixed strategies). This follows easily from the payoff-preserving one-to-one correspondence between the mixed profiles of Γ and the behavior profiles of $\mathcal{E}(\Gamma)$. ■

We have suggested that the notion of a δ-almost-ϵ-PE, is a reasonable "almost" relaxation of (ϵ-)PE, allowing for its computation in PPAD (i.e., using path following algorithms), in the same way that δ-NE (= δ-almost-NE) serves as a relaxation of NE.

We have thusfar not defined a "almost" relaxation for *sequential equilibrium* (SE). Since PE "refines" SE (see Proposition 3), a possible definition is this: "an assessment $(b', \mu^{b'})$, where the behavior profile b' is a δ-almost-ϵ-PE, and where $\mu^{b'}$ is the belief system generated by b' ". This is well-defined, because for $\epsilon > 0$, any δ-almost-ϵ-PE, b', is fully mixed, and thus the belief system $\mu^{b'}$ that it generates is uniquely defined; and we can compute $\mu^{b'}$ efficiently, given b' and $\mathcal{G}$. So, we can take this as our definition of a "almost" relaxation of SE. Theorem 19 then implies that computing such an "almost" SE, given $\mathcal{G}$, and given $\delta > 0$ and $\epsilon > 0$, is PPAD-complete.

6 Conclusions

We have characterized the complexity of approximating various refinements of equilibrium, and "almost equilibrium", for extensive form games of perfect recall with $n \geq 3$ players.

Specifically, we have shown that the complexity of approximate (or almost) equilibrium computation for extensive form games of perfect recall, with $n \geq 3$ players, including for fundamental refinements such as sequential and (quasi-)perfect equilibrium, is the same as that of approximate (or almost) Nash equilibrium computation for normal form games with 3 players. Namely, these problems are, respectively, FIXP_a-complete and PPAD-complete.

Although our results establish that approximating a PE for a n-player EFGPR, is in FIXP_a, our results *do not* imply that computing an actual (*real-valued*) PE for an n-player EFGPR is in FIXP. We leave this as an open question, although the more relevant question, from the point of view of the standard (Turing) model of computation, is containment in FIXP_a (in PPAD) for approximation (respectively, "almost" computation), which we have established.
Some natural open questions suggest themselves:

1. The complexity of approximating a *proper equilibrium* for n-player NFGs. Proper equilibrium, defined by Myerson in [29], is an important refinement of PE for NFGs[32] , which Myerson showed always exists for any NFG.

   It is defined as follows: for an NFG, Γ, and for $\epsilon > 0$, a mixed strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$ is called a ϵ*-proper equilibrium* if it is (a.): fully mixed, and (b.): for every two pure strategies c, c' of any player i, if $U_i(\sigma \mid \pi_i^c) < U_i(\sigma \mid \pi_i^{c'})$ then $\sigma_i(c) \leq \epsilon \cdot \sigma_i(c')$. A *proper equilibrium* is defined to be a limit point of a sequence of ϵ_k-proper equilibria, where $\epsilon_k > 0$ for all $k \in \mathbb{N}$, and where $\lim_{k\to\infty} \epsilon_k = 0$.

   There are connections between proper equilibrium for NFGs and QPEs of EFGPRs. In particular, van Damme [45] showed that a proper equilibrium for an NFG, Γ, induces a QPE in every EFGPR whose (standard) normal form is Γ. However, the other direction does not hold: there are EFGPRs with a QPE (or PE) which is not induced by a proper equilibrium in a corresponding normal form game. Sørensen [44] has given a Lemke-like algorithm for computing a proper equilibrium for 2-player NFGs.[33] Can we approximate a proper equilibrium for n-player NFGs in FIXP_a?

2. One can adapt Myerson's definition of (ϵ-)proper equilibrium in a natural way, to define a notion of *extensive form* (ϵ-)*proper equilibrium* (PropE) as well as (ϵ-)*quasi-proper equilibrium* (QPropE) for EFGPRs. PropE refines PE, and likewise QPropE refines QPE, for EFGPRs. Such refinements for EFGPRs were already alluded to briefly by van Damme in [45][34], but we are unaware of any subsequent study of them. Myerson's existence proof of proper equilibrium for NFGs can be suitably adapted to show existence of both a PropE and a QPropE for any EFGPR. Can we approximate a PropE, and a QPropE, for n-player EFGPRs in FIXP_a?

We believe the answer to both of the above questions is "Yes".
Even if the answers are "yes", it is not entirely clear what the suitable "δ-almost" relaxations of (qausi-)proper equilibrium should be. We need such relaxations to place the problems in PPAD, i.e., to enable discrete path following algorithms that compute a suitably refined "almost equilibrium". One natural attempt is to define such a relaxation as follows: a δ*-almost-*ϵ*-proper equilbrium* for NFGs is a mixed strategy profile σ which is (a.): fully mixed, and (b.): for every two pure strategies c, c' for any player i, if $U_i(\sigma \mid \pi_i^c) < U_i(\sigma \mid \pi_i^{c'}) - \delta$ then $\sigma_i(c) \leq \epsilon \cdot \sigma_i(c')$. It remains to be seen whether this definition is the "right" one, and in particular whether computing such a δ-almost relaxation can be placed in PPAD.

(**Note added during late revision:** In a very recent paper, Hansen and Lund [16] have answered question (1.) above, affirmatively, proving that approximating a proper equilbrium for an n-player NFG is in FIXP_a. In fact, their proof makes crucial use of the notion of δ-almost ϵ-proper equilibrium which we have suggested above, and does so in a novel and interesting way.)

[32]Peter Bro Miltersen, in conversation with the author, has referred to proper equilibrium as *"the mother of all"* refinements of equilibrium for NFGs.

[33]For NFGs with $n \geq 3$ players, Yamamoto [50] outlined a procedure for approximating a proper equilibrium based on a *continuous* homotopy path following approach, but as indicated by Sørensen in [44], even for 2-player NFGs it is unclear under what conditions Yamamoto's procedure is guaranteed to converge to an approximate proper equilibrium.

[34]As van Damme remarks in [45], his main result actually shows that every proper equilibrium of an NFG, Γ, induces a QPropE in every EFGPR which has Γ as its (standard) normal form.

We want to again highlight that we believe our results can potentially provide a "practical" computation method for computing a "almost" (ϵ-perfect, and ϵ-quasi-perfect) equilibrium for EFGPRs, with $n \geq 3$ players, by applying Scarf-like discrete path following algorithms on the "small" algebraic fixed point functions that we have developed for n-player EFGPRs. We believe this is a promising computational approach that should be implemented and explored experimentally.

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