Ring Exploration with Myopic Luminous Robots

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Abstract

We investigate exploration algorithms for autonomous mobile robots evolving in uniform ring-shaped networks. Different from the usual Look-Compute-Move (LCM) model, we consider two characteristics: myopia and luminosity. Myopia means each robot has a limited visibility. We consider the weakest assumption for myopia: each robot can only observe its neighboring nodes. Luminosity means each robot maintains a non-volatile visible light. We consider the weakest assumption for luminosity: each robot can use only two colors for its light. The main interest of this paper is to clarify the impact of luminosity on exploration with myopic robots.

As a main contribution, we prove that 1) in the fully synchronous model, two and three robots are necessary and sufficient to achieve perpetual and terminating exploration, respectively, and 2) in the semi-synchronous and asynchronous models, three and four robots are necessary and sufficient to achieve perpetual and terminating exploration, respectively. These results clarify the power of lights for myopic robots since, without lights, five robots are necessary and sufficient to achieve terminating exploration in the fully synchronous model, and no terminating exploration algorithm exists in the semi-synchronous and asynchronous models.

We also show that, in the fully synchronous model (resp., the semi-synchronous and asynchronous models), the proposed perpetual exploration algorithm is universal, that is, the algorithm solves perpetual exploration from any solvable initial configuration with two (resp., three) robots and two colors. On the other hand, we show that, in the fully synchronous model (resp., the semi-synchronous and asynchronous models), no universal algorithm exists for terminating exploration, that is, no algorithm may solve terminating exploration from any solvable initial configuration with three (resp., four) robots and two colors.

Keywords: autonomous mobile robots, deterministic exploration, discrete environments, limited visibility, visible light

1 Introduction

1.1 Background and Motivation

Studies about cooperation of autonomous mobile robots have attracted a lot of attention recently in the field of Distributed Computing. The main goal of those works is to characterize the minimum capabilities of robots that permit to achieve a given task. Since the pioneering work of Suzuki and Yamashita [24], many results have been published in their Look-Compute-Move (LCM) model. In the LCM model, each robot repeats executing cycles of look, compute, and move phases. At the beginning of each cycle, the robot observes positions of other robots (look phase). According to its observation, the robot computes whether it moves somewhere or stays idle (compute phase). If the robot decides to move, it moves to the target position by the end of the cycle (move phase). To consider minimum capabilities, most studies assume that robots are identical (i.e., robots execute the same algorithm and cannot be distinguished), oblivious (i.e., robots have no memory of their past actions), and silent (i.e., robots cannot communicate with other robots explicitly). Indeed, communication among robots is done only in an implicit way by observing positions of other robots and moving to a new position. Previous works considered problem solvability of LCM robots in continuous environments (aka two- or three-dimensional Euclidean space) [17, 18, 24, 25], while others considered discrete environments (aka graph networks) [4, 6, 16, 22, 23].

In this paper, we focus on robots evolving in graph networks. The most fundamental tasks in graph networks are gathering and exploration. The goal of gathering is to make all robots meet at a non-predetermined single node. Gathering has been studied for rings [4, 6, 22, 23], grids and trees [5]. Two types of exploration tasks have been well studied: perpetual exploration requires robots to visit nodes so that every node is visited infinitely many times by a robot, and terminating exploration requires robots to terminate after every node is visited by a robot at least once. For example, perpetual exploration has been studied for rings [1] and grids [2], and terminating exploration has been studied for rings [14, 16], trees [15], grids [12], tori [13], and arbitrary networks [3].

All aforementioned works in graph networks make the assumption that each robot observes all other robots in the networks. That is, each robot has a sensor that can obtain a global snapshot. However, this powerful ability somewhat contradicts the principle of very weak mobile entities. For this reason, recent studies consider the more realistic case of myopic robots [8, 10, 19, 20]. A myopic robot has limited visibility, *i.e.*, it can see nodes (and robots on them) only within a certain fixed distance ϕ . Datta et al. studied terminating exploration of rings for $\phi = 1$ [8] and $\phi = 2, 3$ [9]. Guilbault and Pelc studied gathering in bipartite graphs with $\phi = 1$ [19], and infinite lines with $\phi > 1$ [20]. Not surprisingly, in the weakest setting, *i.e.*, $\phi = 1$, robots can only achieve few tasks. It is shown [8] that, when $\phi = 1$ holds, five robots are necessary and sufficient to achieve terminating exploration in the fully synchronous (FSYNC) model. On the other hand, no terminating exploration algorithm exists in the semi-synchronous (SSYNC) and asynchronous (ASYNC) models. Also, gathering [19] is possible when $\phi = 1$ only if robots initially form a star.

Since most results for myopic robots with $\phi = 1$ are negative, a natural question is which additional assumptions can improve task solvability. In this paper, we focus on a non-volatile visible light [7] as an additional assumption. A robot endowed with such a light is called a luminous robot. Each luminous robot is equipped with a light device that can emit a constant number of colors to other robots, a single color at a time. The light color is non-volatile, so it can be used as a constant-space memory. For non-myopic luminous robots, the power of lights is well understood [7, 11, 21].

Table 1: Ring exploration with myopic robots.

					#robots	
Reference	Exploration	Synchrony	ϕ	#colors	necessary	sufficient
[8]	terminating	FSYNC	1	1	5	5
[8]	terminating	SSYNC & ASYNC	1	1	impossible	
[10]	terminating	SSYNC & ASYNC	2	1	5	7
[10]	terminating	SSYNC & ASYNC	3	1	5	5
This paper	perpetual	FSYNC	1	2	2	2
This paper	terminating	FSYNC	1	2	3	3
This paper	perpetual	SSYNC & ASYNC	1	2	3	3
This paper	terminating	SSYNC & ASYNC	1	2	4	4

For example, if each robot has a five colors light, the difference between the asynchronous model and the semi-synchronous model disappears [7]. However, to the best of our knowledge, the impact of lights on myopic robots has not been studied yet.

1.2 Our Contributions

We focus on ring exploration and the impact of lights on myopic robots with $\phi = 1$. We consider the weakest assumption for lights: each robot can use only two colors for its light. Table 1 summarizes our contributions and related works. Note that robots with no light are equivalent to robots with a single color light.

As a main contribution, we prove that (i) in the fully synchronous model, two and three robots are necessary and sufficient to achieve perpetual and terminating exploration, respectively, and (ii) in the semi-synchronous and asynchronous models, three and four robots are necessary and sufficient to achieve perpetual and terminating exploration, respectively. These results clarify the power of lights for myopic robots since, without lights, five robots are necessary and sufficient to achieve terminating exploration in the fully synchronous model, and no terminating exploration algorithm exists in the semi-synchronous and asynchronous models. Interestingly, even if robots can observe nodes up to distance three $(i.e., \phi = 3)$, five robots are required to achieve terminating exploration without light. This means that there exist some tasks that myopic luminous robots with small visibility can achieve, but that non-luminous robots with larger visibility cannot.

Similarly to previous works for myopic robots, all algorithms proposed in this paper assume some specific initial configurations because most configurations are not solvable. For example, when myopic robots are deployed so that no robot can observe other robots, they cannot achieve exploration. However, our perpetual exploration algorithms achieve the best possible property, that is, they are universal. This means that, in the fully synchronous model (resp., the semi-synchronous and asynchronous models), the proposed algorithm solves perpetual exploration from any solvable initial configuration with two (resp., three) robots and two colors. As for terminating exploration, we show that no universal algorithm exists. That is, in the fully synchronous model (resp., the semi-synchronous and asynchronous models), no algorithm may solve terminating exploration from any solvable initial configuration with three (resp., four) robots and two colors.

Due to space limitation, we omit some of proofs. The omitted proofs are given in the appendix.

2 Preliminaries

2.1 System model

The system consists of n nodes and k mobile robots. The nodes $v_0, v_1, \ldots, v_{n-1}$ form an undirected and unoriented ring-shaped graph, where a link exists between v_i and v_{i+1} , for i < n, and between v_{n-1} and v_0 . For simplicity we consider mathematical operations on node indices as operations modulo n. Neither nodes nor links have identifiers or labels, and consequently robots cannot distinguish nodes and links. Robots do not know n, the size of the ring. Robots occupy some nodes of the ring. The distance between two nodes is the number of links in a shortest path between the nodes. The distance between two robots a and b is the distance between two nodes occupied by a and b. Two robots a and b are neighbors if the distance between a and b is one. A set a0 of robots is connected if the induced subgraph of nodes occupied by the robots in a1 is connected; otherwise, a2 is disconnected.

Robots we consider have the following characteristics and capabilities. Robots are *identical*, that is, robots execute the same deterministic algorithm and cannot be distinguished based on their appearance (in particular, their do *not* have unique identifiers). Robots are *luminous*, that is, each robot has a light (or state) that is visible to itself and other robots. A robot can choose the color of its lights from a discrete set Col. When the set Col is finite, we denote by κ the number of available colors (*i.e.*, $\kappa = |Col|$). Robots have no other persistent memory and cannot remember the history of past actions. Robots cannot communicate with other robots explicitly, however they can communicate implicitly by observing positions and colors of other robots (for collecting information), and by changing their color and moving (for sending information). Each robot r can observe positions and colors of robots within a fixed distance ϕ ($\phi > 0$) from its current position. Since robots are identical, they share the same ϕ . If $\phi = \infty$, robots can observe all other robots in the ring. If $\phi = 1$, robots are *myopic*, that is, they can only observe robots that are located at neighboring nodes.

Each robot executes an algorithm by repeating three-phases cycles: Look, Compute, and Move (L-C-M). During the *Look* phase, the robot observes positions and colors of robots within distance φ. During the Compute phase, the robot computes its next color and movement according to the observation in the Look phase. The robot may change its color at the end of the Compute phase. If the robot decides to move, it moves to a neighboring node during the *Move* phase. To model asynchrony of executions, we introduce the notion of scheduler that decides when each robot executes phases. When the scheduler makes robot r execute some phase, we say the scheduler activates the phase of r or simply activates r. We consider three types of synchronicity: the FSYNC (full-synchronous) model, the SSYNC (semi-synchronous) model, and the ASYNC (asynchronous) model. In the FSYNC model, the scheduler executes full cycles of all robots synchronously and concurrently. In the SSYNC model, the scheduler selects a non-empty subset of robots and executes full cycles of the selected robots synchronously and concurrently. In the ASYNC model, the scheduler executes cycles of robots asynchronously. Note that in the ASYNC model, a robot r can move based on an outdated view observed previously by r. Throughout the paper we assume that the scheduler is fair, that is, each robot is activated infinitely often. We consider the scheduler as an adversary. That is, we assume that the scheduler is omniscient (it knows robot positions, colors, algorithms, etc.), and tries to activate robots in such a way that they fail executing the task.

In the sequel, $M_i(t)$ denotes the multiset of colors of robots located in node v_i at instant t. If v_i is not occupied by any robot at t, then $M_i(t) = \emptyset$ holds, and v_i is free at instant t.

Then, v_i is a tower at instant t if $|M_i(t)| \ge 2$. A configuration C(t) of the system at instant t is defined as $C(t) = (M_0(t), M_1(t), \ldots, M_{n-1}(t))$. If t is clear from the context, we simply write $C = (M_0, M_1, \ldots, M_{n-1})$. If there exists an index x such that $M_{x+i} = M_{x-i}$ holds for any i, or if $M_{x+i} = M_{x-(i+1)}$ holds for any i (i.e., there exists at least one axis of symmetry in the configuration), configuration C is called symmetric.

When a robot observes its environment, it gets a view up to distance ϕ . Consider a robot r on node v_i ; then, r obtains two views: the forward view and the backward view. The forward and backward views of r are defined as $V_f = (c_r, M_{i-\phi}, \ldots, M_{i-1}, M_i, M_{i+1}, \ldots, M_{i+\phi})$, and $V_b = (c_r, M_{i+\phi}, \ldots, M_{i+1}, M_i, M_{i-1}, \ldots, M_{i-\phi})$, respectively, where c_r denotes r's color. Since we assume unoriented rings (where robots may not share the same notion of left and right), each robot cannot distinguish its forward view from its backward view. If the forward view and the backward view of r are identical, then r's view is symmetric. In this case, r cannot distinguish between the two directions when it moves, and the scheduler decides which direction r moves to. If r observes no other robot in its view, r is isolated.

2.2 Algorithm, execution, and problem

An algorithm is described as a set of rules. Each rule is represented in the following manner < Label > :< Guard > ::< Action >. The guard < Guard > is a possible view obtained by a robot. If a forward or backward view of robot r matches a guard in an algorithm, we say r is enabled. We also say the corresponding rule < Label > is enabled. If a robot is enabled, the robot may change its color and move based on the corresponding action < Action > during the Compute and Move phases.

For an infinite sequence of configurations $E = C_0, C_1, \ldots, C_t, \ldots$, we say E is an execution from initial configuration C_0 if, for every instant t, C_{t+1} is obtained from C_t after some robots execute phases. We say C_i is the i-th configuration of execution E.

A problem \mathcal{P} is defined as a set of executions: An execution E solves problem \mathcal{P} if $E \in \mathcal{P}$ holds. An algorithm \mathcal{A} solves problem \mathcal{P} from initial configuration C_0 if any execution from initial configuration C_0 solves problem \mathcal{P} . We simply say an algorithm \mathcal{A} solves problem \mathcal{P} if there exists an initial configuration C_0 such that \mathcal{A} solves \mathcal{P} from C_0 . For configuration C and problem \mathcal{P} , C is solvable for \mathcal{P} if there exists an algorithm (specific to C) that solves \mathcal{P} from initial configuration C. Let $C_s(\mathcal{P})$ be a set of all configurations solvable for \mathcal{P} . We say algorithm \mathcal{A} is universal with respect to problem \mathcal{P} if \mathcal{A} solves \mathcal{P} from any initial configuration in $C_s(\mathcal{P})$.

2.3 Exploration problems

In this paper, we consider perpetual exploration problem and terminating exploration problem in case of $\phi = 1$.

Definition 1 (Perpetual exploration problem) Perpetual exploration is defined as a set of executions E such that every node is infinitely many times visited by some robot in E.

Definition 2 (Terminating exploration problem) Terminating exploration is defined as a set of executions E such that 1) every node is visited by at least one robot in E and 2) there exists a suffix of E such that no robots are enabled.

2.4 Descriptions

Let $C = (M_0, \ldots, M_{n-1})$ be a configuration. We say $C' = (M'_0, \ldots, M'_{n'-1})$ is a sub-configuration of C if there exists x such that $M_{x+i} = M'_i$ holds for any i ($0 \le i \le n'-1$). In this case, we say n' is the length of sub-configuration C'. We sometimes describe a sub-configuration $C' = (M'_0, \ldots, M'_{n'-1})$ by listing all colors in M'_i as the i-th column. That is, when $M'_i = \{c_1^i, \ldots, c_{|M'_i|}^i\}$ holds for each i ($0 \le i \le n'-1$), we describe C' as follows:

$$\begin{array}{cccc} c_1^0 & c_1^1 & & c_1^{n'-1} \\ c_2^0 & c_2^1 & & c_2^{n'-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{|M_0'|}^0 & c_{|M_1'|}^1 & & c_{|M_{n'-1}'|}^{n'-1} \end{array}$$

When $M_i' = \emptyset$ holds, we write \emptyset as the *i*-th column. If *h* free nodes exist successively, we sometimes write \emptyset^h instead of writing *h* columns with \emptyset . For simplicity, when C' is a sub-configuration of C and all robots appear in C', we use C' instead of C to represent configuration C. We also use this description to represent views of robots.

Throughout the paper, we consider the case of $\phi = 1$. We describe a rule in an algorithm in the following manner:

$$\mathcal{R}_{rule}: egin{array}{cccc} c_{-1,1} & c_{0,1} & c_{1,1} \ c_{-1,2} & c_{0,2} & c_{1,2} \ dots & dots & dots \ c_{-1,m_{-1}} & (c_{0,m_0}) & c_{1,m_1} \end{array} :: c_{new}, Movement$$

Notation \mathcal{R}_{rule} is a label of the rule. The middle part represents a guard. This represents a view $V = (c_{0,m_0}, M_{-1}, M_0, M_1)$, where $M_i = \{c_{i,1}, \ldots, c_{i,m_i}\}$ holds for $i \in \{-1,0,1\}$. Intuitively, each column represents colors of robots on a single node and a color within parentheses represents its current color. If a forward or backward view of robot r is equal to V, r is enabled. In this case, r can execute an action represented by c_{new} , Movement. Notation c_{new} represents a new color of the robot, and Movement represents the movement. Notation Movement can be \bot , \leftarrow , \rightarrow , or $\leftarrow \lor \rightarrow$:

1) \bot implies a robot does not move, 2) \leftarrow (resp., \rightarrow) implies a robot moves toward the node such that a set of robot colors is M_{-1} (resp., M_1), and 3) $\leftarrow \lor \rightarrow$ implies a robot moves toward one of two directions (the scheduler decides the direction). When the view V described in a guard is symmetric, Movement should be either \bot or $\leftarrow \lor \rightarrow$. As an example, consider the following rule.

$$\mathcal{R}_{ex}: egin{array}{c} \mathsf{G} \ (\mathsf{W}) \ \mathsf{G} \end{array} :: \ \mathsf{G},
ightarrow$$

Robot r is enabled by \mathcal{R}_{ex} if 1) the color of r is W, 2) the current node is occupied by two robots with colors G and W, 3) one neighboring node is occupied by no robot, and 4) another neighboring node is occupied by a robot with color G. If r is enabled by \mathcal{R}_{ex} , r changes its color to G and moves toward the node occupied by a robot with color G.

Algorithm 1 Fully-Synchronous Perpetual Exploration for k=2

Initial configurations

GW and WG

Rules

 $0\mathsf{GW}: \quad \emptyset \ (\mathsf{G}) \ \mathsf{W} :: \ \mathsf{G}, \leftarrow \\
0\mathsf{WG}: \quad \emptyset \ (\mathsf{W}) \ \mathsf{G} :: \ \mathsf{W}, \rightarrow$

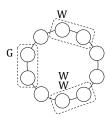


Figure 1: An example of an independent territory set. Notations W and G represent robots with colors W and G, respectively.

3 Full-synchronous Robots

3.1 A universal perpetual exploration algorithm for two robots with two colors

In this subsection, we provide a perpetual exploration algorithm for two robots with two colors. Note that, since one robot cannot achieve perpetual exploration clearly because the direction of its movement is decided by the scheduler, two robots are necessary to achieve perpetual exploration. A set of colors is $Col = \{G, W\}$. The algorithm is given in Algorithm 1. In the initial configuration, two robots with colors G and G stay at neighboring nodes. In this algorithm, the robot with color G moves against the other robot, and the robot with color G moves toward the other robot. This implies two robots move in the same direction. Since they move synchronously, the views of the two robots are not changed. Hence, the two robots continue to move and achieve perpetual exploration. Clearly we have the following theorem.

Theorem 3 In case of $\phi = 1$ and k = 2, Algorithm 1 solves perpetual exploration from initial configurations GW and WG for $n \ge 2$ in the FSYNC model.

In the following, we show that other initial configurations are unsolvable for $n \geq 6$. This implies Algorithm 1 is universal with respect to perpetual exploration for $n \geq 6$ in case of $\phi = 1$ and k = 2. To prove the impossibilities, we first prove Lemma 4. This lemma will be used for many impossibility proofs not only in the FSYNC model but also in the SSYNC and ASYNC models. For configuration C, we define $V_r(C)$ as a set of nodes occupied by at least one robot. We say a set of two neighboring nodes $T = \{v_i, v_{i+1}\}$ is a territory of robots on node v if $v \in T$ holds. We say a territory set T is independent if, for every pair of territories $T_1, T_2 \in T$, the distance between any node in T_1 and any node in T_2 is at least two (See Fig. 1).

Lemma 4 Consider a configuration C such that no two nodes in $V_r(C)$ are neighbors and, for every node $v \in V_r(C)$, robots on v have the same color. If there exists a territory set T such that T is independent and every node in $V_r(C)$ belongs to some territory in T, robots on $v \in V_r(C)$ cannot go out of their territory in T after configuration C in the FSYNC, SSYNC, and ASYNC models.

Proof: Assume that such a territory set \mathcal{T} exists. We prove the lemma by induction. At configuration C, every robot stays at its territory. Consider configuration C' such that every robot stays at its territory. Since no two nodes in different territories share the same neighbor, each robot observes no robots on its neighbor nodes. This means the view of the robot is symmetric and consequently the scheduler can decide the direction of its movement. In addition, all robots on a single node have the same color, they make the same behaviors if the scheduler activates them at the same time. Hence, each robot moves to another node in its territory if it decides to move. That is, every robot stays at its territory again. Therefore, the lemma holds.

Now we go back to the FSYNC model, and prove that initial configurations other than GW and WG are unsolvable.

Lemma 5 Assume $n \ge 6$. Let C be a configuration such that two robots are disconnected. In this case, C is unsolvable.

Proof: Let d be the distance between two robots in C. Without loss of generality, we assume that two robots r_1 and r_2 occupy nodes v_0 and v_d at C, respectively. We define territories T_1 and T_2 as follows: If d = 2 holds, we define $T_1 = \{v_0, v_{n-1}\}$ and $T_2 = \{v_d, v_{d+1}\}$, and if d > 2 holds, we define $T_1 = \{v_0, v_1\}$ and $T_2 = \{v_d, v_{d+1}\}$. Since $\mathcal{T} = \{T_1, T_2\}$ is independent, two robots visit only nodes in $T_1 \cup T_2$ from Lemma 4. Therefore, they cannot achieve perpetual exploration.

Lemma 6 Assume $n \geq 6$. Configurations GG and WW are unsolvable.

Proof: Since two robots have the same view, they move in a symmetric manner. If each robot moves toward the other robot, the robots just swap their positions. Hence, to achieve exploration, eventually each robot moves against the other robot. After the movement, the distance between them is three. Similarly to Lemma 5, robots cannot achieve perpetual exploration from the configuration.

From Theorem 3 and Lemmas 5 and 6, a set of solvable configurations is $C_s = \{GW, WG\}$. Therefore, we have the following theorem.

Theorem 7 In case of $\phi = 1$, k = 2, and $Col = \{G, W\}$, Algorithm 1 is universal with respect to perpetual exploration for $n \ge 6$ in the FSYNC model.

3.2 Impossibility of terminating exploration with two robots

In this subsection, we prove that no algorithm solves terminating exploration for k=2.

Theorem 8 In case of $\phi = 1$ and k = 2, no algorithm solves terminating exploration in the FSYNC model. This holds even if robots can use an infinite number of colors.

Proof: Assume that such algorithm \mathcal{A} exists. Consider an execution $E = C_0, C_1, \ldots$ of \mathcal{A} in a n_1 -node ring R_1 ($n_1 \geq 6$). Let t be the minimum instant such that two robots terminate or become disconnected at C_t . Next, for some $n_2 > 2(t+1)$, let us consider an execution $E' = C'_0, C'_1, \ldots$ of \mathcal{A} in a n_2 -node ring R_2 . Clearly, as long as two robots keep connected, they do not recognize the difference between R_1 and R_2 . Hence, in E', two robots move similarly to E until C'_t . If two robots

Algorithm 2 Fully-Synchronous Terminating Exploration for k=3

Initial configurations

WWW, GWW, WWG, and GWG

Rules

 $\begin{array}{lll} 0\mathsf{G}\mathsf{W} : & \emptyset \ (\mathsf{G}) \ \mathsf{W} :: \ \mathsf{G}, \leftarrow \\ 0\mathsf{W}\mathsf{G} : & \emptyset \ (\mathsf{W}) \ \mathsf{G} :: \ \mathsf{W}, \rightarrow \\ 0\mathsf{W}\mathsf{W} : & \emptyset \ (\mathsf{W}) \ \mathsf{W} :: \ \mathsf{G}, \bot \\ \mathsf{G}\mathsf{W}\mathsf{W} : & \mathsf{G} \ (\mathsf{W}) \ \mathsf{W} :: \ \mathsf{W}, \leftarrow \\ \mathsf{G}\mathsf{W}\mathsf{G} : & \mathsf{G} \ (\mathsf{W}) \ \mathsf{G} :: \ \mathsf{W}, \leftarrow \lor \rightarrow \end{array}$

terminate at C'_t , they have visited at most 2(t+1) nodes and thus they do not achieve exploration. If two robots become disconnected at C'_t , we can define an independent territory set at C'_t . From Lemma 4, two robots cannot visit the remaining nodes and thus they cannot achieve exploration. \Box

3.3 A terminating exploration algorithm for three robots with two colors

In this subsection, we give a terminating exploration algorithm for three robots with two colors in case of $n \ge 3$. A set of colors is $Col = \{G, W\}$. The algorithm is given in Algorithm 2.

Executions of Algorithm 2 for $n \geq 5$ are given in Fig. 2. We consider three robots r_1 , r_2 , and r_3 . In the figure, W_i (resp., G_i) represents robot r_i with color W (resp., G_i). Arrows represent that indicated robots are enabled. At configuration WWW, r_1 and r_3 are enabled by rule 0WW (Fig. 2(a)) and change their colors to G_i . At configuration GWG_i (Fig. 2(b)), robots r_1 and r_2 (i.e., a pair of robots G_i) and G_i (i.e., another robot G_i) move to the opposite directions by rules 0GW and G_i and G_i . Note that, since the view of r_2 is symmetric at configuration G_i (Fig. 2(c)) or G_i (Fig. 2(f)). After robot G_i (Fig. 2(f)) or G_i (F

We can easily verify that Algorithm 2 works for n = 3 or n = 4. Hence we have the following theorem.

Theorem 9 In case of $\phi = 1$ and k = 3, Algorithm 2 solves terminating exploration from initial configurations WWW, GWW, WWG, and GWG for $n \ge 3$ in the FSYNC model.

Note that we can construct another algorithm by swapping the roles of colors G and W in Algorithm 2. Clearly this algorithm solves terminating exploration from configurations such that colors G and W are swapped from solvable configurations for Algorithm 2. This implies configurations GGG, WGG, GGW, and WGW are also solvable. Hence, we have the following lemma.

Lemma 10 If k = 3 holds and a set of colors is $\{G, W\}$, configurations WWW, WWG, WGW, WGG, GWW, GWG, GGW, and GGG are solvable for terminating exploration in the FSYNC model.

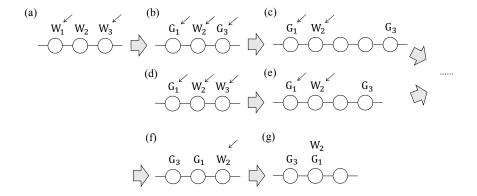


Figure 2: Executions of Algorithm 2

We also prove that there exists no universal algorithm with respect to terminating exploration for three robots with two colors. This validates the assumption that Algorithm 2 starts from some designated initial configuration.

Theorem 11 In case of $\phi = 1$, k = 3, and $\kappa = 2$, no universal algorithm exists with respect to terminating exploration in the FSYNC model.

Proof: (Sketch) To prove the lemma by contradiction, we assume universal algorithm \mathcal{A} exists. We first prove that, by \mathcal{A} , the distance between some pair of two robots becomes at least five in some large ring. Intuitively this holds because, if the three robots always stay within distance four, they cannot distinguish small rings and large rings, and terminate without exploration in large rings.

When the distance becomes at least five, some two robots should be connected. Otherwise, we can define an independent territory set and exploration is impossible from Lemma 4. Intuitively the two connected robots should explore the ring because another robot cannot go out of its territory. This implies the two robots should form sub-configuration GW. In addition, since one of the two robots should move toward another robot to explore the ring, \mathcal{A} should include rule $0\mathsf{WG}X:\emptyset(\mathsf{W})\mathsf{G}::X,\to \mathrm{or}\ 0\mathsf{GW}X:\emptyset(\mathsf{G})\mathsf{W}::X,\to \mathrm{for}\ \mathrm{some}\ X\in Col.$

Assume \mathcal{A} includes rule 0WGX (we can prove another case similarly). Consider a solvable configuration WGW. From this configuration, two robots with color W move toward G at the same time by rule 0WGX. This implies two robots with the same color make a tower in the next configuration. After the configuration, since the two robots move in the same manner, three robots behave as if there were only two robots. Similarly to Theorem 8, we can prove that they cannot achieve terminating exploration. This is a contradiction because \mathcal{A} cannot achieve terminating exploration from a solvable initial configuration.

4 Semi-synchronous and Asynchronous Robots

4.1 Impossibility of perpetual exploration with two robots

In this subsection, we prove that two robots are not sufficient to achieve perpetual exploration in the SSYNC model. Clearly this impossibility result holds in the ASYNC model.

Algorithm 3 Asynchronous Perpetual Exploration for k=3

Initial configurations

WWG, WGG, GWW, GGW,
$$\overset{G}{W}\overset{G}{W}\overset{G}{W}\overset{W}{W}\overset{W}{W}$$
, $\overset{G}{G}\overset{G}{G}$, and $\overset{W}{G}\overset{G}{G}$.

Rules

 $0\mathsf{GW}: \emptyset (\mathsf{G}) \mathsf{W} :: \mathsf{G}, \rightarrow$

 $0TW: \quad \begin{matrix} \mathsf{G} \\ \emptyset \ (\mathsf{W}) \ \mathsf{W} \end{matrix} :: \ \mathsf{G}, \rightarrow$

 $0TG: \begin{array}{c} \mathsf{W} \\ \emptyset \ (\mathsf{G}) \ \mathsf{G} \end{array} :: \ \mathsf{W}, \leftarrow$

 $0WG: \emptyset (W) G :: W, \rightarrow$

Theorem 12 In case of $\phi = 1$ and k = 2, no algorithm solves perpetual exploration in the SSYNC model. This holds even if robots can use an infinite number of colors.

Proof: Assume that such algorithm \mathcal{A} exists for $n \geq 6$. First consider the case that initially two robots r_1 and r_2 are connected. Clearly some robot r_i ($i \in \{1,2\}$) eventually moves against another robot. At that time, we assume that the scheduler activates only r_i . Since another robot does not move, the two robots become disconnected.

From the above fact, two robots are initially disconnected or eventually become disconnected. Let us consider the configuration such that two robots are disconnected. In this case, we can define an independent territory set, and thus robots cannot achieve exploration from Lemma 4. This is a contradiction.

4.2 A universal perpetual exploration algorithm for three robots with two colors

In this subsection, we give a universal perpetual exploration algorithm for three robots with two colors in the SSYNC and ASYNC models. We first give a perpetual exploration algorithm by three robots with two colors in the ASYNC model, and after that we prove the algorithm is universal in the SSYNC and ASYNC models. A set of colors is $Col = \{G, W\}$. The algorithm is given in Algorithm 3.

Executions of Algorithm 3 for $n \geq 4$ are given in Fig. 3. Let us consider configuration WWG, and assume that r_1 , r_2 , and r_3 compose the configuration in this order (Fig. 3(a)). Here only r_3 is enabled with rule 0GW, and r_3 moves toward r_2 . In a configuration in Fig. 3(b), only r_2 is enabled with rule 0TW. If r_2 is activated, r_2 changes its color to G and moves toward r_1 (Fig. 3(c)). Note that, in the ASYNC model, after r_2 changes its color, some robots may observe the intermediate configuration before r_2 moves toward r_1 . However, since no rule matches the intermediate configuration, robots do not move based on the configuration. After r_2 moves from Fig. 3(c) by rule 0TG, the sub-configuration becomes WWG (Fig. 3(e)) but the robots change their positions from Fig. 3(a) to Fig. 3(e). Similarly, robots repeat the behavior from Fig. 3(a) to Fig. 3(e), and they achieve perpetual exploration. From configuration WGG in Fig. 3(d), r_1 moves by rule 0WG and becomes a configuration in Fig. 3(c). After that, they move similarly to the case from a

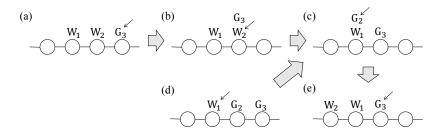


Figure 3: Executions of Algorithm 3

configuration in Fig. 3(a). These executions include configurations

WWG, WGG,
$$\overset{\mathsf{G}}{\mathsf{W}}$$
 $\overset{\mathsf{M}}{\mathsf{W}}$, and $\overset{\mathsf{W}}{\mathsf{G}}$ $\overset{\mathsf{G}}{\mathsf{G}}$,

and consequently from these configurations robots can achieve perpetual exploration. Since remaining configurations

are symmetric to the above configurations, robots can also achieve perpetual exploration from the configurations. Therefore, we have the following theorem.

Theorem 13 In case of $\phi = 1$ and k = 3, Algorithm 3 solves perpetual exploration from initial configurations

$$WWG, WGG, GWW, GGW, WW, WW, GG, W$$
 and W

for $n \geq 3$ in the ASYNC model.

We can also show that other initial configurations are unsolvable for $n \geq 9$ in the SSYNC model. This implies Algorithm 3 is universal with respect to perpetual exploration for $n \geq 9$ in the SSYNC and ASYNC models.

Theorem 14 In case of $\phi = 1$, k = 3, and $Col = \{G, W\}$, Algorithm 3 is universal with respect to perpetual exploration for $n \ge 9$ in the SSYNC and ASYNC models.

4.3 Impossibility of terminating exploration with three robots

In this subsection, we prove that three robots are not sufficient to achieve terminating exploration in the SSYNC model. Clearly this impossibility result holds in the ASYNC model.

Theorem 15 In case of $\phi = 1$ and k = 3, no algorithm solves terminating exploration in the SSYNC model. This holds even if robots can use an infinite number of colors.

Proof: For contradiction, assume that such algorithm \mathcal{A} exists. Consider an execution $E = C_0, C_1, \ldots$ of \mathcal{A} in a n_1 -node ring R_1 ($n_1 \geq 9$). Let t be the minimum instant such that three robots terminate or become disconnected at C_t . Next, for $n_2 = 4(t+1)$, let us consider an execution

Algorithm 4 Asynchronous Terminating Exploration for k = 4

Initial configurations

WWGG, WWWG, WWGW, GGWW, GWWW, WGWW.

Rules

 $0\mathsf{GW}:\quad\emptyset\;(\mathsf{G})\;\mathsf{W}\;::\;\;\mathsf{G},\rightarrow$

 $\mathsf{GGW}: \quad \mathsf{G} \; (\mathsf{G}) \; \mathsf{W} \; :: \; \; \mathsf{G}, \rightarrow$

 $0TW: \quad {\mathsf{G} \atop \emptyset \ (\mathsf{W}) \ \mathsf{W}} :: \ \mathsf{G}, \to$

 $\mathsf{G} T \mathsf{W} : \quad \begin{matrix} \mathsf{G} \\ \mathsf{G} \ (\mathsf{W}) \ \mathsf{W} \end{matrix} :: \ \mathsf{G}, \rightarrow$

 $0\mathsf{WG}:\ \emptyset\ (\mathsf{W})\ \mathsf{G}::\ \mathsf{W},\to$

 $E' = C'_0, C'_1, \ldots$ of \mathcal{A} in a n_2 -node ring R_2 such that the scheduler activates robots similarly to E. Clearly, as long as three robots keep connected, they do not recognize the difference between R_1 and R_2 . Hence, in E', three robots move similarly to E until C'_t . Note that, since robots have visited at most 3(t+1) until C'_t , they must explore the remaining t+1 nodes. If three robots terminate at C'_t , they do not achieve exploration. If three robots become disconnected at C'_t , robots cannot achieve exploration from the proof of Theorem 14.

4.4 A terminating exploration algorithm for four robots with two colors

In this subsection, we give a terminating exploration algorithm for four robots with two colors in case of $n \geq 4$. A set of colors is $Col = \{\mathsf{G}, \mathsf{W}\}$. The algorithm is given in Algorithm 4. Note that rules 0GW, 0TW, 0TG are identical to Algorithm 3. Hence, once three robots construct sub-configurations

$$\mathcal{C}_{pe} = \left\{ \mathsf{WWG}, \left. egin{matrix} \mathsf{G} & \mathsf{W} \\ \mathsf{W} & \mathsf{W} & \mathsf{G} & \mathsf{G} \end{array}
ight\},$$

they explore the ring similarly to Algorithm 3.

Executions of Algorithm 4 for $n \geq 5$ are given in Fig. 4. At configuration WWGG (Fig. 4(a)) only r_3 can move by rule GGW, and the configuration becomes one in Fig. 4(b) after it moves. Since r_1 , r_2 , and r_3 form a sub-configuration in C_{pe} , they explore the ring. Since r_4 does not move, three robots eventually join r_4 from the opposite direction (Fig. 4(c)).

Figure 5 shows executions after a configuration in Fig. 4(c). In this figure, we reassign indices to robots: r_1 , r_2 , r_3 , and r_4 form sub-configuration GWWG in this order. Since r_1 and r_4 are enabled, robots can make several behaviors depending on activation by the scheduler. Notation LC-i (resp., M-i) means the scheduler activates Look and Compute phases (resp., Move phase) of r_i . Although the scheduler can activate Look and Compute phases separately, we combine the two phases because a Look phase does not change a configuration. At configuration GWWG, the scheduler can activate r_1 and r_4 to change the configuration. If the scheduler activates exactly one

Figure 4: An execution from WWGG of Algorithm 4

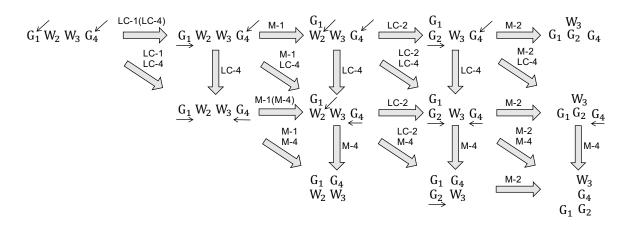


Figure 5: A termination execution of Algorithm 4

robot, we only show the case of r_1 because the case of r_4 is symmetric to r_1 . An arrow below a robot means that the robot decides to move to the direction. In every execution from configuration GWWG, robots eventually terminate. That is, Algorithm 4 solves terminating exploration from initial configuration WWGG.

We consider other initial configurations WWWG and WWGW in Fig. 6. From initial configuration WWWG (Fig. 6(a)), robots eventually form configuration WWGG (Fig. 6(e)) and thus they can solve terminating exploration. From initial configuration WWGW (Fig. 6(f)), robots form a configuration in Fig. 6(g) and the configuration is the same as in Fig. 6(b). Hence, they can solve terminating exploration.

Since configurations GGWW, GWWW, and WGWW are symmetric to WWGG, WWWG, and WWGW, respectively, we have the following theorem.

Theorem 16 In case of $\phi = 1$ and k = 4, Algorithm 4 solves terminating exploration from initial configurations WWGG, WWWG, WWGW, GGWW, GWWW, and WGWW for $n \geq 5$ in the ASYNC model.

Note that we can construct another algorithm by swapping the roles of colors G and W in Algorithm 4. Clearly this algorithm solves terminating exploration from configurations such that colors G and W are swapped from solvable configurations for Algorithm 4. This implies configurations GGGW, GGWG, WGGG, and GWGG are also solvable. Hence, we have the following lemma.

Lemma 17 If k = 4 holds and a set of colors is $\{G, W\}$, configurations WWGG, WWWG, WWGW, GGWW, GWWW, WGWW, GGGW, GGWG, WGGG, and GWGG are solvable for terminating exploration in the ASYNC model.

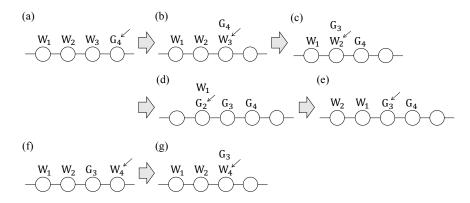


Figure 6: Executions from WWWG and WWGW of Algorithm 4

We also prove that there exists no universal algorithm with respect to terminating exploration for four robots with two colors. This validates the assumption that Algorithm 4 starts from some designated initial configuration.

Theorem 18 In case of $\phi = 1$, k = 4, and $\kappa = 2$, no universal algorithm exists with respect to terminating exploration in the SSYNC and ASYNC models.

Proof: (Sketch) Similarly to Theorem 11, we can show that the distance between some pair of robots become large and some three robots must explore the ring. At that time, we also show that the three robots always form a sub-configuration in

$$\mathcal{C}_{exp} = \left\{ \mathsf{WGG}, \mathsf{GGW}, \mathsf{GWW}, \mathsf{WWG}, \begin{matrix} \mathsf{G} & \mathsf{G} & \mathsf{W} & \mathsf{W} \\ \mathsf{W} & \mathsf{W} & \mathsf{G} & \mathsf{G} & \mathsf{G} \end{matrix} \right\}.$$

Note that this is a set of solvable configurations for perpetual exploration. Since a universal algorithm makes the three robots explore the ring by changing their sub-configuration from one in C_{exp} to another in C_{exp} , we can show that the algorithm should include some set of rules. Lastly, we prove the set of rules makes some solvable initial configuration unsolvable.

5 Conclusions

In this paper, we investigated the possibility of exploration algorithms for myopic luminous robots evolving in uniform ring-shaped networks. Considering weakest possible assumptions for myopia and luminosity, we proved that: (i) in the fully synchronous model, two and three robots are necessary and sufficient to achieve perpetual and terminating exploration, respectively, and ii) in the semi-synchronous and asynchronous models, three and four robots are necessary and sufficient to achieve perpetual and terminating exploration, respectively. These tight results characterize the power of lights for myopic robots since, without lights, five robots are necessary and sufficient to achieve terminating exploration in the fully synchronous model, and no terminating exploration algorithm exists in the semi-synchronous and asynchronous models. We also showed that our perpetual exploration algorithms are universal, and that no universal algorithm exists for terminating exploration.

This paper leaves open many issues with respect to problem solvability for myopic luminous robots. In case of non-myopic luminous robots, the difference between the semi-synchronous model and the asynchronous model disappears. Does this difference still hold for myopic luminous robots? If visibility ϕ is large, robots may be able to use distance to neighboring robots to store information instead of lights. Now, is there some relation between tasks achieved by myopic luminous robots with a large number of colors, and tasks achieved by non-luminous robots with large visibility? Is there a tradeoff between the visibility distance and the number of colors? It is also interesting to consider other tasks and topologies with myopic luminous robots.

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A Proof of Theorem 11

To prove Theorem 11 by contradiction, we assume universal algorithm \mathcal{A} exists. We assume $Col = \{\mathsf{G},\mathsf{W}\}$. We consider a ring with $n > 4 \cdot 10^3$ nodes and an execution $E = C_0, C_1, \ldots$ of \mathcal{A} in the ring. Note that, since three robots can visit at most three new nodes in each configuration, E includes at least $n/3 > (4/3) \cdot 10^3$ configurations.

The outline of the proof is as follow. We first prove that three robots eventually become disconnected (Lemma 19), and at that time two robots compose a sub-configuration GW (Lemma 21). Then, we prove that some rule is necessary to make the two robots explore the ring (Lemma 22), however the rule makes a solvable configuration unsolvable (Lemma 23).

Lemma 19 The distance between some pair of two robots becomes at least five before the 10^3 -th configuration of execution E.

Proof: For contradiction, assume that the distance between any pair of robots is at most four until 10^3 -th configuration of E. For each configuration C_i ($0 \le i \le 10^3$), we define C'_i as a sub-configuration of C_i such that C'_i contains five nodes (i.e., the length of C'_i is five), every robot occupies one of the five nodes, and some robot occupies the first node of C'_i . Since each robot has one of two colors and occupies one of five nodes, the number of such sub-configurations is at most 10^3 . Hence, for some t and u ($t < u \le 10^3$), sub-configurations C'_t and C'_u are identical. In the following, we consider two cases.

The first case is that the five nodes contained in C'_t and C'_u are identical. This implies configurations C_t and C_u are identical. Since \mathcal{A} is deterministic, robots repeat the behavior from C_t to C_u after C_u . Since robots can visit at most $3 \cdot 10^3$ nodes before configuration C_t , they cannot visit remaining nodes after C_u . Thus robots cannot complete exploration.

The second case is that the five nodes contained in C'_t and C'_u are not identical. This means robots change their position from C_t to C_u . However, since no robots exist out of the five nodes, the behaviors of robots depend only on the sub-configuration. Hence, robots repeat the behaviors from C_t to C_u after C_u . This implies robots continue to explore the ring, and thus they cannot terminate.

For both cases, robots cannot achieve terminating exploration. This is a contradiction. \Box

From Lemma 19, the distance between some pair of two robots becomes at least five before the 10^3 -th configuration. Let t_1 be the smallest instant such that the distance between some pair of two robots is at least five at C_{t_1} . Clearly $t_1 \leq 10^3$ holds. Since three robots can visit at most $3(t_1+1)$ nodes before C_{t_1} , at least 10^3-3 unexplored nodes exist at C_{t_1} . The next two lemmas show that, to explore the unexplored nodes, robots have to construct a specific sub-configuration.

Lemma 20 If a tower exists at configuration C_{t_1} , robots break the tower before they visit three unexplored nodes.

Proof: Assume that a tower exists at configuration C_{t_1} . Since the number of robots is three and the distance between some pair of robots is at least five at C_{t_1} , the tower includes two robots and the views of the robots are symmetric. The view of a robot not in the tower is also symmetric. Hence, we can define their territories similarly to Lemma 4. This implies robots do not go out of their territories before they break the tower. Therefore, the lemma holds.

Let t_2 be the smallest instant such that the distance between some pair of two robots is at least five and no tower exists at C_{t_2} . Since at least $10^3 - 3$ unexplored nodes exist at C_{t_1} , at least $10^3 - 7$ unexplored nodes exist at C_{t_2} from Lemma 20.

Lemma 21 Configuration C_{t_2} includes sub-configuration $GW\emptyset^i X$ or $WG\emptyset^i X$ for some $i \geq 3$ and $X \in Col$ (or sub-configuration symmetric to one of them).

Proof: For contradiction, assume that C_{t_2} does not include such sub-configurations. Since the distance between some pair of two robots is at least five and no tower exists, C_{t_2} includes either 1) sub-configuration $Y\emptyset^iZ\emptyset^jX$ for some $i+j\geq 3$ and $X,Y,Z\in Col,$ 2) sub-configuration $YY\emptyset^iX$ for some $i\geq 3$ and $X,Y\in Col,$ or 3) sub-configuration symmetric to Case 1 or 2.

In Case 1, we can define an independent territory set. This implies robots cannot visit remaining unexplored nodes from Lemma 4. Thus, they cannot achieve exploration.

Let us consider Case 2. We assume robots r_1 and r_2 have color Y and r_3 has color X at C_{t_2} . Since the scheduler can make r_3 move forward and backward, r_1 and r_2 should move to achieve exploration. Since the view of r_1 is identical (symmetric) to the view of r_2 , r_1 and r_2 make symmetric behaviors. If r_1 moves toward r_2 , r_1 and r_2 swap their positions and visit no unexplored nodes. If r_1 moves against r_2 , its sub-configuration becomes $Y'\emptyset\emptyset Y'\emptyset^jX'$ for some $j \in \{i-2,i-1,i\}$ and $X',Y' \in Col$. Here Y' is a new color of r_1 and r_2 , and j and X' depend on the behavior of r_3 . For any case, this sub-configuration reduces to Case 1. This implies robots cannot achieve exploration.

We consider Case 3 similarly to Cases 1 and 2. For every case, robots cannot achieve exploration. \Box

From Lemma 21, at configuration C_{t_2} , two robots with colors G and W are away from the other robot. Since the remaining isolated robot cannot explore by itself, two robots with colors G and W explore the remaining part of the ring. The following lemma proves that A includes some rules to realize this behavior.

Lemma 22 We consider the following rules:

• $0WGX : \emptyset(W)G :: X, \rightarrow (for some X \in Col)$

• $0 GWX : \emptyset(G)W :: X, \rightarrow (for some X \in Col)$

Algorithm A includes a rule 0WGX or 0GWX

Proof: For contradiction, assume that \mathcal{A} includes neither 0WGX nor 0GWX. From Lemma 21, at configuration C_{t_2} , two robots with colors G and W are away from the other robot and compose sub-configuration GW or WG. Since sub-configurations GW and WG are symmetric, we consider only sub-configuration GW. To complete exploration, the two robots must explore the remaining part of the ring.

Since \mathcal{A} includes neither 0WGX nor 0GWX, a robot in a sub-configuration GW does not move toward another robot in GW. We consider three cases about the movement of robots in GW:

1. One or both of robots in GW move against the other robots in GW. In this case, these two robots become isolated. At this configuration, we can define an independent territory set, and thus robots cannot achieve exploration from Lemma 4.

- 2. Both robots in GW change their colors and do not move. In this case, robots just swap their colors and thus the views of them are not changed. This implies robots repeatedly change their colors and never move. Therefore, robots cannot achieve exploration.
- 3. One robot in GW changes its color. That is, two robots in GW at C_{t_2} compose sub-configuration WW or GG at C_{t_2+1} . Recall that, from Lemma 21, sub-configurations of C_{t_2} include $\mathsf{GW}\emptyset^i X$ or $\mathsf{WG}\emptyset^i X$ for some $i \geq 3$ and $X \in Col$. This implies sub-configurations of C_{t_2+1} include $YY\emptyset^j X'$ for some $j \in \{i, i+1\}$ and some $X', Y \in Col$. Here X' and j depend on the behavior of a robot with color X at C_{t_2} . As proved in Case 2 of Lemma 21, from sub-configuration $YY\emptyset^j X'$, robots cannot achieve exploration.

For every case, robots cannot achieve exploration. This is a contradiction.

From Lemma 22, \mathcal{A} includes a rule 0WGX or 0GWX. The following lemma shows, if \mathcal{A} includes rule 0WGX or 0GWX, \mathcal{A} cannot solve terminating exploration from some solvable configuration.

Lemma 23 1) If A includes rule 0WGX, A cannot solve terminating exploration from configuration WGW. 2) If A includes rule 0GWX, A cannot solve terminating exploration from configuration GWG.

Proof: Without loss of generality, we consider only statement 1 (Statement 2 is obtained by swapping the roles of two colors). Assume that, initially, robots r_1 , r_2 , and r_3 construct a configuration WGW in this order. Since \mathcal{A} includes rule 0WGX, the next configuration includes a sub-configuration

$$\begin{array}{c} X & X \\ Y & X \end{array}$$
 or $\begin{array}{c} X \\ Y \end{array}$

for some $X, Y \in Col$. That is, r_1 and r_3 assign X to its color, move based on rule $0\mathsf{WG}X$, and makes a tower, and r_2 assigns Y to its color and moves (or stays) based on some rule. Note that r_1 and r_3 stay at the same node with the same color. This implies, since the views of r_1 and r_3 are identical, they move together after this configuration. Similarly to Lemma 19, the distance between two agents becomes at least five. In this configuration, since r_1 and r_3 move together, one node is occupied by r_1 and r_3 with the same color, and another node is occupied by r_2 . Since we can define an independent territory set, robots cannot achieve exploration from Lemma 4. That is, \mathcal{A} cannot solve terminating exploration from configuration WGW.

From Lemmas 22 and 23, \mathcal{A} cannot solve terminating exploration from configuration WGW or GWG. However, from Lemma 10, configurations WGW and GWG are solvable. This contradicts to the assumption that \mathcal{A} is universal. Therefore, Theorem 11 holds.

B Proof of Theorem 14

To prove Theorem 14, we show that initial configurations other than ones in Theorem 13 are unsolvable for $n \ge 9$ in the SSYNC model. First we prove the following simple lemma.

Lemma 24 Consider a configuration C such that, for some node v_i , v_i is occupied by at least one robot, all robots on v_i have the same color, and v_{i-1} and v_{i+1} are occupied by no robot. After configuration C, the following statements hold.

- (1) If no robots appear in v_{i-2} and v_{i+1} , robots on v_i cannot visit nodes other than v_{i-1} and v_i .
- (2) If no robots appear in v_{i+2} and v_{i-1} , robots on v_i cannot visit nodes other than v_{i+1} and v_i .

Proof: We prove only statement (1) because we can similarly prove statement (2). Since robots on v_i have the same color, they have the same view. Consequently, if the scheduler repeatedly activates them at the same time, they continue to make the same behaviors. In addition, since the views of robots on v_i are symmetric, the scheduler decides the directions of their movements. Consequently, robots on v_i move to v_{i-1} if they decide to move. From statement (1), no robots exist on v_{i-2} and v_i , and hence the views of the robots are the same as in the previous configuration. By repeating such behaviors, the robots can visit only v_{i-1} and v_i . Therefore, the lemma holds. \square

In the following, we show that initial configurations other than ones in Theorem 13 are unsolvable.

Lemma 25 Assume $n \ge 6$. Let C be a configuration such that two or three robots with the same color stay at a single node. In this case, C is unsolvable in the SSYNC model.

Proof: We consider three cases 1) three robots with the same color stay at a single node, 2) two robots with the same color stay at a single node and the distance to another robot is at least two, 3) two robots with the same color stay at a single node and the distance to another robot is one, and 4) three robots stay at a single node but one robot has a different color from other two robots. In Cases 1 and 2, robots cannot achieve exploration from Lemma 4.

To consider Cases 3 and 4, assume that r_1 and r_2 have the same color and stay at node v, and r_3 stays at v or a neighbor of v. We consider the scheduler that repeats 1) activation of r_1 and r_2 and 2) activation of r_3 . Since r_1 and r_2 have the same view, they make the same behavior. If r_1 and r_2 join the node with r_3 and they have the same color as r_3 , this reduces to Case 1. If such a situation never happens, to achieve exploration, eventually r_1 and r_2 move against r_3 , or r_3 moves against r_1 and r_2 . Both cases reduce to Case 2. Therefore, the lemma holds.

Lemma 26 Assume $n \geq 9$. Let C be a configuration such that the longest distance between two robots is at least four. In this case, C is unsolvable in the SSYNC model.

Proof: We consider three robots r_1 , r_2 , and r_3 . If the distance between any pair of robots is at least three at C, robots cannot achieve exploration from Lemma 4. To consider the remaining cases, without loss of generality, we assume that the distance between r_1 and r_2 is at most two at C. We assume that r_1 occupies v_0 and r_2 occupies v_0 , v_1 , or v_2 at C. Let v_d be the node occupied by r_3 at C. Since the distance between some pair of two robots is at least four, we assume $d \ge 4$ and d < n - 4 hold without loss of generality.

In the following, we consider the scheduler that activates one robot at each instant. We prove that, under this scheduler, robots cannot achieve exploration from every possible configuration. More concretely, we prove that r_1 and r_2 always stay at v_{-1} (i.e., v_{n-1}), v_0 , v_1 , or v_2 . This implies, since no robots appear in v_{d-1} and v_{d+2} , r_3 cannot visit nodes other than v_d and v_{d+1} from Lemma 24. Therefore, robots cannot achieve exploration.

First consider the case that r_2 occupies v_2 at C. In this case, the view of every robot is symmetric. Consequently, if a robot moves, its direction is decided by the scheduler. This implies

 r_1 cannot move because, if r_1 moves, it moves to v_{-1} (i.e., v_{n-1}), and from Lemma 4, robots cannot achieve exploration. If r_2 moves, it moves to v_1 and this reduces the next case.

Next, consider the case that r_2 occupies v_1 at C. We consider two sub-cases.

- Case that r_1 can move. If r_1 moves to v_{-1} , robots cannot achieve exploration from Lemma 4. Assume that r_1 moves to v_1 , and let C' be the resultant configuration. Then the views of r_1 and r_2 become symmetric at C'. This implies, if they move, the directions of their movements are decided by the schedule. Hence, if r_1 or r_2 moves, it moves to v_0 . This makes the configuration go back to C (possibly, r_1 and r_2 are swapped). If neither r_1 nor r_2 moves at C', clearly robots cannot achieve exploration.
- Case that r_2 can move. If r_2 moves to v_2 , this reduces to the case that r_2 occupied v_2 . Assume that r_2 moves to v_0 , and let C' be the resultant configuration. Then the views of r_1 and r_2 become symmetric at C'. This implies, if they move, the directions of their movements are decided by the schedule. Hence, if r_1 or r_2 moves, it moves to v_1 . This makes the configuration go back to C (possibly, r_1 and r_2 are swapped). If neither r_1 nor r_2 moves at C', clearly robots cannot achieve exploration.

Next, consider the case that r_2 occupies v_0 at C. In this case, the views of r_1 and r_2 are symmetric, and hence, if they move, the directions of their movements are decided by the scheduler. If r_1 or r_2 moves, it moves to v_1 . This reduces to the case that r_1 and r_2 occupy nodes v_0 and v_1 at C. If neither r_1 nor r_2 moves, clearly robots cannot achieve exploration.

From the above discussion, r_1 and r_2 always stay at v_{-1} , v_0 , v_1 , or v_2 . Therefore robots cannot achieve exploration, and consequently the lemma holds.

Lemma 27 Assume $n \geq 9$. Let C be a configuration such that the longest distance between two robots is three. In this case, C is unsolvable in the SSYNC model.

Proof: Consider three robots r_1 , r_2 , and r_3 . Without loss of generality, we assume that, at C, robots r_1 and r_3 occupy v_0 and v_3 , respectively, and r_2 occupies v_0 or v_1 .

First, we claim that r_3 never moves as long as nodes v_2 and v_4 are free. Otherwise, by activating r_3 several times after C, the scheduler can make r_3 move. Since the view of r_3 is symmetric, the direction of the movement is also decided by the scheduler. Hence r_3 moves to v_4 and the resultant configuration is unsolvable from Lemma 26. Therefore, the above claim holds.

Consider the case that r_2 occupies v_0 at C. Since r_3 cannot move, r_1 or r_2 should move. Since the views of r_1 and r_2 are symmetric, the directions of their movements are decided by the scheduler. Hence, r_1 or r_2 moves to v_{-1} , and the resultant configuration is unsolvable from Lemma 26.

Consider the case that r_2 occupies v_1 at C. Since r_3 cannot move, r_1 or r_2 should move. We consider the following three sub-cases.

- If r_1 moves to v_{-1} , the resultant configuration is unsolvable from Lemma 26.
- If r_1 moves to v_1 (resp., if r_2 moves to v_0), r_1 and r_2 occupy v_1 (resp., v_0) at the resultant configuration C'. Since r_3 cannot move at C', r_1 or r_2 should move after C'. Since views of r_1 and r_2 are symmetric, the directions of their movements are decided by the scheduler. Hence, r_1 or r_2 moves to v_0 (resp., v_1), and this makes the configuration go back to C (possibly, r_1 and r_2 are swapped).

• If r_2 moves to v_2 , r_1 , r_2 , and r_3 occupies v_0 , v_2 , v_3 , respectively. Since we do not consider colors here, the resultant configuration is symmetric to C considered in the current case (i.e., r_1 , r_2 , and r_3 occupies v_0 , v_1 , v_3).

For all sub-cases, the scheduler can make the configuration unsolvable or move robots infinitely among nodes v_0 , v_1 , v_2 , and v_3 . Hence, robots cannot achieve exploration in this case.

Lemma 28 Assume $n \geq 9$. Let C be a configuration such that the longest distance between two robots is two and two robots occupy a single node. In this case, C is unsolvable in the SSYNC model.

Proof: We consider three robots r_1 , r_2 , and r_3 . Without loss of generality, we assume that r_1 and r_2 occupy v_1 and r_3 occupies v_3 . Since views of all robots are symmetric, the directions of their movements are decided by the scheduler. Hence, if r_1 or r_2 moves, it moves to v_0 , and if r_3 moves, it moves to v_4 . For both cases, the longest distance between two robots becomes three. Therefore, robots cannot achieve exploration from Lemma 27.

Lemma 29 Assume $n \geq 9$. Let C be a configuration XYX for some $X \in Col$ and $Y \in Col$. In this case, C is unsolvable in the SSYNC model.

Proof: Without loss of generality, we assume that robots r_1 , r_2 , and r_3 have colors X, Y, and X and occupy v_1 , v_2 , and v_3 , respectively. We consider the scheduler that repeats the following activation: Activate r_1 and r_3 at the same time, and then activate r_2 .

First consider the case that r_1 and r_3 move. Since r_1 and r_3 have the same views, they make symmetric behaviors. If r_1 and r_3 move to v_2 , r_1 and r_3 occupy v_2 and they have the same color. Hence, the resultant configuration is unsolvable from Lemma 25. If r_1 and r_3 move to v_0 and v_4 , respectively, the distance between r_1 and r_3 becomes four. Hence, the configuration is unsolvable from Lemma 26.

Next consider the case that r_2 moves. In this case, r_2 joins r_1 or r_3 and makes a tower. The resultant configuration is unsolvable from Lemma 28.

From Lemmas 26 and 27, the longest distance between two robots is at most two. Three robots must be connected from Lemma 28, and they must not form a symmetric configuration from Lemma 29. In addition, robots with the same color must not occupy a single node from Lemma 25. This implies all configurations other than initial configurations described in Theorem 13 are unsolvable. Clearly, if configuration C is unsolvable in the FSYNC model, C is also unsolvable in the ASYNC model. Therefore, we have Theorem 14.

C Proof of Theorem 18

To prove Theorem 18, we show the following lemma.

Lemma 30 Assume that $\phi = 1$, k = 4, and $Col = \{G, W\}$. We define a set of configurations C_{sol} as follows:

 $C_{sol} = \{WWGG, WWWG, WWGW, GGWW, GWWW, WGWW, GGGW, GGWG, WGGG, GWGG\}.$

In the SSYNC model, no algorithm exists that solves terminating exploration from any initial configuration in C_{sol} .

From Lemma 17, C_{sol} is a subset of solvable initial configurations in the SSYNC and ASYNC models. Hence, universal algorithms for the SSYNC model must solve terminating exploration from any initial configuration in C_{sol} in the SSYNC model. In addition, universal algorithms for the ASYNC model must also solve terminating exploration from any initial configuration in C_{sol} in the SSYNC model. However, Lemma 30 implies no such algorithms exist. This implies no universal algorithm exists in the SSYNC and ASYNC model. Therefore, we have Theorem 18.

In the rest of this section, we prove Lemma 30 by contradiction. Assume that there exists an algorithm \mathcal{A} that solves terminating exploration from any initial configuration in \mathcal{C}_{sol} in the SSYNC model. Let $d_1 = 10^3 + 1$, $d_2 = 3d_1$, and $d_3 = (2d_2)^4$. We consider a ring with $n > 5d_3$ nodes and an execution $E = C_0, C_1, \ldots$ of \mathcal{A} in the ring. Here we assume that, in each instant, the scheduler activates at least one robot that changes its position or its color unless \mathcal{A} terminates. That is, $C_i \neq C_{i+1}$ holds for every $i \leq t$, where t is the instant such that \mathcal{A} terminates in C_t . Note that, since four robots can visit at most four new nodes in each configuration, E includes at least $n/4 > (5/4)d_3$ configurations.

The outline of the proof is as follows. In Lemmas 31 and 32, we give unsolvable configurations. Then, we prove that the distance between some pair of two robots becomes at least d_2 (Lemma 33). We also prove that, in Lemmas 35 and 36, that some three robots must explore the ring by always forming a sub-configuration in

$$\mathcal{C}_{exp} = \left\{ \mathsf{WGG}, \mathsf{GGW}, \mathsf{GWW}, \mathsf{WWG}, \begin{matrix} \mathsf{G} & \mathsf{G} & \mathsf{W} & \mathsf{W} \\ \mathsf{W} & \mathsf{W} & \mathsf{W} & \mathsf{G} & \mathsf{G} & \mathsf{G} \end{matrix} \right\}.$$

Note that these sub-configurations are identical to ones used in the perpetual exploration algorithm in Section 4. The above fact limits a possible set of rules used in algorithm \mathcal{A} . Lastly, we prove that, for every possible set of rules, the set of rules makes some configuration in \mathcal{C}_{sol} unsolvable (Lemmas 38 and 39). This contradicts to the assumption that \mathcal{A} solves terminating exploration from any initial configuration in \mathcal{C}_{sol} .

First, we give examples of unsolvable configurations.

Lemma 31 Let C be a configuration that contains a sub-configuration in

$$\mathcal{C}_{sym} = \left\{ \mathit{WGGW}, \mathit{GWWG}, \mathit{WWWW}, \mathit{GGGG}, \dfrac{\mathit{W}}{\mathit{W}} \dfrac{\mathit{G}}{\mathit{G}} \dfrac{\mathit{W}}{\mathit{W}} \dfrac{\mathit{G}}{\mathit{G}} \dfrac{\mathit{W}}{\mathit{G}} \dfrac{\mathit{W}}{\mathit{W}} \dfrac{\mathit{W}}{\mathit{G}} \dfrac{\mathit{W}}{\mathit{G}} \dfrac{\mathit{W}}{\mathit{G}} \dfrac{\mathit{W}}{\mathit{W}} \dfrac{\mathit{W}}{\mathit{G}} \dfrac{\mathit{W}}{\mathit{G}} \dfrac{\mathit{W}}{\mathit{W}} \dfrac{\mathit{W}}{\mathit{W}} \dfrac{\mathit{W}}{\mathit{G}} \dfrac{\mathit{W}}{\mathit{W}} \dfrac{\mathit{W}} \dfrac{\mathit{W}}{\mathit{W}} \dfrac{\mathit{W}}{\mathit{W}} \dfrac{\mathit{W}} \dfrac{\mathit{W}}{\mathit{W}} \dfrac{\mathit{W}} \dfrac{\mathit{W}} \dfrac{\mathit{W}}{\mathit{W}}$$

In this case, C is unsolvable.

Proof: Note that configurations in C_{sym} are symmetric and the axis of symmetry goes through a link between robots. We consider the scheduler that always activates two symmetric robots. Clearly, all robots keep their symmetry. Hence, to explore the ring, one or two robots must continue to move in one direction (and other one or two robots move in another direction). However, as shown in Lemma 24 and Theorem 12, one or two robots cannot continue to move in one direction. Therefore, robots cannot achieve exploration from configurations in C_{sym} .

Lemma 32 Let C be a configuration such that two robots with the same color occupy a single node. In this case, C is unsolvable.

Proof: Let r_1 and r_2 be robots that have the same color and occupy a single node. We consider the scheduler that always activates r_1 and r_2 at the same time. Clearly, since r_1 and r_2 have the same view, they make the same behaviors. This means r_1 and r_2 move as if they were a single robot. Therefore, similarly to Theorem 15, robots cannot achieve exploration from configuration C.

Next, we consider an execution E of algorithm \mathcal{A} , and prove that the distance between some pair of two robots becomes large in E.

Lemma 33 The distance between some pair of two robots becomes at least d_2 before the $d_3 = (2d_2)^4$ -th configuration of execution E.

Proof: For contradiction, assume that the distance between any pair of robots is at most $d_2 - 1$ until d_3 -th configuration of E. For each configuration C_i ($0 \le i \le d_3$), we define C'_i as a sub-configuration of C_i such that C'_i contains d_2 nodes (i.e., the length of C'_i is d_2), every robot occupies one of the d_2 nodes, and some robot occupies the first node of C'_i . Since each robot has one of two colors and occupies one of d_2 nodes, the number of such sub-configurations is at most d_3 . Hence, for some t and u ($t < u \le d_3$), sub-configurations C'_t and C'_u are identical. In the following, we consider two cases.

The first case is that the d_2 nodes contained in C'_t and C'_u are identical. This implies configurations C_t and C_u are identical. Since \mathcal{A} is deterministic, robots repeat the behavior from C_t to C_u after C_u . Since robots can visit at most $4d_3$ nodes before configuration C_t , they cannot visit remaining nodes after C_u . Thus robots cannot complete exploration.

The second case is that the d_2 nodes contained in C'_t and C'_u are not identical. This means robots change their positions from C_t to C_u . However, since no robots exist out of the d_2 nodes, the behaviors of robots depend only on the sub-configuration. Hence, robots repeat the behaviors from C_t to C_u after C_u . This implies that robots continue to explore the ring, and thus they cannot terminate.

For both cases, robots cannot achieve terminating exploration. This is a contradiction. \Box

From Lemma 33, the distance between some pair of two robots becomes at least d_2 before the d_3 -th configuration. Let t_1 be the smallest instant such that the distance between some pair of two robots is at least d_2 at C_{t_1} . Clearly $t_1 \leq d_3$ holds. Since four robots can visit at most $4(t_1 + 1)$ nodes before configuration C_{t_1} , at least $d_3 - 4$ unexplored nodes exist at C_{t_1} . Next, we consider instant $t_2 = t_1 + d_1$. Since robots can visit at most $4d_1$ nodes during configurations C_{t_1} to C_{t_2} , at least $d_3 - 4 - 4d_1$ unexplored nodes exist at C_{t_2} .

In Lemmas 34 to 36, we show that there exist three robots that explore the network by forming a sub-configuration in C_{exp} from C_{t_1} to C_{t_2} .

Lemma 34 Let C be a configuration in $\{C_{t_1}, C_{t_1+1}, \ldots, C_{t_2}\}$. At configuration C, there exist three robots such that the distance among them is at most four.

Proof: Recall that the distance between some pair of two robots is at least d_2 at configuration C_{t_1} . Since the distance between two robots decreases by at most two after one cycle, the distance between some pair of two robots is at least $d_2 - 2d_1 \ge d_1$. We consider four robots r_1 , r_2 , r_3 , and r_4 . Let $d \ge d_1$ be the longest distance between two robots. Without loss of generality, we assume the distance between r_1 and r_4 is d, r_1 occupies v_0 , and r_4 occupies v_d at C. We can also assume that

 r_2 and r_3 occupy v_i and v_j ($0 \le i \le j \le d$), respectively, and $i \le d - j$ holds. For contradiction, we assume j > 4.

Consider the case that i=2 or i=3 holds. Since j>4 and $j\leq d-2$ holds, we can define an independent territory set $\{\{v_{-1},v_0\},\{v_2,v_3\},\{v_j,v_{j+1}\},\{v_d,v_{d+1}\}\}$. From Lemma 4, robots cannot explore the ring.

Consider the case that i=0 or i=1 holds, that is, r_1 and r_2 are connected. First, we show that r_1 and r_2 continue to occupy nodes in $\{v_h|-2\leq h\leq 3\}$ as long as no robot appears on v_{-3} or v_4 . In the case of i=0, if r_1 or r_2 moves, it moves v_1 because the direction of its movement is decided by the scheduler. This reduces to the case of i=1. Let us consider the case of i=1. If r_1 moves to v_1 or r_2 moves to v_0 , they make a tower. In this case, after r_1 or r_2 moves again, the configuration goes back to the previous one (possibly, r_1 and r_2 are swapped). Hence, we assume that eventually r_1 moves to v_{-1} or r_2 moves to v_2 . If r_1 moves to v_{-1} , we can define territories $\{v_{-2},v_{-1}\}$ for r_1 and $\{v_1,v_2\}$ for r_2 . Similarly, if r_2 moves to v_2 , we can define territories $\{v_{-1},v_0\}$ for r_1 and $\{v_2,v_3\}$ for r_2 . Hence, as long as no robot appears on v_{-3} or v_4 , r_1 and r_2 continue to occupy nodes in $\{v_h|-2\leq h\leq 3\}$. Let us consider r_3 and r_4 in this case. If r_3 and r_4 are connected at C, we can show the same proposition as r_1 and r_2 . That is, as long as no robot appears on v_{d-4} or v_{d+3} , r_3 and r_4 continue to occupy nodes in $\{v_{h'}|d-3\leq h'\leq d+2\}$. Therefore, in this case, the four robots cannot achieve exploration. If r_3 and r_4 are not connected at C, we can define territories for r_3 and r_4 such that r_3 and r_4 do not go out of their territories. Hence, robots cannot achieve exploration.

Now we consider the remaining case i > 3. Clearly r_1 continues to occupy v_0 or v_{-1} as long as no robot appears on v_{-1} or v_{-1} and v_{-1} or v_{-1} and v_{-1} or v_{-1} . First consider the case that v_{-1} and v_{-1} are connected. As described above, v_{-1} and v_{-1} or v_{-1}

For all cases, robots cannot explore the ring. Hence, $j \leq 4$ holds. That is, the distance among robots r_1 , r_2 , and r_3 is at most four.

From Lemma 34, for each configuration C in $\{C_{t_1}, C_{t_1+1}, \ldots, C_{t_2}\}$, there exist three robots such that the distance among them is at most four. Note that, since the distance from one of the three robots to another robot is at least d_1 , a set of three robots within distance four does not change from C_{t_1} to C_{t_2} . In the following, we define r_1 , r_2 , and r_3 as three robots that stay within distance four from C_{t_1} to C_{t_2} . Let r_4 be another robot. In the following lemma, we show that r_1 , r_2 , and r_3 execute a perpetual exploration algorithm from C_{t_1} to C_{t_2} .

Lemma 35 Consider configuration C^* such that r_1 , r_2 , and r_3 stay on the same node with the same color as configuration C_{t_1} and r_4 does not exist. Then, from configuration C^* , robots r_1 , r_2 , and r_3 achieve a perpetual exploration.

Proof: First we consider configurations from C_{t_1} to C_{t_2} . From Lemma 34, three robots r_1 , r_2 , and r_3 always stay within distance four. Hence, for each configuration C_i for $t_1 \leq i \leq t_2$, we can

define C'_i as a sub-configuration of C_i such that C'_i contains five nodes (i.e., the length of C'_i is five), each of r_1 , r_2 , and r_3 occupies one of the five nodes, and some robot occupies the first node of C'_i . Since each robot has one of two colors and occupies one of five nodes, the number of such sub-configurations is at most $10^3 < d_1$. Hence, for some g and h ($t_1 \le g < h \le t_2$), sub-configurations C'_g and C'_h are identical. We consider two cases.

The first case is that the five nodes contained in C'_g and C'_h are identical. This implies that r_1 , r_2 , and r_3 do not change their positions and colors in C_g and C_h . Hence, unless r_1 , r_2 , or r_3 observes r_4 , they repeat the behavior from C_g to C_h after C_h and consequently continue to visit the same nodes. On the other hand, r_4 is far from the three robots and cannot go out of its territory (i.e., its current and neighboring nodes). This implies that r_1 , r_2 , and r_3 do not observe r_4 , and thus they cannot visit the remaining unexplored nodes.

Let us consider another case, that is, the five nodes contained in C'_g and C'_h are different. This implies r_1 , r_2 , and r_3 change their positions from C_g to C_h , that is, they move forward or backward in the ring. Clearly, the three robots do not observe r_4 from C_{t_1} to C_h . This implies that, if r_4 does not exist, they repeat the behavior from C_g to C_h and continue to explore the ring. That is, they can achieve a perpetual exploration. Therefore, the lemma holds.

Lemma 36 From C_{t_1} to C_{t_2} , robots r_1 , r_2 , and r_3 form a sub-configuration in C_{exp} .

Proof: For contradiction, assume that r_1 , r_2 , and r_3 form a sub-configuration C_s not in C_{exp} . From Lemma 35, if r_4 does not exist, the three robots achieve perpetual exploration from configuration C_s . However, C_s is unsolvable for perpetual exploration from Lemmas 25, 26, 27, 28, and 29. This is a contradiction.

From Lemmas 35 and 36, during configurations C_{t_1} to C_{t_2} , robots r_1 , r_2 , and r_3 continue to form a sub-configuration in C_{exp} and explore the ring. That is, they must change their colors and positions from a sub-configuration to another sub-configuration in C_{exp} . We show transitions among sub-configurations in Fig. 7, and list all rules to realize such transitions as follows. Note that, in Fig. 7, we do not distinguish symmetric sub-configurations such as GWW and WWG.

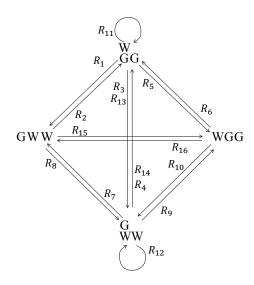


Figure 7: Transitions among configurations in C_{exp}

$$\mathcal{R}_{1} : \underset{\emptyset}{W} (G) G :: W, \leftarrow \qquad \mathcal{R}_{2} : \emptyset(W)W :: G, \rightarrow \\ \mathcal{R}_{3} : \underset{\emptyset}{W} (G) G :: W, \rightarrow \qquad \mathcal{R}_{4} : \underset{\emptyset}{G} (W) W :: G, \rightarrow \\ \mathcal{R}_{5} : \underset{\emptyset}{G} (W) G :: W, \leftarrow \qquad \mathcal{R}_{6} : \emptyset(W)G :: W, \rightarrow \\ \mathcal{R}_{7} : \underset{\emptyset}{W} (G) W :: G, \leftarrow \qquad \mathcal{R}_{8} : \emptyset(G)W :: G, \rightarrow \\ \mathcal{R}_{9} : \underset{\emptyset}{G} (W) W :: G, \leftarrow \qquad \mathcal{R}_{10} : \emptyset (G) G :: W, \rightarrow \\ \mathcal{R}_{11} : \underset{\emptyset}{G} (W) G :: W, \rightarrow \qquad \mathcal{R}_{12} : \underset{\emptyset}{W} (G) W :: G, \rightarrow \\ \mathcal{R}_{13} : \underset{\emptyset}{W} (G) G :: W, \bot \qquad \mathcal{R}_{14} : \underset{\emptyset}{W} (W) W :: G, \bot \\ \mathcal{R}_{15} : G (W) W :: G, \bot \qquad \mathcal{R}_{16} : G (G) W :: W, \bot$$

The next lemma shows \mathcal{A} includes neither \mathcal{R}_{15} nor \mathcal{R}_{16} .

Lemma 37 Algorithm \mathcal{A} includes neither \mathcal{R}_{15} nor \mathcal{R}_{16} .

Proof: Assume that \mathcal{A} includes \mathcal{R}_{15} for contradiction. Consider a configuration WGWW $\in \mathcal{C}_{sol}$. This configuration transits to WGGW by \mathcal{R}_{15} . From Lemma 31, configuration WGGW is unsolvable. This is a contradiction. Similarly we can show that \mathcal{A} does not include \mathcal{R}_{16} .

Since three robots that form a sub-configuration in C_{exp} have to explore the ring, they repeat state transitions by the above rules. This means A includes a set of rules that form a (simple) cycle in Fig. 7. We list all such sets of rules as follows. Note that, from Lemma 37, each set includes neither \mathcal{R}_{15} nor \mathcal{R}_{16} .

$$\begin{split} &\{\mathcal{R}_{11}\}, \, \{\mathcal{R}_{12}\} \\ &\{\mathcal{R}_{1}, \mathcal{R}_{2}\}, \, \{\mathcal{R}_{3}, \mathcal{R}_{4}\}, \, \{\mathcal{R}_{3}, \mathcal{R}_{14}\}, \, \{\mathcal{R}_{13}, \mathcal{R}_{4}\}, \, \{\mathcal{R}_{13}, \mathcal{R}_{14}\} \\ &\{\mathcal{R}_{5}, \mathcal{R}_{6}\}, \, \{\mathcal{R}_{7}, \mathcal{R}_{8}\}, \, \{\mathcal{R}_{9}, \mathcal{R}_{10}\} \\ &\{\mathcal{R}_{2}, \mathcal{R}_{3}, \mathcal{R}_{7}\}, \, \{\mathcal{R}_{2}, \mathcal{R}_{13}, \mathcal{R}_{7}\}, \, \{\mathcal{R}_{8}, \mathcal{R}_{4}, \mathcal{R}_{1}\}, \, \{\mathcal{R}_{8}, \mathcal{R}_{14}, \mathcal{R}_{1}\}, \\ &\{\mathcal{R}_{10}, \mathcal{R}_{4}, \mathcal{R}_{5}\}, \, \{\mathcal{R}_{10}, \mathcal{R}_{14}, \mathcal{R}_{5}\}, \, \{\mathcal{R}_{6}, \mathcal{R}_{3}, \mathcal{R}_{9}\}, \, \{\mathcal{R}_{6}, \mathcal{R}_{13}, \mathcal{R}_{9}\}, \\ &\{\mathcal{R}_{2}, \mathcal{R}_{5}, \mathcal{R}_{10}, \mathcal{R}_{7}\}, \, \{\mathcal{R}_{8}, \mathcal{R}_{9}, \mathcal{R}_{6}, \mathcal{R}_{1}\} \end{split}$$

However, \mathcal{A} cannot include some of the above sets. For example, let us consider a set $\{\mathcal{R}_2, \mathcal{R}_5, \mathcal{R}_{10}, \mathcal{R}_7\}$. Starting from sub-configuration GWW, the three robots change their sub-configuration as follows.

$$\mathsf{GWW} \xrightarrow{\mathcal{R}_2} \mathsf{G} \overset{\mathsf{W}}{\mathsf{G}} \emptyset \xrightarrow{\mathcal{R}_5} \mathsf{GGW} \xrightarrow{\mathcal{R}_{10}} \emptyset \overset{\mathsf{G}}{\mathsf{W}} \mathsf{W} \xrightarrow{\mathcal{R}_7} \mathsf{GWW}$$

Note that these sub-configurations include the same nodes. This implies that, when the robots form GWW again, they do not change their positions. Hence, if \mathcal{A} includes this set, the three robots cannot explore the ring. On the other hand, let us consider a set $\{\mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_7\}$. Starting from sub-configuration GWW, the three robots change their sub-configuration as follows.

$$\emptyset \mathsf{GWW} \xrightarrow{\mathcal{R}_2} \emptyset \ \mathsf{G} \ \mathsf{G} \ \emptyset \xrightarrow{\mathcal{R}_3} \emptyset \ \mathsf{W} \ \mathsf{W} \ \emptyset \xrightarrow{\mathcal{R}_7} \mathsf{GWW} \emptyset$$

In this case, when the robots form GWW again, they change their positions. Hence, by this set of rules, the three robots can explore the ring. By considering all sets of rules, the following four sets of rules allow robots to explore the ring:

$$\{\mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_7\}, \{\mathcal{R}_8, \mathcal{R}_4, \mathcal{R}_1\}, \{\mathcal{R}_{10}, \mathcal{R}_4, \mathcal{R}_5\}, \{\mathcal{R}_6, \mathcal{R}_3, \mathcal{R}_9\}.$$

That is, \mathcal{A} includes at least one of the above four sets of rules. However, in the following lemmas, each set of rules makes some configuration in \mathcal{C}_{sol} unsolvable.

Lemma 38 If A includes $\{R_2, R_3, R_7\}$ or $\{R_{10}, R_4, R_5\}$, A cannot solve terminating exploration from some configuration in C_{sol} .

Proof: First, we consider the case that \mathcal{A} includes $\{\mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_7\}$. Let us consider a configuration WGWW $\in \mathcal{C}_{sol}$. From rules \mathcal{R}_2 and \mathcal{R}_3 , the configuration changes as follows.

$$\mathsf{WGWW} \xrightarrow{\mathcal{R}_2} \mathsf{W} \overset{\mathsf{W}}{\mathsf{G}} \overset{\mathsf{W}}{\mathsf{G}} \xrightarrow{\mathcal{R}_3} \mathsf{W} \overset{\mathsf{W}}{\mathsf{G}} \mathsf{W}$$

Let C_a be the last configuration. Since robots cannot move from C_a by rules \mathcal{R}_2 , \mathcal{R}_3 , \mathcal{R}_7 , \mathcal{A} includes some other rules to move robots from C_a . We consider all possibilities in the following.

• \mathcal{A} includes $\mathcal{R}'_1: \frac{\mathsf{W}}{\mathsf{W}(\mathsf{G})\;\mathsf{W}} :: \mathsf{G}, \leftarrow \vee \rightarrow$. In this case, robots change their states as follows.

$$\begin{array}{c} W \\ W \ G \ W \end{array} \xrightarrow{\mathcal{R}_1'} W \ W \ W \ W \xrightarrow{G} \begin{array}{c} \mathcal{R}_2 \\ W \ W \ W \end{array}$$

From Lemma 31, the last configuration is unsolvable.

• \mathcal{A} includes $\mathcal{R}_2': \frac{\mathsf{W}}{\mathsf{W}(\mathsf{G})\mathsf{W}} :: \mathsf{W}, \leftarrow \vee \rightarrow$. In this case, robots change their states as follows.

$$\overset{\mathsf{W}}{\mathsf{W}}\overset{\mathsf{W}}{\mathsf{G}}\overset{\mathsf{W}}{\mathsf{W}}\overset{\mathcal{R}_2'}{\longrightarrow}\overset{\mathsf{W}}{\mathsf{W}}\overset{\mathsf{W}}{\mathsf{W}}$$

From Lemma 32, the last configuration is unsolvable.

• \mathcal{A} includes $\mathcal{R}_3': \frac{\mathsf{W}}{\mathsf{W}(\mathsf{G})\;\mathsf{W}} :: \mathsf{W}, \perp$. In this case, robots change their states as follows.

$$\begin{array}{c} W \\ W G W \end{array} \xrightarrow{\mathcal{R}_3'} \begin{array}{c} W \\ W W W \end{array}$$

From Lemma 32, the last configuration is unsolvable.

• \mathcal{A} includes $\mathcal{R}'_4: \overset{\mathsf{G}}{\mathsf{W}(\mathsf{W})} \overset{\mathsf{G}}{\mathsf{W}} :: \mathsf{G}, \leftarrow \vee \rightarrow$. In this case, robots change their states as follows.

$$\begin{array}{c} W \\ W \\ G \\ W \end{array} \xrightarrow{\mathcal{R}'_4} W \\ G \\ G \\ \xrightarrow{W} W \\ G \\ W \end{array} \xrightarrow{\mathcal{R}_3} W \\ W \\ G \\ W \\ G \\ W \\ \end{array}$$

That is, the robots repeatedly change their states while keeping their positions. This implies that they cannot achieve exploration.

• \mathcal{A} includes $\mathcal{R}_{5}': \overset{\mathsf{G}}{\mathsf{W}}(\mathsf{W}) \overset{\mathsf{U}}{\mathsf{W}} :: \mathsf{W}, \leftarrow \vee \rightarrow$. In this case, robots change their states as follows.

$$\begin{array}{c} W \\ W G W \end{array} \xrightarrow{\mathcal{R}_5'} W G W$$

From Lemma 32, the last configuration is unsolvable.

ullet \mathcal{A} includes $\mathcal{R}_{6}': \overset{\mathsf{G}}{\mathsf{W}}(\mathsf{W}) \overset{\mathsf{G}}{\mathsf{W}} :: \mathsf{G}, \bot$. In this case, robots change their states as follows.

$$\begin{array}{c} W \\ W \\ G \\ W \end{array} \xrightarrow{\mathcal{R}_{6}'} \begin{array}{c} G \\ W \\ G \\ W \end{array}$$

From Lemma 32, the last configuration is unsolvable.

• \mathcal{A} includes $\mathcal{R}'_7: \underset{\emptyset}{\mathsf{W}} (\mathsf{W}) \overset{\mathsf{W}}{\mathsf{G}} :: \mathsf{W}, \rightarrow.$ In this case, robots change their states as follows.

$$\begin{array}{c} W \\ W G W \end{array} \xrightarrow{\mathcal{R}_7'} \begin{array}{c} W \\ W \\ G W \end{array}$$

From Lemma 32, the last configuration is unsolvable.

• \mathcal{A} includes $\mathcal{R}'_8: \underset{\emptyset}{\overset{\mathsf{W}}{\bigvee}} (\mathsf{W}) \overset{\mathsf{W}}{\mathsf{G}} :: \mathsf{G}, \rightarrow.$ In this case, robots change their states as follows.

$$\begin{array}{c} W \\ W G W \end{array} \xrightarrow{\mathcal{R}'_8} \begin{array}{c} W \\ G \\ G W \end{array}$$

From Lemma 32, the last configuration is unsolvable.

- \mathcal{A} includes \mathcal{R}'_9 : \emptyset (W) G :: W, \leftarrow . Recall that, from C_{t_1} to C_{t_2} , three robots change their states by rules \mathcal{R}_2 , \mathcal{R}_3 , and \mathcal{R}_7 . Let us consider a configuration such that three robots form G W W. In the next configuration, three robots can form G by rule \mathcal{R}'_9 . This contradicts to Lemma 36.
- \mathcal{A} includes $\mathcal{R}'_{10}: \begin{subarray}{c} W \\ \emptyset \end{subarray} : G, \leftarrow.$ Let us consider a configuration such that three robots form $\begin{subarray}{c} G \\ W \end{subarray}$ to explore the ring. In the next configuration, three robots can form $\begin{subarray}{c} G \\ W \end{subarray}$ by rule \mathcal{R}'_{10} . This contradicts to Lemma 36.
- \mathcal{A} includes $\mathcal{R}'_{11}: \begin{subarray}{c} W \\ \emptyset \begin{subarray}{c} W \\ O \end{subarray}$: G, \bot . Let us consider a configuration such that three robots form $\begin{subarray}{c} G \\ W \end{subarray}$ to explore the ring. After that, the three robots change their states as follows.

That is, the robots repeatedly change their states while keeping their positions. This implies that they cannot explore the ring, and thus this contradicts to Lemma 35.

From the above discussion, for any rule included in \mathcal{A} , some configuration in \mathcal{C}_{sol} becomes unsolvable or exploration of three robots becomes impossible. Therefore, if \mathcal{A} includes $\{\mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_7\}$, \mathcal{A} cannot solve terminating exploration from some configuration in \mathcal{C}_{sol} .

We can prove the case of $\{\mathcal{R}_{10}, \mathcal{R}_4, \mathcal{R}_5\}$ similarly because, in rules $\mathcal{R}_{10}, \mathcal{R}_4$, and \mathcal{R}_5 , the roles of colors are just swapped from rules $\mathcal{R}_2, \mathcal{R}_3$, and \mathcal{R}_7 .

Lemma 39 If A includes $\{R_8, R_4, R_1\}$ or $\{R_6, R_3, R_9\}$, A cannot solve terminating exploration from some configuration in C_{sol}

Proof: First, we consider the case that \mathcal{A} includes $\{\mathcal{R}_8, \mathcal{R}_4, \mathcal{R}_1\}$. Let us consider configuration $\mathsf{GGWW} \in \mathcal{C}_{sol}$. Assume that robots r_1 , r_2 , r_3 , and r_4 form the sub-configuration GGWW in this order. Since robots cannot move from this configuration by rules \mathcal{R}_8 , \mathcal{R}_4 , and \mathcal{R}_1 , \mathcal{A} includes some other rules to move robots from the configuration. We consider all possibilities in the following.

First consider the case that r_1 can move. That is, \mathcal{A} includes a rule such that the guard is $\emptyset(\mathsf{G})\mathsf{G}$.

- \mathcal{A} includes $\mathcal{R}_1'': \emptyset(\mathsf{G})\mathsf{G} :: \mathsf{G}, \to$. In this case, robots change configuration $\mathsf{G}\mathsf{GWW} \in \mathcal{C}_{sol}$ to configuration $\mathsf{G}\mathsf{GWW}$, which is unsolvable from Lemma 32.
- \mathcal{A} includes $\mathcal{R}_2'': \emptyset(\mathsf{G})\mathsf{G}:: \mathsf{W}, \to$. In this case, from rules \mathcal{R}_2'' and \mathcal{R}_8 , robots change configuration $\mathsf{GWGG} \in \mathcal{C}_{sol}$ to configuration GW , which is unsolvable from Lemma 31.
- \mathcal{A} includes $\mathcal{R}_3'': \emptyset(\mathsf{G})\mathsf{G}:: \mathsf{G}, \leftarrow$. In this case, from rules \mathcal{R}_3'' and \mathcal{R}_8 , robots change their states from configuration $\mathsf{GWGG} \in \mathcal{C}_{sol}$ as follows.

$$\mathsf{GWGG} \xrightarrow{\mathcal{R}_3''} \mathsf{GWG}\emptyset \mathsf{G} \xrightarrow{\mathcal{R}_8} \overset{\mathsf{G}}{\mathsf{G}} \\ \mathsf{W} \emptyset \emptyset \mathsf{G}$$

The last configuration is unsolvable from Lemma 32.

• \mathcal{A} includes $\mathcal{R}_4'': \emptyset(G)G :: W, \leftarrow$. In this case, from rules \mathcal{R}_4'' and \mathcal{R}_8 , robots change their states from configuration $\mathsf{GWGG} \in \mathcal{C}_{sol}$ as follows.

$$\mathsf{GWGG} \xrightarrow{\mathcal{R}_4''} \mathsf{GWG}\emptyset \mathsf{W} \xrightarrow{\mathcal{R}_8} \overset{\mathsf{G}}{\mathsf{G}} \\ \mathsf{W} \ \emptyset \ \emptyset \ \mathsf{W}$$

The last configuration is unsolvable from Lemma 32.

• \mathcal{A} includes $\mathcal{R}_5'': \emptyset(\mathsf{G})\mathsf{G} :: \mathsf{W}, \bot$. In this case, robots change configuration $\mathsf{GGGW} \in \mathcal{C}_{sol}$ to configuration WGGW , which is unsolvable from Lemma 31.

Next consider the case that r_2 can move. That is, \mathcal{A} includes a rule such that the guard is $\mathsf{G}(\mathsf{G})\mathsf{W}$.

- \mathcal{A} includes $\mathcal{R}_6'': \mathsf{G}(\mathsf{G})\mathsf{W}:: \mathsf{G}, \to$. In this case, from rules \mathcal{R}_6'' and \mathcal{R}_8 , robots change configuration G ration $\mathsf{G}\mathsf{G}\mathsf{W}\mathsf{G} \in \mathcal{C}_{sol}$ to configuration G , which is unsolvable from Lemma 32. $\mathsf{G} \emptyset \mathsf{W}$
- \mathcal{A} includes \mathcal{R}_7'' : $\mathsf{G}(\mathsf{G})\mathsf{W}::\mathsf{W},\to$. In this case, robots change configuration $\mathsf{G}\mathsf{G}\mathsf{W}\mathsf{G}\in\mathcal{C}_{sol}$ to configuration $\mathsf{G}\emptyset\mathsf{W}\mathsf{G}$, which is unsolvable from Lemma 32.
- \mathcal{A} includes $\mathcal{R}_8'': \mathsf{G}(\mathsf{G})\mathsf{W} :: \mathsf{G}, \leftarrow$. In this case, robots change configuration $\mathsf{G}\mathsf{G}\mathsf{W}\mathsf{G} \in \mathcal{C}_{sol}$ to configuration $\mathsf{G}\mathsf{G}\mathsf{W}\mathsf{G}$, which is unsolvable from Lemma 32.

when some robot moves from the configuration, two robots move to the middle node of the two towers and create a tower. Since they have the same color, the configuration is unsolvable from Lemma 32.

• \mathcal{A} includes \mathcal{R}''_{10} : $\mathsf{G}(\mathsf{G})\mathsf{W}$:: W, \perp . In this case, robots change configuration $\mathsf{G}\mathsf{G}\mathsf{W}\mathsf{G} \in \mathcal{C}_{sol}$ to configuration $\mathsf{G}\mathsf{W}\mathsf{W}\mathsf{G}$, which is unsolvable from Lemma 31.

Next consider the case that r_3 can move. That is, \mathcal{A} includes a rule such that the guard is $\mathsf{G}(\mathsf{W})\mathsf{W}$. In the first four cases, we consider configurations from C_{t_1} to C_{t_2} . During these configurations, three robots change their states by rules \mathcal{R}_8 , \mathcal{R}_4 , and \mathcal{R}_1 . In particular, we consider a sub-configuration GWW in \mathcal{C}_{exp} .

- \mathcal{A} includes $\mathcal{R}''_{11}: \mathsf{G}(\mathsf{W})\mathsf{W}:: \mathsf{G}, \to$. From a sub-configuration GWW , the robots change their states to $\overset{\mathsf{G}}{\mathsf{G}} \emptyset \overset{\mathsf{G}}{\mathsf{W}}$. This sub-configuration is not in \mathcal{C}_{exp} , which contradicts to Lemma 36.
- \mathcal{A} includes $\mathcal{R}''_{12}: \mathsf{G}(\mathsf{W})\mathsf{W}:: \mathsf{W}, \to$. From a sub-configuration GWW , the robots change their states to $\overset{\mathsf{W}}{\mathsf{G}} \overset{\mathsf{W}}{\mathsf{W}}$. This sub-configuration is not in \mathcal{C}_{exp} , which contradicts to Lemma 36.
- \mathcal{A} includes $\mathcal{R}''_{13}: \mathsf{G}(\mathsf{W})\mathsf{W}:: \mathsf{G}, \leftarrow$. From a sub-configuration GWW , the robots change their states to $\overset{\mathsf{G}}{\mathsf{G}} \underset{\emptyset}{\emptyset} \mathsf{W}$. This sub-configuration is not in \mathcal{C}_{exp} , which contradicts to Lemma 36.
- \mathcal{A} includes $\mathcal{R}''_{14}: \mathsf{G}(\mathsf{W})\mathsf{W}:: \mathsf{W}, \leftarrow$. From a sub-configuration GWW , the robots change their states to $\overset{\mathsf{W}}{\mathsf{G}} \emptyset \mathsf{W}$. This sub-configuration is not in \mathcal{C}_{exp} , which contradicts to Lemma 36.
- \mathcal{A} includes \mathcal{R}''_{15} : $\mathsf{G}(\mathsf{W})\mathsf{W}$:: G, \perp . In this case, robots change configuration $\mathsf{W}\mathsf{G}\mathsf{W}\mathsf{W} \in \mathcal{C}_{sol}$ to configuration $\mathsf{W}\mathsf{G}\mathsf{G}\mathsf{W}$, which is unsolvable from Lemma 31.

Lastly consider the case that r_4 can move. That is, \mathcal{A} includes a rule such that the guard is $\emptyset(W)W$. Similarly to the previous case, we consider a sub-configuration GWW in \mathcal{C}_{exp} .

• \mathcal{A} includes $\mathcal{R}_{16}'': \emptyset(W)W :: G, \to$. From rules \mathcal{R}_{16}'' and \mathcal{R}_1 , robots change their states from a sub-configuration GWW as follows.

$$\mathsf{GWW} \xrightarrow{\mathcal{R}_{16}''} \overset{\mathsf{W}}{\mathsf{G}} \overset{\mathsf{W}}{\mathsf{G}} \overset{\mathcal{R}_1}{\emptyset} \xrightarrow{\mathsf{GWW}}$$

That is, the robots repeatedly change their states while keeping their positions. This implies that they cannot explore the ring, and thus this contradicts to Lemma 35.

- \mathcal{A} includes $\mathcal{R}''_{17}: \emptyset(\mathsf{W})\mathsf{W}:: \mathsf{W}, \to$. From a sub-configuration GWW, the robots change their states to $\overset{\mathsf{W}}{\mathsf{G}}$ W. This sub-configuration is not in \mathcal{C}_{exp} , which contradicts to Lemma 36.
- \mathcal{A} includes $\mathcal{R}''_{18}:\emptyset(W)W::G,\leftarrow$. From a sub-configuration GWW, the robots change their states to GW \emptyset G. This sub-configuration is not in \mathcal{C}_{exp} , which contradicts to Lemma 36.

- \mathcal{A} includes $\mathcal{R}''_{19}: \emptyset(W)W:: W, \leftarrow$. From a sub-configuration GWW, the robots change their states to GW \emptyset W. This sub-configuration is not in \mathcal{C}_{exp} , which contradicts to Lemma 36.
- \mathcal{A} includes $\mathcal{R}_{20}'': \emptyset(W)W :: G, \bot$. From a sub-configuration GWW, the robots change their states to GWG. This sub-configuration is not in \mathcal{C}_{exp} , which contradicts to Lemma 36.

From the above discussion, for any rule included in \mathcal{A} , some configuration in \mathcal{C}_{sol} becomes unsolvable or exploration of three robots becomes impossible. Therefore, if \mathcal{A} includes $\{\mathcal{R}_8, \mathcal{R}_4, \mathcal{R}_1\}$, \mathcal{A} cannot solve terminating exploration from some configuration in \mathcal{C}_{sol}

We can prove the case of $\{\mathcal{R}_6, \mathcal{R}_3, \mathcal{R}_9\}$ similarly because, in rules $\mathcal{R}_6, \mathcal{R}_3$, and \mathcal{R}_9 , the roles of colors are just swapped from rules $\mathcal{R}_8, \mathcal{R}_4$, and \mathcal{R}_1 .

From Lemmas 38 and 39, \mathcal{A} cannot solve terminating exploration from some configuration in \mathcal{C}_{sol} . Therefore, we have Lemma 30. As described above, this implies Theorem 18.