

CONTINUOUS RANDOMNESS VIA TRANSFORMATIONS OF 2-RANDOM SEQUENCES

CHRISTOPHER P. PORTER

ABSTRACT. Reimann and Slaman initiated the study of sequences that are Martin-Löf random with respect to a continuous measure, establishing fundamental facts about NCR, the collection of sequences that are not Martin-Löf random with respect to any continuous measure. In the case of sequences that are random with respect to a computable, continuous measure, the picture is fairly well-understood: such sequences are truth-table equivalent to a Martin-Löf random sequence. However, given a sequence that is random with respect to a continuous measure but not with respect to any computable measure, we can ask: how close to effective is the measure with respect to which it is continuously random?

In this study, we take up this question by examining various transformations of 2-random sequences (sequences that are Martin-Löf random relative to the halting set \emptyset') to establish several results on sequences that are continuously random with respect to a measure that is computable in \emptyset' . In particular, we show that (i) every noncomputable sequence that is computable from a 2-random sequence is Martin-Löf random with respect to a continuous, \emptyset' -computable measure and (ii) the Turing jump of every 2-random sequence is Martin-Löf random with respect to a continuous, \emptyset' -computable measure. From these results, we obtain examples of sequences that are not proper, i.e., not random with respect to any computable measure, but are random with respect to a continuous, \emptyset' -computable measure. Lastly, we consider the behavior of 2-randomness under a wider class of effective operators (c.e. operators, pseudojump operators, and operators defined in terms of pseudojump inversion), showing that these too yield sequences that are Martin-Löf random with respect to a continuous, \emptyset' -computable measure.

1. INTRODUCTION

The study of algorithmically random sequences with respect to noncomputable measures was initiated by Levin in [Lev76] and was significantly advanced by Reimann and Slaman [RS15, RS21] and Day and Miller [DM13]. Reimann and Slaman focused in particular on Martin-Löf randomness with respect to continuous measures, showing in particular that all but countably many sequences are Martin-Löf random with respect to a continuous measure on Cantor space. This contrasts significantly with the case of randomness with respect to a *computable* continuous measure, as every sequence that is Martin-Löf random with respect to such a measure is truth-table equivalent to an unbiased Martin-Löf random sequence (hereafter, “random” will be short for “Martin-Löf random”).

The present study is motivated by the question: given a sequence that is random with respect to a continuous measure, how close to effective is the continuous measure with respect to which it is random? Clearly being random with respect to a computable, continuous measure is the best we can hope for. But what if our sequence is not *proper*, that is, not random with respect to a computable measure? The aim of this brief study is to move one level up in the arithmetical hierarchy to find such sequences that are random with respect to a measure that is computable in the halting set \emptyset' .

We will focus on two general approaches to generating nonproper sequences that are random with respect to a \emptyset' -computable measure. First, in Section 2 we establish a useful connection between Turing reductions from 2-random sequences and randomness with respect to a \emptyset' -computable measure. In particular, we will prove that every noncomputable sequence that can be computed by a 2-random sequence is random with respect to a continuous, \emptyset' -computable measure, which improves a result due to Reimann and Slaman [RS21] (who showed that every noncomputable sequence computable from a 3-random sequence is random with respect to a continuous measure). From this result, we will be able to obtain examples of nonproper sequences that are random with respect to a continuous, \emptyset' -computable measure.

In Section 3, we will then consider the behavior of 2-random sequences under a broader class of effective operators beyond Turing functionals, including the Turing jump and pseudajump operators. In particular, we will show that the jump of a 2-random sequence is a nonproper sequence that is random with respect to a continuous, \emptyset' -computable measure.

For related work, see Hirschfeldt and Terwijn [HT08] on Δ_2^0 measures. Demuth also studied \emptyset' -computable measures in [Dem88a] and [Dem88b]. Cenzer and Porter [CP18] also studied several notions of randomness for members of Π_1^0 classes that are given in terms of certain \emptyset' -computable measures. See also [LR19] for recent work on sequences that are not random with respect to any continuous measure.

Before turning to our main results, we briefly review a few concepts and fix our notation. In what follows, λ stands for the Lebesgue measure. For binary strings $\sigma, \tau \in 2^{<\omega}$, we use the notation $\sigma \preceq \tau$ to indicate that σ is an initial segment of τ . We similarly define $\sigma \prec X$ for $\sigma \in 2^{<\omega}$ and $X \in 2^\omega$. Moreover, we write the concatenation of σ and τ as $\sigma \frown \tau$. We write the empty string as ϵ . Given $\sigma \in 2^{<\omega}$, $[\![\sigma]\!] = \{X \in 2^\omega : \sigma \prec X\}$ is the *cylinder set* determined by σ .

A map $\Phi : \subseteq 2^\omega \rightarrow 2^\omega$ is a Turing functional if there is an oracle Turing machine that when given $X \in \text{dom}(\Phi)$ as an oracle computes the characteristic function of some $Y \in 2^\omega$; in this case, we write $\Phi(X) \downarrow = Y$. For $k \in \omega$, we will write $\Phi(X; k)$ to be the output the computation on input k . Of course, there may be some k such that $\Phi(X; k)$ is undefined; in this case, we will consider $\Phi(X)$ to be undefined. We can define the domain of Φ to be $\text{dom}(\Phi) = \{X \in 2^\omega : \Phi(X) \downarrow\}$. We can also relativize any such functional to some $Z \in 2^\omega$ to obtain a Z -computable functional.

Recall that a measure on 2^ω is determined by the values it assigns to the cylinder sets. A measure μ on 2^ω is computable if the value $\mu([\![\sigma]\!])$ is a computable real number uniformly in $\sigma \in 2^{<\omega}$. Similarly, for $Z \in 2^\omega$ we can define a Z -computable measure μ by using Z as an oracle to compute the values $\mu([\![\sigma]\!])$ for $\sigma \in 2^{<\omega}$.

If μ is a Z -computable measure on 2^ω and $\Phi : \subseteq 2^\omega \rightarrow 2^\omega$ is a Z -computable functional defined on a set of μ -measure one, then the *pushforward measure* μ_Φ defined by setting

$$\mu_\Phi([\![\sigma]\!]) = \mu(\Phi^{-1}([\![\sigma]\!]))$$

for each $\sigma \in 2^{<\omega}$ is a Z -computable measure.

Recall that for a fixed computable measure μ on 2^ω and $Z \in 2^\omega$, a μ -Martin-Löf test relative to Z (or simply a μ -test relative to Z) is a uniformly $\Sigma_1^0[Z]$ sequence $(U_i)_{i \in \omega}$ of subsets of 2^ω with $\mu(U_n) \leq 2^{-i}$ for every $i \in \omega$. $X \in 2^\omega$ passes such a test $(U_i)_{i \in \omega}$ if $X \notin \bigcap_{i \in \omega} U_i$ and X is μ -Martin-Löf random relative to Z if X passes every μ -Martin-Löf test relative to Z . The set of all such sequences X is denoted by MLR_μ^Z . For each choice of μ and Z as above,

there is a single, *universal*, μ -test relative to Z , $(U_i^Z)_{i \in \omega}$ such that $X \in \text{MLR}_\mu^Z$ if and only if X passes $(U_i^Z)_{i \in \omega}$. Lastly, we say that a sequence is *proper* if it is Martin-Löf random with respect to a computable measure.

In the case that μ is a noncomputable measure, we have to be careful in defining randomness with respect to μ . In particular, we must relativize our tests to some sequence $R \in 2^\omega$ that encodes our measure μ ; such a sequence is called a *representation* of μ . Specific details about representations of measures can be found, for instance in [DM13] or [RS21]. For our purposes, we do not need the full machinery of the representation of measures. Whereas in the general approach to randomness with respect to a noncomputable measure μ , a sequence is μ -Martin-Löf random if it is Martin-Löf random with respect to *some* representation $R_\mu \in 2^\omega$ of μ , in the present study, we only need show that a given sequence is random with respect to a \emptyset' -computable measure, so it suffices to consider our tests relative to the oracle \emptyset' .

An *atom* of a measure μ is a sequence $A \in 2^\omega$ such that $\mu(\{A\}) > 0$. A measure is *continuous* if it has no atoms; otherwise it is *atomic*. A routine relativization of a result due to Kautz [Kau91] yields the following.

Lemma 1.1. *For $Z \in 2^\omega$, $A \in 2^\omega$ is an atom of some Z -computable measure if and only if $A \leq_T Z$.*

One can further show that if $A \leq_T Z$ and A is Martin-Löf random with respect to a Z -computable measure μ , then A must be an atom of μ .

We will make use of a pair of results concerning the interaction between Turing functionals and Martin-Löf randomness:

Theorem 1.2. *For $Z \in 2^\omega$, let $\Phi: 2^\omega \rightarrow 2^\omega$ be a Z -computable functional and let μ be a Z -computable measure satisfying $\mu(\text{dom}(\Phi)) = 1$.*

- (i) (*Randomness preservation* [ZL70]) *If $X \in \text{MLR}_\mu^Z$ then $\Phi(X) \in \text{MLR}_{\mu_\Phi}^Z$.*
- (ii) (*No randomness from nonrandomness* [She86]) *If $Y \in \text{MLR}_{\mu_\Phi}^Z$, then there is some $X \in \text{MLR}_\mu^Z$ such that $\Phi(X) = Y$.*

Our study is primarily concerned with 2-randomness, that is, Martin-Löf randomness relative to the halting set \emptyset' , but we will make use of the fact that every 2-random sequence is weakly 2-random. Recall that a sequence $X \in 2^\omega$ is weakly 2-random if for every Π_2^0 class P with $\lambda(P) = 0$, we have $X \notin P$. Two useful facts about every weakly 2-random sequence (and hence every 2-random sequence) that we will use are as follows. First, if X is weakly 2-random and Φ is a Turing functional with $X \in \text{dom}(\Phi)$, then $\lambda(\text{dom}(\Phi)) > 0$ (since the domain of a Turing functional is a Π_2^0 class). Second, as shown by Downey, Nies, Weber, and Yu [DNWY06], $X \in 2^\omega$ is weakly 2-random if and only if X is Martin-Löf random and X forms a minimal pair with \emptyset' in the Turing degrees (that is, X does not compute any noncomputable Δ_2^0 sets).

For more background on algorithmic randomness, see [Nie09], [DH10], [SUV17], or the surveys contained in [FP20].

2. COMPUTING FROM 2-RANDOM SEQUENCES

As noted in the previous section, Reimann and Slaman [RS21] proved that every noncomputable sequence below a 3-random sequence (i.e., a sequence that is Martin-Löf random

with respect to \emptyset'') is random with respect to a continuous measure. We improve this result as follows.

Theorem 2.1. *Every noncomputable sequence Turing below a 2-random sequence is Martin-Löf random with respect to a continuous, \emptyset' -computable measure.*

Note that we are not claiming that a noncomputable sequence below a 2-random sequence is 2-random with respect to a \emptyset' -computable measure. In the terminology laid out, for instance, in [RS21], 2-randomness with respect to a noncomputable measure requires that we consider the jump of a representation of our measure; in our case, we only need \emptyset' as an oracle.

Our proof of Theorem 2.1 relies on a combination of several tools. First, we will make use of a class of Turing functionals, first isolated by Barmpalias, Day, and Lewis-Pye [BDLP14] in their study of the typical Turing degree, which are referred to as *special* Turing functionals. Here a Turing functional is special if its range does not include any computable sequences. The key result we will use is the following (the result we draw upon in [BDLP14] is slightly more general):

Lemma 2.2 (Barmpalias, Day, Lewis-Pye [BDLP14]). *If X is a 2-random sequence and Y is a noncomputable sequence such that $\Phi(X) = Y$ for some Turing functional Φ , then $\Psi(X) = Y$ for some special Turing functional Ψ .*

Next, we will use a classical result due to Sacks.

Theorem 2.3 (Sacks [Sac63]). *For $X \in 2^\omega$ and a Turing functional Φ , if $\lambda(\Phi^{-1}(X)) > 0$, then X is computable.*

Lastly, we will use the following lemma.

Lemma 2.4 (Functional Extension Lemma). *For $Z \in 2^\omega$, let Φ^Z be a Z -computable functional that is total on a $\Pi_1^0[Z]$ class P . Then there is a total Z -computable functional Ψ^Z that agrees with Φ^Z on P . Moreover, we can define Ψ^Z so that if $X \in 2^\omega \setminus P$, $\Psi^Z(X)$ is a finite modification of X .*

Proof. Given $Z \in 2^\omega$, Φ^Z , and P as above, we define Ψ^Z in terms of a Z -computable approximation of P given by clopen sets $(P_s)_{s \in \omega}$ (where $P_{s+1} \subseteq P_s$ for every $s \in \omega$ and $P = \bigcap_{s \in \omega} P_s$). For $X \in 2^\omega$, we set

$$\Psi^Z(X; n) = \begin{cases} \Phi^Z(X; n) & \text{if } (\exists s \geq n)(\Phi_s^Z(X \upharpoonright s; n) \downarrow \ \& \ X \in P_s) \\ X(n) & \text{otherwise} \end{cases}.$$

Note that if $X \in P$, since Φ^Z is total on P , such a stage s exists. In this case, $\Psi^Z(X; n)$ agrees with $\Phi^Z(X; n)$. Moreover, given $X \in 2^\omega \setminus P$, there is some m such that for all $s \geq m$, $X \notin P_s$ (and we can Z -computably detect when this occurs). Thus we will have $\Psi^Z(X; n) = X(n)$ for all $n \geq m$, as desired. □

We now turn to the proof of Theorem 2.1. Here we draw on a technique used to show that every 2-random sequence X is generalized low, i.e., $X \oplus \emptyset' \equiv_T X'$ (a generalization of which is due to Kautz [Kau91]; see also the proof of [Sim07, Lemma 4.4]).

Proof of Theorem 2.1. Given a noncomputable sequence $Y \leq_T X$ where X is 2-random, let Φ be a Turing functional that witnesses this reduction. By Lemma 2.2, we can assume that Φ is a special Turing functional. By our discussion in Section 1, since X is weakly 2-random, we have $\lambda(\text{dom}(\Phi)) > 0$.

Write $\text{dom}(\Phi) = \bigcap_{n \in \omega} S_n$, where $S_n = \{Z \in 2^\omega : \Phi(Z; n) \downarrow\}$. We define a function $f : \omega \rightarrow \omega$ such that for each $n \in \omega$, $f(n)$ is the least stage s such that $\lambda(S_n \setminus S_{n,s}) \leq 2^{-n}$. Clearly $f \leq_T \emptyset'$. Then we set $V_n = S_n \setminus S_{n,f(n)}$ for each $n \in \omega$, and we further set $W_n = \bigcup_{i > n} V_i$. Since $(W_n)_{n \in \omega}$ is uniformly $\Sigma_1^0[\emptyset']$ and $\lambda(W_n) \leq 2^{-n}$ for every $n \in \omega$, $(W_n)_{n \in \omega}$ is a Martin-Löf test relative to \emptyset' .

Since $X \in \text{dom}(\Phi)$, $X \in S_n$ for every $n \in \omega$. Moreover, since X is 2-random, $X \notin W_j$ for some $j \in \omega$, which implies that $X \notin V_i$ for every $i > j$. It follows that for every $i > j$, $X \notin S_i \setminus S_{i,f(i)}$, so that $X \in S_{i,f(i)}$ for all but finitely many $i \in \omega$. Thus there is some $k \in \omega$ such that $X \in S_{i,f(i)+k}$ for all $i \in \omega$. Then $S = \bigcap_{i \in \omega} S_{i,f(i)+k}$ is a $\Pi_1^0[\emptyset']$ subclass of $\text{dom}(\Phi)$.

Again using the fact that X is 2-random, for the universal \emptyset' -Martin-Löf test $(U_i^{\emptyset'})_{i \in \omega}$, there is some $j \in \omega$ such that $X \in S \cap (2^\omega \setminus U_j^{\emptyset'})$. Moreover, since X is not contained in any $\Pi_1^0[\emptyset']$ -classes of measure 0, it follows that $\lambda(S \cap (2^\omega \setminus U_j^{\emptyset'})) > 0$. We set $P = S \cap (2^\omega \setminus U_j^{\emptyset'})$, a $\Pi_1^0[\emptyset']$ subset of $\text{dom}(\Phi)$ of positive measure that contains only 2-random sequences.

By the Functional Extension Lemma (Lemma 2.4) applied to the case that $Z = \emptyset'$, since Φ is total on P , there is a \emptyset' -computable functional $\Psi^{\emptyset'}$ that agrees with Φ on P . Moreover, we can further assume in the case that $Y \notin P$, $\Psi^{\emptyset'}(Y)$ is a finite modification of Y . Hereafter, we will write $\Psi^{\emptyset'}$ as Ψ .

Since $X \in P$ is 2-random, it follows by randomness preservation (Theorem 1.2(i)) that $\Psi(X)$ is 2-random with respect to λ_Ψ , the \emptyset' -computable measure induced by Ψ . We claim that λ_Ψ is continuous. Suppose for the sake of contradiction that λ_Ψ is not continuous. Then there is some $A \in 2^\omega$ such that $\lambda_\Psi(\{A\}) = \lambda(\Psi^{-1}(\{A\})) > 0$. We have two cases to consider.

Case 1: $\lambda(\Psi^{-1}(\{A\}) \cap P) > 0$. Since Φ agrees with Ψ on P , $\lambda(\Phi^{-1}(\{A\}) \cap P) > 0$ and hence A is computable by Sacks' Theorem. However, since P only contains 2-random sequences and Φ was chosen to be special, no sequence in P computes a computable sequence via Φ . So this case is impossible.

Case 2: $\lambda(\Psi^{-1}(\{A\}) \setminus P) > 0$. Then by the definition of Ψ , each sequence in $\Psi^{-1}(\{A\}) \setminus P$ is a finite modification of A . As the set of finite modifications of a fixed sequence has Lebesgue measure 0, it follows that $\lambda(\Psi^{-1}(\{A\}) \cap P) = 0$, which contradicts our assumption.

As both cases lead to absurdity, it follows that λ_Ψ is continuous as desired. \square

Theorem 2.1 has a several immediate consequences. Recall that $G \in 2^\omega$ is 1-generic if for every $\Sigma_1^0 S \subseteq 2^{<\omega}$, there is some $\sigma \prec G$ such that either $\sigma \in S$ or for all $\tau \succeq \sigma$, $\tau \notin S$.

Corollary 2.5. *There are 1-generic sequences that are Martin-Löf random with respect to a continuous, \emptyset' -computable measure.*

Proof. As shown by Kautz [Kau91], every 2-random computes a 1-generic sequence. Since no 1-generic sequence is computable, we can apply Theorem 2.1 to obtain the result. \square

This provides us with the first example of a sequence that is not proper but is random with respect to a continuous, \emptyset' -computable measure, as Muchnik [MSU98] proved that no 1-generic sequence is random with respect to a computable measure.

Note further that Corollary 2.5 is not true of all 1-generic sequences. There are Δ_2^0 1-generic sequences, and by the remark after Lemma 1.1, if a Δ_2^0 sequence Y is random with respect to a \emptyset' -computable measure μ , then Y must be an atom of μ (and hence μ cannot be continuous). Moreover, Corollary 2.5 fails to hold of any 2-generic sequence (a notion obtained by relativizing the definition of 1-genericity to \emptyset'): by a direct relativization of Muchnik's result mentioned in the previous paragraph, no 2-generic sequence is 2-random with respect to a \emptyset' -computable measure. Using this latter fact, we obtain an alternative proof of the following result due to Nies, Stephan, and Terwijn [NST05] as a Corollary of Theorem 2.1.

Corollary 2.6. *Every 2-random sequence forms a minimal pair with every 2-generic sequence.*

Proof. Suppose there is some noncomputable $Z \in 2^\omega$ that is computable from some 2-random sequence X and from some 2-generic sequence Y . By Jockusch [Joc80] (who attributes the result to Martin), the collection of 2-generic sequences are downward dense, i.e., every noncomputable sequence computable from a 2-generic computes a 2-generic sequence. So without loss of generality, we can assume that Z is 2-generic (since if Z is not 2-generic, it computes a 2-generic sequence that is still below both X and Y). Moreover, by Theorem 2.1, since Z is computable from a 2-random sequence, it is 2-random with respect to a \emptyset' -computable measure. But this contradicts the relativization of Muchnik's theorem. \square

The next corollary of Theorem 2.1 shows us that one can \emptyset' -computably recover unbiased 2-randomness from any noncomputable sequence computable from a 2-random sequence.

Corollary 2.7. *For every noncomputable sequence X computable from some 2-random sequence, $X \oplus \emptyset'$ computes a 2-random sequence.*

Proof. Let X be a noncomputable sequence that is computable from some 2-random sequence. Thus by Theorem 2.1, X is 2-random with respect to a \emptyset' -computable measure. As shown independently by Levin [ZL70] and Kautz [Kau91] (as well as by Schnorr and Fuchs [SF77]), for every sequence Z that is random with respect to some computable measure, there is some Martin-Löf random sequence Y such that $Y \leq_T Z$ (in fact, $Y \equiv_T Z$). Relativizing this result to \emptyset' , we get that for every sequence Z that is 2-random with respect to a \emptyset' -computable measure, there is some 2-random sequence Y such that $Y \leq_T Z \oplus \emptyset'$. Applying this result to X as given above yields the desired conclusion. \square

We conclude this section with one last corollary of Theorem 2.1 and an open question. Recall that NCR is the collection of sequences that are not random with respect to a continuous measure (first introduced by Reimann and Slaman in [RS15]). An immediate consequence of Theorem 2.1 is the following.

Corollary 2.8. *No 2-random sequence computes a noncomputable member of NCR.*

We cannot weaken this result to hold for Demuth randomness. We do not provide a definition of Demuth randomness here (see, for instance, [DH10, Section 7.6]). For our purpose, the key fact is that the collection of 2-random sequences is a proper subset of the collection of Demuth random sequences. Let Y be a Demuth random sequence that is not weakly 2-random. Then by a result of Hirschfeldt and Miller, Y computes a noncomputable c.e. set A (see [DH10, Corollary 7.2.12]). As shown by Kučera and Nies [KN11], such a

c.e. set must be K -trivial (see [Nie09, Section 5.2] for a definition of K -triviality). Lastly, Barmpalias, Greenberg, Montalbán, and Slaman [BGMS12] showed that every K -trivial sequence is in NCR. Putting all of these pieces together, this yields a Demuth random sequence that computes a noncomputable member of NCR. A similar argument does not work for weak 2-randomness, as no weakly 2-random sequence computes a noncomputable Δ_2^0 sequence (and every K -trivial sequence is Δ_2^0). We thus can ask:

Question 2.9. *Can a weakly 2-random sequence compute a noncomputable member of NCR?*

3. THE JUMP AND PSEUDOJUMPS OF A 2-RANDOM SEQUENCE

In this section, we continue our study of continuous randomness by studying the behavior of 2-random sequences under a broader class of effective operators beyond Turing functionals. Here we consider the Turing jump, c.e. operators, pseudojump operators, and operators defined in terms of pseudojump inversion. As we will see, these too yield sequences that are continuously random. We first consider the jump of a 2-random sequence.

Theorem 3.1. *For $X \in 2^\omega$, if X is 2-random, then X' is Martin-Löf random with respect to a continuous, \emptyset' -computable measure.*

Proof. We proceed with a proof similar to that of Theorem 2.1, with several modifications. First, we set $S_n = \{Z \in 2^\omega : \Phi_n(Z; n) \downarrow\}$ for each $n \in \omega$. Then as in the proof of Theorem 2.1, we define a function $f \leq_T \emptyset'$ such that for each $n \in \omega$, $f(n)$ is the least stage s such that $\lambda(S_n \setminus S_{n,s}) \leq 2^{-n}$. Then we set $V_n = S_n \setminus S_{n,f(n)}$ for each $n \in \omega$, and we further set $W_n = \bigcup_{i>n} V_i$, yielding $(W_n)_{n \in \omega}$, a Martin-Löf test relative to \emptyset' . Since $X \notin W_j$ for some $j \in \omega$, it follows that $X \notin V_i$ for all $i > j$.

Now for each $n \in \omega$, $n \in X'$ if and only if $X \in S_n$. Moreover, for all $n > j$, $X \notin S_n \setminus S_{n,f(n)}$. Thus, for all $n > j$ such that $X \in S_n$, we must have $X \in S_{n,f(n)}$. We can thus conclude that for all but finitely many n , $n \in X'$ if and only if $X \in S_{n,f(n)}$. Then there is some $k \in \omega$ such that for all $n \in \omega$, $n \in X'$ if and only if $X \in S_{n,f(n)+k}$.

We then define a total \emptyset' -computable functional Φ as follows:

$$\Phi(Z; n) = \begin{cases} 1 & \text{if } Z \in S_{n,f(n)+k} \\ 0 & \text{otherwise.} \end{cases}$$

It is immediate that $\Phi(X) = X'$. Note further that for any 2-random sequence Y and all but finitely many n , since $n \in Y'$ if and only if $Y \in S_{n,f(n)+k}$, it follows that $\Phi(Y) \equiv_T Y'$.

Clearly Φ is total. Let λ_Φ be the \emptyset' -computable measure induced by Φ . Then by randomness preservation, $X' = \Phi(X)$ is Martin-Löf random with respect to λ_Φ . We verify that λ_Φ is continuous. Suppose otherwise, so that $\lambda_\Phi(\{A\}) > 0$ for some $A \in 2^\omega$. Then $\lambda(\{Y \in 2^\omega : A \leq_T Y \oplus \emptyset'\}) > 0$. It follows from the relativization of Sacks' Theorem (due to Stillwell [Sti72]) that $A \leq_T \emptyset'$. In addition, since $\lambda(\Phi^{-1}(\{A\})) > 0$, $\Phi^{-1}(\{A\})$ must contain some 2-random sequence. However, since no 2-random sequence computes a noncomputable Δ_2^0 sequence, it follows that A must be computable. But for each 2-random sequence $Y \in \Phi^{-1}(\{A\})$, we have $A = \Phi(Y) \equiv_T Y'$, which is impossible. Thus λ_Φ cannot have any atoms. □

Theorem 3.1 provides additional examples of sequences that are not proper but random with respect to a continuous, \emptyset' -computable measure, namely the jump of every 2-random sequence. To establish this, we just need to prove the following.

Lemma 3.2. *For every $X \in 2^\omega$, X' is not proper.*

Proof. We recall one definition and two facts. First, a sequence Z has diagonally noncomputable (DNC) degree if there is some $f \leq_T Z$ such that $f(n) \neq \varphi_n(n)$ for all $n \in \omega$. As shown by Kučera [Kuč85], every Martin-Löf random sequence has DNC degree. Moreover, Arslanov's Completeness Criterion [Ars81] says that no intermediate c.e. set has DNC degree.

Now, suppose that X' is proper for some $X \in 2^\omega$. Let C be a noncomputable, incomplete c.e. set. As $C \leq_m \emptyset' \leq_m X'$, we have $C \leq_{tt} X'$. By randomness preservation, the property of being proper is closed downwards under tt -reducibility, and thus it follows that C is proper. However, every noncomputable proper sequence is Turing equivalent to a Martin-Löf random sequence (as shown independently by Levin [ZL70] and Kautz [Kau91]) and hence has DNC degree, which contradicts Arslanov's Completeness Criterion. \square

Using Theorem 3.1, we can further consider the behavior of 2-random sequences under c.e. operators and pseudojump operators. Recall that a c.e. operator is given by considering the domain of an oracle Turing machine with a fixed oracle; for $e \in \omega$, the e -th c.e. operator is given by the map $X \mapsto W_e^X$. By a relativization of Post's theorem that the halting set is 1-complete, for every $e \in \omega$ and $X \in 2^\omega$, we have $W_e^X \leq_1 X'$. We use this fact to prove the following.

Proposition 3.3. *Suppose that for $e \in \omega$, we have $W_e^Y \not\leq_T \emptyset'$ for every 2-random sequence Y . If X is a 2-random sequence, then W_e^X is Martin-Löf random with respect to a continuous, \emptyset' -computable measure.*

Proof. Fix $e \in \omega$ as in the statement of the proposition. As $W_e^X \leq_1 X'$, this 1-reduction induces a total Turing functional Ψ . Given a 2-random $X \in 2^\omega$, let μ be a \emptyset' -computable measure such that X' is \emptyset' -Martin-Löf random with respect to μ (which is guaranteed to exist by Theorem 3.1). By randomness preservation, W_e^X is random with respect to the induced \emptyset' -computable measure μ_Ψ . We claim that μ_Ψ is continuous. Suppose not. Then $\mu_\Psi(\{A\}) > 0$ for some $A \in 2^\omega$. Since μ_Ψ is \emptyset' -computable and A is a μ_Ψ -atom, it follows from Lemma 1.1 that A is Δ_2^0 . By Theorem 1.2(ii) (no randomness from nonrandomness) applied to both Ψ and the \emptyset' -computable functional induced by taking the jump of X , the set $\{Y : W_e^Y = A\}$ must contain a 2-random sequence Z , i.e., $W_e^Z = A$. However, this contradicts our assumption that $W_e^Y \not\leq_T \emptyset'$ for every 2-random sequence Y . Thus μ_Ψ must be continuous. \square

A *pseudojump operator* is given by a map of the form $X \mapsto X \oplus W_e^X$ for some $e \in \omega$ (see [JS83], [JS84]). We now obtain a result similar to Proposition 3.3 for pseudojump operators, except that we do not need the additional assumption that $W_e^Y \not\leq_T \emptyset'$ for every 2-random sequence Y .

Proposition 3.4. *For every $e \in \omega$ and every 2-random sequence X , $X \oplus W_e^X$ is Martin-Löf random with respect to a continuous, \emptyset' -computable measure.*

Proof. As in the proof of Proposition 3.3, there is a total \emptyset' -functional Ψ that maps each 2-random sequence $Y \in 2^\omega$ to $Y \oplus W_e^Y$. Let μ be a \emptyset' -computable measure such that X' is

\emptyset' -Martin-Löf random with respect to μ . Then as we argued in the proof of Proposition 3.3, given a 2-random sequence $X \in 2^\omega$, $X \oplus W_e^X$ is \emptyset' -Martin-Löf random with respect to the induced measure μ_Ψ , which is \emptyset' -computable. Lastly, μ_Ψ is continuous, as μ is continuous and Ψ is one-to-one. \square

Finally, we consider pseudojump inversion. The pseudojump inversion theorem is as follows:

Theorem 3.5 (Jockusch/Shore [JS83]). *Let $e \in \omega$. For every $A \geq_T \emptyset'$, there is some $B \in 2^\omega$ such that $B \oplus W_e^B \equiv_T A$.*

Here we consider pseudojump inversion as defining an effective operator. One can observe that the proof of the pseudojump inversion theorem (see [JS83, Theorem 2.1]) gives, for each $e \in \omega$, a total \emptyset' -computable functional Ξ such that for every $A \geq_T \emptyset'$, setting $\Xi(A) = B$, we have $B \oplus W_e^B \equiv_T A$. Let us review the details, which are relevant for our discussion.

Fix $e \in \omega$. Given $A \in 2^\omega$, we define a sequence of strings $(\tau_i)_{i \in \omega}$ such that $\tau_i \preceq \tau_{i+1}$ for all $i \in \omega$. First we set $\tau_0 = \epsilon$. For each $i \in \omega$, given τ_{2i} , we use \emptyset' to determine whether there is some $\tau \succeq \tau_{2i}$ such that $i \in W_e^\tau$. If so, we let τ_{2i+1} be the length-lexicographically least such τ ; otherwise, we set $\tau_{2i+1} = \tau_{2i}$. We then set $\tau_{2i+2} = \tau_{2i+1} \frown A(i)$. Then $B = \bigcup_{i \in \omega} \tau_i$ is the desired sequence. Based on this construction, we can prove:

Theorem 3.6. *For every $e \in \omega$ and every 2-random sequence X , if Ξ inverts the pseudojump operator with index e , then $\Xi(X')$ is Martin-Löf random with respect to a continuous, \emptyset' -computable measure.*

Proof. Clearly the operator Ξ as described above is total and \emptyset' -computable. For a 2-random $X \in 2^\omega$, let μ be a \emptyset' -computable measure such that X' is \emptyset' -Martin-Löf random with respect to μ . Applying Ξ to the jump of a 2-random sequence yields a sequence that is \emptyset' -Martin-Löf random with respect to the induced \emptyset' -computable measure μ_Ξ .

To verify that μ_Ξ is continuous, we claim that Ξ is one-to-one. For $A_0, A_1 \in 2^\omega$, suppose that $\Xi(A_0) = \Xi(A_1)$. For $j \in \{0, 1\}$, let $(\tau_i^j)_{i \in \omega}$ be the sequence of strings from the above construction when applied to A_j . We show by induction that $\tau_i^0 = \tau_i^1$ for every $i \in \omega$, from which it follows that $A_0 = A_1$.

- Base case: $\tau_0^0 = \epsilon = \tau_0^1$.
- Inductive step: For $k \in \omega$, suppose that $\tau_k^0 = \tau_k^1$. We have two cases to consider:

Case 1: $k = 2n$ for some $n \in \omega$. Then for $j \in \{0, 1\}$, either τ_{2n+1}^j is the length-lexicographically least $\tau \succeq \tau_k^0$ such that $n \in W_e^\tau$ or $\tau_{2n+1}^j = \tau_{2n}^j$. Either way, we have $\tau_{k+1}^0 = \tau_{k+1}^1$.

Case 2: $k = 2n + 1$ for some $n \in \omega$. Then under the assumption that $\Xi(A_0) = \Xi(A_1)$ and $\tau_k^0 = \tau_k^1$, it follows that $\tau_{k+1}^0 = \tau_k^0 \frown A_0(n) = \tau_k^1 \frown A_1(n) = \tau_{k+1}^1$.

It follows by induction that $\tau_k^0 = \tau_k^1$ for all $k \in \omega$, and hence that $A_0 = A_1$. As Ξ is one-to-one, it follows that μ_Ξ is continuous. \square

Note that, for $e \in \omega$ and $X \in 2^\omega$, in the case that $X \oplus W_e^X$ is proper, it follows that W_e^X is also proper (since $W_e^X \leq_{tt} X \oplus W_e^X$). Thus, we can broaden our analysis of sequences that are not proper but are random with respect to a continuous, \emptyset' -computable measure by answering the following questions.

Question 3.7. For which $e \in \omega$ do we have that W_e^X is not proper for any 2-random sequence X ?

Question 3.8. For which pseudojump operators is it the case that the sequence obtained by pseudojump inversion applied to the jump of a 2-random sequence yields a nonproper sequence?

ACKNOWLEDGEMENTS

The research in this article was supported by NSA Mathematical Sciences Program Young Investigator Grant # H98230-16-1-0310. Thank you to Laurent Bienvenu for early conversations that led to the main idea that motivated this study, and to the anonymous reviewers for helpful feedback that improved the presentation of the paper.

REFERENCES

- [Ars81] Marat Mirzaevich Arslanov. Some generalizations of a fixed-point theorem. *Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika*, (5):9–16, 1981. [8](#)
- [BDLP14] George Barmpalias, Adam R. Day, and Andy E. M. Lewis-Pye. The typical Turing degree. *Proc. Lond. Math. Soc. (3)*, 109(1):1–39, 2014. [4](#)
- [BGMS12] George Barmpalias, Noam Greenberg, Antonio Montalbán, and Theodore A. Slaman. K -trivials are never continuously random. In *Proceedings of the 11th Asian Logic Conference*, pages 51–58. World Sci. Publ., Hackensack, NJ, 2012. [7](#)
- [CP18] Douglas Cenzer and Christopher P. Porter. The random members of a Π_1^0 class. *Theory Comput. Syst.*, 62(7):1637–1671, 2018. [2](#)
- [Dem88a] Osvald Demuth. Reducibilities of sets based on constructive functions of a real variable. *Comment. Math. Univ. Carolin.*, 29(1):143–156, 1988. [2](#)
- [Dem88b] Osvald Demuth. Remarks on the structure of tt-degrees based on constructive measure theory. *Comment. Math. Univ. Carolin.*, 29(2):233–247, 1988. [2](#)
- [DH10] Rodney G. Downey and Denis R. Hirschfeldt. *Algorithmic randomness and complexity*. Theory and Applications of Computability. Springer, New York, 2010. [3](#), [6](#)
- [DM13] Adam R. Day and Joseph S. Miller. Randomness for non-computable measures. *Trans. Amer. Math. Soc.*, 365(7):3575–3591, 2013. [1](#), [3](#)
- [DNWY06] Rodney G. Downey, Andre Nies, Rebecca Weber, and Liang Yu. Lowness and Π_2^0 nullsets. *J. Symbolic Logic*, 71(3):1044–1052, 2006. [3](#)
- [FP20] Johanna N.Y. Franklin and Christopher P. Porter. *Algorithmic Randomness: Progress and Prospects*, volume 50. Cambridge University Press, 2020. [3](#)
- [HT08] Denis R. Hirschfeldt and Sebastiaan A. Terwijn. Limit computability and constructive measure. In *Computational prospects of infinity. Part II. Presented talks*, volume 15 of *Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap.*, pages 131–141. World Sci. Publ., Hackensack, NJ, 2008. [2](#)
- [Joc80] Carl G. Jockusch, Jr. Degrees of generic sets. In *Recursion theory: its generalisation and applications (Proc. Logic Colloq., Univ. Leeds, Leeds, 1979)*, volume 45 of *London Math. Soc. Lecture Note Ser.*, pages 110–139. Cambridge Univ. Press, Cambridge-New York, 1980. [6](#)
- [JS83] Carl G. Jockusch and Richard A. Shore. Pseudojump operators. i. the r.e. case. *Transactions of the American Mathematical Society*, 275(2):599–609, 1983. [8](#), [9](#)
- [JS84] Carl G. Jockusch and Richard A. Shore. Pseudo-jump operators. ii: Transfinite iterations, hierarchies and minimal covers. *The Journal of Symbolic Logic*, 49(4):1205–1236, 1984. [8](#)
- [Kau91] Steven M. Kautz. *Degrees of random sets*. PhD thesis, Cornell University, 1991. [3](#), [4](#), [5](#), [6](#), [8](#)
- [KN11] Antonín Kučera and André Nies. Demuth randomness and computational complexity. *Ann. Pure Appl. Logic*, 162(7):504–513, 2011. [6](#)
- [Kuč85] Antonín Kučera. Measure, Π_1^0 -classes and complete extensions of PA. In *Recursion Theory Week*, pages 245–259. Springer, 1985. [8](#)
- [Lev76] L. A. Levin. Uniform tests for randomness. *Dokl. Akad. Nauk SSSR*, 227(1):33–35, 1976. [1](#)

- [LR19] Mingyang Li and Jan Reimann. Turing degrees and randomness for continuous measures. *arXiv preprint arXiv:1910.11213*, 2019. [2](#)
- [MSU98] Andrei A. Muchnik, Alexei L. Semenov, and Vladimir A. Uspensky. Mathematical metaphysics of randomness. *Theoret. Comput. Sci.*, 207(2):263–317, 1998. [5](#)
- [Nie09] André Nies. *Computability and randomness*, volume 51 of *Oxford Logic Guides*. Oxford University Press, Oxford, 2009. [3](#), [7](#)
- [NST05] André Nies, Frank Stephan, and Sebastiaan A. Terwijn. Randomness, relativization and Turing degrees. *J. Symbolic Logic*, 70(2):515–535, 2005. [6](#)
- [RS15] Jan Reimann and Theodore A. Slaman. Measures and their random reals. *Trans. Amer. Math. Soc.*, 367(7):5081–5097, 2015. [1](#), [6](#)
- [RS21] Jan Reimann and Theodore A. Slaman. Effective randomness for continuous measures. *arXiv preprint arXiv:1808.10102v3*, 2021. [1](#), [2](#), [3](#), [4](#)
- [Sac63] Gerald E. Sacks. *Degrees of unsolvability*. Princeton University Press, Princeton, N.J., 1963. [4](#)
- [SF77] C.P. Schnorr and P. Fuchs. General random sequences and learnable sequences. *J. Symbolic Logic*, 42(3):329–340, 1977. [6](#)
- [She86] Alexander Shen. One more definition of random sequence with respect to computable measure. In *First World Congress of the Bernoulli Society on Math. Statistics and Probability theory, Tashkent*, 1986. [3](#)
- [Sim07] Stephen G. Simpson. Mass problems and almost everywhere domination. *MLQ Math. Log. Q.*, 53(4-5):483–492, 2007. [4](#)
- [Sti72] John Stillwell. Decidability of the “almost all” theory of degrees. *J. Symbolic Logic*, 37:501–506, 1972. [7](#)
- [SUV17] A. Shen, V. A. Uspensky, and N. Vereshchagin. *Kolmogorov complexity and algorithmic randomness*, volume 220 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2017. [3](#)
- [ZL70] A. K. Zvonkin and L. A. Levin. The complexity of finite objects and the basing of the concepts of information and randomness on the theory of algorithms. *Uspehi Mat. Nauk*, 25(6(156)):85–127, 1970. [3](#), [6](#), [8](#)