Complexity of the emptiness problem for graph-walking automata and for tilings with star subgraphs^{*}

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Abstract

This paper proves the decidability of the emptiness problem for two models which recognize graphs: graph-walking automata, and tilings of graphs by star subgraphs (star automata). Furthermore, it is proved that the non-emptiness problem for graph-walking automata (that is, whether a given automaton accepts at least one graph) is NEXP-complete. For star automata, which generalize nondeterministic tree automata to the case of graphs, it is proved that their non-emptiness problem is NP-complete.

1 Introduction

The main result of this paper is the decidability of the emptiness problem for graphwalking automata and its computational complexity.

A graph-walking automaton is a model of a robot in a maze. It has finitely many states, and it deterministically walks on graphs with labelled nodes and labelled edge end-points. The automaton decides by which edge to move depending on the label of the current node and on its current state. The automaton can also decide to accept or to reject, and so it defines a graph language: the set of graphs it accepts.

Graph-walking automata were first introduced by Michael Rabin, who stated the conjecture that for each graph-walking automaton, even if it is additionally allowed to use finitely many pebbles, there is a graph that it cannot fully explore. Budach [3] proved this conjecture for graph-walking automata without pebbles. Later Fraigniaud et al. [4] gave an easier proof of this fact. Rollik [11] proved that not only pebbles, but even co-operation of several interacting automata would not help to traverse every graph, thus proving Rabin's conjecture. Kunc and Okhotin [8] showed that every graph-walking automaton can be transformed to an automaton which halts on every input, to an automaton which accepts only at the initial node, and to a reversible automaton, which all accept the same set of graphs. Later Martynova and Okhotin [9] reduced the number of states needed for these transformations, and obtained asymptotically tight lower bounds.

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Overall, graph-walking automata have been studied for a long time, and it is natural to ask whether their emptiness problem is decidable, and if it is, then in which complexity class it lies.

There are several results on decidability and computational complexity of the emptiness problem for simpler kinds of finite automata that traverse an input object: for deterministic two-way finite automata (2DFA), the emptiness problem is PSPACE-complete (this follows from the work of Kozen [7, Lemma 3.2.3]), whereas for deterministic treewalking automata the analogous problem is EXP-complete, as proved by Bojańczyk [1].

Another kind of finite automata are nondeterministic automata that recognize a given object by *tiling* it with neighbourhoods of states. Such are one-way nondeterministic finite automata (NFA), for which the non-emptiness problem is NL-complete (this is one of the classical problems presented by Jones [6]). For trees, such are nondeterministic tree automata, whose emptiness problem is P-complete, as shown by Veanes [13].

Tiling models were also considered for graphs. Thomas [12] introduced graph acceptors: in this model, a graph is accepted, if it can be covered with tiles (subgraphs) from a fixed finite set, so that each node is in the inner part of some tile, states in overlapping tiles are the same, and some further constraints on the number of occurrences of every tile hold. For this general model, Thomas proved undecidability of the emptiness problem by recognizing the set of rectangular grids and simulating a Turing machine on the grids. Thomas also considered *elementary acceptors*: a special case in which every tile is a star, that is, a node with all its neighbours. For elementary acceptors, Thomas proved that the language of grids cannot be recognized. However, the decidability of the emptiness problem for elementary acceptors remains open.

Besides the emptiness problem for graph-walking automata, another problem considered in this paper is the emptiness problem for *star automata*, that is, for elementary acceptors of Thomas without additional constraints on the number of occurrences of tiles. Star automata are at the same time a special case of the model by Thomas, and a generalization of nondeterministic tree automata to graphs.

In this paper, it is proved that the non-emptiness problem for graph-walking automata is decidable, and furthermore, NEXP-complete, while for star automata this problem is decidable and NP-complete.

The basic definitions of automata are given in Section 2. Graph-walking automata and star automata are defined over a *signature*, which is an alphabet for graphs. A signature defines finite sets of possible node labels and possible labels of edge end-points (called directions). Also, for each node label, there is a set of directions used in all nodes with this label.

The decidability of the emptiness problem and upper bounds on its complexity are obtained for graph-walking automata and for star automata using similar methods. A simpler problem called *signature non-emptiness* is considered first: does there exist at least one graph over a given signature? Its decidability is proved in Section 3 by reducing it to finding a non-negative integer solution to a certain system of linear equations. From this, it is inferred that the non-emptiness problem for signatures can be solved in NP. Furthermore, if a signature is non-empty, that is, if there is at least one graph over this signature, then the number of nodes in the smallest such graph does not exceed $2mr \min\{r^r, k^{2r-2}\}$, where m is the number of node labels in the signature, 2r is the number of directions, and k is the maximum degree of a node.

It turns out that both checking non-emptiness of a graph-walking automaton and

checking non-emptiness of a star automaton can be reduced to checking non-emptiness of a certain signature, which is constructed for a given automaton.

For star automata, such a reduction is presented in Section 4. It gives a proof that the non-emptiness problem for star automata is in NP. Also it gives an upper bound $sn^2k^{kn^2-1}$ on the number of nodes in the smallest accepted graph, where n is the number of states in the star automaton, s is the number of stars, and k is the number of directions in the signature.

In Section 5, a graph-walking automaton is reduced to a signature. The reduction proves that its non-emptiness problem is in NEXP, as well as gives an upper bound $m4^{n(k+1)}k^{k4^n-1}$ on the number of nodes in the smallest accepted graph, where n is the number of states, k is the number of directions, and m is the number of node labels.

In Section 6, all the above non-emptiness problems are proved to be hard in their complexity classes. NP-hardness of the signature non-emptiness problem is obtained by reducing 3-colourability to this problem. This also gives NP-hardness for non-emptiness of star automata. To prove NEXP-hardness of non-emptiness of graph-walking automata, it is shown that a graph-walking automaton can recognize the set of graphs containing a rectangular grid of exponential size in the number of its states. On this grid, the computation of a nondeterministic Turing machine is then simulated.

Note that the complexity classes for related problems, such as whether a graph-walking automaton accepts all graphs over its signature (the universality problem), or whether the intersection of languages of two automata is empty, can be inferred from the result for the non-emptiness problem. Indeed, since every graph-walking automaton can be transformed to an automaton that halts on every input, and the transformation given by Kunc and Okhotin [8] can be done in polynomial time, the emptiness problem for graph-walking automata is equivalent to the universality problem. As for the intersection emptiness problem, Martynova and Okhotin [10] obtained a transformation for the intersection of two graph-walking automata, which can be done in polynomial time too. Thus, the universality problem and the intersection emptiness problem for graph-walking automata are both co-NEXP-complete.

2 Graph-walking and star automata

In this section, graph-walking automata and star automata are formally defined. All definitions for graph-walking automata are inherited from the paper by Kunc and Okhotin [8]. Star automata are a variant of *elementary acceptors* by Thomas [12] without constraints on the number of tiles, and are given in a different notation for uniformity with graphwalking automata.

Graph-walking automata are defined over a signature. A signature specifies the sets of labels of nodes and edge end-points in the graphs, and thus defines the set of all labelled graphs that can be used as inputs for a graph-walking automaton.

Definition 1 ([8]). A signature S is a quintuple $S = (D, -, \Sigma, \Sigma_0, (D_a)_{a \in \Sigma})$, where:

- D is a finite set of directions, which are labels attached to edge end-points;
- a bijection $-: D \to D$ provides an opposite direction, with -(-d) = d for all $d \in D$;
- Σ is a finite set of node labels;

- $\Sigma_0 \subseteq \Sigma$ is a subset of possible labels of the initial node;
- $D_a \subseteq D$, for every $a \in \Sigma$, is the set of directions used in nodes labelled with a.

Graphs are defined over a signature like strings are defined over an alphabet.

Definition 2. A graph over a signature $S = (D, -, \Sigma, \Sigma_0, (D_a)_{a \in \Sigma})$ is a quadruple $(V, v_0, +, \lambda)$, where:

- V is a finite set of nodes;
- $v_0 \in V$ is the initial node;
- edges are defined by a partial function $+: V \times D \to V$, such that if v + d is defined, then (v + d) + (-d) is defined and equals v; also denote v - d = v + (-d);
- node labels are assigned by a total mapping $\lambda: V \to \Sigma$, such that
 - i. v + d is defined if and only if $d \in D_{\lambda(v)}$, and
 - ii. $\lambda(v) \in \Sigma_0$ if and only if $v = v_0$.

The set of all graphs over the signature S is denoted by L(S).

The function + defines the edges of the graph. If u + d = v, then the nodes u and v in the graph are connected with an edge with its end-points labelled with directions d (on the side of u) and -d (on the side of v). Multiple edges and loops are possible: if v + d = v and $d \neq -d$, then it is a loop at the node v with two ends labelled with directions d and -d. If v + d = v and d = -d, then it is a loop at the node v with one end, labelled with d.

A graph-walking automaton is defined similarly to a 2DFA, with an input graph instead of an input string.

Definition 3. A (deterministic) graph-walking automaton (GWA) over a signature $S = (D, -, \Sigma, \Sigma_0, (D_a)_{a \in \Sigma})$ is a quadruple $A = (Q, q_0, F, \delta)$, where

- Q is a finite set of states;
- $q_0 \in Q$ is the initial state;
- $F \subseteq Q \times \Sigma$ is a set of acceptance conditions;
- $\delta: (Q \times \Sigma) \setminus F \to Q \times D$ is a partial transition function, with $\delta(q, a) \in Q \times D_a$ for all q and a where δ is defined.

When an automaton operates on a graph, at every moment it knows its current state and sees only the label of the current node. The transition function gives the new state and the direction to one of the neighbouring nodes, in which the automaton moves. If the current pair of a state and a node label is in F, then the automaton accepts. If the pair is not in F and no transition is defined for it, then the automaton rejects. It may also continue walking indefinitely, it this case it is said to loop.

Formally, an automaton's configuration on a graph $G = (V, v_0, +, \lambda)$ is a pair (q, v), with $q \in Q$ and $v \in V$. A computation of an automaton A on a graph G is the following uniquely defined sequence of configurations. The computation starts in the initial configuration (q_0, v_0) . For every configuration (q, v) in the computation, if $\delta(q, \lambda(v))$ is defined and equals (q', d), then the next configuration after (q, v) is (q', v+d). Otherwise, the configuration (q, v) is the last one in the computation; if $(q, \lambda(v)) \in F$, then the automaton *accepts* in the configuration (q, v), otherwise it rejects. If the computation is an infinite sequence, then the automaton is said to *loop*.

A graph-walking automaton A defines the language L(A), this is the set of graphs it accepts.

The methods used in this paper to prove the decidability of the emptiness problem for graph-walking automata and to determine its computational complexity can also be applied to another related model. These are *star automata*, which are defined as follows.

Definition 4. Let $S = (D, -, \Sigma, \Sigma_0, (D_a)_{a \in \Sigma})$ be a signature and let some linear order be fixed on the set of directions D. A star automaton A_* over the signature S is a pair (Q, T), where

- Q is a finite set of states;
- T is a finite set of stars, where a star is a sequence of the form $(a, q, q_1, \ldots, q_{|D_a|})$, where a is a node label, q is used for the state in the current node, q_1, \ldots, q_{D_a} are used for states in the neighbours of the current node in all directions from D_a .

A graph G is accepted by the star automaton A_* , if there is a choice of states $(q(v))_{v \in V}$ in all nodes such that the following condition holds for each node $v \in V$. Let a be the label of the node v, let d_1, \ldots, d_{D_a} be the directions from D_a listed in the order. Then, the star in the node v is the sequence $s(v) = (a, q(v), q(v + d_1), \ldots, q(v + d_{|D_a|}))$. And every such star should belong to the set of automaton's stars T. Such a sequence $(q(v))_{v \in V}$ is called a computation of the star automaton A_* on the graph G. There can be several computations.

3 The non-emptiness problem for signatures is in NP

In this section, the decidability of the non-emptiness problem for signatures is proved; more precisely, an NP-algorithm that solves this problem is constructed. Furthermore, for non-empty signatures, an upper bound on the number of nodes in the minimal graph over a given signature is obtained.

It turns out that to prove that a signature is non-empty it is not necessary to find an actual graph. It is sufficient to find only a collection of nodes without the edge structure of the graph; such a collection is described by a vector with every coordinate giving the number of nodes with a certain label. A vector can be turned into a graph if it satisfies a few conditions.

Definition 5. Let $S = (D, -, \Sigma, \Sigma_0, (D_a)_{a \in \Sigma})$ be a signature. A vector of non-negative integers $(x_a)_{a \in \Sigma}$, where x_a is the number of nodes with the label a, is called *balanced*, if it satisfies the following two balance conditions:

1. an initial node exists and is unique: $\sum_{a_0 \in \Sigma_0} x_{a_0} = 1$,

2. for each direction $d \in D$, such that $d \neq -d$, all nodes together need the same number of edges by d and by -d:

$$\sum_{a \in \Sigma: \ d \in D_a} x_a = \sum_{a \in \Sigma: \ -d \in D_a} x_a.$$

The next lemma shows that every balanced vector gives rise to a graph, and hence one can work with balanced vectors instead of graphs.

Lemma 1. Let $S = (D, -, \Sigma, \Sigma_0, (D_a)_{a \in \Sigma})$ be a signature. Let x_a , for each node label $a \in \Sigma$, be a non-negative integer.

A graph over the signature S with exactly x_a nodes labelled with a, for all $a \in \Sigma$, exists if and only if the vector $(x_a)_{a \in \Sigma}$ is balanced.

Furthermore, there is an algorithm that, given a signature S and a balanced vector $(x_a)_{a \in \Sigma}$, constructs a graph over S with exactly x_a nodes with each label $a \in \Sigma$, and does so in time linear in the sum of sizes of the signature and of the constructed graph.

Proof. For every graph G over S, let $(x_a)_{a\in\Sigma}$ be the vector of quantities of nodes for all labels. It is claimed that the vector $(x_a)_{a\in\Sigma}$ is balanced. The first balance condition holds, because every graph has exactly one initial node. Now to the second condition. Let $d \in D$ be one of the directions, with $d \neq -d$. Then, every edge v + d = u in the graph links the two edge end-points: in the direction d at the node v, and in the direction -d at the node u. Thus, the total number $\sum_{a\in\Sigma: d\in D_a} x_a$ of edge end-points labelled with d in the graph equals the number $\sum_{a\in\Sigma: -d\in D_a} x_a$ of edge end-points labelled with -d, and the second balance condition holds.

Conversely, let $(x_a)_{a\in\Sigma}$ be a balanced vector. A graph $G = (V, v_0, +, \lambda)$ with exactly x_a nodes for each node label a is constructed by the following algorithm.

- First, the set of nodes V and the labelling function λ are defined: for each node label $a \in \Sigma$ in the signature, x_a new nodes labelled with a are added to the set V.
- The initial node is the node with a label from the set Σ_0 , the first balance condition states that such a node exists and is unique.
- Now the edges shall be defined so, that each node v labelled with a will have edges exactly in the directions from D_a . For each direction $d \in D$, let I_d be the set of all nodes v with $d \in D_{\lambda(v)}$.

For such directions $d \in D$, that d = -d, the algorithm makes loops: for every node $v \in I_d$ it adds a loop v + d = v.

For each pair of opposite directions $d \neq -d$, the algorithm takes nodes from I_d and I_{-d} , and links them with (d, -d)-edges. By the second balance condition, $|I_d| = |I_{-d}|$, thus, every node gets all the edges it needs.

Now, to check whether a signature is non-empty, that is, whether there is at least one graph over this signature, one can just check whether there is at least one balanced vector for this signature.

For a signature $S = (D, -, \Sigma, \Sigma_0, (D_a)_{a \in \Sigma})$, balanced vectors $(x_a)_{a \in \Sigma}$ with the minimal possible sum of coordinates will be called *minimal balanced vectors*.

How large could be the sum of the coordinates of a minimal balanced vector? The next theorem gives an upper bound on this sum, that is, on the minimal number of nodes in the graph over a signature.

Theorem 1. Let $S = (D, -, \Sigma, \Sigma_0, (D_a)_{a \in \Sigma})$ be a non-empty signature, and assume that $|D| \ge 2$ and that D_a is non-empty for all $a \in \Sigma$. Let $r = \frac{1}{2}|D|$, $m = |\Sigma|$ and k = 1 $\max_{a \in \Sigma} |D_a|.$

Then, there is a graph over the signature S with at most $2mr\min\{r^r, k^{2r-2}\}$ nodes.

Note that the bound $2mrk^{2r-2}$ can be useful for signatures with many directions, but with a small maximum degree of nodes. Later on, such signatures will be produced by the reductions of the emptiness problems for graph-walking automata and for star automata to the emptiness problem for signatures.

First, the conditions and the claims of Theorem 1 are reformulated in the language of linear algebra.

By Lemma 1, to prove Theorem 1 it is sufficient to prove that there is such a balanced vector $(x_a)_{a \in \Sigma}$ that $\sum_{a \in \Sigma} x_a \leq 2mr \min\{r^r, k^{2r-2}\}.$

Let n be the number of pairs of opposite directions $\{d, -d\}$, with $d, -d \in D$ and $d \neq -d$, in the signature S. It is convenient to rewrite n linear equations in the second balance condition as one vector equation. Let $\{d_1, -d_1, \ldots, d_n, -d_n\}$ be all such directions in D that $d \neq -d$, here the directions d_i and $-d_i$ are opposite, for $i = 1, \ldots, n$.

For each node label $a \in \Sigma$, the contribution of one node labelled with a to the balance of directions in a graph is given by a column vector v_a of height n. The *i*-th element of the vector v_a , for $i \in \{1, \ldots, n\}$, is defined as follows:

$$v_{a,i} = \begin{cases} 1 & \text{if } d_i \in D_a, -d_i \notin D_a \\ -1 & \text{if } -d_i \in D_a, d_i \notin D_a \\ 0 & \text{if } d_i \notin D_a, -d_i \notin D_a \text{ or } d_i \in D_a, -d_i \in D_a \end{cases}$$
(1)

Thus, the *i*-th element of the vector v_a is the contribution of an *a*-labelled node to the difference of the number of directions d_i and $-d_i$ in a graph.

Then, the second balance condition for the vector of quantities of labels $(x_a)_{a\in\Sigma}$ can be written in the following form:

$$\sum_{a \in \Sigma} x_a v_a = 0$$

If n = 0, then all directions are of the form d = -d, and one initial node with the loops is a correct graph. Let $n \ge 1$. As $n \le r$, it is sufficient to prove an upper bound $2mn\min\{n^n, k^{2n-2}\}$. Let $(x_a)_{a\in\Sigma}$ be a balanced vector with the minimal possible sum of the coordinates. Among the initial node labels, only one has a non-zero coefficient. Fix this initial label a_0 and let the vector $-v_{a_0}$ be denoted by b. Then, the coefficients for other initial labels are zeros and $\sum_{a \in (\Sigma \setminus \Sigma_0)} x_a = (\sum_{a \in \Sigma} x_a) - 1$. Then, to prove the theorem, it is sufficient to find such a non-negative integer solution $(x_a)_{a \in (\Sigma \setminus \Sigma_0)}$ to the equation $\sum_{a \in (\Sigma \setminus \Sigma_0)} x_a v_a = b$, that $\sum_{a \in (\Sigma \setminus \Sigma_0)} x_a \leq 2mn \min\{n^n, k^{2n-2}\} - 1$. Some vectors v_a for different non-initial labels can coincide. Let v_1, \ldots, v_ℓ be all vectors

from the set $\{v_a \mid a \in (\Sigma \setminus \Sigma_0)\}$ without repetitions and without a zero vector. Note

that $\ell < m$. Then, it is sufficient to find a non-negative integer solution $(x_i)_{i=1}^{\ell}$ to the equation $\sum_{i=1}^{\ell} x_i v_i = b$, with $\sum_{i=1}^{\ell} x_i \leq 2\ell n \min\{n^n, k^{2n-2}\}$.

What is known about vectors v_1, \ldots, v_ℓ ? These are column vectors of height n, with all elements in $\{0, 1, -1\}$. Each vector has at most k non-zero elements, since each node label $a \in \Sigma$ has at most k directions in D_a . To apply the methods of linear algebra, these vectors are considered over the field of real numbers: $v_1, \ldots, v_\ell \in \mathbb{R}^n$. Therefore, Theorem 1 is reduced to the following lemma.

Lemma 2. Let $v_1, \ldots, v_{\ell} \in \{0, 1, -1\}^n$ be distinct non-zero column vectors of height n, where $n \ge 1$; let $b \in \{0, 1, -1\}^n$ be a column vector. Let k be the maximum number of non-zero elements in the vector. Then, if the linear equation $\sum_{i=1}^{\ell} x_i v_i = b$ has at least one non-negative integer solution, then there exists such a non-negative integer solution $(x_i)_{i=1}^{\ell}$, that $\sum_{i=1}^{\ell} x_i \le 2\ell n \min\{n^n, k^{2n-2}\}$.

The proof of Lemma 2 will use a classical bound on matrix determinants, as well as its corollaries. Hadamard obtained the following upper bound for matrices of -1 and 1.

Theorem A (Hadamard [5]). Let $n \ge 1$ be an integer and let A be an $n \times n$ matrix, with all elements in $\{-1, 1\}$. Then, $|\det A| \le n^{\frac{n}{2}}$.

Hadamard also proved that if n is a power of 2, then the bound $n^{\frac{n}{2}}$ is achieved on some matrices.

The upper bound by Hadamard can be generalized from the case of elements in $\{-1, 1\}$ to any real numbers with absolute value not greater than 1. This is established in the following corollary. Also, I include a trivial upper bound for matrices with a small number of non-zeros in columns.

Corollary 1. Let $n \ge 1$ be an integer and let A be an $n \times n$ matrix, with all elements real and not exceeding 1 in absolute value. Then, $|\det A| \le n^{\frac{n}{2}}$.

If for some integer k, with $k \ge 1$, it is known that each column in the matrix A, maybe except one, has at most k non-zero elements, then $|\det A| \le k^{n-1}$.

Proof. First, the upper bound $n^{\frac{n}{2}}$ is proved for matrices without conditions on the number of non-zero elements. Among all $n \times n$ matrices with all elements real and not exceeding 1 in absolute value, let A be the one with the maximum absolute value of the determinant. It can be assumed that det $A \ge 0$, because otherwise one can multiply the first column by -1 and negate the determinant. The determinant of A is given by the following formula.

$$\det A = \sum_{\sigma \in S_n} (-1)^{\operatorname{sign}(\sigma)} \prod_{i=1}^n a_{i,\sigma(i)}.$$

If all elements of A are in $\{-1, 1\}$, then $|\det A| \leq n^{\frac{n}{2}}$ by the bound by Hadamard (Theorem A). Now let a_{ij} be any element in A with absolute value less than 1. The formula for the determinant can be represented as $\det A = ba_{ij} + c$, where b and c do not depend on a_{ij} . If b is positive, then a_{ij} can be changed to 1, making the determinant of A greater; otherwise, a_{ij} can be replaced with -1, without decreasing the determinant. Thus, all elements with absolute value less than 1 can be replaced one by one without decreasing the determinant, and so the Hadamard's bound $|\det A| \leq n^{\frac{n}{2}}$ holds for the matrix A.

Now to the second part of the corollary. Let the matrix A have at most k non-zero elements in each column, maybe except one column. By induction on n it is proved that

the determinant of every such matrix is at most k^{n-1} . For a 1×1 matrix, the determinant does not exceed 1 in absolute value. For an $n \times n$ matrix, with n > 1, as the absolute value of the determinant does not change when the columns in the matrix are permuted, one can assume that the last column has the greatest number of non-zero elements. By expanding along the first column, the determinant equals a sum of not more than k determinants of $(n-1) \times (n-1)$ matrices with the same properties, which by induction hypothesis are not greater than k^{n-2} in absolute value. And these matrices are taken with coefficients not greater than 1 in absolute value. Thus, $|\det A| \leq k \cdot k^{n-2} = k^{n-1}$.

The upper bounds on determinants are used to estimate the coefficients in linear equations.

Lemma 3. Let $n \ge 1$ be an integer, let $v_1, \ldots, v_t \in \{0, 1, -1\}^n$, with $t \ge 1$, be column vectors of height n, which are linearly independent in \mathbb{R}^n . Let k be the maximum number of non-zero elements in a vector, and let $N = \min\{n^{\frac{n}{2}}, k^{n-1}\}$. Let some vector $u \in \mathbb{R}^n$, with the maximum absolute value of its elements c, be represented as a linear combination: $\alpha_1 v_1 + \ldots + \alpha_t v_t = u$.

Then, $|\alpha_i| \leq cN$, for all $i \in \{1, \ldots, t\}$. Furthermore, if all elements in the vector u are integers, then all coefficients α_i , for $i \in \{1, \ldots, t\}$, are rational, and after multiplying the equation by their least common denominator one obtains the equation $\beta_1 v_1 + \ldots + \beta_t v_t + \beta_{t+1} u = 0$, with all coefficients β_i , for $i \in \{1, \ldots, t+1\}$, integer and not exceeding cN in absolute value.

Proof. If u is a zero vector, then all coefficients in the linear combination are zeros. Now let u be not a zero vector. The vectors v_1, \ldots, v_t are linearly independent, so the system of equations $x_1v_1 + \ldots + x_tv_t = u$ has at most one solution. Thus, the solution $(\alpha_1, \ldots, \alpha_t)$ is unique. To solve this system of equations using Cramer's rule, one needs the matrix of coefficients $V = (v_1, \ldots, v_t)$ to be square.

Since the vectors v_1, \ldots, v_t are linearly independent, $t \leq n$. First, consider the case of t < n. The matrix (v_1, \ldots, v_t, u) has the column rank t, because the columns v_1, \ldots, v_t are linearly independent, and the column u is their linear combination. It is known that the column rank equals the row rank, so there are t linearly independent rows in the matrix (V, u), all other rows are their linear combinations. That is, in the system of equations $x_1v_1 + \ldots + x_tv_t = u$, all equations are linear combinations of some t linearly independent equations. Taking only these t linearly independent equations one obtains a system $x_1v'_1 + \ldots + x_tv'_t = u'$, with all vectors of height t. The set of solutions has not changed, so $(\alpha_1, \ldots, \alpha_t)$ remains the only solution. Let $V' = (v'_1, \ldots, v'_t)$ be the matrix of coefficients of the new system of equations, it is a non-degenerate square matrix. If t = n, then the matrix V is already square and non-degenerate; in this case let V' = V, u' = u.

Now the new system of equations can be solved by Cramer's rule. Let $V'_i = (v'_1, \ldots, v'_{i-1}, u', v'_{i+1}, \ldots, v'_t)$ be the matrix, obtained from V' by replacing of the *i*-th column with the column vector u', for each $i = 1, \ldots, t$. Then, Cramer's rule claims that the unique solution to the system is $\alpha_i = \frac{\det V'_i}{\det V'}$, for $i = 1, \ldots, t$. Now one needs to estimate the determinants of the matrices V' and V'_i , for $i = 1, \ldots, t$.

Now one needs to estimate the determinants of the matrices V' and V'_i , for i = 1, ..., t. The matrix V' has all its elements in $\{0, 1, -1\}$. Also, each column of V' has at most k non-zeros. So Corollary 1 gives $|\det V'| \leq \min\{n^{\frac{n}{2}}, k^{n-1}\} = N$. Since all elements of V' are integers and the matrix is non-degenerate, det V' is a non-zero integer. Now consider the matrix V'_i , for some i = 1, ..., t. Let V''_i be the matrix obtained from V'_i by dividing the *i*-th column, which equals u, by c. Then, all elements of V''_i are not greater than 1 in absolute value. And each column has at most k non-zero elements, maybe, except the *i*-th column. By Corollary 1, the determinant of V''_i is estimated as follows: $|\det V''_i| \leq \min\{n^{\frac{n}{2}}, k^{n-1}\} = N$. Thus, the determinant of the matrix V'_i , which has one column multiplied by c, is bounded like this: $|\det V'_i| \leq cN$.

So, $|\alpha_i| = |\frac{\det V'_i}{\det V'}| \leq |\det V'_i| \leq cN$, for all i = 1, ..., t. If all elements of the vector u are integers, then all the determinants $\det V'_i$ are integers as well. Then all α_i , for i = 1, ..., t, are rational. And after muliplying the equation by their least common denominator, which is not greater than $|\det V'|$ in absolute value, one gets all new coefficients β_i , for i = 1, ..., t + 1, not greater in absolute value than $\max\{|\det V'_1|, ..., |\det V'_t|, |\det V'|\} \leq cN$.

Now it is time to prove the lemma, to which Theorem 1 has been reduced.

Proof of Lemma 2. Let $N = \min\{n^{\frac{n}{2}}, k^{n-1}\}$ be the upper bound from Corollary 1 on the determinants of $n \times n$ matrices with real elements not exceeding 1 in absolute value, and with at most k non-zero elements in each column, maybe, except one.

Let (x_1, \ldots, x_ℓ) be a non-negative integer solution to the system of linear equations $\sum_{i=1}^{\ell} x_i v_i = b$, with the minimum sum $\sum_{i=1}^{\ell} x_i$, and among these, with the minimum number of coordinates greater than N. The goal is to prove, that $\sum_{i=1}^{\ell} x_i \leq 2\ell n \min\{n^n, k^{2n-2}\}$.

Step 1 is to prove that all vectors v_i , for $i = 1, ..., \ell$, with $x_i > N$, are linearly independent over the field \mathbb{R} .

For the sake of a contradiction, suppose that these vectors are linearly dependent. Then a linear dependence involving the least number of vectors is chosen. The vectors v_1, \ldots, v_ℓ are rearranged, so that the vectors from the dependence go in the beginning: let v_1, \ldots, v_{t+1} be the vectors from this minimal linear dependence. It is known that $t \ge 2$, because all vectors v_1, \ldots, v_ℓ are distinct and there is no zero vector among them.

The vectors v_1, \ldots, v_t are linearly independent, whereas v_1, \ldots, v_{t+1} are linearly dependent. Then, the vector v_{t+1} is uniquely represented as a linear combination of the others: $v_{t+1} = \alpha_1 v_1 + \ldots + \alpha_t v_t$, where $\alpha_1, \ldots, \alpha_t \in \mathbb{R}$.

The vector v_{t+1} has all its elements integer and the maximum absolute value of its elements is 1; the vectors v_1, \ldots, v_t satisfy all conditions of Lemma 3. Thus, by Lemma 3, all coefficients $\alpha_1, \ldots, \alpha_t$ are rational, and after multiplying the linear combination by their least common denominator one gets the new linear combination $\beta_1 v_1 + \ldots + \beta_{t+1} v_{t+1} = 0$, with all coefficients integer and not exceeding N in absolute value.

Since the chosen linear dependence has the minimal number of vectors, $\beta_i \neq 0$, for all $i = 1, \ldots, t + 1$. If $\sum_{i=1}^{t+1} \beta_i < 0$, then the dependence $\beta_1 v_1 + \ldots + \beta_{t+1} v_{t+1} = 0$ can be multiplied by -1, so one can assume, that $\sum_{i=1}^{t+1} \beta_i \ge 0$. Consider the case when $\sum_{i=1}^{t+1} \beta_i > 0$. Then, let (y_1, \ldots, y_ℓ) be a vector defined by

Consider the case when $\sum_{i=1}^{t+1} \beta_i > 0$. Then, let (y_1, \ldots, y_ℓ) be a vector defined by $y_i = x_i - \beta_i$, for $i = 1, \ldots, t+1$, and $y_i = x_i$, for $i = t+2, \ldots, \ell$. Then, $\sum_{i=1}^{\ell} y_i v_i = b$, that is, (y_1, \ldots, y_ℓ) is another solution to the system of equations. All y_i are non-negative integers, because x_1, \ldots, x_{t+1} are greater than N, and $\beta_1, \ldots, \beta_{t+1}$ are integer and not greater than N in absolute value. And, $\sum_{i=1}^{\ell} y_i < \sum_{i=1}^{\ell} x_i$. This contradicts the minimality of the sum of the coordinates in the solution (x_1, \ldots, x_ℓ) .

of the coordinates in the solution (x_1, \ldots, x_ℓ) . Now let $\sum_{i=1}^{t+1} \beta_i = 0$. Then one can similarly subtract $(\beta_1, \ldots, \beta_{t+1})$ from (x_1, \ldots, x_{t+1}) several times until some coefficient among the first t+1 becomes not greater than N. Such subtractions will not break the equation, will not make any coordinate negative, will not change the sum of the coordinates in the solution, but will decrease the number of coordinates which are greater than N. This contradicts the minimality of the number of such coordinates among the solutions with the minimal sum of the coordinates.

Step 1 is done. Now it is known that all vectors among v_1, \ldots, v_ℓ which have the corresponding coefficients in the solution (x_1, \ldots, x_ℓ) greater than N are linearly independent. Let these vectors be put first, so that they are v_1, \ldots, v_ℓ .

Step 2 is to prove that x is the desired solution, that is, that $\sum_{i=1}^{\ell} x_i \leq 2\ell n \min\{n^n, k^{2n-2}\}.$

The sum to be estimated is: $\sum_{i=1}^{\ell} x_i = \sum_{i=1}^{t} x_i + \sum_{i=t+1}^{\ell} x_i$. The second sum is bounded by $\sum_{i=t+1}^{\ell} x_i \leq (\ell - t)N$, as it has all coefficients not greater than N. If the first sum is non-empty (t > 0), then the first t variables are bounded as follows. The system of equations is rewritten in the following way: $x_1v_1 + \ldots + x_tv_t = b - (x_{t+1}v_{t+1} + \ldots + x_\ell v_\ell)$. Here the vectors v_1, \ldots, v_t are linearly independent, whereas the sum on the right-hand side is a column vector of height n, with all elements not greater than ℓN in absolute value (if t > 0, then $\sum_{i=t+1}^{\ell} x_i < \ell N$). By applying Lemma 3, with $u = b - (x_{t+1}v_{t+1} + \ldots + x_\ell v_\ell)$, one obtains $|x_i| \leq \ell N^2$, for all $i = 1, \ldots, t$. As $t \leq n$,

$$\sum_{i=1}^{\ell} x_i = \sum_{i=1}^{t} x_i + \sum_{i=t+1}^{\ell} x_i \leqslant t\ell N^2 + (\ell - t)N \leqslant n\ell N^2 + \ell N \leqslant \\ \leqslant 2\ell nN^2 = 2\ell n \min\{n^n, k^{2n-2}\}.$$

Theorem 1, which has just been proved, gives the upper bound $2mr \min\{r^r, k^{2r-2}\}$ on the number of nodes in the minimal graph over a non-empty signature, which depends on its parameters: on the number of node labels $m = |\Sigma|$, on the number of directions 2r = |D| and on the maximum possible degree of a node $k = \max\{|D_a| \mid a \in \Sigma\}$. This bound, and also Lemma 1, that allows one to work with balanced vectors instead of graphs, help to construct an NP-algorithm, that solves the non-emptiness problem for signatures.

Theorem 2. There is an NP-algorithm that takes a signature as an input and determines whether there is at least one graph over this signature or not.

Proof. The size of an input $S = (D, -, \Sigma, \Sigma_0, (D_a)_{a \in \Sigma})$ is not less than $|\Sigma| + |D|$. In the degenerate case of $|D| \leq 1$, it is sufficient to check for one-node graphs. Any initial labels a_0 with D_{a_0} empty form correct graphs; any such non-initial labels can be omitted.

With the trivial cases removed, by Theorem 1, if a graph over the signature S exists, then there is a graph with at most exponentially many nodes in |D| and $|\Sigma|$. Then, by Theorem 1 and by Lemma 1, the signature is non-empty if and only if there exists a balanced vector $(x_a)_{a\in\Sigma}$, with the sum of coordinates not greater than this exponential upper bound.

Thus, the nondeterministic algorithm guesses a vector $(x_a)_{a\in\Sigma}$, with sum of the coordinates not greater than exponential, and writes it down in polynomial time. It remains to check whether the guessed vector is balanced: that is, whether only one label among

the initial node labels has a non-zero coefficient, and whether for each pair of opposite directions $(d, -d) \in D$, with $d \neq -d$, the following equation holds:

$$\sum_{a \in \Sigma: \ d \in D_a} x_a = \sum_{a \in \Sigma: \ -d \in D_a} x_a.$$

This can all be checked in polynomial time, because the number of terms in these sums is polynomial, and each term is not greater than exponential.

If the algorithm guessed the vector, which is balanced, then the signature is non-empty and the algorithm answers "yes". Otherwise, it answers "no". \Box

In fact, the non-emptiness problem for signatures is NP-complete, this is shown later in Section 6.

4 Reducing a star automaton to a signature

This section proves the decidability of the emptiness problem for star automata. An NP-algorithm is constructed, which, for a given star automaton, determines whether it accepts at least one graph. Moreover, an upper bound on the number of nodes in the smallest accepted graph is proved in this section.

It turns out that the emptiness problem for star automata can be reduced in polynomial time to the emptiness problem for signatures, which was proved to be in NP.

Theorem 3. There exists a polynomial-time algorithm that takes as an input a signature $S = (D, -, \Sigma, \Sigma_0, (D_a)_{a \in \Sigma})$ with k directions and a star automaton $A_* = (Q, T)$ over S with n states and s stars, and computes a signature $S' = (D', -, \Sigma', \Sigma'_0, (D'_{a'})_{a' \in \Sigma'})$ with kn^2 directions and with s node labels, with the following property. There exists a bijective function f that maps a graph G over S and a computation $C = (q(v))_{v \in V}$ of the automaton A_* on this graph to a graph G' = f(G, C) over the signature S', which has the same set of nodes and the same edge structure as the graph G (the only difference between G and G' is in node labels and in directions).

Proof. Node labels and directions of the new signature S' will contain information on old node labels and directions, and also some additional information that encodes the computation of the star automaton A_* on a graph. More precisely, node labels will additionally encode stars in nodes that appear in the computation, whereas directions will encode the states of the star automaton at the two ends of an edge.

The new signature S' is constructed as follows.

- Node labels are all the stars of the automaton A_* , that is, $\Sigma' = T$.
- Initial node labels are all the stars of A_* , in which the first component is an initial node label from the old signature, that is, $\Sigma'_0 = \{(a, q, q_1, \dots, q_{|D_a|}) \in T \mid a \in \Sigma_0\}.$
- The set of directions is $D' = D \times Q \times Q$, where the direction (d, q_1, q_2) means that in the old graph the direction d was here, and in the encoded computation the state at the current node is q_1 and the state at the opposite end of the edge is q_2 .
- The relation of the opposite direction is: $-(d, q_1, q_2) = (-d, q_2, q_1)$, for all $(d, q_1, q_2) \in D'$.



Figure 1: Left: computation of a star automaton A_* on a graph G. Right: augmented graph G' that encodes both G and this computation.

• For each star $t = (a, q, q_1, \dots, q_{|D_a|}) \in \Sigma'$, where $d_1, \dots, d_{|D_a|}$ are ordered directions from D_a , the set of directions for the node label t is defined by $D'_t = \{ (d_i, q, q_i) \mid i = 1, \dots, |D_a| \}$.

Such a signature S' can be computed from S and A_* in polynomial time. There are exactly kn^2 directions and exactly s node labels in the signature S'.

It will be proved now, that there is a one-to-one correspondence between graphs over S'and pairs (G, C) of a graph over S and a computation of A_* on this graph. An example of such a correspondence is shown in Figure 1. For a star automaton A_* with stars (a, q, p, p), (a, p, p, q), (b, p, q, p, q), (a_0, q, p) , its computation on a graph G is given on the left. On the right, there is a graph G' that encodes stars in node labels and states at the two ends of an edge in directions.

Let $G = (V, v_0, +, \lambda)$ be a graph over S, and let $C = (q(v))_{v \in V}$ be a computation of the star automaton A_* on this graph. Then the graph $f(G, C) = G' = (V', v'_0, +, \lambda')$ over the signature S' that encodes the graph G and the computation C is constructed as follows.

- The set of nodes and the initial node are the same: $V' = V, v'_0 = v_0$.
- The edges in the graph G' connect the same nodes as in G, but all the directions are augmented with the states at the ends of an edge. If v + d = u in the graph G, then v + (d, q(v), q(u)) = u in the graph G', and these are all edges in G'. Then, the ends of each edge are labelled with opposite directions.
- The node labels in G' are stars in nodes. For each node $v \in V$ with some label $\lambda(v) = a$, the node label in the graph G' is $\lambda'(v) = (a, q(v), q(v+d_1), \ldots, q(v+d_{|D_a|}))$, where $d_1, \ldots, d_{|D_a|}$ are ordered directions from D_a . Then, $\lambda'(v) \in T = \Sigma'$, because $(q(v))_{v \in V}$ is a computation. And the directions in G', used at the node v, are all the directions from $D_{\lambda'(v)}$. And only the initial node has an initial label.

This transformation maps different pairs (G, C) to different graphs G', because no information is lost. Conversely, for each graph G' over the signature S' there is a unique corresponding pre-image (G, C), where G is obtained by dropping some information from all labels, and node labels explicitly give states and stars in a computation. Each edge in G' checks that the states at the nodes it connects are consistent with the stars. \Box

Now the results proved for signatures in the previous section will be transferred to star automata.

Corollary 2. The non-emptiness problem for star automata, that is, whether a given star automaton accepts at least one graph or not, can be solved in NP.

Proof. By Theorem 3, for a star automaton A_* that works over some signature S, one can construct in polynomial time such a signature S' of polynomial size, that graphs over S' are bijectively mapped to the computations of A_* on graphs over S.

A graph is accepted by the star automaton A_* if there exists at least one computation of A_* on it. Thus, to check whether the star automaton is non-empty, one can just check whether the signature S' is non-empty. By Theorem 2, the latter can be done in nondeterministic polynomial time.

The upper bound on the number of nodes in the minimal graph over a signature (Theorem 1) can be transferred to star automata as well.

Corollary 3. Let $S = (D, -, \Sigma, \Sigma_0, (D_a)_{a \in \Sigma})$ be a signature with $k \ge 2$ directions, and with $|D_a| \ge 1$ for all $a \in \Sigma$. Let $A_* = (Q, T)$ be a star automaton with n states and with s stars over this signature. If A_* accepts at least one graph, then the accepted graph with the minimal number of nodes has at most $sn^2k^{kn^2-1}$ nodes.

Proof. Let A_* accept at least one graph. The signature S' is constructed from the signature S and from the star automaton A_* by Theorem 3. The graphs over S' correspond to the computations of A_* on graphs over S with the same number of nodes.

Then, the number of nodes in the minimal accepted graph for A_* equals the number of nodes in the minimal graph over the signature S'. This signature has kn^2 directions and s node labels, the maximum degree of a node does not exceed k (because the function f from Theorem 3 does not change the edge structure of a graph). Then, Theorem 1 gives the following upper bound on the number of nodes in the minimal graph: $2s\frac{1}{2}kn^2\min\{(\frac{1}{2}kn^2)^{\frac{1}{2}kn^2}, k^{kn^2-2}\}$. It can be bounded by a simpler expression:

$$2s\frac{1}{2}kn^{2}\min\left\{\left(\frac{1}{2}kn^{2}\right)^{\frac{1}{2}kn^{2}},k^{kn^{2}-2}\right\}\leqslant skn^{2}k^{kn^{2}-2}=sn^{2}k^{kn^{2}-1}.$$

5 Reducing a graph-walking automaton to a signature

In Section 4, the emptiness problem for star automata was reduced to the emptiness problem for signatures. In this section such a reduction is made for the emptiness problem for graph-walking automata. Note that whereas a computation of a star automaton is a way to choose states in nodes, and the graph is accepted by a star automaton if there is at least one computation on this graph, graph-walking automata are different. In a graph-walking automaton, the computation on a graph is a sequence of configurations (q, v) of the automaton on a graph, where q is the current state, and v is the node which the automaton visits at the moment. This sequence in defined uniquely for each graph. The graph is accepted if the computation is accepting, that is, ends with an accepting configuration.

One way to reduce a graph-walking automaton to a signature is to simulate it by a star automaton. The next theorem shows that if some set of graphs is recognized by a graph-walking automaton, then this set of graphs can be defined by some star automaton. There is an analogous result for trees: star automata on trees are *nondeterministic tree automata*, graph-walking automata on trees are *deterministic tree-walking automata*, and, as noted by Bojańczyk and Colcombet [2], the inclusion of the class of languages defined even by nondeterministic tree-walking automata into the class defined by tree automata is a folklore result.

Theorem 4. For every n-state graph-walking automaton $A = (Q, q_0, F, \delta)$ over some signature $S = (D, -, \Sigma, \Sigma_0, (D_a)_{a \in \Sigma})$ with k directions and m node labels, there exists a star automaton $A_* = (P,T)$ with $(k + 1)^n$ states and at most $m(k + 1)^{n(k+1)}$ stars, defined over the same signature S, which accepts exactly the same graphs as A. The star automaton A_* has size exponential in the size of A and is constructed in exponential time.

This theorem is given without a proof, because the next theorem gives a direct reduction of a graph-walking automaton to a signature that provides a better upper bound on the number of nodes in the minimal accepted graph.

Theorem 5. There exists an algorithm that takes as an input some n-state graph-walking automaton $A = (Q, q_0, F, \delta)$ over some signature $S = (D, -, \Sigma, \Sigma_0, (D_a)_{a \in \Sigma})$ with k directions and m node labels, and computes such a signature $S' = (D', -, \Sigma', \Sigma'_0, (D'_{a'})_{a' \in \Sigma'})$ with $k4^n$ directions and with not more than $m4^{nk}$ node labels, that the following condition holds.

There exist two functions f and g. The function $f: L(A) \to L(S')$ injectively maps graphs over S, accepted by the automaton A, to graphs over S', and the function $g: L(S') \to L(A)$ is a surjection, such that g(f(G)) = G. If G' = f(G) or G = g(G'), then the graphs G and G' have the same sets of nodes and the same edge structure, only node labels and directions are different.

The size of the resulting signature is exponential in the size of the input, and the algorithm works in time exponential in the size of the input.

Proof. New node labels and directions of the signature S' encode node labels and directions of the signature S and some additional information about the behavior of the automaton A in the vicinity of the node or edge end-point.

The new directions are $D' = D \times 2^Q \times 2^Q = \{ (d, Q_{in}, Q_{out}) \mid d \in D; Q_{in}, Q_{out} \subseteq Q \}$. Every new direction (d, Q_{in}, Q_{out}) is an old direction d with two sets of states attached: Q_{in} encodes the states in which the automaton came in its computation on a graph to the current edge end-point moving in the direction -d, whereas Q_{out} consists of states, in which the automaton comes to the opposite end of the edge, moving in the direction d.

The opposite direction is $-(d, Q_{in}, Q_{out}) = (-d, Q_{out}, Q_{in})$, for each $(d, Q_{in}, Q_{out}) \in D'$.



Figure 2: Left: the accepting computation of some graph-walking automaton A on some graph G over the signature S. Right: the graph G' over the signature S' that encodes the graph and the accepting computation.

Each node label in S' contains an old node label and all information about the new directions in the node. But not every combination of new directions at a node makes a new label. The goal is to ensure that each graph over S' encodes a graph over S that is accepted by A, along with an accepting computation of A on this graph. For this, some combinations that cannot appear in accepting computations of the automaton A will be left out.

The set of node labels Σ' is a subset of

 $\widehat{\Sigma'} = \{ (a, E) \mid a \in \Sigma, E = \{ (d, Q_{in,d}, Q_{out,d}) \}_{d \in D_a}, \text{ where } Q_{in,d}, Q_{out,d} \subseteq Q \text{ for all } d \in D_a \}.$

It will be specified later, which elements of the set $\widehat{\Sigma'}$ are in Σ' and which are not.

The set of directions of a new node label (a, E) is $D'_{(a,E)} = E$. The label (a, E) is initial if and only if the label a is initial. Note that for each node label $a \in \Sigma$, there is only one direction $(d, Q_{in,d}, Q_{out,d})$ with the first component d in the set E, for each direction $d \in D_a$.

Figure 2 gives an example of how a graph G over S accepted by the automaton A can be converted to a graph G' over the signature S' by adding to each direction the information on the states in which the automaton crosses the edge, and by adding to each node label the information contained in all new directions at the node.

To complete the definition of the signature S', it remains to say, which pairs (a, E) from the set $\widehat{\Sigma}'$ are in the set Σ' , that is, are node labels of S'. Some pairs (a, E), which represent situations that cannot occur in any accepting computations of A, will be left out, and leaving them out will ensure that every graph over the signature S' encodes some graph G and an accepting computation of A on G.

A pair (a, E) is in Σ' if and only if the following conditions hold.

1. The sets $Q_{in,d}$ and $Q_{in,e}$ cannot intersect for directions $d \neq e$, where $d, e \in D_a$. If the label *a* is initial, then for each $d \in D_a$ it is prohibited to have $q_0 \in Q_{in,d}$.

Indeed, the automaton A cannot come to the node in the state q twice in the accepting computation, otherwise it will repeat a configuration and loop. By similar reasons the automaton cannot return to the initial node in the state q_0 in the accepting computation.

Denote by Q_{in} the set of all states in which the automaton A visits the node, according to the information in the node label (a, E). If $a \notin \Sigma_0$, then $Q_{in} = (\bigcup_{d \in D_a} Q_{in,d})$, if $a \in \Sigma_0$, then $Q_{in} = (\bigcup_{d \in D_a} Q_{in,d}) \cup \{q_0\}$.

2. For each state $q_1 \in Q_{in}$, either a transition $\delta(q_1, a)$ or acceptance $(q_1, a) \in F$ should be defined. If the transition $\delta(q_1, a) = (q_2, d)$ for some q_2 and d is defined, then this transition should be encoded, that is $q_2 \in Q_{out,d}$ should hold.

Indeed, if the automaton A in the accepting computation visits some node in the state q_1 , then it either accepts, or makes a transition, it cannot reject.

- 3. For each $d \in D_a$ and for each state $q_2 \in Q_{out,d}$, there must be a way to move from the current node in the state q_2 in the direction d. That is, there should exist a state $q_1 \in Q_{in}$, with $\delta(q_1, a) = (q_2, d)$.
- 4. For every two distinct states $p_1, q_1 \in Q_1$, with the transitions at the label *a* defined, the transitions should be distinct: $\delta(p_1, a) \neq \delta(q_1, a)$.

Indeed, the automaton in the accepting computation cannot come to the same configuration twice, otherwise it loops.

The signature S' has $k4^n$ directions. There are at most $m4^{kn}$ node labels, as in a label (a, E) there are m ways to choose an old label a, and 4^n ways to choose sets $Q_{in,d}$ and $Q_{out,d}$ for each direction $d \in D_a$.

All the directions with their opposite directions, and all the labels from $\widehat{\Sigma}'$ with their sets of directions can be written down in time linear in their length, that is, exponential in the length of the input. Checking whether a label $(a, E) \in \widehat{\Sigma}'$ satisfies all conditions, can be done in linear time in the length of the label.

The signature S' has been constructed, and it remains to prove the correspondence between the graphs over S accepted by the automaton A, and all the graphs over S', and to construct the functions f and g which define this correspondence.

Let the automaton A accept some graph $G = (V, v_0, +, \lambda)$ over the signature S. Then, the graph $f(G) = G' = (V', v'_0, +, \lambda')$ over the signature S' is constructed as follows.

- The set of nodes and the initial node are the same: $V' = V, v'_0 = v_0$.
- The edges in G' are the same as in G, but with additional information encoded in the directions. Let some edge e with directions (d, -d) connect the nodes u and vin the graph G, that is, u + d = v in G. Let $Q_{in} \subseteq Q$ be a set of states in which the automaton A in its computation comes to the node u from the node v by the edge e, let $Q_{out} \subseteq Q$ be a set of states, in which the automaton A arrives to the node vfrom the node u by the edge e. Then, the corresponding edge in G' is defined by $u + (d, Q_{in}, Q_{out}) = v$ and $v + (-d, Q_{out}, Q_{in}) = u$. These are all edges in G'.

The node labels in G' are the node labels from G, but with added information on the new directions. Let a node v in G have label a, and accordingly edges in directions from D_a. These directions in the graph G' are augmented with the information about the automaton's moves, forming the set E of new directions. Then the node label of the node v in the graph G' is (a, E). The label (a, E) is in Σ', because it encodes the moves of the automaton in the accepting computation (and only labels encoding situations impossible in accepting computations were not included in Σ'). The node v has edges in directions from E = D'_(a,E). And only the initial node has an initial label, because the new labels' being initial depends only on the component a of (a, E).

Now it remains to check, that each graph over S' corresponds to some graph over S that is accepted by A.

What is the general form of a graph G' over S'? In the first components of directions and node labels, it encodes some graph g(G') = G over S (and this is a definition of g). Then g(f(G)) = G by definition. The other components of directions and node labels encode some information about moves of the automaton. It will be shown that all moves from the computation of the automaton A on a graph G must be encoded, and that looping or rejecting cannot be encoded. Then, for each graph G' over S', the corresponding graph G = g(G') must be accepted by the automaton A. Note that, besides all moves from the accepting computation, the graph G' may additionally encode some cycles of transitions that do not intersect with the accepting computation. So an accepted graph G over Smay have several pre-images G', such that g(G') = G.

It remains to prove that each graph G' over the signature S' must encode all moves the automaton A makes in its computation on the graph G = g(G'), and possibly some moves not in this computation, and that the computation of A on G must be accepting.

Fix a graph G' over the signature S', let G = g(G'), and let $C = C_0, C_1, \ldots, C_N$ be the computation of the automaton A on the graph G, where C_N is the last configuration, or $N = \infty$ if the automaton loops. It should be proved that C is accepting and is encoded in G'.

This is proved by induction on *i* that either the configuration C_i is accepting, or the next configuration C_{i+1} exists and it is different from all previous configurations, and the move from configuration C_i to C_{i+1} is encoded in G'.

Let $i \in \{0, 1, 2, ..., N\}$, and let the claim be proved for all j < i.

Denote the *i*-th configuration by (q, v). Let (a, E) be the label of the node v in G'. Then, one can define Q_{in} for the label (a, E) as in the conditions on Σ' . If i = 0, then $a \in \Sigma_0$ and $q = q_0 \in Q_{in}$. Otherwise, the move from C_{i-1} to C_i is encoded in G', and $q \in Q_{in}$ as well. Then, by the second condition, as $q \in Q_{in}$, either $(q, a) \in F$, or $\delta(q, a) = (r, d)$, for some $r \in Q$, $d \in D$, and the transition is encoded as $r \in Q_{out,d}$. In the latter case r will be in $Q_{in,-d}$ for the node v + d. It remains to check that C_i is different from all previous configurations. If i = 0, then this is true. Now, let (p, u) be the previous configuration, with $\delta(p, \lambda(u)) = (q, d)$ and with u + d = v. Then $q \in Q_{in,-d}$ for the label (a, E) of the node v. The first condition gives that the automaton could not have entered the node v in the state q from another direction earlier in the computation, and that (q, v) cannot be the initial configuration. And if the previous direction is the same, then the 4-th condition prohibits entering (q, v) earlier from a previous state other than p. Then, only (p, u) can be the previous configuration for (q, v), and, by the induction hypothesis, (p, u) is unique in C_0, \ldots, C_{i-1} . Then, (q, v) is unique in C_0, \ldots, C_i .

Thus, the computation of A on G is encoded in G', this computation cannot loop, cannot reject, so it is accepting.

Using Theorem 5 that reduces graph-walking automata to signatures, one can solve the non-emptiness problem for graph-walking automata in nondeterministic exponential time.

Corollary 4. The problem of whether a given graph-walking automaton accepts at least one graph is in NEXP.

Proof. First, the algorithm from Theorem 5 is applied to a given signature S and to a given graph-walking automaton A over this signature, and it constructs a signature S', such that there exist functions $f: L(A) \to L(S')$ and $g: L(S') \to L(A)$. Then, L(A) is non-empty if and only if L(S') is non-empty. The size of the signature S' is exponential in the size of S and A, and this signature is constructed in exponential time. Checking whether L(S') is non-empty can be done in nondeterministic polynomial time in the size of S', that is, in nondeterministic exponential time in the sum of sizes of S and A.

Actually, the non-emptiness problem for graph-walking automata is NEXP-complete, that will be proved in Section 6.

An upper bound on the number of nodes in the minimal graph accepted by a graphwalking automaton can be derived from the analogous bound for signatures.

Corollary 5. Let $S = (D, -, \Sigma, \Sigma_0, (D_a)_{a \in \Sigma})$ be a signature with $k \ge 2$ directions, with m node labels, and with $|D_a| \ge 1$ for each $a \in \Sigma$. Let $A = (Q, q_0, F, \delta)$ be a graph-walking automaton over S with n states. Then, if A accepts at least one graph, then the number of nodes in the smallest accepted graph is at most $m4^{n(k+1)}k^{k4^n-1}$.

Proof. Let A accept at least one graph. By Theorem 5, there is a signature S', and functions $f: L(A) \to L(S')$ and $g: L(S') \to L(A)$ that do not change the number of nodes in a graph. So the minimal number of nodes for graphs over S accepted by A is equal to the minimal number of nodes in graphs over S'.

The signature S' has $k4^n$ directions, at most $m4^{kn}$ node labels, and the maximum degree of a node at most k. The latter is because f preserves edge structure of graphs. Then, by Theorem 1, the minimal graph over the signature S' has the number of nodes at most

$$m4^{nk}k4^n \min\left\{\left(\frac{1}{2}k4^n\right)^{\frac{k4^n}{2}}, k^{k4^n-2}\right\} \leqslant m4^{nk}k4^nk^{k4^n-2} = m4^{n(k+1)}k^{k4^n-1}.$$

 \Box

6 Computational complexity of emptiness problems

It has been proved that the non-emptiness problems for signatures and for star automata are both in NP, and that the non-emptiness problem for graph-walking automata is in NEXP. In this section, all these problems are proved to be complete in their complexity classes.

NP-hardness of the non-emptiness problem for signatures is proved by a reduction of graph 3-colourability to this problem.



Figure 3: From 3-colourability to signature non-emptiness: mapping a graph with a colouring to a graph over a signature.

Theorem 6. The problem of whether there is at least one graph over a given signature is NP-hard.

Proof. The 3-colourability problem for a connected graph G = (V, E) is to check whether its nodes can be coloured in $\{1, 2, 3\}$, so that every edge connects differently coloured nodes.

For an input graph G, one should construct such a signature S_G in polynomial time, that there exists a graph over S_G if and only if the graph G can be coloured correctly.

The signature S_G will be constructed so, that graphs over it correspond to correct colourings of the graph G = (V, E).

Nodes of G can have any of the three colours, and for each node and for each colour there is a corresponding node label. Furthermore, for every edge with two distinct colours on its ends, there is a separate node label representing this edge with these colours, that is, an unordered pair of two coloured nodes.

$$\Sigma = \left\{ \ (v,i) \ \middle| \ v \in V, \ i = 1,2,3 \ \right\} \cup \left\{ \ \{(u,i), \ (v,j)\} \ \middle| \ (u,v) \in E, \ i,j \in \{1,2,3\}, i \neq j \ \right\}.$$

The condition of the colouring to be correct is checked by not having labels of the form $\{(u, i), (v, i)\}$, representing edges with the same colour at both ends.

Fix any node $v_0 \in V$, and let all labels (v_0, i) , with i = 1, 2, 3, be initial.

The set of directions is organized so that for every edge (u, v) in the graph G, node labels (u, i) and (v, j), which correspond to the nodes u and v in the graph G, would require a connection through an intermediate node that corresponds to the edge (u, v) in G, and which gathers information on the colours of the nodes u and v.

$$D = \{ \pm(u, i, v) \mid u, v \in V, (u, v) \in E, i = 1, 2, 3 \}.$$

The opposite direction to +(u, i, v) is given by -(u, i, v), for all $u, v \in V$ with $(u, v) \in E$, and for all i = 1, 2, 3.

Each node of a graph over this signature which represents one of the nodes of G should be connected with the nodes representing all the edges coming out of this node.

$$D_{(u,i)} = \{ +(u, i, v) \mid v \in V, (u, v) \in E \}, \text{ for all } u \in V, i = 1, 2, 3.$$

 $D_{\{(u,i),\,(v,j)\}}=\{-(u,i,v),\,-(v,j,u)\},\,\,\text{for all}\,\,u,v\in V,\,(u,v)\in E,\,i,j\in\{1,2,3\},\,i\neq j$

It remains to prove that the signature S_G is as desired, that is, there is a graph over S_G if and only if there is a correct 3-colouring of G.

First of all, if a coloring $c: V \to \{1, 2, 3\}$ exists, then a graph G_c over S_G representing this colouring is constructed with the set of nodes $V \cup E$, where each node $v \in V$ has label (v, c(v)), each node $(u, v) \in E$ has label $\{(u, c(u)), (v, c(v))\}$. For every edge $(u, v) \in E$ in the graph G, the graph G_c has edges from u to (u, v) and from (u, v) to v, with the appropriate directions, as illustrated in Figure 3.

Conversely, let \widehat{G} be any graph over the signature S_G . It is claimed that in this case there exists a correct 3-colouring of G, and moreover, $\widehat{G} = G_c$ for some correct 3-colouring c of G.

First, it is proved that for each node $v \in V$ of the graph G, there is exactly one node in \widehat{G} with a label of the form (v, i), for some i. Consider the shortest simple path from v_0 to v in G (it exists because G is connected); the proof is by induction on the length of this path. The base case is a path of length 0: here the node corresponding to v_0 exists because \widehat{G} must have an initial node, and it is unique because the initial node is unique. For the induction step, let u be the next to the last node on the path, with $(u, v) \in E$. By the induction hypothesis, in \widehat{G} , there is a unique node of the form (u, i), for some i. This node emits a unique edge in the direction +(u, i, v), which must lead to a node labelled with $\{(u, i), (v, j)\}$, for some j, which in turn emits a unique edge in the direction -(v, j, u) that ends in a node labelled with (v, j)—so this node exists. If there were another node in \widehat{G} labelled with (v, k), for any k, then, by the same reasoning, it would be connected to some node labelled with (u, ℓ) through some intermediate node; this node must be the same as the above node labelled with (u, i), because such a node is unique. However, there is a unique path simulating the edge (u, v), hence this node labelled with (v, k) must coincide with the above node labelled with (v, j).

Therefore, \widehat{G} has the set of nodes $V \cup E$, which replicates the structure of G, with every edge split by an intermediate node. Then, it must be G_c for some colouring c. This colouring is correct, because each intermediate node checks that the colours at both ends of the corresponding edge are distinct. Then, correct colourings of the graph G correspond to graphs over S_G .

Note that the intermediate nodes that split the edges of G are necessary, because node labels cannot accumulate information on the colours of all the neighbours of a node, as this would require an exponential number of node labels.

The non-emptiness problem for star automata is NP-complete as well. Its membership in NP was established above, and its NP-hardness follows from the NP-hardness of nonemptiness of signatures.

Theorem 7. The problem of checking whether a given star automaton accepts at least one graph is NP-hard.

Proof. Non-emptiness for signatures was proved in Theorem 6 to be NP-hard. Now the NP-hardness of the non-emptiness problem for star automata is proved by reducing the non-emptiness problem for signatures to it, as follows.

Let S be a given signature. Consider the automaton A_* over it, that has one state, and, for each node label, has a star with this state at the centre and with this state at all rays. This star automaton accepts all graphs, so its non-emptiness is equivalent to non-emptiness of the signature S. And this automaton A_* has size polynomial in the size of S. Now it is time to prove the NEXP-completeness of the non-emptiness problem for graph-walking automata. It was proved in Corollary 4, that this problem is in NEXP. For NEXP-hardness it will be proved that a signature and a graph-walking automaton can define a set of graphs containing a square grid of size exponential in the number of states of the automaton and in the size of the signature. And then a nondeterministic Turing machine working in exponential time will be simulated on such grids.

Theorem 8. The problem of whether there is at least one graph accepted by a given graph-walking automaton is NEXP-hard.

Proof. Fix some NEXP-complete problem and some nondeterministic Turing machine M that solves this problem in exponential time. It can be assumed that the Turing machine is one-tape with the tape infinite to the right, and that the machine never moves to the left from the first position of the tape, in which an input string begins. The number of states, the number of transitions in the transition function, the sizes of the input aphabet and of the work alphabet are constant, as the Turing machine M is fixed.

The problem whether a given string w over the input alphabet is accepted by the Turing machine M is NEXP-complete. So to prove the theorem it is enough to reduce in polynomial time this problem about M to the non-emptiness problem for a graph-walking automaton. That is, such a deterministic polynomial-time algorithm is needed, that for a given string w it constructs a signature S_w and a graph-walking automaton A_w so that a graph accepted by A_w will exist if and only if there exists an accepting computation of the machine M on the string w.

Let $f: \mathbb{N} \to \mathbb{N}$ be a polynomial-time computable function that, for each length ℓ of an input string, gives a number $f(\ell)$, bounded by a polynomial in ℓ , such that $f(\ell) \ge \max\{\ell, 2\}$, and that the Turing machine M halts on every string of length at most ℓ in not more than $2^{f(\ell)} - 1$ steps. Then, each computation of M on each string of length at most ℓ can be written on a grid of length $2^{f(\ell)} \times 2^{f(\ell)}$.

The signature $S_w = (D, -, \Sigma, \Sigma_0, (D_a)_{a \in \Sigma})$ depends only on the length of w and is constructed as follows.

Let n = f(|w|), so that each computation of M on a string w can be written on a grid of size $2^n \times 2^n$; the number n = f(|w|) can be computed in polynomial time and is polynomial in the length of w.

The signature S_w is composed of three parts: $D = D_1 \cup D_2 \cup D_3$, $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$, all sets here are disjoint. And for each node label $a \in \Sigma_i$, it should hold that $D_a \subseteq D_i$, for $i \in \{1, 2, 3\}$. In every graph over S_w all nodes are divided into three sets: $V = V_1 \cup V_2 \cup V_3$, where V_i consists of the nodes with labels in Σ_i , for i = 1, 2, 3. There are two special pairs of opposite directions: $+d \in D_1$ and $-d \in D_2$, and $+d' \in D_2$ and $-d' \in D_3$. For every other direction, the opposite direction lies in the same set. Thus, nodes in V_1 and nodes in V_2 can be connected only by (+d, -d)-edges; similarly, nodes in V_2 and nodes in V_3 cannot be connected with an edge.

The idea is that nodes with labels in Σ_2 form a grid on which the Turing machine working on w will be simulated. Each node label from Σ_2 will have both directions -dand +d'. Labels from Σ_1 will allow the nodes in V_1 to form only a full binary tree of height 2n that emits exactly 2^{2n} edges in the direction +d from its leaves, thus ensuring that in every graph the number of nodes in V_2 is exactly 2^{2n} . Labels from Σ_3 will be used to attach a chain of length 2n to every node with label in Σ_2 , with the chain consisting of zeros and ones. The automaton A_w will check that nodes in V_2 form a $2^n \times 2^n$ grid, and that chains attached to these nodes correctly encode the row number and the column number in the grid for each node. Next, the automaton A_w will check that some accepting computation of the Turing machine M on the string w is encoded on the grid. Figure 4 shows a graph over some signature S_w with n = 2, that defines a correct grid on nodes with labels in Σ_2 .

The only initial node label in the signature S_w is $a_0 \in \Sigma_1$. The first part Σ_1 and D_1 should be defined so that the nodes with labels in Σ_1 can form only one graph: a full binary tree of height 2n with 2^{2n} leaves. The set of node labels is $\Sigma_1 = \{a_0, a_1, b_1, a_2, b_2, \ldots, a_{2n}, b_{2n}\}$, and the set of directions is $D_1 = \{\pm \ell_1, \pm r_1, \pm \ell_2, \pm r_2, \ldots, \pm \ell_{2n}, \pm r_{2n}\} \cup \{+d\}$. Here the label a_0 is initial, it is used for the root of a tree (level 0), the labels a_i and b_i are used for left and right children of the *i*-th level. The node label a_0 has the set of directions $D_{a_0} = \{+\ell_1, +r_1\}$, that is, the root has two edges to the two nodes of level 1. Labels a_i and b_i , for $i \in \{1, \ldots, 2n - 1\}$, have the sets of directions $D_{a_i} = \{-\ell_i, +\ell_{i+1}, +r_{i+1}\}$ and $D_{b_i} = \{-r_i, +\ell_{i+1}, +r_{i+1}\}$. So the *i*-th level generates twice as many nodes on level i+1. The node labels of the last level 2n (for the leaves of the tree) have sets of directions $D_{a_{2n}} = \{-\ell_{2n}, +d\}$ and $D_{b_{2n}} = \{-r_{2n}, +d\}$, that is, each leaf emits one edge in the direction +d, which is used for connection with nodes in V_2 .

Thus, in every graph over the signature S the initial node is labelled with $a_0 \in \Sigma_1$ and all nodes in V_1 form a full binary tree with 2^{2n} leaves and each leaf emits an edge in the direction +d.

The part Σ_3 , D_3 is constructed to allow only chains of nodes of length 2n with one direction -d' in each chain, with zeros and ones in nodes. This part of the signature is defined by $\Sigma_3 = \{0_1, \ldots, 0_{2n}\} \cup \{1_1, \ldots, 1_{2n}\}, D_3 = \{\pm d_1, \ldots, \pm d_{2n-1}\} \cup \{-d'\}$. And $D_{0_1} = D_{1_1} = \{-d', +d_1\}; D_{0_i} = D_{1_i} = \{-d_{i-1}, +d_i\}$, for $i \in \{1, \ldots, 2n-1\}$; and $D_{0_{2n}} = D_{1_{2n}} = \{-d_{2n-1}\}$.

Then, each node in V_2 has a chain attached to it in the direction +d'. Every such chain consists of nodes with labels in Σ_3 , has length 2n and encodes a number from 0 to $2^{2n} - 1$ in a sequence of zeros and ones in nodes. Let some node v in a graph have a label in Σ_2 . Then, the *coordinates* of v are the pair of numbers (i_v, j_v) , for $i_v, j_v \in \{0, \ldots, 2^n - 1\}$, where the number i_v is defined by the first n bits in the chain of nodes in V_3 attached to v, and the number j_v is defined by the second n bits. The number i_v is meant to be the number of the row in the grid where v is located, and j_v is meant to be the number of the column. Note that the coordinates of the node v are by definition just a pair of numbers, encoded in a chain, even if these numbers do not correspond to the actual position of the node v in a grid.

Now to the main part of the signature: Σ_2 and D_2 . There are 6 directions in D_2 : two of them, -d and +d', are used for connection with V_1 and V_3 , and 4 directions are used for a grid: ± 1 are horizontal (+1 is right, -1 is left), and ± 2 are vertical (+2 is up, -2 is down), so $D_2 = \{\pm 1, \pm 2\} \cup \{-d, +d'\}$. The set of node labels is $\Sigma_2 = \text{Pos} \times \text{Alph} \times \text{Head}$, that is, each node label in Σ_2 is of the form (*pos*, *alph*, *head*). The first component *pos* gives the type of a position of a node in a grid: in one of 4 corners, on the side or in the



Figure 4: A graph that defines a correct $2^n \times 2^n$ grid, for n = 2. The graph has three levels: the tree on the nodes in V_1 at the top, the grid on the nodes in V_2 in the middle, and chains on nodes in V_3 at the bottom, which encode the coordinates of nodes in the grid (the upper two bits encode the row number, and the lower two bits encode the column number).

centre. So there are 9 variants of the first component of a node label:

$$Pos = \{LU, CU, RU, \\ LC, CC, RC, \\ LD, CD, RD\}$$

where the first letter of pos gives the type of horizontal position (L, C or R), and the second letter gives the type of vertical position (D, C or U). The set of directions D_a for each node label $a \in \Sigma_2$ depends only on the component pos of the label a: the directions -d and +d' are always in D_a ; the direction +1 is in D_a if the node is not at the right border of a grid, that is, if $pos \notin \{RU, RC, RD\}$; the direction -1 is in D_a if and only if $pos \notin \{LU, LC, LD\}$; similarly $+2 \in D_a$ if and only if $pos \notin \{LU, CU, RU\}$; and $-2 \in D_a$ if and only if $pos \notin \{LD, CD, RD\}$.

The components *alph* and *head* of node labels in Σ_2 will be used for simulating configurations of the Turing machine M on rows of a grid. Let Γ be the work alphabet of M, it contains the input alphabet, the new blank symbol and maybe some other symbols; let Q be a finite set of states of the Turing machine. Then, Alph = Γ , that is, the component *alph* gives one of the symbols in the work alphabet of M, and Head = $Q \cup \{0\}$, where $0 \notin Q$, that is, the component *head* gives either a state of the Turing machine M if the head is simulated at the current position, or *head* = 0 if there is no head in this position.

This signature S_w is constructed in time linear in n.

Now a graph-walking automaton A_w over the signature S_w should be constructed, so that it accepts only graphs, in which nodes in V_2 form a correct grid, and the components *alph* and *head* of the labels in these nodes encode a correct accepting computation of the Turing machine M on the string w. The work of the automaton A_w on a graph is divided into two phases: checking the grid and checking the encoding of the Turing machine's computation on that grid.

In the first phase the automaton does not distinguish the components *alph* and *head* in labels in Σ_2 , its actions on a node labelled with $(pos, alph, head) \in \Sigma_2$ depend only on the component *pos*.

The goal of the first phase is to check that nodes with labels in Σ_2 form a $2^n \times 2^n$ grid on directions ± 1 and ± 2 , and that the coordinates (i_v, j_v) of each node v in V_2 are numbers of its row and its column in a grid. For convenience, the automaton also checks that the leftmost path in the tree on nodes in V_1 leads to a node in V_2 with coordinates (0, 0). If all these conditions hold for a graph, then this graph is said to define a correct grid.

The automaton checks whether a graph defines a correct grid as follows.

- 1. At the beginning, the automaton checks that the leftmost path in the tree on nodes with labels in Σ_1 leads to a node with a label in Σ_2 that has coordinates (0,0). The automaton starts at the initial node at the root of the tree, then it moves to the left child until it comes to a node with label in Σ_2 . Then it checks that all nodes in the attached chain contain zeros. This can be done with a constant number of states.
- 2. The automaton checks, for each node v with label in Σ_2 , that the component *pos* of the label agrees with coordinates (i_v, j_v) given in the chain of nodes from V_3 attached

to the node v. For that, it should be checked that pos = XY, where

$$X = \begin{cases} L & \text{if } j_v = 0\\ C & \text{if } 0 < j_v < 2^n - 1\\ R & \text{if } j_v = 2^n - 1 \end{cases} \qquad Y = \begin{cases} D & \text{if } i_v = 0\\ C & \text{if } 0 < i_v < 2^n - 1\\ U & \text{if } i_v = 2^n - 1 \end{cases}$$

When the automaton visits some node $v \in V_2$, it can check this condition for the node v using a constant number of states and return to the node. Indeed, it needs just to check several conditions of the form that all bits of the first or the second n bits of a chain are all zeros or are all ones.

To do such a check for each node in V_2 , the automaton needs to visit somehow all nodes in V_2 . This can be done by traversing the tree on nodes in V_1 . This tree can be traversed with a constant number of states. The leaves in this tree correspond to nodes in V_2 , each leaf is connected by a (+d, -d) edge to some node in V_2 , and each node in V_2 is connected to some leaf. Thus, the automaton checks for each leaf in a tree that its neighbour in V_2 has the component *pos* agree with the coordinates. This can be done using a constant number of states.

3. Then the automaton checks that directions $\pm 1, \pm 2$ in the grid lead to correct nodes. That is, for each node v in V_2 with coordinates (i_v, j_v) , the following conditions must hold. If an edge in the direction -1 exists $(j_v > 0)$, then it should lead to a node with coordinates $(i_v, j_v - 1)$. If an edge by +1 exists $(j_v < 2^n - 1)$, then it should lead to a node with coordinates $(i_v, j_v + 1)$. Similarly, the direction +2 must increase the coordinate i_v , and the direction -2 must decrease it.

When the automaton visits some node v in V_2 , it can check these conditions using O(n) states and return to the node v. Indeed, to check the equality of two vectors of length n containing 0s and 1s, the automaton can compare them bit by bit remembering only the position of the current bit in a vector and the value of this bit. To check that the number encoded in the first vector is greater by 1 than the number encoded in the second vector, the automaton can check that the binary representations of the vectors are of the form $x10^i$ and $x01^i$, with $x \in \{0,1\}^*$ and $i \ge 0$, and this can be checked bit by bit.

To make these checks for all nodes in V_2 the automaton traverses the tree on the nodes in V_1 as at the previous step.

4. If the automaton did not reject at the previous steps, then it returns to the node with coordinates (0,0) and starts the second phase.

If the automaton rejects at the first phase, then the graph does not define a correct grid. It is claimed that the checks the automaton makes are sufficient, that is, that if the automaton starts the second phase, then the graph defines a correct grid. Let the automaton start the second phase on some graph G.

First, it is shown that all nodes with labels in Σ_2 have distinct coordinates and that every pair of coordinates (i, j), for $i, j \in \{0, \ldots, 2^n - 1\}$, occurs somewhere.

The node with coordinates (0,0) exists because such a node is on the leftmost path. For each node with some coordinates (i, j), the automaton has checked that its neighbours in directions $\pm 1, \pm 2$ exist and have coordinates (i, j + 1), (i, j - 1), (i + 1, j), (i - 1, j), as long as these coordinates are between 0 and $2^n - 1$. Then, for all $i, j \in \{0, \ldots, 2^n - 1\}$, there is a node in V_2 with coordinates (i, j). As the tree on nodes in V_1 is defined uniquely, $|V_2| = 2^{2n}$ in every graph. So the node with each pair of coordinates is unique.

Note that the automaton has no way to distinguish a node from its copy locally, so it is important that counting arguments give uniqueness to each pair of coordinates.

Then, as a node with each pair of coordinates exists and is unique, and coordinates increase or decrease along the directions in the grid, the graph defines a correct grid.

The states and transitions used by the automaton in the first phase can be written down in time quadratic in n, as both the number of states and the number of node labels in the signature are linear in n.

In the second phase, the automaton checks that some accepting computation of the Turing machine M on the string w is encoded in the grid.

The automaton should check that the initial row encodes the initial configuration of the Turing machine M on the string w, that the next row encodes one of possible next configurations, and so on, up to an accepting configuration. Rows after the accepting configuration are allowed to contain anything.

How are configurations encoded in rows? The Turing machine M works in exponential time, and the number n was chosen so that every computation on w contains at most $2^n - 1$ steps, and that $|w| \leq n$. Thus, the head of the Turing machine never visits positions beyond $2^n - 1$ on the tape, and during the computation the symbols at these positions are blank symbols. So the tape contents in a configuration can be thought of as a string of length 2^n . This string is encoded in the nodes of a row in the components *alph* of node labels, one symbol of the string per node. The position of the head is encoded by having the component *head* non-zero only in one node; in this node, the component *head* encodes a state of the Turing machine.

The automaton works in the second phase as follows.

- 1. The automaton A_w starts the second phase on a graph at the node with coordinates (0,0), and the graph is known to define a correct grid of size $2^n \times 2^n$.
- 2. First, the automaton checks the encoding of the initial configuration. It goes through the first |w| nodes in the first row remembering in a state the number of moves j it made, and for each node it checks that the component *alph* of the node label is the j-th symbol of w. Then it continues moving to the right using one state for that, and checking that the components *alph* in all other nodes in the first row contain blank symbols. While moving from (0,0) to $(0,2^n-1)$ the automaton additionally checks that in the node (0,0) the component *head* contains one of the initial states of the Turing machine, and that in all other nodes of the first row the component *head* of the label is 0.
- 3. For each row $i \in \{0, \ldots, 2^n 1\}$, starting from the row i = 0, the automaton makes the following two actions.

First, the automaton checks whether the current configuration is accepting. It finds the node in which the head is encoded, and if (head, alph) is an accepting pair of M, then the automaton immediately accepts.

If the configuration encoded in the i-th row is not accepting, then the automaton checks that the next row encodes one of the possible next configurations. This check

can be done using a constant number of states as follows. In the neighbourhood of the head in the *i*-th row, the automaton checks that a transition is correctly made; elsewhere, the automaton checks that the tape symbols are unchanged, and no extra heads appear. Once the check is complete, the automaton moves to the next row.

Working as described above, the automaton accepts a graph in the second phase if and only if one of the accepting computations of M on w is encoded on the grid, and otherwise it rejects. The automaton A_w can be constructed in time polynomial in n, and the NEXPcomplete problem of whether the Turing machine M accepts a given string w or not is reduced to the problem of whether the graph-walking automaton A_w over S_w accepts at least one graph. Thus, non-emptiness for graph-walking automata is NEXP-hard. \Box

7 Conclusion

In this paper it has been shown that the emptiness problems for signatures, for star automata and for graph-walking automata are decidable. And the computational complexity classes for these problems were determined: the non-emptiness problems for signatures and for star automata are NP-complete, whereas non-emptiness for graph-walking automata is NEXP-complete. Table 1 compares these new results about automata on graphs with the previous results for similar automata on strings and on trees.

Note that the reduction of graph-walking automata to signatures works even in the case of nondeterministic graph-walking automata. In this case, the conditions on incoming and outgoing states encoded in a label should be replaced with the conditions that the incoming states are all different, and that there is a bijection between the incoming and the outgoing states, with a transition possible for each pair. So the non-emptiness for nondeterministic graph-walking automata is NEXP-complete as well.

| | strings | trees | graphs |
|-------------|---------------------|------------------|------------------------------|
| walking | (2DFA) | (DTWA) | (DGWA) |
| | PSPACE-complete [7] | EXP-complete [1] | NEXP-complete (Cor 4, Thm 8) |
| tilings by | (NFA) | (tree automata) | (star automata) |
| edges/stars | NL-complete [6] | P-complete [13] | NP-complete (Cor 2, Thm 7) |

Table 1: Complexity of the non-emptiness problem for different families of automata.

In this paper, several upper bounds on the number of nodes in minimal accepted graphs have been obtained. Bounds have been proved for graph-walking automata (Corollary 5), for star automata (Corollary 3), and simply for graphs over a signature (Theorem 1). It will be good to prove some lower bounds, and maybe to improve the upper bounds given in this paper.

Star automata in this paper are a special case of elementary acceptors of Thomas [12], they are elementary acceptors without conditions on the number of occurrences of each star. Is the emptiness problem for elementary acceptors of Thomas also decidable? This remains an open question.

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