# Complexity of Manipulation and Bribery in Premise-Based Judgment Aggregation with Simple Formulas 

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#### Abstract

Judgment aggregation is a framework to aggregate individual opinions on multiple, logically connected issues into a collective outcome. These opinions are cast by judges, which can be for example referees, experts, advisors or jurors, depending on the application and context. It is open to manipulative attacks such as Manipulation where judges cast their judgments strategically. Previous works have shown that most computational problems corresponding to these manipulative attacks are NP-hard. This desired computational barrier, however, often relies on formulas that are either of unbounded size or of complex structure.

We revisit the computational complexity for various Manipulation and Bribery problems in premise-based judgment aggregation, now focusing on simple and realistic formulas. We restrict all formulas to be clauses that are (positive) monotone, Horn-clauses, or have bounded length. For basic variants of Manipulation, we show that these restrictions make several variants, which were in general known to be NP-hard, polynomial-time solvable. Moreover, we provide a P vs. NP dichotomy for a large class of clause restrictions (generalizing monotone and Horn clauses) by showing a close relationship between variants of Manipulation and variants of Satisfiability. For Hamming distance based Manipulation, we show that NP-hardness even holds for positive monotone clauses of length three, but the problem becomes polynomial-time solvable for positive monotone clauses of length two. For Bribery, we show that NP-hardness even holds for positive monotone clauses of length two, but it becomes polynomial-time solvable for the same clause set if there is a constant budget.


Key words - Judgment Aggregation; Social Choice Theory; Computational Complexity

## 1 Introduction

Justine is the head of a committee deciding on financial support for new startup companies. For her decisions, she uses publicly available evaluations of experts (judges) with respect to a set of basic features such as cult potential $(c)$, marketability ( $m$ ), high profitability ( $h$ ), and strong competitors' existence ( $s$ ). As a brilliant mathematician and economist, Justine developed a model that can reliably predict the success of the startup by putting the features into logical relation. For example, she defined two further composed features "market entering potential" as $e:=\neg s \vee c$ and "short-term risk" as $r:=\neg m \vee \neg h$. Since the expert's evaluations are different, she needs to aggregate them to feed her model. Her first idea was to take the majority on each feature, but she recognizes that she may obtain the following evaluations from three experts:

| expert 1: | $s$ | $\wedge$ | $\neg C$ | $\wedge$ | $m$ | $\wedge$ | $h$ | $\wedge$ | $\neg e$ | $\wedge$ | $\neg r$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| expert 2: | $S$ | $\wedge$ | c | $\wedge$ | $\neg m$ | $\wedge$ | $h$ | $\wedge$ | $e$ | $\wedge$ | $r$ |
| expert 3: | $\neg S$ | $\wedge$ | $\neg C$ | $\wedge$ | $m$ | $\wedge$ | $\neg h$ | $\wedge$ | $e$ | $\wedge$ | $r$ |

where the majority opinions claim strong competitors' existence ( $s$ ), marketability ( $m$ ), and high profitability $(h)$, and disclaim cult potential $(c)$. However, the majority opinions also claim market entering potential ( $e:=\neg s \vee c$ ) and short-term risk ( $r:=\neg m \vee \neg h$ ); an obviously paradoxical situation. Justine does a quick literature review and identifies her aggregation problem as "judgment aggregation" and the observed paradox as a variant of the well-known doctrinal paradox [26]. To avoid this paradox and since the experts are anyway better in evaluating basic features than evaluating composed features, she decides to adapt the concept of premise-based judgment aggregation rules [17] for her decision process: Basic features form the premises, and composed features are conclusions (which logically connect premises). The aggregation process is performed only on the premises and conclusions are deduced from them. That is, the outcome in the above example is $s, \neg c, m, h$, and, hence, also $\neg e$ and $\neg r$.

Justine is happy with the aggregation process, but she is worried about the reliability of the results. (1) For example, what if an expert made a mistake? Can she compute efficiently whether a set of important features remains stable even if some expert provided a wrong evaluation? (2) What if an expert evaluated strategically or untruthfully due to bribery or lobbyism? Is it difficult for an expert to compute a successful strategy? A quick literature review identifies all questions posed by Justine as variants of Manipulation and Bribery, which are computationally intractable.

Although the intractability of strategic evaluation and bribery seems to be good news, Justine is skeptical about the relevance of these results for her application. In her model, all formulas are length-two Horn clauses. All intractability results found, however, use rather complex or long formulas as conclusions, and, hence, do not apply in her situation. So Justine is stuck with the literature's state of the art and cannot decide whether her model (with simple formulas) is vulnerable towards manipulation or bribery.

In this paper, we help Justine (and all others in similar situations) and provide a fine-grained computational complexity analysis of Manipulation and Bribery for judgment aggregation with simple formulas. Our results are essentially good news for Justine: Concerning question (1), we show that checking the robustness of an outcome (i.e. whether a single judge/mistake can change it) turns out to become polynomial-time solvable for simple formulas. Concerning question (2), we show that most manipulative actions (such as strategic evaluation or bribery) remain computationally hard even for structurally simple formulas.

### 1.1 Related Work

For a detailed introduction to the general topic of judgment aggregation we refer to excellent surveys in the field [5, 19, 25, 27, 28]. We focus here on a description of work related to computational complexity of strategic behavior for premise-based judgment aggregation rules, which is most relevant to our work. We will point the interested reader to specific surveys for broader information.

Dietrich and List [16] introduced strategic behavior to judgment aggregation. They introduced the concept of strategy-proofness to analyze whether certain strategic behavior can influence the outcome (ignoring the computational complexity of manipulative attacks). Following Bartholdi III et al. [1, 2], intractability of manipulative attacks is usually seen as "barrier against manipulation" and, hence, a desired property. Endriss et al. [20] were the first who analyzed the computational complexity of strategic behavior in judgment aggregation and showed that it is NP-hard for a judge to decide whether she can cast a judgment set that influences the collective outcome in a beneficial way (assuming Hamming distance based preferences over judgment sets), even for the simple premise-based majority rule, where a premise is accepted if it is accepted by more than half of judges. We also consider, among others, Hamming distance based preferences, but for uniform premise-base quota rules, which is a family of rules that contains their premise-based majority rule as prominent special case. Moreover, Endriss et al. [20] allow arbitrarily complex formulas to obtain their results. Baumeister et al. [3, 4, 7]
continued this line of research and where the first who extended the results to the more general uniform premise-base quota rules. They also initiated the analysis of further variants of strategic behavior for judgment aggregation, including further variants of Manipulation or cases where an external agent influences the structure (Control) or the opinions of the judges (Bribery), showing NP-hardness for most considered problems. However, these NP-hardness results usually rely on complex or size-unbounded formulas, leaving open the complexity of cases with simple and realistic formulas, which is the focus of our paper. Similar to Endriss et al. [20], they also did not impose any systematic restriction on the complexity of the formula. We continue this line of research but we focus on practically meaningful simple formula and analyze to which extent desired computational complexity transfers. Besides premise-based rules, de Haan [14] studied the complexity of Manipulation, Bribery and Control for the Kemeny judgment aggregation rule, and Baumeister et al. [8] investigated the complexity of manipulation by changing the processing order in sequential rules. For a broader overview on strategic behavior in judgment aggregation, we refer to a recent survey [6].

Our work also fits well into the line of research initiated by the seminal paper of Faliszewski et al. [23] showing that the barrier against manipulative attacks sometimes disappears in context of restricted domains. In context of voting one usually considers restricted preference domains (see Elkind et al. [18] for a recent survey) whereas we focus on restricted formulas.

More generally, fine-grained analysis has also been considered for outcome determination in judgment aggregation, which is usually computationally intractable for many judgment aggregation rules (e.g., Kemeny rule) [22]. de Haan [13] investigated the parameterized complexity of outcome determination of the Kemeny rule by considering structural parameters, including the maximum size of formulas, which is also one restriction considered in this paper. de Haan 15] studied the influence of different formula restrictions (e.g., Horn and Krom formulas) on the complexity of outcome determination in judgment aggregation. Finally, the problem of agenda safety in judgment aggregation, which determines for a given set of formulas and a given aggregation procedure whether the procedure's outcome is always consistent for any combination of judgment sets, has also been considered from the perspective of parameterized complexity [21].

### 1.2 Contributions and Organization

We analyze the computational complexity of variants of Manipulation and Bribery in premisebased judgment aggregation with simple formulas. We restrict all formulas to clauses and systematically study the effect of the number of positive (negative) literals in a clause and clause length on the computational complexity. In particular, we consider Horn clauses (implicationlike conclusions which for instance are fundamental in logic programming [11, 29]), (positive) monotone clauses, and clauses of bounded length. In Section 2 we describe the formal model and introduce our notation.

In Section 3 we revisit the computational complexity for basic variants of Manipulation, showing that the restriction to clauses makes several variants, which were in general known to be NP-hard [4], polynomial-time solvable. Our main result in this section is a P vs. NP dichotomy for a large class of clause restrictions (generalizing monotone and Horn clauses) by showing a close relationship between variants of Manipulation and variants of Satisfiability. For details, we refer to Table 12 in our conclusion (Section (6).

In Section 4 we revisit Hamming distance based Manipulation. Our main result in this section is that for positive monotone clauses the problem becomes polynomial-time solvable for clauses of length $\ell=2$ but remains NP-hard when $\ell=3$. This is particularly surprising since Satisfiability is trivial for positive monotone clauses even for unbounded length. The latter result is reached by showing NP-hardness of a natural variant of Vertex Cover which we believe to be interesting on its own. The NP-hardness also holds for monotone or Horn clauses of length $\ell=2$.

In Section 5 we revisit the computational complexity of two variants of Bribery, where we
show that the NP-hardness even holds for positive monotone clauses of length $\ell=2$, probably the most basic case. We then consider the restricted case with a fixed budget, and show that in this case the problem becomes polynomial-time solvable for positive monotone clauses of length $\ell=2$. For general clause sets, we show an interesting relation between Hamming distance based Manipulation and Bribery with the same clause set: the NP-hardness of Hamming distance based Manipulation implies the NP-hardness of the corresponding Bribery.

## 2 Model and Preliminaries

### 2.1 Premise-Based Judgment Aggregation

We adopt the judgment aggregation framework described by Baumeister et al. [4] and Endriss et al. [20] and slightly simplify the notation for premise-based rules.

The topics to be evaluated are collected in the agenda $\Phi=\Phi_{p} \uplus \Phi_{c}$ that consists of a finite set of premises $\Phi_{p}$ (propositional variables) as well as a finite set of conclusions $\Phi_{C}$ (propositional formulas built from the premises using standard logical connectives $\neg, \vee$, and $\wedge$ ) ${ }^{1}$ In this paper we only study disjunctive agendas, i.e., there is no formulas containing $\wedge$ (see more details in Section (2.3). The agenda does not contain any doubly negated formulas and is closed under complementation, that is, $\neg \alpha \in \Phi$ if and only if $\alpha \in \Phi$. An evaluation on the agenda is expressed as a judgment set $J \subseteq \Phi$. A judgment set is complete if each premise and conclusion is contained either in the negated or non-negated form and consistent if there is an assignment that satisfies all premises and conclusions in the judgment set simultaneously. The set of all complete and consistent subsets of $\Phi$ is denoted by $\mathcal{J}(\Phi)$.

Let $N=\{1, \ldots, n\}$ be a set of $n>1$ judges. A profile is a vector of judgment sets $\boldsymbol{J}=$ $\left(J_{1}, \ldots, J_{n}\right) \in \mathcal{J}(\Phi)^{n}$. We denote by $\left(\boldsymbol{J}_{-i}, J_{i}^{\prime}\right)$ the profile that is like $\boldsymbol{J}$, except that $J_{i}$ has been replaced by $J_{i}{ }^{\prime}$. A judgment aggregation procedure for agenda $\Phi$ and judges $N=\{1, \ldots, n\}$ is a function $F: \mathcal{J}(\Phi)^{n} \rightarrow 2^{\Phi}$ that maps a profile $\boldsymbol{J}$ to a single judgment set, which is called a collective judgment set. A procedure is called complete (consistent) if the collective judgment set under the procedure is always complete (consistent).

The most natural procedure is probably the majority rule, which accepts a formula if and only if it is accepted by more than half of the individual judges. Dietrich and List [17] introduced the quota rule as a generalization of the majority rule, where each formula has an given acceptance threshold. As shown in the introductory example, the majority rule (and the quota rule) does not satisfy consistency. In this paper, we consider the uniform premise-based quota rule [4], which first applies the quota rule to the premises and then accepts all conclusions that are satisfied by these collectively accepted premises. The formal definition follows.

Definition 1 (Uniform Premise-based Quota Rule for $q \in[0,1)$ ). A uniform premise-based quota rule $U P Q R_{q}: \mathcal{J}(\Phi)^{n} \rightarrow 2^{\Phi}$ divides the premises $\Phi_{p}$ into two disjoint subsets $\Phi_{q}$ and $\Phi_{\bar{q}}$, where $\Phi_{q}$ consists of all premises in the non-negated form and $\Phi_{\bar{q}}$ consists of all premises in the negated form. For each $\boldsymbol{J} \in \mathcal{J}(\Phi)^{n}$ the outcome $\operatorname{UPQR}_{q}(\boldsymbol{J})$ is the collective judgment set that contains every premise from $\Phi_{q}$ that appears at least $\lfloor q n+1\rfloor$ times in the profile $\boldsymbol{J}$, every premise from $\Phi_{\bar{q}}$ that appears at least $\lceil n-q n\rceil$ times in the profile, as well as all conclusions that are satisfied by these premises.

Notice that since $\lfloor q n+1\rfloor+\lceil n-q n\rceil=n+1$, it is guaranteed that for each premise $\alpha \in \Phi_{p}$, we have either $\alpha \in \operatorname{UPQR}_{q}(\boldsymbol{J})$ or $\neg \alpha \in \operatorname{UPQR}_{q}(\boldsymbol{J})$. Then by the way conclusions are selected, the outcome $\mathrm{UPQR}_{q}(\boldsymbol{J})$ is always complete and consistent.

The following example formally restates the introductory example.

[^0]Example. The premise set $\Phi_{p}$ contains two parts $\Phi_{q}=\{s, c, m, h\}$ and $\Phi_{\bar{q}}=\{\neg s, \neg c, \neg m, \neg h\}$. The conclusion set $\Phi_{c}$ contains $\neg s \vee c, \neg m \vee \neg h$ and their negations. The profile is given as follows:

| Judgment Set | $s$ | $c$ | $m$ | $h$ |  | $\neg s \vee c$ | $\neg m \vee \neg h$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $J_{1}$ | 1 | 0 | 1 | 1 |  | 0 | 0 |
| $J_{2}$ | 1 | 1 | 0 | 1 |  | 1 | 1 |
| $J_{3}$ | 0 | 0 | 1 | 0 |  | 1 | 1 |
| $\mathrm{UPQR}_{1 / 2}$ | 1 | 0 | 1 | 1 | $\Rightarrow$ | 0 | 0 |

In the table we use 1 or 0 to represent whether the formula is contained in the judgment set or not. As an example, $J_{1}=\{s, \neg c, m, h, \neg(\neg s \vee c), \neg(\neg m \vee \neg h)\}$. The collective judgment set is obtained by applying $\mathrm{UPQR}_{q}$ with $q=1 / 2$. Thus, to be included in the outcome, every positive premise needs to be accepted by $\left\lfloor\frac{3}{2}+1\right\rfloor=2$ judges, and every negative premise needs to be accepted by $\left\lceil 3-\frac{3}{2}\right\rceil=2$ judges. We first have that $s, \neg c, m$ and $h$ are included in the outcome. Or we just say $s=1, c=0, m=1$ and $h=1$. Then we get that $\neg s \vee c=0$ and $\neg m \vee \neg h=0$, which means $\neg(\neg s \vee c)$ and $\neg(\neg m \vee \neg h)$ are included in the outcome.

### 2.2 Decision Variables

In order to analyze the influence of judges on the outcome $\mathrm{UPQR}_{q}(\boldsymbol{J})$, we call a variable $x$ decided by judge $i$ if the outcome with respect to $x$ is decided by the judgment set of judge $i$, i.e., for any judgment set $J^{*} \in \mathcal{J}(\Phi)$, it holds that $x \in \operatorname{UPQR}_{q}\left(J_{-i}, J^{*}\right)$ if and only if $x \in J^{*}$. In the above example, variables $c$ and $m$ are decided by the third judge while $s$ and $h$ are not.

According to the definition of $\mathrm{UPQR}_{q}$, we get the following characterization for variables decided by a judge.

Observation 1. A variable $x \in \Phi_{q}$ (resp. $x \in \Phi_{\bar{q}}$ ) is decided by judge $i$ if and only if except for $j u d g e i$ there are exactly $\lfloor q n\rfloor$ (resp. $\lceil n-q n-1\rceil$ ) judges that accept $x$.

It follows that if variable $x$ is not decided by judge $i$, then except for judge $i$ the number of judges that accept $x$ is either at most $\lfloor q n\rfloor-1$ or at least $\lfloor q n\rfloor+1$ (resp. at most $\lceil n-q n-2\rceil$ or at least $\lceil n-q n\rceil$ ). In both cases the outcome of $x$ is independent of the judgment set of judge $i$.

Observation 2. If variable $x$ is not decided by judge i, then judge $i$ cannot change the outcome of $x$, i.e., for any judgment set $J^{*} \in \mathcal{J}(\Phi)$, it holds that $x \in U P Q R_{q}\left(J_{-i}, J^{*}\right)$ if and only if $x \in U P Q R_{q}(J)$.

We will use these observations in the analysis of many variants of Manipulation. We call a variable a decision variable if the variable is decided by the manipulator. We do not consider the corresponding definition for conclusions since for our problems only those conclusions whose outcome can be changed by the judge need to be considered and the rest can simply be deleted from the problem input.

### 2.3 Clause Restrictions

We restrict the conclusions to be clauses and their negations, where a clause is defined as a disjunction of literals. In particular, we consider positive monotone clauses which are clauses with no negative literals, monotone clauses which are clauses with only positive literals or with only negative literals, Horn clauses which are clauses with at most one positive literal, and clauses of bounded length, where the length of a clause is the number of literals contained in it. Moreover, we generalize these restrictions and define classes of clause restrictions based on a classification with respect to the number of positive and negative literals in a clause.

Definition 2. A clause set $\mathcal{C}$ is called a standard-form clause set if $\mathcal{C}$ is a union of some $\mathcal{S}_{i}^{j}$, where $\mathcal{S}_{i}^{j}$ is the set of clauses which contain exactly i literals and exactly $j$ of them are negative. Denote $\mathcal{S}_{k}^{0}$ and $\mathcal{S}_{k}^{k}$ as $\mathcal{M}_{k}^{+}$and $\mathcal{M}_{k}^{-}$.

This classification is useful as most clause classes we care about can be defined as the union of some $\mathcal{S}_{i}^{j}$. For example, positive monotone clauses can be denoted by $\bigcup_{i=1}^{\infty} \mathcal{S}_{i}^{0}=\bigcup_{i=1}^{\infty} \mathcal{M}_{i}^{+}$, Horn clauses can be denoted by $\bigcup_{i=1}^{\infty}\left(\mathcal{M}_{i}^{-} \cup \mathcal{S}_{i}^{i-1}\right)$, and clauses of length 3 can be denoted by $\bigcup_{j=0}^{3} \mathcal{S}_{3}^{j}$.

We remark that our results in this paper can be directly translated to the case where we restrict the conclusions to be conjunctions of literals, and their negations, since the negation of a clause is a conjunction of literals and the agenda is closed under complementation. In other words, for each pair of clause $\alpha$ of its negation $\neg \alpha$ in the agenda, we can rewrite $\neg \alpha$ as a conjunction of literals and $\alpha$ as the negation of a conjunction of literals, such that all conclusions in the agenda are conjunctions of literals or their negations.

### 2.4 Satisfiability

We will build connections between variants of Manipulation and variants of Satisfiability. A formula is said to be satisfiable if there exists a value assignment that assigns every variable appearing in the formula 1 or 0 such that the formula is valued as 1 . Satisfiability is the problem of deciding whether a given formula is satisfiable. 3-SAT is a restricted variant of Satisfiability where each formula is a conjunction of clauses of length 3 . We define more general variants of Satisfiability based on the clause set.

Definition 3. Let $\mathcal{C}$ be a clause set. $\mathcal{C}$-SAT is the problem of deciding whether a given formula $C_{1} \wedge \cdots \wedge C_{m}$ with $C_{i} \in \mathcal{C}$ is satisfiable or not.

Now 3-SAT corresponds to $\bigcup_{j=0}^{3} \mathcal{S}_{3}^{j}$-Sat.

## 3 Basic Manipulation Problems

In this section, we analyze the computational complexity of problems modeling simple variants of strategic behavior of some judge. The core idea is that a judge might cast an untruthful judgment set in order to influence the collective judgment set towards some desired judgment set. Note that we provide alternative, simpler (yet equivalent) problem definitions compared to those known from the literature [4]. In contrast to Baumeister et al. [4] who focus on the assumption on the preferences of the manipulator over all possible outcomes, which requires rather technical concepts of preference relations between judgment sets, we take a different approach and directly model the requirements on the preferred outcome. For example, the simplest variant of manipulation from Baumeister et al. [4], UPQR-U-Possible-MAnipulation, actually models the question whether the collective outcome is "robust against one judge providing a faulty judgment set" as asked by Justine in the introduction. Formally, we consider the following problems.

UPQR Manipulation basic variants
(Problem names from [4] listed below.)
Input: An agenda $\Phi$, a profile $\boldsymbol{J}=\left(J_{1}, \ldots, J_{n}\right) \in \mathcal{J}(\Phi)^{n}$, the manipulator's desired consistent (possibly incomplete) set $J \subseteq J_{n}$, and a uniform rational threshold $q \in[0,1$ ).

UPQR-Robustness-Manipulation (=UPQR-U-Possible-Manipulation [4])
Question: $\exists J^{*} \in \mathcal{J}(\Phi): \operatorname{UPQR}_{q}(\boldsymbol{J}) \cap J \neq \operatorname{UPQR}_{q}\left(\boldsymbol{J}_{-n}, J^{*}\right) \cap J$ ?
UPQR-Possible-Manipulation (=UPQR-CR-Possible-Manipulation [4])
Question: $\exists J^{*} \in \mathcal{J}(\Phi):\left(\operatorname{UPQR}_{q}\left(\boldsymbol{J}_{-n}, J^{*}\right) \cap J\right) \backslash\left(\operatorname{UPQR}_{q}(\boldsymbol{J}) \cap J\right) \neq \emptyset$ ?
UPQR-Necessary-Manipulation (=UPQR-CR-Necessary-Manipulation [4])
Question: $\exists J^{*} \in \mathcal{J}(\Phi): \operatorname{UPQR}_{q}(\boldsymbol{J}) \cap J \subsetneq \operatorname{UPQR}_{q}\left(\boldsymbol{J}_{-n}, J^{*}\right) \cap J$ ?
UPQR-Exact-Manipulation (=UPQR-TR-Negessary-Manipulation [4])
Question: $\exists J^{*} \in \mathcal{J}(\Phi): J \subseteq \operatorname{UPQR}_{q}\left(J_{-n}, J^{*}\right)$ ?
Intuitively, the manipulator only cares about the formulas in the desired set $J$, which is a subset of the manipulator's judgment set. UPQR-Robustness-MAniPULATION asks whether the manipulator can achieve a different outcome with respect to $J$. UPQR-Possible-MAnipuLATION asks whether the manipulator can achieve an outcome that contains a formula from $J$ which is not contained in the truthful outcome. UPQR-NECESSARY-MANIPULATION asks whether the manipulator can achieve an outcome that contains a formula from $J$ which is not contained in the truthful outcome, and meanwhile contains all formulas that are in both $J$ and the truthful outcome. UPQR-Exact-Manipulation asks whether the manipulator can achieve an outcome that contains all formulas from $J$.

Baumeister et al. [4] showed that all the four variants of UPQR Manipulation with the desired set being incomplete are NP-complete. However, a complex formula in conjunctive normal form is needed in the conclusion set in these reductions. In this section, we give a more refined analysis by considering how restricting the conclusions to different standard-form clause sets influences the computational complexity of UPQR MANIPULATION for all four basic variants. For completeness, before presenting the tractable cases in Section 3.2 and intractable cases in Section 3.3, we first provide the original problem definitions in Baumeister et al. [4] and argue that our problem definitions are equivalent in Section 3.1.

### 3.1 Relation to Baumeister et al. [4]

Before we start the actual analysis our our problems, we show the equivalence between our problem definitions and those in Baumeister et al. [4].

### 3.1.1 Preferences over judgment sets

In order to define their manipulation problems, Baumeister et al. [4] used the following the notations for their concepts of preferences over judgment sets. Let U be the set of all weak orders over $\mathcal{J}(\Phi)$. For a weak order $\succeq$ over $\mathcal{J}(\Phi)$ and for all $X, Y \in \mathcal{J}(\Phi)$, define $X \succ Y$ by $X \succeq Y$ and $Y \nsucceq X$, and define $X \sim Y$ by $X \succeq Y$ and $Y \succeq X$. We say $X$ is weakly preferred to $Y$ whenever $X \succeq Y$, and we say $X$ is preferred to $Y$ whenever $X \succ Y$. Given some (possibly incomplete) judgment set $J$, define

1. the set of unrestricted $J$-induced (weak) preferences as the set $\mathrm{U}_{J}$ of weak orders $\succeq$ in U such that for all $X, Y \in \mathcal{J}(\Phi), X \sim Y$ whenever $X \cap J=Y \cap J$;
2. the set of top-respecting J-induced (weak) preferences as $\mathrm{TR}_{J} \subseteq \mathrm{U}_{J}$ such that $\succeq \in \mathrm{TR}_{J}$ if and only if for all $X \in \mathcal{J}(\Phi)$ with $X \cap J \neq J$, it holds that $J \succ X$;
3. the set of closeness-respecting $J$-induced (weak) preferences as $\mathrm{CR}_{J} \subseteq \mathrm{U}_{J}$ such that $\succeq$ $\in \mathrm{CR}_{J}$ if and only if for all $X, Y \in \mathcal{J}(\Phi)$ with $Y \cap J \subseteq X \cap J$, we have $X \succeq Y$.

Table 1: Relation between original problem definitions in [4] and our definitions. We ignore $\mathrm{UPQR}_{q}$-U-Necessary-Manipulation as it is known to be possibly strategy proof [4].

| Definitions in [4] |  | Our definitions |
| :--- | :--- | :--- |
| UPQR $_{q}$-U-Possible-Manipulation | $\leftrightarrow$ | UPQR-Robustness-Manipulation |
| UPQR $_{q}$-TR-Necessary-Manipulation | $\leftrightarrow$ | UPQR-Exact-Manipulation |
| UPQR $_{q}$-TR-Possible-Manipulation | $\leftrightarrow$ | UPQR-Robustness-Manipulation |
| UPQR $_{q}$-CR-Necessary-Manipulation | $\leftrightarrow$ | UPQR-Necessary-Manipulation |
| UPQR $_{q}$-CR-Possible-Manipulation | $\leftrightarrow$ | UPQR-Possible-Manipulation |

Let $J, X$, and $Y$ be three judgment sets for the same agenda $\Phi$, where $J$ is possibly incomplete and let $T_{J} \in\left\{\mathrm{U}_{J}, \mathrm{TR}_{J}, \mathrm{CR}_{J}\right\}$ be a type of $J$-induced preferences. We say judgment set $X$ is necessarily preferred to judgment set $Y$ for type $T_{J}$ if $X \succ Y$ for all $\succeq \in T_{J}$ and judgment set $X$ is possibly preferred to judgment set $Y$ for type $T_{J}$ if there is some $\succeq \in T_{J}$ with $X \succ Y$.

### 3.1.2 Original Problem Definitions

For each $T \in\{\mathrm{U}, \mathrm{TR}, \mathrm{CR}\}$ and any rational quota $q \in[0,1)$, Baumeister et al. [4] defined their Manipulation problems as follows.
$\mathrm{UPQR}_{q}-T$-Necessary-Manipulation (resp. $\mathrm{UPQR}_{q}-T$-Possible-Manipulation)
Input: An agenda $\Phi$, a profile $\boldsymbol{J}=\left(J_{1}, \ldots, J_{n}\right) \in \mathcal{J}(\Phi)^{n}$, the manipulator's desired consistent (possibly incomplete) set $J \subseteq J_{n}$.
Question: Does there exist a judgment set $J^{*} \in \mathcal{J}(\Phi)$ such that $\operatorname{UPQR}_{q}\left(J_{-n}, J^{*}\right) \succ$ $\operatorname{UPQR}_{q}(\boldsymbol{J})$ for all $\succeq \in T_{J}$ (resp. for some $\succeq \in T_{J}$ )?

### 3.1.3 Alternative Characterizations

Since the problem definitions in its various variants together with the respective $J$-induces preference models are rather complicated to parse, we provided alternative, more direct questions that are helpful to analyze and understand the considered problems at the beginning of this section. In the following we show that our problem definitions are equivalent to the ones in Baumeister et al. [4] (see also Table (1). Note that some variants have trivial solutions and can thus be ignored in our analysis. Moreover, provide a "motivation"-paragraph for each relevant manipulation variant, where we discuss potential application settings.
$\mathbf{U P Q R}{ }_{q}$ - U-Necessary-Manipulation: This case is known to be possibly strategy proof, that is, no manipulated outcome would be necessarily preferred to truthful outcome [4]. Consequently, we ignore this problem variant.
$\mathbf{U P Q R}_{q}$ - U-Possible-Manipulation: Recall that unrestricted $J$-induced (weak) preferences only require that two outcomes $X$ and $Y$ must be equally good, when they are identical with respect to $J$. That is, when $X$ and $Y$ not identical with respect to $J$, then both $X \succ Y$ and $Y \succ X$ are possible. Therefore, a manipulator that wants to obtain a possibly better outcome only needs to archive an outcome that is different to the original outcome with respect to the desired set.
Thus, $\mathrm{UPQR}_{q}$-U-Possible-Manipulation asks whether the manipulating judge can achieve a different outcome (that contains at least one more or one less formula from the desired set compared to the truthful outcome) with respect to the desired set.

Formally, we have: $\exists J^{*} \in \mathcal{J}(\Phi): \mathrm{UPQR}_{q}\left(\boldsymbol{J}_{-n}, J^{*}\right) \succ \mathrm{UPQR}_{q}(\boldsymbol{J})$ for some $\succeq \in \mathrm{U}_{J} ? \Leftrightarrow$ $\exists J^{*} \in \mathcal{J}(\Phi): \mathrm{UPQR}_{q}\left(\boldsymbol{J}_{-n}, J^{*}\right) \cap J \neq \mathrm{UPQR}_{q}(\boldsymbol{J}) \cap J ?$
Thus, $\mathrm{UPQR}_{q}$-U-Possible-Manipulation is equivalent to UPQR-Robustness-Manipulation.
Motivation: This problem can be seen as a question for stability or robustness [10]. Interpreting the desired set as "set of relevant propositions", it asks whether a particular judge can change the collective outcome with respect to relevant propositions e.g. by changing her mind or making a mistake. Possible applications are when we know that some specific judge (formally the manipulator) is not certain or perfectly qualified for a reliable evaluation and we want to know whether we can trust the aggregated outcome with respect to specific important formulas (formally the desired set).

UPQR $_{q}$-TR-Necessary-Manipulation: Recall that top-respecting $J$-induced (weak) preferences only require that $J$ is preferred to every outcome that does not already contain $J$. That is, an outcome $X$ is necessarily better than an outcome $Y$ when $Y$ did not contain $J$ but $X$ does.
Thus, $\mathrm{UPQR}_{q}$-TR-Necessary-Manipulation asks whether the manipulator can achieve an outcome that contains the desired set, and in addition the truthful outcome does not contain the whole desired set.
Formally, we have: $\exists J^{*} \in \mathcal{J}(\Phi): \operatorname{UPQR}_{q}\left(\boldsymbol{J}_{-n}, J^{*}\right) \succ \operatorname{UPQR}_{q}(\boldsymbol{J})$ for all $\succeq \in \mathrm{TR}_{J} ? \Leftrightarrow$ $\exists J^{*} \in \mathcal{J}(\Phi): J \subseteq \operatorname{UPQR}_{q}\left(\boldsymbol{J}_{-n}, J^{*}\right)$ and $J \nsubseteq \mathrm{UPQR}_{q}(\boldsymbol{J})$ ?
Since it is trivial to check the second condition $J \nsubseteq \mathrm{UPQR}_{q}(\boldsymbol{J}), \mathrm{UPQR}_{q}$-TR-NecessaryManipulation is essentially equivalent to UPQR-Exact-Manipulation.
Motivation: The motivation for this manipulation variant is straightforward. Whenever one is aware of the precise goal of a potential manipulator, one can ask whether the manipulator can fully reach that goal.
$\mathbf{U P Q R}_{q}$-TR-Possible-Manipulation: Recall that top-respecting $J$-induced (weak) preferences only require that $J$ is preferred to every outcome that does not already contain $J$. That is, an outcome $X$ is only required to be better than an outcome $Y$ when $Y$ did not contain $J$ but $X$ does. Thus, when $Y$ does not contain $J$, then $X$ is possibly better than $Y$ whenever it is different from $Y$ with respect to $J$.
Thus, $\mathrm{UPQR}_{q}$-TR-Possible-Manipulation asks whether the manipulator can can achieve a different outcome with respect to the desired set, and in addition the truthful outcome does not contain the desired set.
Formally, we have: $\exists J^{*} \in \mathcal{J}(\Phi): \operatorname{UPQR}_{q}\left(\boldsymbol{J}_{-n}, J^{*}\right) \succ \operatorname{UPQR}_{q}(\boldsymbol{J})$ for some $\succeq \in \mathrm{TR}_{J} ? \Leftrightarrow$ $\exists J^{*} \in \mathcal{J}(\Phi): \mathrm{UPQR}_{q}\left(\boldsymbol{J}_{-n}, J^{*}\right) \cap J \neq \mathrm{UPQR}_{q}(\boldsymbol{J}) \cap J$ and $J \nsubseteq \mathrm{UPQR}_{q}(\boldsymbol{J}) ?$
Since it is trivial to check the second condition $J \nsubseteq \operatorname{UPQR}_{q}(\boldsymbol{J}), \mathrm{UPQR}_{q}$-TR-PossibleManipulation is essentially the same as UPQR-Robustness-Manipulation.

UPQR $_{q}$-CR-Necessary-Manipulation: Recall that closeness-respecting $J$-induced (weak) preferences require that an outcome $X$ must be better than an outcome $Y$ when the set of formulas from $J$ included in $X$ is a superset of the set of formulas from $J$ included in $Y$. That is, an outcome $X$ is necessarily better than an outcome $Y$ when $X$ contain all formulas from $J$ also included in $Y$ plus some more.
Thus, $\mathrm{UPQR}_{q}-\mathrm{CR}-$ Necessary-Manipulation asks whether the manipulator can achieve an outcome that contains at least one conclusion from the desired set that was not in the truthful outcome, and at the same time contains all formulas from the desired set that were part of the truthful outcome.

Formally, we have: $\exists J^{*} \in \mathcal{J}(\Phi): \mathrm{UPQR}_{q}\left(\boldsymbol{J}_{-n}, J^{*}\right) \succ \mathrm{UPQR}_{q}(\boldsymbol{J})$ for all $\succeq \in \mathrm{CR}_{J} ? \Leftrightarrow$ $\exists J^{*} \in \mathcal{J}(\Phi): \operatorname{UPQR}_{q}(\boldsymbol{J}) \cap J \subsetneq \operatorname{UPQR}_{q}\left(\boldsymbol{J}_{-n}, J^{*}\right) \cap J ?$
This is the same as UPQR-Necessary-Manipulation.
Motivation: This variant is relevant when the goal set (desired set) of the manipulator is known and the valuation of the manipulator is strictly monotone (that is, obtaining one more desired formula is an improvement). Strict monotonicity is useful to model a risk-averse or conservative manipulator who would only manipulate if no formula from the desired set has to be given up.
$\mathbf{U P Q R}_{q}$-CR-Possible-Manipulation: Recall that closeness-respecting $J$-induced (weak) preferences only requires that an outcome $Y$ must be better than an outcome $X$ when the set of formulas from $J$ included in $Y$ is a superset of the set of formulas from $J$ included in $X$. That is, an outcome $X$ is possibly better than $Y$ whenever it contains some formula from $J$ which is not already included in $Y$.
Thus, $\mathrm{UPQR}_{q}$-CR-Possible-Manipulation asks whether the manipulator can achieve an outcome that contains at least one formula from the desired set that was not in the truthful outcome.
Formally, we have: $\exists J^{*} \in \mathcal{J}(\Phi): \operatorname{UPQR}_{q}\left(\boldsymbol{J}_{-n}, J^{*}\right) \succ \operatorname{UPQR}_{q}(\boldsymbol{J})$ for some $\succeq \in \mathrm{CR}_{J}$ ? $\Leftrightarrow$ $\exists J^{*} \in \mathcal{J}(\Phi):\left(\operatorname{UPQR}_{q}\left(\boldsymbol{J}_{-n}, J^{*}\right) \cap J\right) \backslash\left(\operatorname{UPQR}_{q}(\boldsymbol{J}) \cap J\right) \neq \emptyset ?$
This is the same as UPQR-Possible-Manipulation.
Motivation: This variant can be very useful if we have limited information about the manipulator. In case we know the absolute goal (formally the desired set of formulas) but not the respective priorities, we should consider this manipulator model since it allows to detect potentially dangerous situations where a manipulator may "exchange" one formula of the desired set from the truthful outcome by another, more prioritized formula.

### 3.1.4 Minor Definition Details

While we have seen that the manipulation variants we consider resemble those known in the literature, our actual definition of the decision problems still slightly differ from those in the literature [4] as we put the threshold value $q$ as part of the input. For our polynomial-time algorithms, the quota is only interesting for computing which variables can be decided by the manipulator. Thus, it has no influence on the computational complexity. Our hardness reductions usually assume that $q=1 / 2$ and $n=3$, but they can all be adapted to work for any rational quota $q$, due to the following lemma.

Lemma 1. For any input instance $I$ consisting of an agenda $\Phi$, a profile $\boldsymbol{J}=\left(J_{1}, J_{2}, J_{3}\right) \in$ $\mathcal{J}(\Phi)^{3}$, the manipulator's desired set $J \subseteq J_{3}$, and a threshold $\frac{1}{2}$, for any $q \in[0,1)$, we can compute in polynomial time an "equivalent" instance $I^{\prime}$ consisting of the same agenda $\Phi$, a profile $\boldsymbol{J}^{\prime}=\left(J_{1}^{\prime}, \ldots, J_{n}^{\prime}\right) \in \mathcal{J}(\Phi)^{n}$ with $J_{n}^{\prime}=J_{3}$ and $n=\max \left\{\left\lceil\frac{1}{q}\right\rceil,\left\lceil\frac{1}{1-q}\right\rceil\right\}+1$, the manipulator's desired set $J^{\prime}=J$, and the threshold $q$, such that $U P Q R_{\frac{1}{2}}\left(\boldsymbol{J}_{-3}, J^{*}\right)=U P Q R_{q}\left(\boldsymbol{J}_{-n}^{\prime}, J^{*}\right)$ for any $J^{*} \in \mathcal{J}(\Phi)$.

Proof. We first make some observations about the desired instance $I^{\prime}$. Denote $\tau=\lfloor q n+1\rfloor$. From $n=\max \left\{\left\lceil\frac{1}{q}\right\rceil,\left\lceil\frac{1}{1-q}\right\rceil\right\}+1$ we get $q n>q\left\lceil\frac{1}{q}\right\rceil \geq 1$ and hence $q n+1>2$. Similarly, from $n=\max \left\{\left\lceil\frac{1}{q}\right\rceil,\left\lceil\frac{1}{1-q}\right\rceil\right\}+1$ we get $(1-q) n>(1-q)\left\lceil\frac{1}{1-q}\right\rceil \geq 1$ and hence $q n+1<n$. Thus we have $q n+1 \in(2, n)$, and then $\tau=\lfloor q n+1\rfloor \in[2, n-1]$. For any variable $x \in \Phi$, if $x \in J_{i}^{\prime}$ for $1 \leq i \leq n-1$, then $x \in \operatorname{UPQR}_{q}\left(J_{-n}^{\prime}, J^{*}\right)$ for any $J^{*} \in \mathcal{J}(\Phi)$, and hence $x$ is not decided by the manipulator. Similarly, if $\neg x \in J_{i}^{\prime}$ for $1 \leq i \leq n-1$, then $\neg x \in \mathrm{UPQR}_{q}\left(\boldsymbol{J}_{-n}^{\prime}, J^{*}\right)$ for any $J^{*} \in \mathcal{J}(\Phi)$, and hence $x$ is not decided by the manipulator. On the other hand, if except for the manipulator there are exactly $\tau-1$ judges that accept $x$, then $x$ is decided by the manipulator.

Based on these observations, we construct the profile $\boldsymbol{J}^{\prime} \in \mathcal{J}(\Phi)^{n}$ with $J_{n}^{\prime}=J_{3}$ as follows. For each variable $x \in \Phi$ that is not decided by the manipulator in $I$, if $x \in \operatorname{UPQR}_{\frac{1}{2}}(\boldsymbol{J})$, then we make $x \in J_{i}^{\prime}$ for $1 \leq i \leq n-1$ such that $x \in \operatorname{UPQR}_{q}\left(\boldsymbol{J}^{\prime}\right)$ and $x$ is not decided by the manipulator; otherwise we make $\neg x \in J_{i}^{\prime}$ for $1 \leq i \leq n-1$ such that $\neg x \in \operatorname{UPQR}_{q}\left(\boldsymbol{J}^{\prime}\right)$ and $x$ is not decided by the manipulator. For each variable $x \in \Phi$ that is decided by the manipulator in $I$, from the remaining $n-1$ judges we make exactly $\tau-1$ judges accept $x$ such that $x$ is decided by the manipulator. This finishes the premise part. The conclusion part follows trivially since all judgment sets should be complete and consistent.

Finally, we show $\operatorname{UPQR}_{\frac{1}{2}}\left(J_{-3}, J^{*}\right)=\operatorname{UPQR}_{q}\left(J_{-n}^{\prime}, J^{*}\right)$ for any $J^{*} \in \mathcal{J}(\Phi)$. It suffices to show that $\operatorname{UPQR}_{\frac{1}{2}}\left(J_{-3}, J^{*}\right) \cap \Phi_{p}=\operatorname{UPQR}_{q}\left(J_{-n}^{\prime}, J^{*}\right) \cap \Phi_{p}$ for any $J^{*} \in \mathcal{J}(\Phi)$. For each variable $x \in \Phi$, if $x$ is not decided by the manipulator in $I$, then by construction $x$ is not decided by the manipulator in $I^{\prime}$ and $x \in \operatorname{UPQR}_{\frac{1}{2}}\left(\boldsymbol{J}_{-3}, J^{*}\right)$ if and only if $x \in \operatorname{UPQR}_{q}\left(\boldsymbol{J}_{-n}^{\prime}, J^{*}\right)$. If $x$ is decided by the manipulator in $I$, then by construction $x$ is also decided by the manipulator in $I^{\prime}$, and hence

$$
x \in \mathrm{UPQR}_{\frac{1}{2}}\left(\boldsymbol{J}_{-3}, J^{*}\right) \Leftrightarrow x \in J^{*} \Leftrightarrow x \in \mathrm{UPQR}_{q}\left(\boldsymbol{J}_{-n}^{\prime}, J^{*}\right) .
$$

We remark that the constructed equivalent instance $I^{\prime}$ in Lemma 1 is required to have at least $n=\max \left\{\left\lceil\frac{1}{q}\right\rceil,\left\lceil\frac{1}{1-q}\right\rceil\right\}+1$ judges such that we can create variables not decided by the manipulator. From the perspective of parameterized complexity analysis, this leaves two special cases: (1) instances with high quotas and small numbers of judges where a variable is accepted if all judges accept it and (2) instances with low quotas and small numbers of judges where a variable is accepted if at least one judge accepts it. We discuss how to adapt our hardness reductions for these two cases in Appendix B

### 3.2 Tractable Cases of Manipulation

We start our analysis with UPQR-Robustness-Manipulation and UPQR-Possible-ManipuLATION which turn out to be linear-time solvable when the conclusions are just simple clauses. We first show that the manipulator cannot change the outcome of a premise which is in the desired set but not in the truthful outcome due to the monotonicity of quota rules.

Lemma 2. For any manipulated judgment set $J^{*} \in \mathcal{J}(\Phi)$, we have

$$
\left(U P Q R_{q}\left(\boldsymbol{J}_{-n}, J^{*}\right) \backslash U P Q R_{q}(\boldsymbol{J})\right) \cap\left(J \cap \Phi_{p}\right)=\emptyset .
$$

Proof. Assume towards a contradiction that there exists a judgment set $J^{*}$ such that

$$
A=\left(\operatorname{UPQR}_{q}\left(\boldsymbol{J}_{-n}, J^{*}\right) \backslash \operatorname{UPQR}_{q}(\boldsymbol{J})\right) \cap\left(J \cap \Phi_{p}\right) \neq \emptyset
$$

Let $x$ be a variable such that $x \in A$ or $\neg x \in A$. We show that $x \in A$ will lead to a contradiction. The other case can be proved analogously. Suppose $x \in A$, then we have $x \in J \subseteq J_{n}$ and $x \notin$ $\mathrm{UPQR}_{q}(\boldsymbol{J})$. By definition, this implies that variable $x$ is not decided by the manipulator. By Observation 2, the outcome of $x$ should be independent of the manipulator, which contradicts with $x \in \operatorname{UPQR}_{q}\left(\boldsymbol{J}_{-n}, J^{*}\right) \backslash \mathrm{UPQR}_{q}(\boldsymbol{J})$.

Lemma 3. UPQR-Robustness-Manipulation and UPQR-Possible-Manipulation with conclusions being clauses are solvable in linear time.

Proof. We first show the result for UPQR-Possible-Manipulation. According to the definition of UPQR-Possible-MAnipulation and Lemma 2, the question is whether there exists a manipulated judgment set $J^{*} \in \mathcal{J}(\Phi)$ such that $\left(\operatorname{UPQR}_{q}\left(J_{-n}, J^{*}\right) \backslash \mathrm{UPQR}_{q}(\boldsymbol{J})\right) \cap\left(J \cap \Phi_{c}\right) \neq \emptyset$.

That is, at least one target conclusion from the desired set which was not in the truthful outcome should be included in the manipulated outcome. We first compute the set $D$ of decision variables in linear time. Then, for every target conclusion from $J \backslash \mathrm{UPQR}_{q}(J)$, we check one by one whether it can be satisfied by just changing values of variables in $D$. The manipulation is successful if and only if at least one target conclusion from $J \backslash \mathrm{UPQR}_{q}(\boldsymbol{J})$ is satisfiable. Since every conclusion is a clause (disjunction of literals) or a negation of a clause (conjunction of literals), checking its satisfiability can be done in time linear in the clause size.

For UPQR-Robustness-Manipulation, the question is whether the manipulator can change the outcome of a formula in the desired set $J$. We first compute the set $D$ of decision variables in linear time. If there exists a variable $x \in D$ such that $x \in J$ or $\neg x \in J$, then this instance can be easily manipulated as the manipulator can change the outcome of a premise in $J$. Otherwise, if for every variable $x \in D$, we have that $x \notin J$ and $\neg x \notin J$, then a successful manipulation has to influence the outcomes of conclusions in $J$. That is, we need to find a manipulated judgment set $J^{*}$ such that $\mathrm{UPQR}_{q}\left(\boldsymbol{J}_{-n}, J^{*}\right) \cap\left(J \cap \Phi_{c}\right) \neq \mathrm{UPQR}_{q}(\boldsymbol{J}) \cap\left(J \cap \Phi_{c}\right)$. For every conclusion from $J \backslash \operatorname{UPQR}_{q}(\boldsymbol{J})$, we check one by one whether it can be satisfied by just changing values of variables in $D$. For every conclusion from $J \cap \mathrm{UPQR}_{q}(\boldsymbol{J})$, we check one by one whether its negation can be satisfied by just changing values of variables in $D$. Since every conclusion is a clause (disjunction of literals) or a negation of a clause (conjunction of literals), checking its satisfiability can be done in time linear in the clause size.

Next, we show that UPQR-Necessary-Manipulation boils down to solving a related Satisfiability problem.

Lemma 4. UPQR-Necessary-Manipulation and UPQR-Exact-Manipulation with conclusions chosen from clause set $\mathcal{C} 2$ can be solved by solving at most $\left|\Phi_{c}\right|$ instances of $\mathcal{C}$-Sat.

Proof. We first show the result for UPQR-Necessary-Manipulation. In UPQR-NecessaryManipulation we should find a manipulated judgment set $J^{*} \in \mathcal{J}(\Phi)$ such that $\operatorname{UPQR}_{q}(\boldsymbol{J}) \cap$ $J \subsetneq \operatorname{UPQR}_{q}\left(\boldsymbol{J}_{-n}, J^{*}\right) \cap J$. That is, the manipulated result $\mathrm{UPQR}_{q}\left(\boldsymbol{J}_{-n}, J^{*}\right)$ should not only contain one more formula $C^{*}$ from $J \backslash \mathrm{UPQR}_{q}(\boldsymbol{J})$, but also contain all formulas in $Q_{0}=$ $\mathrm{UPQR}_{q}(\boldsymbol{J}) \cap J$. According to Lemma 2, this formula $C^{*}$ cannot be a premise, and hence, $C^{*}$ should be a conclusion. Moreover, since the manipulated result should contain all formulas in $Q_{0}$, specifically, all premises in $Q_{0}$, the manipulator is only allowed to change the values of variables from $D^{*}=D \backslash\left\{x \mid x \in Q_{0} \vee \neg x \in Q_{0}\right\}$, where $D$ is the set of decision variables. So the problem is to check whether there exists a conclusion $C^{*} \in J \backslash \operatorname{UPQR}_{q}(\boldsymbol{J})$ such that a set $Q=\left(Q_{0} \cap \Phi_{c}\right) \cup\left\{C^{*}\right\}$ of conclusions can be satisfied by just controlling the values of variables from $D^{*}$.

We can simply try all possible $C^{*} \in J \backslash \operatorname{UPQR}_{q}(\boldsymbol{J})$ and for each $C^{*}$ check the set $Q=$ $\left(Q_{0} \cap \Phi_{c}\right) \cup\left\{C^{*}\right\}$. Each conclusion in $Q$ is either a clause from $\mathcal{C}$ or a negation of a clause from $\mathcal{C}$. To satisfy a negative clause (conjunction of literals), the values of all variables in this clause are fixed. For all negative clauses in $Q$, we first check whether they are consistent, which can be done in linear time. If all negative clauses in $Q$ are consistent, then we get the value for all variables in them. Next, we need to check for each such variable whether the corresponding value can be satisfied, that is, either the value is the original value before the manipulation or the variable is in $D^{*}$. If not, these negative clauses can not be satisfied. After this, we only need to check whether the remaining positive clauses in $Q$ can be satisfied. Since all clauses in $Q$ are chosen from $\mathcal{C}$, the remaining problem forms an instance of $\mathcal{C}$-Sat.

For UPQR-Exact-Manipulation, the question is whether there exists a manipulated judgment set $J^{*} \in \mathcal{J}(\Phi)$ such that $J \subseteq \operatorname{UPQR}_{q}\left(J_{-n}, J^{*}\right)$. According to Lemma 2 if $J \cap \Phi_{p} \nsubseteq$ $\operatorname{UPQR}_{q}(J)$, then for any $J^{*}$, we have $J \cap \Phi_{p} \nsubseteq \operatorname{UPQR}_{q}\left(J_{-n}, J^{*}\right)$, which means there is no

[^1]successful manipulation. Hence in the following we can assume that $J \cap \Phi_{p} \subseteq \mathrm{UPQR}_{q}(\boldsymbol{J})$. Then the goal is to satisfy all conclusions in $J \cap \Phi_{c}$ by changing the values of variables in $D \backslash\{x \mid x \in$ $J \vee \neg x \in J\}$, where $D$ is the set of decision variables. This can be reduced to an instance of $\mathcal{C}$-Sat in the same way as in UPQR-Necessary-Manipulation.

As a corollary, every polynomial-time algorithm for $\mathcal{C}$-SAT can be adapted to these two variants of Manipulation.

Corollary 1. Let $\mathcal{C}$ be a clause set. If $\mathcal{C}$-Sat is in $P$, then UPQR-Necessary-Manipulation and UPQR-Exact-Manipulation with conclusions chosen from $\mathcal{C}$ are in $P$.

In the next section we show the other direction holds for all standard-form clause sets.

### 3.3 Intractable Cases of Manipulation: Manipulation vs. Satisfiability

In this section, we give a full characterization for the computational complexity of UPQR-Necessary-Manipulation by showing that $\mathcal{C}$-Sat and UPQR-Necessary-Manipulation with conclusions chosen from $\mathcal{C}$ are actually equivalent under polynomial-time Turing reductions when $\mathcal{C}$ is a standard-form clause set (see Definition 2 for the definition of standard-form clause set).

Theorem 1. Let $\mathcal{C}$ be a standard-form clause set, then UPQR-Necessary-Manipulation with conclusions chosen from $\mathcal{C}$ and $\mathcal{C}$-Sat are equivalent under polynomial-time Turing reductions.

In order to prove Theorem [1 we first identify the type of standard-form clause sets $\mathcal{C}$ for which $\mathcal{C}$-Sat is NP-hard.

Lemma 5. Let $\mathcal{C}$ be a standard-form clause set, then $\mathcal{C}$-Sat is NP-hard if and only if

1. there is a pair of $i, j$ with $i \geq 3$ and $0<j<i$ such that $\mathcal{M}_{2}^{+} \cup \mathcal{M}_{2}^{-} \cup \mathcal{S}_{i}^{j} \subseteq \mathcal{C}$, or
2. there is a pair of $k_{1}, k_{2}$ with $\max \left\{k_{1}, k_{2}\right\} \geq 3$ and $\min \left\{k_{1}, k_{2}\right\} \geq 2$ such that $\mathcal{M}_{k_{1}}^{+} \cup \mathcal{M}_{k_{2}}^{-} \subseteq$ $\mathcal{C}$.

Proof. $\Rightarrow$ If $\mathcal{C}$ does not contain any $\mathcal{M}_{k_{1}}^{+}$with $k_{1} \geq 2$, then $\mathcal{C}$-SAT can solved in linear time as follows. Given a formula $f$ of $\mathcal{C}$-Sat, we first take out all clauses in $f$ that consists of only one positive literal (clauses from $\mathcal{M}_{1}^{+}$) and set all these variables to 1 . If this makes any one of the remaining clauses in $f$ unsatisfied, then $f$ is unsatisfiable. Otherwise, all remaining clauses contain at least one negative variable, which can be satisfied simultaneously by setting all remaining variables to 0 . Similarly, if $\mathcal{C}$ does not contain any $\mathcal{M}_{k_{2}}^{-}$with $k_{2} \geq 2$, then $\mathcal{C}$-Sat can solved in linear time. Therefore, $\mathcal{C}$ must contain at least one $\mathcal{M}_{k_{1}}^{+}$with $k_{1} \geq 2$ and at least one $\mathcal{M}_{k_{2}}^{-}$with $k_{2} \geq 2$. In addition, if $\mathcal{C}$ does not contain any clause set which contains clauses of length at least 3 , then $\mathcal{C}$-SAT is solvable in polynomial time since 2-SAT is solvable in polynomial time. Therefore, $\mathcal{C}$ must contain some $\mathcal{S}_{i}^{j}$ with $i \geq 3$ and $1<j<i$ or some $\mathcal{M}_{k}^{+}$ (or $\mathcal{M}_{k}^{-}$) for some $k \geq 3$.
$\Leftarrow$ For the first case, we show $\left(\mathcal{M}_{2}^{+} \cup \mathcal{M}_{2}^{-} \cup \mathcal{S}_{i}^{j}\right)$-SAT is NP-hard by reducing from $\left(\mathcal{M}_{i}^{+} \cup \mathcal{M}_{2}^{-}\right)$SAT with $i \geq 3$, which is NP-hard according to Lemma 14 in Appendix A. It suffices to show that for any clause $x_{1} \vee \cdots \vee x_{i} \in \mathcal{M}_{i}^{+}$, there is an equivalent formula in $\left(\mathcal{M}_{2}^{+} \cup \mathcal{M}_{2}^{-} \cup \mathcal{S}_{i}^{j}\right)$-SAT. This is true since

$$
x_{1} \vee \cdots \vee x_{i} \Leftrightarrow\left(\neg y_{1} \vee \cdots \vee \neg y_{j} \vee x_{j+1} \vee \cdots \vee x_{i}\right) \wedge\left(y_{1} \vee x_{1}\right) \wedge \cdots \wedge\left(y_{j} \vee x_{j}\right),
$$

where the formula on the right side is a conjunction of clauses from $\mathcal{M}_{2}^{+} \cup \mathcal{S}_{i}^{j}$.
For the second case, since $\left(\mathcal{M}_{k_{1}}^{+} \cup \mathcal{M}_{k_{2}}^{-}\right)$-SAT and $\left(\mathcal{M}_{k_{2}}^{+} \cup \mathcal{M}_{k_{1}}^{-}\right)$-SAT are equivalent under linear-time reductions, we can assume $k_{1} \leq k_{2}$. First, according to Lemma 14 in Appendix A,
$\left(\mathcal{M}_{2}^{+} \cup \mathcal{M}_{k_{2}}^{-}\right)$-SAT with $k_{2} \geq 3$ is NP-hard. Then to show other cases, we construct a reduction from $\left(\mathcal{M}_{k_{1}-1}^{+} \cup \mathcal{M}_{k_{2}}^{-}\right)$-SAT to $\left(\mathcal{M}_{k_{1}}^{+} \cup \mathcal{M}_{k_{2}}^{-}\right)$-SAT. It suffices to show that for any clause $X_{k_{1}-1}=$ $x_{1} \vee \cdots \vee x_{k_{1}-1}$, there is an equivalent formula in $\left(\mathcal{M}_{k_{1}}^{+} \cup \mathcal{M}_{k_{2}}^{-}\right)$-SAT. This is true since

$$
X_{k_{1}-1} \Leftrightarrow\left(X_{k_{1}-1} \vee y_{1}\right) \wedge \cdots \wedge\left(X_{k_{1}-1} \vee y_{k_{2}}\right) \wedge\left(\neg y_{1} \vee \cdots \vee \neg y_{k_{2}}\right)
$$

Combining Theorem 1 and Lemma 5 we get a full characterization for the computational complexity of UPQR-NECESSARY-MANIPULATION with conclusions chosen from a standardform clause set $\mathcal{C}$. As a corollary, the NP-hardness of UPQR-NECESSARY-MANIPULATION holds even if conclusions are clauses with length 3 or monotone clauses.

Corollary 2. Let $\mathcal{C}=\cup_{j=0}^{3} \mathcal{S}_{3}^{j}$ or $\mathcal{C}=\cup_{k=1}^{\infty}\left(\mathcal{M}_{k}^{+} \cup \mathcal{M}_{k}^{-}\right)$. Then, UPQR-NECESSARY-MANIPuLATION with conclusions chosen from $\mathcal{C}$ is NP-hard.

According to Lemma 5, we need to consider two cases. We cover these two cases in the following by Lemma 6, Lemma 7, and Lemma 8. The main idea for the proofs of these lemmas is as follows. By definition, an instance of UPQR-NECESSARY-MANIPULATION with conclusions chosen from $\mathcal{C}$ is a yes-instance if and only if there is one target conclusion $C^{*} \in J \backslash \operatorname{UPQR}_{q}(\boldsymbol{J})$ such that $C^{*}$ and all formulas in $Q_{0}=\operatorname{UPQR}_{q}(\boldsymbol{J}) \cap J$ can be included in the manipulated outcome simultaneously. Since all formulas in $Q_{0}$ are in the original outcome $\operatorname{UPQR}_{q}(\boldsymbol{J})$, we already know that all formulas in $Q_{0}$ can be satisfied simultaneously. The question is whether it is possible to satisfy one more clause $C^{*} \notin Q_{0}$. Therefore, Theorem 1 implies that the additional information "all formulas in $Q_{0}$ can be satisfied simultaneously" does not help to efficiently determine whether all conclusions in $Q=Q_{0} \cup\left\{C^{*}\right\}$ can be satisfied simultaneously. Our reductions in the following lemmas will reflect this phenomenon.

We first prove a weaker version of Theorem 1 in the following Lemma 6, where the clauses sets in Satisfiability and Manipulation are similar but not the same.
Lemma 6. If $\left(\mathcal{M}_{k_{1}}^{+} \cup \mathcal{M}_{k_{2}}^{-}\right)$-Sat is NP-hard, then UPQR-NECESSARY-Manipulation with conclusions chosen from a closely related standard-form clause set $\mathcal{C}=\mathcal{M}_{k_{1}+1}^{+} \cup \mathcal{M}_{k_{2}}^{-}$is NP-hard.
Proof. We present a polynomial-time reduction from $\left(\mathcal{M}_{k_{1}}^{+} \cup \mathcal{M}_{k_{2}}^{-}\right)$-Sat to UPQR-NECESSARYManipulation with conclusions chosen from $\mathcal{C}=\mathcal{M}_{k_{1}+1}^{+} \cup \mathcal{M}_{k_{2}}^{-}$. Given an instance

$$
f=C_{1}^{+} \wedge \cdots \wedge C_{m_{1}}^{+} \wedge C_{1}^{-} \wedge \cdots \wedge C_{m_{2}}^{-}
$$

of $\left(\mathcal{M}_{k_{1}}^{+} \cup \mathcal{M}_{k_{2}}^{-}\right)$-SAT, where $C_{i}^{+} \in \mathcal{M}_{k_{1}}^{+}$and $C_{i}^{-} \in \mathcal{M}_{k_{2}}^{-}$, we construct an instance of UPQR-Necessary-Manipulation with $q=\frac{1}{2}$ as follows (see also Table 21). The agenda contains all variables $x_{1}, \ldots, x_{n}$ that appear in $f$ and their negations. In addition, we create $y_{1}, \ldots, y_{k_{2}}$ and their negations in premises. Then we add $C_{i}^{+} \vee y_{1}$ for $1 \leq i \leq m_{1}, C_{i}^{-}$for $1 \leq i \leq m_{2}$, $\neg y_{1} \vee \cdots \vee \neg y_{k_{2}}$ and their negations as conclusions. The set of judges is $N=\{1,2,3\}$. The manipulator is the third judge and his desired set $J$ consists of all positive conclusions. Note that the manipulator is decisive for variables $x_{1}, \ldots, x_{n}$ and $y_{1}$. We now show that $f$ is satisfiable if and only if the manipulation is feasible.
$\Rightarrow$ Suppose that $f$ is satisfiable, then there is a value assignment $x_{i}^{*}, 1 \leq i \leq n$ such that all clauses in $f$ are satisfied. So the manipulator can set $x_{i}=x_{i}^{*}, 1 \leq i \leq n$ to satisfy conclusions $C_{i}^{+} \vee y_{1}$ for $1 \leq i \leq m_{1}$ and $C_{i}^{-}$for $1 \leq i \leq m_{2}$. The remaining conclusion $\neg y_{1} \vee \cdots \vee \neg y_{k_{2}}$ can be satisfied by setting $y_{1}=0$. Thus the manipulation is feasible.
$\Leftarrow$ Suppose that the manipulation is feasible. Since all positive conclusions except for $\neg y_{1} \vee$ $\cdots \vee \neg y_{k_{2}}$ are already in the truthful outcome $\mathrm{UPQR}_{1 / 2}(\boldsymbol{J})$, to make a successful manipulation, the manipulator has to make the manipulated outcome contain all positive conclusions. Specifically, for conclusion $\neg y_{1} \vee \cdots \vee \neg y_{k_{2}}$, since $y_{j}=1, j \geq 2$ can not be changed by the manipulator, the manipulator has to set $y_{1}=0$. Then, to satisfy all remaining conclusions $C_{i}^{+} \vee y_{1}$ and $C_{i}^{-}$ is equivalent to setting values for $x_{1}, \ldots, x_{n}$ to satisfy $f=C_{1}^{+} \wedge \cdots \wedge C_{m_{1}}^{+} \wedge C_{1}^{-} \wedge \cdots \wedge C_{m_{2}}^{-}$.

Table 2: Instance of UPQR-Necessary-Manipulation with conclusion set $\mathcal{C}=\mathcal{M}_{k_{1}+1}^{+} \cup \mathcal{M}_{k_{2}}^{-}$ for the proof of Lemma 6

| Judgment Set | $x_{1}$ | $\ldots$ | $x_{n}$ | $y_{1}$ | $y_{2}$ | $\ldots$ | $y_{k_{2}}$ | $C_{i}^{+} \vee y_{1}$ | $C_{i}^{-}$ | $\neg y_{1} \vee \cdots \vee \neg y_{k_{2}}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $J_{1}$ | 1 | $\ldots$ | 1 | 1 | 1 | $\ldots$ | 1 | 1 | 0 | 0 |
| $J_{2}$ | 0 | $\ldots$ | 0 | 0 | 1 | $\ldots$ | 1 | 0 | 1 | 1 |
| $J_{3}$ | 0 | $\ldots$ | 0 | 1 | 0 | $\ldots$ | 0 | 1 | 1 | 1 |
| $\mathrm{UPQR}_{1 / 2}$ | 0 | $\ldots$ | 0 | 1 | 1 | $\ldots$ | 1 | $\Rightarrow$ | 1 | 1 |

Table 3: Instance of UPQR-Necessary-Manipulation with conclusion set $\mathcal{C}=\mathcal{M}_{2}^{+} \cup \mathcal{M}_{3}^{-}$ for the proof of Lemma 7


Note that in Lemma 6 the two clause sets $\mathcal{M}_{k_{1}}^{+} \cup \mathcal{M}_{k_{2}}^{-}$(in Satisfiability) and $\mathcal{M}_{k_{1}+1}^{+} \cup$ $\mathcal{M}_{k_{2}}^{-}$(in Manipulation) are not the same. This leaves a gap when conclusions of UPQR-Necessary-Manipulation are chosen from $\mathcal{M}_{2}^{+} \cup \mathcal{M}_{3}^{-}$(or equivalently $\mathcal{M}_{3}^{+} \cup \mathcal{M}_{2}^{-}$): We cannot adopt Lemma 6, since the corresponding Satisfiability problem is $\left(\mathcal{M}_{1}^{+} \cup \mathcal{M}_{3}^{-}\right)$-SAT (or $\left(\mathcal{M}_{2}^{+} \cup \mathcal{M}_{2}^{-}\right)$-SAT), which is not NP-hard (cf. Lemma 5 ). Next we close this gap by giving a more involved reduction to show the NP-hardness for the case when conclusions are chosen from $\mathcal{M}_{2}^{+} \cup \mathcal{M}_{3}^{-}$.

Lemma 7. UPQR-Necessary-Manipulation with conclusions chosen from $\left(\mathcal{M}_{2}^{+} \cup \mathcal{M}_{3}^{-}\right)$is NP-hard.

Proof. We present a polynomial-time reduction from $\left(\mathcal{M}_{2}^{+} \cup \mathcal{M}_{3}^{-}\right)$-SAT. Given an instance $f_{1} \wedge f_{2}$ of $\left(\mathcal{M}_{2}^{+} \cup \mathcal{M}_{3}^{-}\right)$-SAT, where $f_{1}$ is a conjunction of clauses of the form " $x_{i_{1}} \vee x_{i_{2}}$ " from $\mathcal{M}_{2}^{+}$with $x_{i_{1}}, x_{i_{2}} \in\left\{x_{1}, \ldots, x_{n}\right\}$ and $f_{2}$ is a conjunction of clauses of the form " $\neg x_{i_{1}} \vee \neg x_{i_{2}} \vee \neg x_{i_{3}}$ " from $\mathcal{M}_{3}^{-}$with $x_{i_{1}}, x_{i_{2}}, x_{i_{3}} \in\left\{x_{1}, \ldots, x_{n}\right\}$, we construct an instance of UPQR-NECESSARY-MANIPULation with conclusions chosen from $\mathcal{M}_{2}^{+} \cup \mathcal{M}_{3}^{-}$and $q=\frac{1}{2}$ as follows (see also Table (3). The agenda contains $x_{i}, y_{i}, z_{i}(1 \leq i \leq n), w, v$ and their negations as the premise set. The conclusion set consists of the following conclusions and their negations.

- $w \vee v$.
- $\neg x_{i_{1}} \vee \neg x_{i_{2}} \vee \neg x_{i_{3}}$ for every original clause $\neg x_{i_{1}} \vee \neg x_{i_{2}} \vee \neg x_{i_{3}}$ in $f_{2}$.
- $z_{i_{1}} \vee z_{i_{2}}$ for every original clause $x_{i_{1}} \vee x_{i_{2}}$ in $f_{1}$. Note that original variables $x_{i}(1 \leq i \leq n)$ are replaced by variables $z_{i}(1 \leq i \leq n)$.
- $x_{i} \vee y_{i}, y_{i} \vee z_{i}, \neg x_{i} \vee \neg y_{i} \vee \neg w$ and $\neg y_{i} \vee \neg z_{i} \vee \neg w$ for every $1 \leq i \leq n$.

The set of judges is $N=\{1,2,3\}$. The manipulator is the third judge and his desired set $J$ consists of all positive conclusions. The manipulator is decisive for all variables except for $v$. We now show that $f_{1} \wedge f_{2}$ is satisfiable if and only if the manipulation is feasible.
$\Rightarrow$ Assume that $f_{1} \wedge f_{2}$ is satisfiable, then there is a value assignment $x_{i}^{*}, 1 \leq i \leq n$ such that all clauses in $f_{1} \wedge f_{2}$ are satisfied. So the manipulator can set $x_{i}=z_{i}=x_{i}^{*}, 1 \leq i \leq n$ to satisfy conclusions $\neg x_{i_{1}} \vee \neg x_{i_{2}} \vee \neg x_{i_{3}}$ and $z_{i_{1}} \vee z_{i_{2}}$. All remaining positive conclusions can be satisfied by setting $w=1$ and $y_{i}=-x_{i}^{*}$ for $1 \leq i \leq n$. Thus the manipulation is feasible. Recall that the manipulator is decisive for all variables except for $v$.
$\Leftarrow$ Assume that the manipulation is feasible. Since all positive conclusions, except for $w \vee v$, are already in the truthful outcome $\operatorname{UPQR}_{1 / 2}(\boldsymbol{J})$, the assumption that the manipulation is feasible means that there is a value assignment for all variables with $v=0$ (since $v$ is not decided by the manipulator) such that all positive conclusions can be satisfied. Specifically, for conclusion $w \vee v$, since $v=0$ can not be changed by the manipulator, the manipulator has to set $w=1$. Then we have that

$$
\begin{aligned}
& \left(\neg x_{i} \vee \neg y_{i} \vee \neg w\right) \wedge\left(x_{i} \vee y_{i}\right) \stackrel{w=1}{\Rightarrow} x_{i}=-y_{i} \\
& \left(\neg y_{i} \vee \neg z_{i} \vee \neg w\right) \wedge\left(y_{i} \vee z_{i}\right) \stackrel{w=1}{\Rightarrow} y_{i}=-z_{i}
\end{aligned}
$$

This means in this value assignment $x_{i}=z_{i}$. Since all conclusions $\neg x_{i_{1}} \vee \neg x_{i_{2}} \vee \neg x_{i_{3}}$ and $z_{i_{1}} \vee z_{i_{2}}$ can be satisfied by this value assignment with $x_{i}=z_{i}$, we have that $f_{1} \wedge f_{2}$ is satisfiable.

In a similar way, we can prove the result of Theorem 1 for the first case in Lemma 5. Before that, we show the following symmetric equivalence.
Observation 3. UPQR-NECESSARY-MANIPULATION with conclusions chosen from $\bigcup_{k=1}^{\ell} \mathcal{S}_{i_{k}}^{j_{k}}$ and UPQR-NECESSARY-MANIPULATION with conclusions chosen from $\bigcup_{k=1}^{\ell} \mathcal{S}_{i_{k}}^{i_{k}-j_{k}}$ are equivalent under linear-time reductions.

Proof. For these two problems, any instance of one problem can be transformed into an equivalent instance of the other problem by replacing every variable $x_{i}$ with its negation $\neg x_{i}$, and vice versa, and replacing the quota $q$ with $q^{\prime}$ such that $\lfloor q n+1\rfloor=\left\lceil n-q^{\prime} n\right\rceil$, where $\lfloor q n+1\rfloor$ is the number of judges needed for a variable $x_{i}$ to be included in the outcome of one problem and $\left\lceil n-q^{\prime} n\right\rceil$ is the number of judges needed for its negation $\neg x_{i}$ to be included in the outcome of the other problem. If $q n$ is an integer, we set $q^{\prime}=1-q-\frac{1}{n}$. Otherwise, we set $q^{\prime}=1-q$. It is easy to verify that in both cases we have $q^{\prime} \in[0,1)$ and $\lfloor q n+1\rfloor=\left\lceil n-q^{\prime} n\right\rceil$.
Lemma 8. UPQR-NECESSARY-MANIPULATION with conclusions chosen from $\mathcal{M}_{2}^{+} \cup \mathcal{M}_{2}^{-} \cup \mathcal{S}_{i}^{j}$, where $i \geq 3$ and $0<j<i$, is NP-hard.

Proof. We first prove the result for $i=3$. We present a polynomial-time reduction from $\left(\mathcal{M}_{2}^{+} \cup\right.$ $\left.\mathcal{M}_{2}^{-} \cup \mathcal{S}_{3}^{2}\right)$-Sat to UPQR-NECESSARY-MANIPULATION with conclusions chosen from $\mathcal{M}_{2}^{+} \cup \mathcal{M}_{2}^{-} \cup$ $\mathcal{S}_{3}^{2}$. Then, according to Observation 3, UPQR-NEcESSARY-MANIPULATION with conclusions chosen from $\mathcal{M}_{2}^{+} \cup \mathcal{M}_{2}^{-} \cup \mathcal{S}_{3}^{1}$ is also NP-hard.

Given an instance $f_{1}(X) \wedge f_{2}(X) \wedge f_{3}(X)$ of $\left(\mathcal{M}_{2}^{+} \cup \mathcal{M}_{2}^{-} \cup \mathcal{S}_{3}^{2}\right)$-Sat, where $f_{1}(X)$ is a conjunction of clauses from $\mathcal{M}_{2}^{+}, f_{2}(X)$ is a conjunction of clauses from $\mathcal{M}_{2}^{-}, f_{3}(X)$ is a conjunction of clauses from $\mathcal{S}_{3}^{2}$, and all variables in $f_{1}(X) \wedge f_{2}(X) \wedge f_{3}(X)$ are from $X=\left\{x_{1}, x_{2}, \ldots, x_{n}\right\}$. We denote by $f_{1}(Z)$ the formula after replacing every $x_{k}(1 \leq k \leq n)$ by $z_{k}$ in $f_{1}(X)$. Since

$$
\begin{aligned}
f_{1}(X) & \Leftrightarrow f_{1}(Z) \wedge\left(\bigwedge_{k=1,2, \ldots, n} x_{k}=z_{k}\right) \\
& \Leftrightarrow f_{1}(Z) \wedge\left(\bigwedge_{k=1,2, \ldots, n}\left(\neg x_{k} \vee \neg y_{k}\right) \wedge\left(x_{k} \vee y_{k}\right) \wedge\left(\neg y_{k} \vee \neg z_{k}\right) \wedge\left(y_{k} \vee z_{k}\right)\right)
\end{aligned}
$$

Table 4: Instance of UPQR-NECESSARY-MANIPULATION with conclusion set $\mathcal{M}_{2}^{+} \cup \mathcal{M}_{2}^{-} \cup \mathcal{S}_{i}^{j}$ for the proof of Lemma 8 .

| Judgment Set | $x_{k}$ | $y_{k}$ | $z_{k}$ | $w$ | $v$ | clauses in $f_{1}(Z)$ | clauses in $f_{2}(X)$ | clauses in $f_{3}(X)$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $J_{1}$ | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 |
| $J_{2}$ | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| $J_{3}$ | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 1 |
| UPQR $_{1 / 2}$ | 0 | 1 | 1 | 1 | 1 | $\Rightarrow 1$ | 1 |  |
| Judgment Set | $\neg w \vee \neg v$ |  | $\neg x_{k} \vee \neg y_{k} \vee w$ | $\neg y_{k} \vee \neg z_{k} \vee w$ | $x_{k} \vee y_{k}$ | $y_{k} \vee z_{k}$ |  |  |
| $J_{1}$ | 0 |  | 1 |  | 1 | 1 | 1 |  |
| $J_{2}$ | 1 |  | 1 | 1 | 0 | 0 |  |  |
| $J_{3}$ | 1 |  | 1 |  | 1 | 1 | 1 |  |
| $\mathrm{UPQR}_{1 / 2}$ | 0 |  | 1 |  | 1 |  | 1 | 1 |

we construct an instance of UPQR-NECESSARY-MANIPULATION with $q=\frac{1}{2}$ as follows (see also Table (4). The agenda contains $x_{k}, y_{k}, z_{k}$ for $1 \leq k \leq n, w, v$ and their negations as premises. The conclusion set consists of the following clauses and their negations.

- All clauses in $f_{1}(Z), f_{2}(X)$ and $f_{3}(X)$.
- $\neg w \wedge \neg v$.
- $x_{k} \vee y_{k}, y_{k} \vee z_{k}, \neg x_{k} \vee \neg y_{k} \vee w$ and $\neg y_{k} \vee \neg z_{k} \vee w$ for every $1 \leq k \leq n$.

Similar to the proof for Lemma 7, the manipulator is the third judge who is decisive for all premises except for $v$. The desired set contains all positive clauses. The truthful outcome contains all positive clauses except for $\neg w \wedge \neg v$ with $x_{k}=0, y_{k}=1, z_{k}=1, w=v=1$. Similar to the proof for Lemma 7 we now show that $f_{1}(X) \wedge f_{2}(X) \wedge f_{3}(X)$ is satisfiable if and only if the manipulation is feasible.
$\Rightarrow$ Assume that $f_{1}(X) \wedge f_{2}(X) \wedge f_{3}(X)$ is satisfiable, then there is a value assignment $x_{k}^{*}, 1 \leq$ $k \leq n$ such that all clauses in $f_{1}(X) \wedge f_{2}(X) \wedge f_{3}(X)$ are satisfied. So the manipulator can set $x_{k}=z_{k}=x_{k}^{*}(1 \leq k \leq n)$ to satisfy conclusions in $f_{1}(Z), f_{2}(X)$ and $f_{3}(X)$. All remaining positive conclusions can be satisfied by setting $w=0$ and $y_{k}=-x_{k}^{*}$ for $1 \leq k \leq n$. Thus the manipulation is feasible. Recall that the manipulator is decisive for all variables except for $v$.
$\Leftarrow$ Assume that the manipulation is feasible. Since all positive conclusions, except for $\neg w \vee$ $\neg v$, are already in the truthful outcome $\mathrm{UPQR}_{1 / 2}(\boldsymbol{J})$, the manipulated result must contain all positive conclusions. So there is a value assignment for all variables with $v=1$ such that all positive conclusions are satisfied. Specifically, for conclusion $\neg w \vee \neg v$, since $v=1$ can not be changed by the manipulator, the manipulator has to set $w=0$. Then we have that

$$
\begin{aligned}
& \left(\neg x_{k} \vee \neg y_{k} \vee w\right) \wedge\left(x_{k} \vee y_{k}\right) \stackrel{w=0}{\Rightarrow} x_{k}=-y_{k} \\
& \left(\neg y_{k} \vee \neg z_{k} \vee w\right) \wedge\left(y_{k} \vee z_{k}\right) \stackrel{w=0}{\Rightarrow} y_{k}=-z_{k}
\end{aligned}
$$

This means in this value assignment $x_{k}=z_{k}(1 \leq k \leq n)$. Since all conclusions in $f_{1}(Z)$, $f_{2}(X)$ and $f_{3}(X)$ can be satisfied by this value assignment with $x_{k}=z_{k}(1 \leq k \leq n)$, we have that $f_{1}(X) \wedge f_{2}(X) \wedge f_{3}(X)$ is satisfiable.

When $i>3$, we can assume $j \geq 2$ according to Observation 3. We can make a reduction from $\left(\mathcal{M}_{2}^{+} \cup \mathcal{M}_{2}^{-} \cup \mathcal{S}_{i}^{j}\right)$-Sat similarly as what we did in the above reduction for $i=3$. The only difference is that now clauses $\neg x_{k} \vee \neg y_{k} \vee w$ and $\neg y_{k} \vee \neg z_{k} \vee w$ are not allowed in conclusions since their length is $3 \neq i$. To fix this, we just need to create some dummy variables whose value is fixed and not decided by the manipulator and add them to $\neg x_{k} \vee \neg y_{k} \vee w$ and $\neg y_{k} \vee \neg z_{k} \vee w$
such that they are in $\mathcal{S}_{i}^{j}$. Formally, we first create $u_{1}, u_{2}, \ldots, u_{i-3}$, where in the truthful outcome we have that $u_{1}=u_{2}=\cdots=u_{j-2}=1$ and $u_{j-1}=\cdots=u_{i-3}=0$, and they are not decided by the manipulator. Then we add them into $\neg x_{k} \vee \neg y_{k} \vee w$ and $\neg y_{k} \vee \neg z_{k} \vee w$ in the following way:

$$
\begin{aligned}
& \neg x_{k} \vee \neg y_{k} \vee w \rightsquigarrow\left(\neg x_{k} \vee \neg y_{k} \vee \neg u_{1} \vee \cdots \vee \neg u_{j-2}\right) \vee\left(w \vee u_{j-1} \vee \cdots \vee u_{i-3}\right) \in \mathcal{S}_{i}^{j}, \\
& \neg y_{k} \vee \neg z_{k} \vee w \rightsquigarrow\left(\neg y_{k} \vee \neg z_{k} \vee \neg u_{1} \vee \cdots \vee \neg u_{j-2}\right) \vee\left(w \vee u_{j-1} \vee \cdots \vee u_{i-3}\right) \in \mathcal{S}_{i}^{j} .
\end{aligned}
$$

We are now ready to prove Theorem 1 .
Proof of Theorem 1. According to Corollary 亿 it suffices to show that when $\mathcal{C}$-Sat is NP-hard, then UPQR-Necessary-Manipulation with conclusions chosen from $\mathcal{C}$ is NP-hard. According to Lemma區, we need to consider two cases.

Case 1: $\mathcal{M}_{2}^{+} \cup \mathcal{M}_{2}^{-} \cup \mathcal{S}_{i}^{j} \subseteq \mathcal{C}$, where $i \geq 3$ and $0<j<i$. This case has been proved in Lemma 8

Case 2: $\mathcal{M}_{k_{1}}^{+} \cup \mathcal{M}_{k_{2}}^{-} \subseteq \mathcal{C}$, where $\max \left\{k_{1}, k_{2}\right\} \geq 3, \min \left\{k_{1}, k_{2}\right\} \geq 2$. According to Observation 3, it suffices to consider the case when $k_{1} \leq k_{2}$. If $k_{1} \geq 3$, then according to Lemma 5 $\left(\mathcal{M}_{k_{1}-1}^{+} \cup \mathcal{M}_{k_{2}}^{-}\right)$-Sat is NP-hard. Then according to Lemma 6, UPQR-Necessary-ManipuLATION with conclusions chosen from $\mathcal{M}_{k_{1}}^{+} \cup \mathcal{M}_{k_{2}}^{-}$is NP-hard. If $k_{1}=2$ and $k_{2} \geq 4$, then according to Lemma 5 , $\left(\mathcal{M}_{2}^{+} \cup \mathcal{M}_{k_{2}-1}^{-}\right)$-Sat is NP-hard. Then according to Lemma6, UPQR-Necessary-Manipulation with conclusions chosen from $\mathcal{M}_{2}^{+} \cup \mathcal{M}_{k_{2}}^{-}$is NP-hard. The only remaining case is when conclusions are chosen from $\mathcal{M}_{2}^{+} \cup \mathcal{M}_{3}^{-}$, which is shown to be NP-hard in Lemma 7 .

Next, we show that the same equivalence holds for UPQR-Exact-Manipulation, since all reductions in Lemmas 6 to 8 can also be used for UPQR-Exact-Manipulation.

Theorem 2. Let $\mathcal{C}$ be a standard-form clause set, then UPQR-Exact-Manipulation with conclusions chosen from $\mathcal{C}$ and $\mathcal{C}$-Sat are equivalent under polynomial-time Turing reductions.

Proof. According to Corollary [1 we just need to prove that when when $\mathcal{C}$-Sat is NP-hard, UPQR-Exact-Manipulation with conclusions clause from $\mathcal{C}$ is also NP-hard. In all reductions in Lemmas 6 to 8 , the desired set consists of all positive conclusions and the truthful outcome contains all but one of them, thus to achieve UPQR-Necessary-Manipulation is the same as to achieve UPQR-Exact-Manipulation, i.e., the manipulated outcome should contain all positive conclusions $\left(\operatorname{UPQR}_{q}\left(J_{-n}, J^{*}\right)=J\right)$. That means all these reductions can be directly used to prove that the corresponding UPQR-Exact-Manipulation is NP-hard. Then following the same line in the proof for Theorem 1 , we can prove $\mathcal{C}$-Sat and UPQR-ExactManipulation with conclusions clause from $\mathcal{C}$ are equivalent under polynomial-time Turing reductions.

Finally, we remark that our core contribution when showing our P vs. NP dichotomy (Theorem(1) can be also interpreted as a pure equivalence statement about variants of Satisfiability. Recall that when $\mathcal{C}$-SAT is NP-hard, Theorem 1 implies that the additional information "all conclusions included in the truthful outcome can be satisfied simultaneously" does not help to efficiently determine whether one more conclusion can be satisfied simultaneously. Accordingly, we introduce the following variant of Satisfiability.

## Almost Satisfiable $\mathcal{C}$-Sat

Input: A standard-form clause set $\mathcal{C}$ and a formula $C_{1} \wedge \cdots \wedge C_{m}$ with $C_{i} \in \mathcal{C}(1 \leq i \leq m)$ knowing that $C_{1} \wedge \cdots \wedge C_{m-1}$ is satisfiable.
Question: Is formula $C_{1} \wedge \cdots \wedge C_{m}$ satisfiable?

Proposition 1. Let $\mathcal{C}$ be a standard-form clause set, then $\mathcal{C}$-Sat and Almost Satisfiable $\mathcal{C}$-Sat are equivalent under polynomial-time many-one reductions.

Proof. Note that the equivalence under polynomial-time Turing reductions between these two problems are trivial. Here we show the equivalence under polynomial-time many-one reductions. The reduction from Almost Satisfiable $\mathcal{C}$-Sat to $\mathcal{C}$-Sat is trivial. For the other direction, if for a standard-form clause set $\mathcal{C}, \mathcal{C}$-Sat is in P , then for any instance of $\mathcal{C}$-Sat, we can first decide in polynomial time its satisfiability, and then reduce it to a trivial yes/no-instance of Almost Satisfiable $\mathcal{C}$-Sat. The remaining case is when $\mathcal{C}$-Sat is NP-hard. For this case, we can adopt the same idea used in the proof of Theorem Recall that for the reductions in Lemmas 6 to 8, there exists a value assignment for all variables such that all but one positive conclusions are satisfied (in the truthful outcome), and the goal of the manipulator is to find a value assignment for decision variables such that all positive conclusions are satisfied. The only difference is that in Manipulation we can construct a clause where some variables in this clause are decided by the manipulator, while the rest variables are not. In other words, in Manipulation we can make use of constant 0 or 1 . This, however, is not easy to achieve in $\mathcal{C}$-Sat. Thus we need to prove that for a standard-form clause set $\mathcal{C}$ such that $\mathcal{C}$-Sat is NP-hard, we can use clauses in $\mathcal{C}$ to create a constant 0 and 1 (i.e., to enforce a variable to be 0 or 1 ).

We show that we can create a constant 1. A constant 0 can be created analogously. According to Lemma 5, if $\mathcal{C}$-SAT is NP-hard, then $\mathcal{C}$ has to contain both $\mathcal{M}_{k_{1}}^{+}$and $\mathcal{M}_{k_{2}}^{-}$with $k_{1}, k_{2} \geq 2$. We create our target variable $x$, and a set of $\left(k_{1}-1\right) k_{2}$ variables $Y=\left\{y_{1}, y_{2}, \ldots, y_{\left(k_{1}-1\right) k_{2}}\right\}$. Then we create two groups of clauses such that to satisfy all of them, the value of $x$ must be 1 .

For the first group, we partition $Y$ into $k_{1}-1$ subsets, each with $k_{2}$ consecutive variables (e.g., $\left\{y_{1}, y_{2}, \ldots, y_{k_{2}}\right\}$ ). For each subset, we create a clause in $\mathcal{M}_{k_{2}}^{-}$, which is a disjunction of negations of all variables in this subset (e.g., $\neg y_{1} \vee \cdots \vee \neg y_{k_{2}}$ ). To satisfy every clause, at least one variable from each subset need to be 0 , and hence at least $k_{1}-1$ variables from $Y$ need to be 0 .

For the second group, for any $k_{1}-1$ variables in $Y$, we create a clause in $\mathcal{M}_{k_{1}}^{+}$which is a disjunction of $x$ and all these $k_{1}-1$ variables (e.g., $\left.x \vee y_{1} \cdots \vee y_{k_{1}-1}\right)$. Since there are $\binom{\left(k_{1}-1\right) k_{2}}{k_{1}-1}$ different choices, we create $\binom{\left(k_{1}-1\right) k_{2}}{k_{1}-1}$ such clauses for the second group. Among all these choices, there is at least one choice $y_{i_{1}}, \ldots, y_{i_{k_{1}-1}}$ such that all of them are 0 due to clauses in the first group. Then in the corresponded clause $x \vee y_{i_{1}} \cdots \vee y_{i_{k_{1}-1}}, x$ must be 1 . Therefore, to satisfy all clauses in the first and the second group, we have to set $x=1$. That is, we create a constant $x=1$ by adding $k_{1}-1+\binom{\left(k_{1}-1\right) k_{2}}{k_{1}-1}$ clauses. Note that for a fixed $\mathcal{C}, k_{1}$ and $k_{2}$ are constants.

## 4 Hamming Distance Based Manipulation

We now move on to UPQR-HD-Manipulation, which is the very first variant of ManipuLATION analyzed by Endriss et al. [20] for the majority threshold $q=1 / 2$. In UPQR-HDManipulation, the manipulator wants to make the outcome "closer" to his desired set and the distance between the outcome and his desired set is measured by the Hamming distance. This models the case when each conclusion is equally important to the manipulator, so that the manipulator only cares about the number of formulas in the desired set $J$ achieved by the collective judgment set. The formal definition is given as follow: $]^{3}$.

[^2]
## UPQR-HD-Manipulation

Input: An agenda $\Phi$, a profile $\boldsymbol{J}=\left(J_{1}, \ldots, J_{n}\right) \in \mathcal{J}(\Phi)^{n}$, the manipulator's desired consistent (possibly incomplete) set $J \subseteq J_{n}$, and a uniform rational threshold $q \in[0,1$ ).
Question: Does there exist a judgment set $J^{*} \in \mathcal{J}(\Phi)$ such that $\operatorname{HD}\left(J, \operatorname{UPQR}_{q}\left(\boldsymbol{J}_{-n}, J^{*}\right)\right)<\operatorname{HD}\left(J, \operatorname{UPQR}_{q}(\boldsymbol{J})\right) ?$

Herein, the Hamming distance $\operatorname{HD}(J, S)$ between the possibly incomplete desired set $J$ and a complete collective judgment set $S$ is the number of formulas in $J$ which are not contained in $S$, i.e. $\operatorname{HD}(J, S)=|J \backslash S|$.

Without loss of generality, in this section we assume that $J=J_{n} \cap \Phi_{c}$, that is, the desired set contains all conclusions from $J_{n}$ but no premise: Every instance of UPQR-HD-MANIPULATION can be easily transformed into an equivalent instance with $J=J_{n} \cap \Phi_{c}$ as follows. If for some conclusion $\varphi$ none of $\varphi$ and $\neg \varphi$ appears in $J$, then just delete $\varphi$ and $\neg \varphi$ from the agenda. If there is some premise $x$ with $x \in J$ (or $\neg x \in J$ ), we can remove it from $J$, then create two clauses $x \vee x^{\prime}$ and $\neg\left(x \vee x^{\prime}\right)$ in the conclusions and add $x \vee x^{\prime}$ (or $\neg\left(x \vee x^{\prime}\right)$ ) to $J$, where $x^{\prime}$ is a dummy variable with $x^{\prime} \notin J_{i}$ for all $1 \leq i \leq n$. Note that doing so we just add positive monotone clauses with two literals into the conclusion set.

Baumeister et al. [4] proved that UPQR-HD-MANIPULATION is NP-hard for positive monotone clauses. In this section we show that this problem is NP-hard even for positive monotone clauses of length $\ell=3$ by reducing from a natural variant of Vertex Cover, which could be interesting on its own. When the clause length is 2 , we show the problem is in P for positive monotone clauses, but NP-hard for monotone clauses or Horn clauses.

### 4.1 Condition for a Successful Manipulation

In this section we give a sufficient and necessary condition for a successful manipulation when conclusions are positive monotone clauses. First we classify all variables into the following four different classes with respect to their value in the truthful outcome $\operatorname{UPQR}_{q}(\boldsymbol{J})$ and the judgment set of the manipulator $J_{n}$ :

1. $P_{1}^{1}=\left\{x \in \Phi_{p} \mid x \in J_{n} \wedge x \in \operatorname{UPQR}_{q}(\boldsymbol{J})\right\}$;
2. $P_{1}^{0}=\left\{x \in \Phi_{p} \mid x \notin J_{n} \wedge x \in \operatorname{UPQR}_{q}(\boldsymbol{J})\right\} ;$
3. $P_{0}^{0}=\left\{x \in \Phi_{p} \mid x \notin J_{n} \wedge x \notin \operatorname{UPQR}_{q}(\boldsymbol{J})\right\} ;$
4. $P_{0}^{1}=\left\{x \in \Phi_{p} \mid x \in J_{n} \wedge x \notin \operatorname{UPQR}_{q}(\boldsymbol{J})\right\}$.

Next we give some definitions used in this section.
Definition 4. A variable $x$ is called useful if it is decided by the manipulator and there exists a positive conclusion $\varphi \in J \backslash U P Q R_{q}(\boldsymbol{J})$ containing $x$. A positive monotone clause $\varphi$ is called a good conclusion if $\varphi \in J \backslash U P Q R_{q}(\boldsymbol{J})$ and is called a bad conclusion if $\varphi \notin J \cup U P Q R_{q}(\boldsymbol{J})$, or equivalently, $\neg \varphi \in J \cap U P Q R_{q}(\boldsymbol{J})$ (recall that $J$ is complete with respect to the conclusion set).

According to the definition, changing the value of a useful variable can make the outcome include at least one more desired (good) conclusion $\varphi$ with $\varphi \in J \backslash \mathrm{UPQR}_{q}(\boldsymbol{J})$, but it could also make the outcome include some undesired (bad) conclusion $\varphi$ with $\varphi \notin J \cup \operatorname{UPQR}_{q}(\boldsymbol{J})$, thus lose the desired conclusion $\neg \varphi \in J \cap \operatorname{UPQR}_{q}(\boldsymbol{J})$. We observe that all useful variables are from $P_{0}^{0}$ due to the monotonicity of quota rules.

Observation 4. If $x$ is a useful variable, then $x \in P_{0}^{0}$.

Proof. By definition, variables from $P_{0}^{1} \cup P_{1}^{0}$ are not decided by the manipulator, so they are not useful. For a variable $x \in P_{1}^{1}$, any positive monotone clause $\varphi$ containing $x$ is already in $J \cap \mathrm{UPQR}_{q}(\boldsymbol{J})$, thus changing the value of $x$ from 1 to 0 cannot make the outcome include any good conclusion from $J \backslash \operatorname{UPQR}_{q}(\boldsymbol{J})$. Hence, variables from $P_{1}^{1}$ are not useful. Therefore, all useful variables are from $P_{0}^{0}$.

Example. Consider the following profile $\boldsymbol{J}=\left(J_{1}, J_{2}, J_{3}\right) \in \mathcal{J}(\Phi)^{3}$, where the manipulator is the third judge and its desired set $J=J_{3} \cap \Phi_{c}$.

| Judgment Set | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{3}^{\prime}$ | $x_{4}$ |  | $x_{1} \vee x_{2}$ | $x_{2} \vee x_{3}$ | $x_{3} \vee x_{3}^{\prime}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $J_{1}$ | 1 | 0 | 1 | 1 | 1 |  | 1 | 1 | 1 |
| $x_{3} \vee x_{4}$ |  |  |  |  |  |  |  |  |  |
| $J_{2}$ | 0 | 0 | 0 | 0 | 1 |  | 0 | 0 | 0 |
| $J_{3}$ | 1 | 1 | 0 | 0 | 0 |  | 1 | 1 | 0 |
| $\mathrm{UPQR}_{1 / 2}$ | 1 | 0 | 0 | 0 | 1 | $\Rightarrow$ | 1 | 0 | 0 |

Variables $x_{1}, x_{3}$, and $x_{3}^{\prime}$ are decided by the manipulator, but $x_{1} \in P_{1}^{1}$ is not useful since changing its value from 1 to 0 would only exclude $x_{1} \vee x_{2} \in J$ from the outcome. Conclusion $x_{2} \vee x_{3} \in$ $J \backslash \operatorname{UPQR}_{q}(\boldsymbol{J})$ is good, and changing $x_{3}$ from 0 to 1 will make $x_{3}$ and $x_{2} \vee x_{3}$ included in the outcome. Conclusion $x_{3} \vee x_{3}^{\prime} \notin J \cup \mathrm{UPQR}_{q}(\boldsymbol{J})$ is bad, and changing $x_{3}$ or $x_{3}^{\prime}$ from 0 to 1 will make $x_{3} \vee x_{3}^{\prime}$ included in the outcome, and hence its negation $\neg\left(x_{3} \vee x_{3}^{\prime}\right) \in J$ will be excluded from the outcome.

Now we give a sufficient and necessary condition for a successful manipulation.
Lemma 9. An instance of UPQR-HD-MAnIPULATION with all conclusions being positive monotone clauses is a yes-instance if and only if there is a set $T \subseteq P_{0}^{0}$ of useful variables, such that after changing their values from 0 to 1, the number of good conclusions included in the outcome is strictly larger than the number of bad conclusions included in the outcome:

$$
\left|\left\{\varphi \in J \backslash U P Q R_{q}(\boldsymbol{J}) \mid T_{\varphi} \cap T \neq \emptyset\right\}\right|>\left|\left\{\varphi \notin J \cup U P Q R_{q}(\boldsymbol{J}) \mid T_{\varphi} \cap T \neq \emptyset\right\}\right|,
$$

where $T_{\varphi}$ is the set of variables appearing in clause $\varphi$.
Proof. The "if" direction is obvious and we prove the "only if" direction as follows. According to the definition of useful variables, if an instance is a yes-instance, then the manipulator can achieve a better outcome by only changing the values of useful variables. According to Observation 4, all useful variables are from $P_{0}^{0}$. To prove this lemma, it suffices to show that we just need to consider good conclusions and bad conclusions. If a positive monotone clause $\varphi$ is neither good nor bad, then it must be $\neg \varphi \in J \backslash \operatorname{UPQR}_{q}(\boldsymbol{J})$ or $\varphi \in J \cap \operatorname{UPQR}_{q}(\boldsymbol{J})$. In both cases we have that $\varphi \in \operatorname{UPQR}_{q}(\boldsymbol{J})$. Changing the values of variables in $P_{0}^{0}$ from 0 to 1 will not change the value of $\varphi\left(\varphi\right.$ is still in $\operatorname{UPQR}_{q}(\boldsymbol{J})$ after this change). Therefore, we just need to consider the influence on the number of good conclusions and bad conclusions after changing the values of useful variables.

### 4.2 Positive Monotone Clauses of Length $\ell=2$

In this section we show UPQR-HD-Manipulation with positive monotone clauses of length $\ell=$ 2 is solvable in polynomial time by a reduction to the Weighted Maximum Density SubGRAPH (WMDS) problem, which can be solved in polynomial time by a reduction to the Minimum Cut problem [24].


Figure 1: Illustration of the constructed weighted graph in the proof of Theorem 3 On the left side a vertex represents a variable from $P_{0}^{0}$ or $P_{0}^{1}$. A line between two vertices represents a conclusion containing the two corresponding variables. A red solid line represents a good conclusion and a blue dotted line represents a bad conclusion. We transform it into the vertex weighted graph on the right side, where the vertex set $V$ corresponds to $P_{0}^{0}$ and the weight $w$ for a vertex $v \in V$ is the difference between the number of bad and good conclusions that contain the corresponding variable $x_{v}$.

## Weighted Maximum Density Subgraph

Input: An undirected graph $G=(V, E)$ with nonnegative rational edge weights $w(e)$ and vertex weights $w(v)$, and a nonnegative rational number $k$.
Question: Does there exist a vertex subset $V^{\prime} \subseteq V$ with $\sum_{v \in V^{\prime}} w(v)>0$ such that

$$
\frac{\sum_{e \in E\left(G\left[V^{\prime}\right]\right)} w(e)}{\sum_{v \in V^{\prime}} w(v)}>k,
$$

where $G\left[V^{\prime}\right]$ is the subgraph induced by $V^{\prime}$ ?

Theorem 3. UPQR-HD-MANIPULATION with positive monotone clauses of length $\ell=2$ is solvable in polynomial time.

Proof. According to Lemma 9, we need to find a set $T \subseteq P_{0}^{0}$ of useful variables such that the number of good conclusions containing variables from $S$ is strictly larger than that of bad conclusions. For any good conclusion $\varphi \in J \backslash \operatorname{UPQR}_{q}(\boldsymbol{J})$, since $\varphi \in J \subseteq J_{n}$, there exists one variable $x$ in $\varphi$ such that $x \in J_{n}$. Moreover, since $\varphi \notin \operatorname{UPQR}_{q}(\boldsymbol{J})$ we have $x \notin \operatorname{UPQR}_{q}(\boldsymbol{J})$. Thus $\varphi$ contains at least one variable $x \in P_{0}^{1}$, which is not decided by the manipulator. Hence a good conclusion $\varphi$ of length 2 contains at most one useful variable from $P_{0}^{0}$. However, a bad conclusion $\varphi^{\prime} \notin J \cup \mathrm{UPQR}_{q}(\boldsymbol{J})$ of length 2 may contain two useful variables from $P_{0}^{0}$. Thus, if we change the values of a set $T \subseteq P_{0}^{0}$ of useful variables and sum up the number of included bad conclusions for each variable, then some bad conclusions will be counted twice.

To solve this issue, we create a weighted graph $G=(V, E)$ as follows (see also Figure 1 ): First, for every useful variable $x \in P_{0}^{0}$, create a vertex $v \in V$ and assign it a weight $w(v)=n_{v}-p_{v}$, where $n_{v}$ is the number of bad conclusions containing $x$ and $p_{v}$ is the number of good conclusions containing $x$. Thus $w(v)$ is the increased Hamming distance when a single variable $x$ is changed. Second, for every pair of vertices $u$ and $v$, create an edge between them if there is a bad conclusion $\varphi=x_{u} \vee x_{v}$, where $x_{u}$ and $x_{v}$ are the corresponding variables of $u$ and $v$. Based on the constructed weighted graph, we can reduce an instance of UPQR-HD-MAnipulation to an instance of WMDS problem.

We first do the following preprocessing. If there is a vertex $v \in V$ with $w(v)<0$, then changing this variable alone can strictly decrease the Hamming distance and hence the manipulation is feasible. If there is an edge $e=\{u, v\}$ with $w(u)=w(v)=0$, then there is a bad conclusion


Figure 2: Comparison between different clause classes. A vertex represents a variable from $P_{0}^{0} \cup P_{0}^{1}$, and only variables from $P_{0}^{0}$ could be decided by the manipulator. A line between two vertices represents a conclusion containing the two corresponding variables. A line is solid or dotted if changing the value of one of its endpoints in $P_{0}^{0}$ will change the value of this conclusion, while a line is dashed if changing the value of both endpoints in $P_{0}^{0}$ will change the value of this conclusion.
$\varphi=x_{u} \vee x_{v}$ that is counted in both $w(u)$ and $w(v)$. So changing the value of $x_{u}$ and $x_{v}$ can decrease the Hamming distance by 1 , which means the manipulation is feasible.

In the following we can assume $w(v) \geq 0$ for every $v \in V$ and there is no edge $e=\{u, v\}$ with $w(u)=w(v)=0$. For any vertex subset $V^{\prime} \subseteq V$, changing the value of the corresponding variables can increase the Hamming distance by $\sum_{v \in V^{\prime}} w(v)-\left|E\left(G\left[V^{\prime}\right]\right)\right|$, where $G\left[V^{\prime}\right]$ is the subgraph induced by $V^{\prime}$. If $\sum_{v \in V^{\prime}} w(v)=0$, then according to the above assumption, we have $\left|E\left(G\left[V^{\prime}\right]\right)\right|=0$ and the Hamming distance has not changed. Therefore manipulation is feasible if and only if there is a vertex subset $V^{\prime}$ with $\sum_{v \in V^{\prime}} w(v)>0$ such that $\sum_{v \in V^{\prime}} w(v)-$ $\left|E\left(G\left[V^{\prime}\right]\right)\right|<0$ or equivalently,

$$
\frac{\left|E\left(G\left[V^{\prime}\right]\right)\right|}{\sum_{v \in V^{\prime}} w(v)}>1
$$

This is just an instance of the WMDS problem where every edge has weight 1 , which can be solved in polynomial time [24].

### 4.3 Positive Monotone Clauses of Length $\ell=3$

In this section we show that UPQR-HD-MANIPULATION with positive monotone clauses of length $\ell=3$ is NP-hard. The main difference between $\ell=2$ and $\ell=3$ is that when $\ell=2$, every good conclusion must contain a variable from $P_{0}^{1}$, and hence contains at most one useful variable from $P_{0}^{0}$. This makes it easy to count the number of included good conclusions. When $\ell \geq 3$, however, in addition to one variable from $P_{0}^{1}$, a good conclusion can contain two useful variables from $P_{0}^{0}$. Hence, useful variables are not independent with respect to good conclusions. See Figure 2 for the comparison between $\ell=2$ and $\ell=3$. Intuitively, when $\ell=3$, we need to find a vertex subset of $P_{0}^{0}$ in Figure 2 to cover more red solid edges than blue dotted edges. This leads us to define the following closely related graph problem.


Figure 3: Illustration of the constructed instance in the proof of Lemma 10. Blue dotted edges are edges in $E^{-}$and red solid edges are edges in $E^{+}$. The manipulator just need to consider variables in $\left\{x_{1}, \ldots, x_{n}, y_{1}, y_{2}\right\}$.

## Positive Vertex Cover

Input: An undirected graph $G=\left(V, E^{+} \cup E^{-}\right)$with $E^{+} \cap E^{-}=\emptyset$.
Question: Is there a vertex subset $V^{\prime} \subseteq V$ which covers strictly more edges in $E^{+}$than in $E^{-}$? Herein, an edge is covered by a vertex subset if the vertex subset contains at least one endpoint of the edge.

We show that Positive Vertex Cover is NP-hard and then provide a simple reduction from Positive Vertex Cover to UPQR-HD-Manipulation with positive monotone clauses of length $\ell=3$.

## Lemma 10. Positive Vertex Cover is NP-hard.

Proof. We construct a reduction from Cubic Vertex Cover, where given an undirected 3regular graph and an integer $k$, the task is to determine whether there exists a vertex cover of size at most $k$. Let $\left(G_{0}=\left(V_{0}, E_{0}\right), k\right)$ be an instance of Cubic Vertex Cover. Denote $n=\left|V_{0}\right|$. Then $\left|E_{0}\right|=\frac{3 n}{2}$. Since a vertex cover for a cubic graph needs at least $\frac{n}{2}$ vertices to cover all $\frac{3 n}{2}$ edges, we can assume $\frac{n}{2} \leq k \leq n$. We create an instance $G=\left(V, E^{+} \cup E^{-}\right)$ of Positive Vertex Cover as follows (see also Figure (3). First, for every original vertex $v_{i}$ in $V_{0}$, create a vertex $x_{i}$ in $V$, and for every edge $\left\{v_{i}, v_{j}\right\}$ in $E_{0}$, create an edge $\left\{x_{i}, x_{j}\right\}$ in $E^{+}$. Then, create three more vertices $y_{1}, y_{2}, y_{3}$ in $V$ and create edges $\left\{x_{i}, y_{1}\right\},\left\{x_{i}, y_{2}\right\}$ and $\left\{x_{i}, y_{3}\right\}$ for every $1 \leq i \leq n$ in $E^{-}$. Add an edge $\left\{y_{1}, y_{2}\right\}$ in $E^{+}$. Next, create vertices $z_{1}, \ldots, z_{n-p}$ in $V$, where $p=\frac{\overline{3} n}{4}-\frac{k}{2}$, and create edges $\left\{z_{j}, y_{1}\right\}$ and $\left\{z_{j}, y_{2}\right\}$ for every $1 \leq j \leq n-p$ in $E^{+}$. Finally, create vertices $w_{1}, w_{2}, w_{3}$ in $V$, and create edges $\left\{z_{j}, w_{1}\right\},\left\{z_{j}, w_{2}\right\}$ and $\left\{z_{j}, w_{3}\right\}$ for every $1 \leq j \leq n-p$ in $E^{-}$.

Notice that every vertex in $\left\{w_{1}, w_{2}, w_{3}, y_{3}\right\}$ only covers edges in $E^{-}$, thus it's always better to not choose them. Similarly, every vertex in $\left\{z_{1}, \ldots, z_{n-p}\right\}$ covers 3 edges in $E^{-}$and two edges in $E^{-}$, thus it's always better to not choose them. Therefore, our choice is constrained in $\left\{x_{1}, \ldots, x_{n}, y_{1}, y_{2}\right\}$. The following lemma describes when should we choose $y_{1}$ or $y_{2}$, which is the key point in this proof.

Lemma 11. If the number $k^{\prime}$ of vertices chosen from $\left\{x_{1}, \ldots, x_{n}\right\}$ is less than $p$, then it is always better to not choose $y_{1}$ or $y_{2}$. Otherwise, it is always better to choose both $y_{1}$ and $y_{2}$.

Proof. We prove the result for $y_{1}$ and the result for $y_{2}$ can be proved analogously. Notice that after choosing $k^{\prime}$ vertices from $\left\{x_{1}, \ldots, x_{n}\right\}$, the number of uncovered edges incident on $y_{1}$
from $E^{-}$is $n-k^{\prime}$, while the number of uncovered edges incident on $y_{1}$ from $E^{+}$is $n-p+1$. If $k^{\prime}<p$, then $n-k^{\prime} \geq n-p+1$, and hence it is better to not choose $y_{1}$. If $k^{\prime} \geq p$, then $n-k^{\prime}<$ $n-p+1$, and hence it is better to choose $y_{1}$.

Let $V^{\prime}$ be the set of vertices we will choose in the constructed instance. Denote the number of edges in $E^{+}$covered by $V^{\prime}$ and the number of edges in $E^{-}$covered by $V^{\prime}$ by $n_{1}$ and $n_{2}$, respectively. We show in the following that there is a vertex cover of size $k$ in $G_{0}$ if and only if there is a vertex subset $V^{\prime} \subseteq V$ such that $n_{1}>n_{2}$.
$\Rightarrow$ Suppose that there is a vertex cover of size $k$ in $G_{0}$. Let $V^{\prime}$ consists of the corresponding $k$ vertices in $V$ and two more vertices $y_{1}$ and $y_{2}$. Then vertices in $V^{\prime}$ cover all edges in $E^{+}$. That is, all edges corresponding to $E_{0}$, edges $z_{j} y_{1}$ and $z_{j} y_{2}$ for every $1 \leq j \leq n-p$ and $y_{1} y_{2}$. Thus

$$
n_{1}=\frac{3 n}{2}+2(n-p)+1=2 n+k+1
$$

On the other hand, in $E^{-}$, edges $x_{i} y_{1}$ and $x_{i} y_{2}$ for every $1 \leq i \leq n$ and $k$ edges from $\left\{x_{i} y_{3} \mid 1 \leq\right.$ $i \leq n\}$ are covered, so

$$
n_{2}=2 n+k<n_{1}
$$

$\Leftarrow$ Suppose that there is a vertex subset $V^{\prime} \subseteq V$ such that $n_{1}>n_{2}$. Let $k^{\prime}=\mid V^{\prime} \cap$ $\left\{x_{1}, \ldots, x_{n}\right\} \mid$. We first show that $k^{\prime} \geq p$. Assume towards a contradiction that $k^{\prime}<p$. According to Lemma 11, it is better to not choose $y_{1}$ or $y_{2}$. Therefore, we can assume that $V^{\prime} \subseteq\left\{x_{1}, \ldots, x_{n}\right\}$. Since every vertex in $\left\{x_{1}, \ldots, x_{n}\right\}$ covers 3 edges in $E^{-}$, and every edge in $E^{-}$is covered by at most one vertex in $\left\{x_{1}, \ldots, x_{n}\right\}$, we have $n_{2}=3 k^{\prime}$. Since every vertex in $\left\{x_{1}, \ldots, x_{n}\right\}$ covers 3 edges in $E^{+}$, we have $n_{1} \leq 3 k^{\prime}=n_{2}$, which is a contradiction. In the following, we can assume that $k^{\prime} \geq p$.

Since $k^{\prime} \geq p$, according to Lemma 11, it is better to choose $y_{1}$ and $y_{2}$. Thus we can assume that $y_{1} \in V^{\prime}$ and $y_{2} \in V^{\prime}$. Let $m$ be the number of edges corresponding to $E_{0}$ covered by vertices in $V^{\prime}$. Now we compute $n_{1}$ and $n_{2}$. For $n_{1}$, vertices in $V^{\prime}$ cover edges $z_{j} y_{1}$ and $z_{j} y_{2}$ for every $1 \leq j \leq n-p, y_{1} y_{2}$ and $m$ edges in $E_{0}$, so

$$
n_{1}=2(n-p)+1+m
$$

For $n_{2}$, edges $x_{i} y_{1}$ and $x_{i} y_{2}$ for every $1 \leq i \leq n$ and $k^{\prime}$ of edges $x_{i} y_{3}$ are covered, so

$$
n_{2}=2 n+k^{\prime}
$$

Since $n_{1}>n_{2}$, we have that $m>k^{\prime}+2 p-1=\frac{3 n}{2}+k^{\prime}-k-1$ or

$$
m \geq \frac{3 n}{2}+k^{\prime}-k
$$

Since $m \leq\left|E_{0}\right|=\frac{3 n}{2}$, we have that $k^{\prime} \leq k$. If $k^{\prime}=k$, then $m=\frac{3 n}{2}$. Thus $V^{\prime} \cap\left\{x_{1}, \ldots, x_{n}\right\}$ is a vertex cover of size $k$ for $G_{0}$. If $k^{\prime}<k$, then except for the $k^{\prime}$ edges in $V^{\prime} \cap\left\{x_{1}, \ldots, x_{n}\right\}$, we can always find $k-k^{\prime}$ more vertices to cover $k-k^{\prime}$ more edges in $G_{0}$ such that $m+k-k^{\prime} \geq \frac{3 n}{2}$. Therefore, there is a vertex cover of size $k$ for $G_{0}$.

Now we can show the NP-hardness of UPQR-HD-MANIPULATION with positive monotone clauses of length $\ell=3$ by a simple reduction from Positive Vertex Cover.

Theorem 4. UPQR-HD-MANIPULATION with positive monotone clauses of fixed length $\ell(\geq 3)$ is NP-hard.

Proof. We prove the result for $\ell=3$. Other cases when $\ell>3$ can be shown by slightly adapting the following proof. We present a reduction from Positive Vertex Cover. Given a graph $G=\left(V,\left(E^{+} \cup E^{-}\right)\right)$with $E^{+} \cap E^{-}=\emptyset$, we construct an instance of UPQR-HD-MANIPULATION

Table 5: Instance of UPQR-HD-MANIPULATION with positive monotone clauses of length $\ell=3$ for the proof of Theorem 4

| Judgment Set | $x_{1}$ | $\ldots$ | $x_{n}$ | $y$ | $z$ |  | $x_{i} \vee x_{j} \vee y$ | $x_{i} \vee x_{j} \vee z$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $J_{1}$ | 1 | $\ldots$ | 1 | 0 | 0 |  | 1 | 1 |
| $J_{2}$ | 0 | $\ldots$ | 0 | 0 | 0 | 0 | 0 |  |
| $J_{3}$ | 0 | $\ldots$ | 0 | 1 | 0 |  | 1 | 0 |
| $\mathrm{UPQR}_{1 / 2}$ | 0 | $\ldots$ | 0 | 0 | 0 | $\Rightarrow$ | 0 | 0 |

with $q=\frac{1}{2}$ as follows (see also Table (5). The agenda contains $x_{i}$ for each $v_{i} \in V(1 \leq i \leq n)$, $y, z$ and their negations as the premise set. The conclusions set consists of clauses $x_{i} \vee x_{j} \vee y$ for every edge $\left\{v_{i}, v_{j}\right\}$ in $E^{+}, x_{i} \vee x_{j} \vee z$ for every edge $\left\{v_{i}, v_{j}\right\}$ in $E^{-}$and their negations. The third judge is the manipulator and the desired set $J=J_{3} \cap \Phi_{c}$. Note that except for $y$ and $z$, all variables $x_{i}$ are decided by the manipulator. A successful manipulation is an action that changes the values of strictly more conclusions in $\left\{x_{i} \vee x_{j} \vee y \mid\left\{v_{i}, v_{j}\right\} \in E^{+}\right\}$than that in $\left\{x_{i} \vee x_{j} \vee z \mid\left\{v_{i}, v_{j}\right\} \in E^{-}\right\}$. Therefore, the manipulation is feasible if and only if there is a vertex subset which covers strictly more edges in $E^{+}$than $E^{-}$.

### 4.4 Monotone or Horn Clauses of Length $\ell=2$

In this section we study two remaining cases: Monotone clauses of length $\ell=2$ and Horn clauses of length $\ell=2$. For these two cases, we cannot use the characterization of a successful manipulation for monotone clauses given in Lemma 9. Actually, different from positive monotone clauses of length $\ell=2$, we will show that these two clause structures can be used to encode hard problems.

### 4.4.1 Monotone Clauses of Length $\ell=2$

For monotone clauses we may have both $x_{i} \vee x_{j}$ and $\neg x_{i} \vee \neg x_{j}$ in the conclusions. Consider the following example, where the third judge is the manipulator and the desired set is $J=$ $\left\{\neg x_{1} \vee \neg x_{2}\right\}$.

| Judgment Set | $x_{1}$ | $x_{2}$ |  |
| :--- | :--- | :--- | :--- |
| $J_{1}$ | 1 | 1 |  |
| $J_{1} \vee \neg x_{2}$ |  |  |  |
| $J_{2}$ | 0 | 0 | 1 |
| $J_{3}$ | 0 | 0 |  |
| $\mathrm{UPQR}_{1 / 2}$ | 0 | 0 | $\Rightarrow$ |

Conclusion $\neg x_{1} \vee \neg x_{2} \in J \cap \operatorname{UPQR}_{q}(\boldsymbol{J})$ will be excluded from the outcome only when both $x_{1}$ and $x_{2}$ have been changed. Recall that for positive monotone clauses, changing one variable is enough to include a bad conclusion (see also Figure 2 for the comparison). Thus, for monotone clauses of length $\ell=2$ we have a new kind of "bad" conclusions, which can be used to encode hard problems.
Theorem 5. UPQR-HD-MANIPULATION with monotone clauses of length $\ell=2$ is NP-hard.
Proof. We present a reduction from Clique on regular graphs. Given an instance $(G=$ $(V, E), k, d)$ of Clique on regular graphs, where $d$ is the vertex degree and $k$ is the size of a desired clique, we build an instance of UPQR-HD-MANIPULATION with $q=\frac{1}{2}$ as follows (see also Table 6 for the profile). Denote $n=|V|$. The agenda contains $x_{i}(1 \leq i \leq n), y_{i}(1 \leq i \leq n)$, $x^{*}, y^{*}$ and their negations as the premise set. The conclusion set consists of the following clauses and their negations.

Table 6: Instance of UPQR-HD-Manipulation with monotone clauses of length $\ell=2$ for the proof of Theorem 5

| Judgment Set | $x_{i}$ | $x^{*}$ | $y_{i}$ | $y^{*}$ |  | $\neg x_{i} \vee \neg x_{j}$ | $x_{i} \vee x_{i^{\prime}}$ | $x_{i} \vee x^{*}$ | $x_{i} \vee y_{i}$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :--- | :--- | :--- | :--- |
| $x^{*} \vee y^{*}$ |  |  |  |  |  |  |  |  |  |
| $J_{1}$ | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| $J_{2}$ | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| $J_{3}$ | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 1 |
| $\mathrm{UPQR}_{1 / 2}$ | 0 | 0 | 0 | 0 | $\Rightarrow$ | 1 | 0 | 0 | 0 |

- $\neg x_{i} \vee \neg x_{j}(1 \leq i<j \leq n)$.
- $x_{i} \vee x_{i^{\prime}}$ for each edge $\left\{v_{i}, v_{i^{\prime}}\right\} \in E$.
- $x_{i} \vee x^{*}$ for each $1 \leq i \leq n$.
- $d+1$ copies $\sqrt{4}^{2} x_{i} \vee y_{i}$ for each $1 \leq i \leq n$.
- $n-k+1$ copies of $x^{*} \vee y^{*}$.

There are three judges and their judgment sets are shown in Table 6. The manipulator is the third judge and his desired set $J=J_{3} \cap \Phi_{c}$. The manipulator is decisive for all $x_{i}(1 \leq i \leq n)$ and $x^{*}$.

The effects of changing $k^{\prime}$ variables from $\left\{x_{i} \mid 1 \leq i \leq n\right\}$ are as follows. On the one hand, $k^{\prime}(d+1)$ of $x_{i} \vee y_{i}$ will be changed, which is good. On the other hand, from the first three groups $\neg x_{i} \vee \neg x_{j}, x_{i} \vee x_{i^{\prime}}$, and $x_{i} \vee x^{*}, p \geq k^{\prime}(d+1)$ of them will be changed, which is bad. Note that $p=k^{\prime}(d+1)$ holds if the corresponding $k^{\prime}$ vertices form a clique in $G$, and in this case these two effects cancel each other out. Now, if we continue to change $x^{*}$, then $n-k+1$ of $x^{*} \vee y^{*}$ will be changed, which is good, and $n-k^{\prime} \leq n-k$ more of $x_{i} \vee x^{*}$ will be changed, which is bad. The key point is that if $k^{\prime} \geq k$, then we have $n-k+1>n-k \geq n-k^{\prime}$. Therefore, a clique of size $k^{\prime} \geq k$ in $G$ corresponds to a successful manipulation that changes the corresponding $k^{\prime}$ variables of $x_{i}$ and $x^{*}$ such that overall the Hamming distance will be decreased by 1 .

Now we show that there is a clique of size $k^{\prime} \geq k$ in $G$ if and only if the manipulation is feasible. The "only if" direction is clear from the above analysis, and we just need to show the "if" direction. Suppose there exists a successful manipulation. From the above analysis we have that $x^{*}$ has to be changed and the number $k^{\prime}$ of changed variables from $\left\{x_{i} \mid 1 \leq i \leq n\right\}$ should satisfy $k^{\prime} \geq k$. Let $V^{\prime}$ be the set of $k^{\prime}$ vertices in $G$ corresponding to the $k^{\prime}$ variables. We claim that there are $k$ vertices from $V^{\prime}$ that form a clique in $G$. To prove this claim, let us consider the induced subgraph $G\left[V^{\prime}\right]$. Suppose towards a contradictions that $G\left[V^{\prime}\right]$ does not have a clique of size $k$, then there are at least $k^{\prime}-k+1$ pairs of non-adjacent vertices in in $G\left[V^{\prime}\right]$. To see this, notice that we can delete one vertex from each pair of non-adjacent vertices to get a clique. Since $G\left[V^{\prime}\right]$ does not have a clique of size $k$, there should be at least $k^{\prime}-k+1$ pairs of non-adjacent vertices. Then, after the manipulation, from the first three groups $\neg x_{i} \vee \neg x_{j}$, $x_{i} \vee x_{i^{\prime}}$, and $x_{i} \vee x^{*}$, at least

$$
k^{\prime} d+\left(k^{\prime}-k+1\right)+n=k^{\prime}(d+1)+(n-k+1)
$$

of them will be changed. On the other hand, $k^{\prime}(d+1)$ of $x_{i} \vee y_{i}$ and $n-k+1$ of $w \vee z_{i}$ will be changed. Thus, the Hamming distance does not change, which is a contradiction.

[^3]
### 4.4.2 Horn Clauses of Length $\ell=2$

For Horn clauses, we have conclusions of the form $\neg x_{i} \vee x_{j}$. This allows variables from $P_{1}^{1}$ to be "useful". To see this, consider the following example where the third judge is the manipulator and $J=\left\{\neg x_{1} \vee x_{2}\right\}$ is the desired set.

| Judgment Set | $x_{1}$ | $x_{2}$ |  |
| :--- | :--- | :--- | :--- |
|  | $\neg x_{1} \vee x_{2}$ |  |  |
| $J_{1}$ | 1 | 0 | 0 |
| $J_{2}$ | 0 | 0 | 1 |
| $J_{3}$ | 1 | 1 |  |
| UPQR $_{1 / 2}$ | 1 | 0 | 1 |

Changing the value of $x_{1} \in P_{1}^{1}$ from 1 to 0 can make $\neg x_{1} \vee x_{2} \in J \backslash \mathrm{UPQR}_{q}(\boldsymbol{J})$ included in the outcome. Thus, for Horn clauses of length $\ell=2$ we have a new kind of useful variables, which allows to encode hard problems.

Theorem 6. UPQR-HD-Manipulation with Horn clauses of length $\ell=2$ is NP-hard.
Proof. We adopt the reduction in the proof for Theorem 5 with some modifications such that all clauses in the modified reduction are Horn clauses. We copy the whole judgment profile, except for conclusions $\neg x_{i} \vee \neg x_{j}(1 \leq i<j \leq n)$, which are not allowed in Horn clause. To overcome this issue, we create a new variable $x_{i}^{\prime} \in P_{1}^{1}$ for every variable $x_{i} \in P_{0}^{0}(1 \leq i \leq n)$, and add conclusions $x_{i}{ }^{\prime} \vee x_{j}{ }^{\prime}$ for all $1 \leq i<j \leq n$. Notice that here $x_{i}{ }^{\prime} \vee x_{j}{ }^{\prime}$ will play the role of $\neg x_{i} \vee \neg x_{j}$ in the original reduction. That is, changing $x_{i}{ }^{\prime}$ or $x_{j}^{\prime}$ alone is not enough to change the result of $x_{i}{ }^{\prime} \vee x_{j}{ }^{\prime}$, and changing the value of both $x_{i}{ }^{\prime}$ and $x_{j}{ }^{\prime}$ can change the result of $x_{i}{ }^{\prime} \vee x_{j}{ }^{\prime}$.

It remains to show that we can use a gadget to enforce that any successful manipulation has to either choose both of $x_{i}$ and $x_{i}{ }^{\prime}$, or choose none of them, for every $1 \leq i \leq n$. To this end, for each pair of $x_{i}$ and $x_{i}{ }^{\prime}$, we add new variables $e_{i}^{j} \in P_{0}^{0}(1 \leq j \leq N)$ and $f_{i}^{j} \in P_{0}^{1}\left(1 \leq j \leq \frac{N}{2}\right)$, where $N$ is a large even number whose value will be determined later. In addition, we add clauses $x_{i} \vee e_{i}^{j}, x_{i}{ }^{\prime} \vee e_{i}^{j}$ for all $1 \leq j \leq N$, and $x_{i} \vee t_{j}, \neg x_{i}{ }^{\prime} \vee t_{j}$ for all $1 \leq j \leq \frac{N}{2}$. See Table 7 for the judgment sets of the gadget for the pair of $x_{i}$ and $x_{i}{ }^{\prime}$ and see Figure 4 for an illustration. Note that $x_{i}, x_{i}{ }^{\prime}$ and all $e_{i}^{j}(1 \leq j \leq N)$ are decided by the manipulator.


Figure 4: Gadget for the pair of $x_{i}$ and $x_{i}{ }^{\prime}$.

Table 7: Gadget for the pair of $x_{i}$ and $x_{i}{ }^{\prime}$ in the proof of Theorem 6.

| Judgment Set | $x_{i}$ | $x_{i}{ }^{\prime}$ | $e_{i}^{j}$ | $f_{i}^{j}$ |  | $x_{i} \vee e_{i}^{j}$ | $x_{i}{ }^{\prime} \vee e_{i}^{j}$ | $x_{i} \vee f_{i}^{j}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $J_{1}$ | 1 | 1 | 1 | 0 |  | $\neg x_{i}{ }^{\prime} \vee f_{i}^{j}$ |  |  |
| $J_{2}$ | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 |
| $J_{3}$ | 0 | 1 | 0 | 1 |  | 0 | 1 | 0 |
| $\mathrm{UPQR}_{1 / 2}$ | 0 | 1 | 0 | 0 | $\Rightarrow$ | 0 | 1 | 1 |

Now we show the correctness of this gadget. That is, any successful manipulation has to change both $x_{i}$ and $x_{i}{ }^{\prime}$ or none of them, and for the first case, there exists a way to manipulate within the gadget such that within the gadget there is no influence on the Hamming distance. If we change $x_{i}, x_{i}{ }^{\prime}$ and all $e_{i}^{j}(1 \leq j \leq N)$, then all $N$ conclusions from $\left\{x_{i} \vee f_{i}^{j}, \neg x_{i}{ }^{\prime} \vee f_{i}^{j} \mid 1 \leq j \leq\right.$ $\left.\frac{N}{2}\right\}$ will be newly included into the outcome, and on the other hand, $N$ conclusions from $\left\{x_{i} \vee e_{i}^{j} \mid\right.$ $1 \leq j \leq N\}$ will also be newly included, while all conclusions from $\left\{x_{i}{ }^{\prime} \vee e_{i}^{j} \mid 1 \leq j \leq N\right\}$ remain to be included. Therefore, the influence on the Hamming distance within the gadget is $N-N=0$. In other words, there exists a way to manipulate within the gadget such that we change both $x_{i}$ and $x_{i}{ }^{\prime}$, and there is no influence on the Hamming distance. However, if we just change only one of $x_{i}$ and $x_{i}{ }^{\prime}$, then we get only $\frac{N}{2}$ conclusions from $\left\{x_{i} \vee f_{i}^{j}, \neg x_{i}{ }^{\prime} \vee f_{i}^{j} \left\lvert\, 1 \leq j \leq \frac{N}{2}\right.\right\}$, but lose $N$ conclusions from $\left\{x_{i} \vee e_{i}^{j}, x_{i}{ }^{\prime} \vee e_{i}^{j} \mid 1 \leq j \leq N\right\}$ no matter how we change variables in $\left\{e_{i}^{j} \mid 1 \leq j \leq N\right\}$. We can choose the value for $N$ to be large enough such that the difference $\frac{N}{2}$ in this gadget can not be reverted by any choice of other variables.

## 5 Bribery

In this section, we consider two variants of BRIBERY introduced by Baumeister et al. [4], where an external agent pays some judges to change their judgment sets so that the outcome changes towards some desired judgment set.

## UPQR-Bribery (resp. UPQR-Microbribery)

Input: An agenda $\Phi$, a profile $\boldsymbol{J}=\left(J_{1}, \ldots, J_{n}\right) \in \mathcal{J}(\Phi)^{n}$, a (possibly incomplete) consistent judgment set $J$ desired by the briber, a budget $k \in \mathbb{Z}^{+}$, and a uniform rational threshold $q \in$ $[0,1)$.
Question: Is it possible to change up to $k$ individual judgment sets in $\boldsymbol{J}$ (resp. $k$ premise entries in $\boldsymbol{J})$ such that for the resulting new profile $\boldsymbol{J}^{\prime}$ it holds that $\operatorname{HD}\left(\mathrm{UPQR}_{q}\left(\boldsymbol{J}^{\prime}\right), J\right)<$ $\operatorname{HD}\left(\operatorname{UPQR}_{q}(\boldsymbol{J}), J\right) ?$

Baumeister et al. [4] proved that both UPQR-Bribery and UPQR-Microbribery are NP-hard when conclusions are positive monotone clauses. However, there is no bound on the length of conclusions used in the reductions. We study how the length of conclusions influence the computational complexity of these bribery problems.

### 5.1 UPQR-Bribery

We start with UPQR-Bribery and first consider the case when the budget $k$ is a fixed constant. For this case, we can guess different choices of $k$ judges to bribe. For each choice it is easy to determine the set of variables that can be changed and then the main problem is to determine which subset of variables to change. This is very similar to UPQR-HD-Manipulation where the manipulator needs to find a subset of decision variables to change.

Recall that in Section 4 we show that UPQR-HD-MANIPULATION with positive monotone clauses of length $\ell=2$ is solvable in polynomial time via a reduction to WMDS. Using similar
ideas, we show that for a fixed budget $k$, UPQR-Bribery with the same clause set is also solvable in polynomial time. The main difference is that for UPQR-HD-Manipulation the manipulator cannot decide the outcome of variables from $P_{1}^{0} \cup P_{0}^{1}$ (notations from Section 4.1) and hence just need to consider variables from $P_{0}^{0}$ according to Observation 4, while this restriction does not hold for UPQR-Bribery. Consequently, for UPQR-Bribery we need to consider more cases, as discussed in the following.
Theorem 7. UPQR-Bribery with conclusions being positive monotone clauses of length $\ell=2$ is solvable in polynomial time when the budget $k$ is a fixed constant.
Proof. First of all, for any instance of UPQR-Bribery we can transform it into an equivalent instance such that $J$ containing no premise and $\{\varphi, \neg \varphi\} \cap J \neq \emptyset$ for every conclusion $\varphi \in \Phi_{c}$, using the same method as in Section 4. In the following we can assume that $J$ contains only conclusions and for any conclusion $\varphi \in \Phi_{c}$ we have that $\varphi \in J$ or $\neg \varphi \in J$.

We can try all possible choices of bribing $k$ judges from all $n$ judges and there are $\binom{n}{k}$ different choices. For each choice, we can find the set $D$ of variables whose value can be changed. For any $x \in D$, if all positive clauses containing $x$ are included in the desired set $J$, then it is always better to make $x$ included in the outcome. Otherwise, if there exists a positive clause $\varphi$ with $\neg \varphi \in J$ that contains $x$, then we put $x \in D$ into a set $T_{1}$ if $x \in \operatorname{UPQR}_{q}(\boldsymbol{J})$ and put $x \in D$ into a set $T_{0}$ if $x \notin \operatorname{UPQR}_{q}(\boldsymbol{J})$. We just need to consider variables in $T_{1} \cup T_{0}$.

We first observe that conclusions containing two variables from $T_{1} \cup T_{0}$ are not in the desired set $J$.

Lemma 12. For every conclusion $x_{i} \vee x_{j}$ with $x_{i}, x_{j} \in T_{1} \cup T_{0}$, we have $x_{i} \vee x_{j} \notin J$.
Proof. Let $\bar{J}$ be any complete and consistent extension of $J$. For any variable $x \in T_{1} \cup T_{0}$, we have $x \notin \bar{J}$ since there is a positive monotone clause $\varphi \notin J$ that contains $x$. Thus, $x_{i} \notin \bar{J}$ and $x_{j} \notin \bar{J}$, which imply $x_{i} \vee x_{j} \notin J$.

Next, we consider our problem in the following three cases:

1. $T_{1}=\emptyset$.

In this case we only need to consider variables in $T_{0}$. We classify all conclusions $x_{i} \vee x_{j}$ into three cases according to the number of variables from $T_{0}$ included. If both $x_{i}$ and $x_{j}$ are not in $T_{0}$, then the value of $x_{i} \vee x_{j} \in J$ is fixed and we do not need care about this conclusion. If only one variable, say $x_{i}$, is in $T_{0}$, then the value of the other variable $x_{j}$ is fixed and we can take it as a constant. If $x_{j} \in \operatorname{UPQR}_{q}(\boldsymbol{J})$, then $x_{i} \vee x_{j}$ will be included into the outcome for any $x_{i}$ and hence we do not need care about this conclusion. The interesting case is when $x_{j} \notin \mathrm{UPQR}_{q}(\boldsymbol{J})$, where the result of $x_{i} \vee x_{j}$ depends only on $x_{i}$. If both $x_{i}$ and $x_{j}$ are in $T_{0}$, then we know that $x_{i} \vee x_{j} \notin \mathrm{UPQR}_{q}(\boldsymbol{J})$ and $x_{i} \vee x_{j} \notin J$ (from Lemma 121). In this case changing any one variable will make $x_{i} \vee x_{j}$ included into the outcome, which is bad. To sum up, for each variable $x \in T_{0}$, we need to consider conclusions $x_{i} \vee x_{j}$ with $x_{j} \notin T_{0} \cup \mathrm{UPQR}_{q}(\boldsymbol{J})$ or $x_{j} \in T_{0}$. We can assign each variable $x_{i} \in T_{0}$ a weight to record the difference between the number of newly included desired conclusions and the number of lost desired conclusions if we change the value of $x_{i}$ as follows:

$$
\begin{aligned}
w\left(x_{i}\right)= & \mid\left\{x_{i} \vee x_{j} \mid x_{j} \notin T_{0} \cup \operatorname{UPQR}_{q}(\boldsymbol{J}) \text { and } x_{i} \vee x_{j} \in J\right\} \mid \\
& -\mid\left\{x_{i} \vee x_{j} \mid x_{j} \notin T_{0} \cup \operatorname{UPQR}_{q}(\boldsymbol{J}) \text { and } x_{i} \vee x_{j} \notin J\right\} \mid \\
& -\left|\left\{x_{i} \vee x_{j} \mid x_{j} \in T_{0}\right\}\right|
\end{aligned}
$$

For any subset $T \subseteq T_{0}$, let $f(T)$ be the difference between the number of newly included desired conclusions and the number of lost desired conclusions after changing the values of variables in $T$, then

$$
f(T)=\sum_{x_{i} \in T} w\left(x_{i}\right)+g(T),
$$

where $g(T)=\mid\left\{x_{i} \vee x_{j} \mid x_{i} \in T\right.$ and $\left.x_{j} \in T\right\} \mid$ is the number of lost desired conclusions counted twice in both $w\left(x_{i}\right)$ and $w\left(x_{j}\right)$. Our goal is to find a subset $T \subseteq T_{0}$ such that $f(T)>0$, that is,

$$
\frac{g(T)}{-\sum_{x_{i} \in T} w\left(x_{i}\right)}>0
$$

Using the same method as in the proof of Theorem 3, we can reduce this problem to WMDS, which is solvable in polynomial time. Note that in WMDS we do not require non-negativity for vertex weights.
2. $T_{0}=\emptyset$.

In this case we only need to consider variables in $T_{1}$. Similar to the first case when $T_{0}=\emptyset$, we can assign each variable $x_{i} \in T_{1}$ a weight to record the difference between the number of newly included desired conclusions and the number of lost desired conclusions if we change the value of $x_{i}$ as follows:

$$
\begin{aligned}
w\left(x_{i}\right)= & \mid\left\{x_{i} \vee x_{j} \mid x_{j} \notin T_{1} \cup \operatorname{UPQR}_{q}(\boldsymbol{J}) \text { and } x_{i} \vee x_{j} \notin J\right\} \mid \\
& -\mid\left\{x_{i} \vee x_{j} \mid x_{j} \notin T_{1} \cup \operatorname{UPQR}_{q}(\boldsymbol{J}) \text { and } x_{i} \vee x_{j} \in J\right\} \mid
\end{aligned}
$$

Note that here we do not count conclusions $x_{i} \vee x_{j}$ with $x_{j} \in T_{1}$, since for this kind of conclusions, we need to change both $x_{i}$ and $x_{j}$ to be 0 to change the outcome of $x_{i} \vee x_{j} \in$ $\operatorname{UPQR}_{q}(\boldsymbol{J}) \backslash J$. Instead, we count this part for any subset $T \subseteq T_{1}$ as $g(T)=\mid\left\{x_{i} \vee x_{j} \mid\right.$ $x_{i} \in T$ and $\left.x_{j} \in T\right\} \mid$. Then, $f(T)$, the difference between the number of newly included desired conclusions and the number of lost desired conclusions after changing the values of variables in $T$, can be defined as follows:

$$
f(T)=\sum_{x_{i} \in T} w\left(x_{i}\right)+g(T)
$$

Our goal is to find a $T$ such that $f(T)>0$, that is,

$$
\frac{g(T)}{-\sum_{x_{i} \in T} w\left(x_{i}\right)}>0
$$

Again, using the same method as in the proof of Theorem 3, we can reduce this problem to WMDS, which is solvable in polynomial time.
3. $T_{1} \neq \emptyset$ and $T_{0} \neq \emptyset$.

We first consider two restricted subproblems where we only change variables from $T_{0}$ or $T_{1}$. These two subproblems are just the two cases considered above, which are solvable in polynomial time. If we can succeed in one of them, then we are done. Otherwise, there is no feasible bribery if we only change variables from $T_{0}$ or $T_{1}$. We claim that in this case there is no feasible bribery.
Suppose, towards a contradiction, that we can decrease the Hamming distance by $d>0$ through changing the value of variables in a set $T \subseteq T_{1} \cup T_{0}$, then $T \cap T_{0} \neq \emptyset$ and $T \cap T_{1} \neq \emptyset$. Denote $d_{0}$ the decreased Hamming distance if we only change the value of variables in $T \cap T_{0}$, and $d_{1}$ the decreased Hamming distance if we only change the value of variables in $T \cap T_{1}$. Since both two subproblems have no solution, we have $d_{0} \leq 0$ and $d_{1} \leq 0$. Now we consider the relation between $d, d_{0}$ and $d_{1}$. To this end, we just need to consider conclusions in $Q=\left\{x_{i} \vee x_{j} \mid x_{i} \in T \cap T_{0}, x_{j} \in T \cap T_{1}\right\}$, for which we have the following observations:
(a) $x_{i} \vee x_{j} \in \mathrm{UPQR}_{q}(\boldsymbol{J})$ since $x_{j} \in T_{1}$ implies $x_{j} \in \mathrm{UPQR}_{q}(\boldsymbol{J})$.
(b) $x_{i} \vee x_{j} \notin J$ by $x_{i}, x_{j} \in T_{0} \cup T_{1}$ and Lemma 12,

Table 8: Profiles of variables in manipulation (left) and bribery (right).

| Judgment Set | $x \in D$ | $x \in V^{+}$ | $x \in V^{-}$ | Judgment Set | $x \in D$ | $x \in V^{+}$ | $x \in V^{-}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $J_{1}$ | 1 | 1 | 0 | $J_{1}$ | 1 | 1 | 0 |
| $J_{2}$ | 0 | 1 | 0 | $J_{2}$ | 0 | 1 | 0 |
| $J_{3}$ | 0/1 | 0 | 1 | $J_{3}$ | 0/1 | 1 | 0 |
| $\mathrm{UPQR}_{1 / 2}$ | 0/1 | 1 | 0 | $\mathrm{UPQR}_{1 / 2}$ | 0/1 | 1 | 0 |
| $J_{M}$ | 0/1 | 0 | 1 | $J_{B}$ | 0/1 | 0 | 1 |

(c) Changing $x_{i}$ alone from 0 to 1 cannot change the value of $x_{i} \vee x_{j}$ since it is still 1 .
(d) Changing $x_{j}$ alone from 1 to 0 will make $x_{i} \vee x_{j}=0 \vee 0=0$.
(e) Changing both $x_{i}$ and $x_{j}$ cannot change the value of $x_{i} \vee x_{j}$ since it is still 1 .

From (a) and (b) we know that change the value of $x_{i} \vee x_{j} \in Q$ from 1 to 0 can decrease the Hamming distance by 1. From (c) and (e) we know that conclusions from $Q$ has no influence on $d_{0}$ and $d$, while from (d) we know that these $|Q|$ conclusions are counted in $d_{1}$. Thus, $d=d_{0}+\left(d_{1}-n_{\delta}\right) \leq 0$. This is a contradiction.

Next, we consider more general clause classes with bounded length. We present a simple linear-time reduction from UPQR-HD-MANIPULATION to UPQR-Bribery, which allows to transfer the hardness results for UPQR-HD-MAnIPULATION established in Section 4 to UPQRBribery.

Lemma 13. For any clause set $\mathcal{C}$, there is linear-time many-one reduction from UPQR-HDMANIPULATION with conclusions chosen from $\mathcal{C}$ to UPQR-BRIBERY with conclusions chosen from $\mathcal{C}$ and the budget $k=1$.

Proof. For an instance of UPQR-HD-MAnIPULATION with conclusions chosen from $\mathcal{C}$, let $\boldsymbol{J}_{\boldsymbol{M}}$ be its profile and $J_{M}=J_{n}$ be the desired set of the manipulator. We first compute the set $D$ of variables decided by the manipulator. Then we partition all variables into three sets: $D$, $V^{+}$and $V^{-}$, where $V^{+}$is set of variables which are in $\operatorname{UPQR}_{q}\left(\boldsymbol{J}_{\boldsymbol{M}}\right)$ and not decided by the manipulator, and $V^{-}$is set of variables which are not in $\mathrm{UPQR}_{q}\left(\boldsymbol{J}_{\boldsymbol{M}}\right)$ and not decided by the manipulator.

Now we construct an instance of UPQR-BRIBERY with conclusions chosen from $\mathcal{C}$ as follows. The agenda remains the same and the desired set for the briber is $J_{B}=J_{M}$. The profile $\boldsymbol{J}_{\boldsymbol{B}}$ for the instance of UPQR-BRIBERY is the same as $\boldsymbol{J}_{\boldsymbol{M}}$ for all variables in $D$. While for each variable $x \in V^{+}, x$ is contained in every judgment set, and for each variable $x \in V^{-}, x$ is not contained in any judgment set. Finally, we set $k=1$. See Table 8 for the two profiles for variables in $\boldsymbol{J}_{\boldsymbol{M}}$ and $\boldsymbol{J}_{\boldsymbol{B}}$ in an example with three judges.

With budget $k=1$ the briber can only change the values of variables in $D$, the same as the manipulator. In addition, we have $\operatorname{UPQR}_{q}\left(\boldsymbol{J}_{\boldsymbol{M}}\right)=\mathrm{UPQR}_{q}\left(\boldsymbol{J}_{\boldsymbol{B}}\right)$ and $J_{M}=J_{B}$. Therefore the instance of Manipulation is a yes-instance if and only if the instance of Bribery is a yes-instance.

Combining Lemma 13 with Theorems 4 to 6, we get the following NP-hard results for UPQRBribery when conclusions have length 2 or 3 . Note that all reductions in the proofs for Theorems 4 to 6 have only three judges.
Corollary 3. UPQR-BRIBERY with conclusions being Horn clauses of length $\ell=2$ (resp. monotone clauses of length 2 or positive monotone of length $\ell \geq 3$ ) is $N P$-hard, even when there are only 3 judges and the budget $k=1$.

Table 9: Instance with only positive monotone clauses of length $\ell=2$ in the proof of Theorem 8 .

| Judgment Set | $x_{1}$ | $\ldots$ | $x_{n}$ | $y_{1}$ | $y_{2}$ |  | $x_{i} \vee y_{1}$ | $x_{i} \vee y_{2}$ | $x_{1} \vee x_{j}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $J_{1}$ |  |  |  | 0 | 0 |  |  |  |  |
| $\ldots$ |  | $L$ |  | $\ldots$ | $\ldots$ |  |  | $\star$ |  |
| $J_{m}$ |  |  |  | 0 | 0 |  |  |  |  |
| $\mathrm{UPQR}_{1 / 2}$ | 0 | $\ldots$ | 0 | 0 | 0 | $\Rightarrow$ | 0 | 0 | 0 |
| $J$ | 0 | $\ldots$ | 0 | 1 | 1 |  | 1 | 1 | 0 |

Theorem7 and Corollary 3 show that when the budget $k$ if a fixed constant, UPQR-Bribery and UPQR-HD-MANIPULATION have the same computational complexity results for many different clause classes. Next we show that different from UPQR-HD-Manipulation, the complexity of UPQR-Bribery could also come from the choice of $k$ judges to bribe: When the budget $k$ is not fixed, UPQR-Bribery becomes NP-hard even for the most basic case when conclusions are positive monotone clauses of length $\ell=2$. We show this result by a reduction from Optimal-Lobbying defined as follows, which is NP-hard and W[2]-hard with respect to the budget $k$ 12].

Optimal-LobBying
Input: An $m \times n(0,1)$ matrix $L$ and a positive integer $k$.
Question: Is there a choice of $k$ rows in $L$ such that by changing the entries of these rows each column of the resulting matrix has a strict majority of ones.

Theorem 8. UPQR-BRIBERY with conclusions being positive monotone clauses of length $\ell=2$ is NP-hard and W[2]-hard with respect to the budget $k$.

Proof. Let $(L, k)$ be an instance of Optimal-Lobbying. Without loss of generality, we can assume that each column of $L$ has a strict majority of zeros and $k \leq\left\lfloor\frac{m}{2}\right\rfloor$. We construct an instance of UPQR-Bribery with $q=\frac{1}{2}$ as follows (see also Table 9). The agenda contains variables $x_{i}(1 \leq i \leq n), y_{1}, y_{2}$ and their negations as the premise set. The conclusion set consists of $x_{i} \vee y_{1}, x_{i} \vee y_{2}(1 \leq i \leq n), x_{1} \vee x_{j}(2 \leq j \leq n)$ and their negations. There are $m$ judges and their judgment set with respect to $x_{i}(1 \leq i \leq n)$ are based on $L$, i.e., $x_{i} \in J_{j}$ if and only if $L_{i, j}=1$. Premises $y_{1}$ and $y_{2}$ are not contained in any judgment set. The briber's desired set $J$, which is complete, is shown in the last row of Table 9. Note that since $k \leq\left\lfloor\frac{m}{2}\right\rfloor$, the briber could only change the values of variables of $x_{i}(1 \leq i \leq n)$.

Now we show that instance $(L, k)$ is a yes-instance of Optimal-LobBYing if and only if the constructed instance is a yes-instance of UPQR-BRIBERY.
$\Rightarrow$ Assume that instance $(L, k)$ is a yes-instance, then the briber can bribe the corresponding $k$ judges such that the values of all $x_{i}(1 \leq i \leq n)$ become 1 . Consequently, all $x_{i} \vee y_{1}, x_{i} \vee y_{2}$ $(1 \leq i \leq n)$ will be included into the outcome, and meanwhile all $x_{i}(1 \leq i \leq n)$ and $x_{i} \vee x_{j}$ $(2 \leq j \leq n)$ will also be included into the outcome. So the Hamming distance will be decreased by $2 n-(2 n-1)=1$ and this is successful bribery.
$\Leftarrow$ Assume that the bribery is feasible. Let $S \subseteq\left\{x_{1}, \ldots, x_{n}\right\}$ be the set of changed variables in a successful bribery. Our goal is to show that $S=\left\{x_{1}, \ldots, x_{n}\right\}$, which would imply that instance $(L, k)$ is a yes-instance. Suppose towards a contradiction that $S \neq\left\{x_{1}, \ldots, x_{n}\right\}$. Then, on one hand, desired conclusions in $\left\{x_{i} \vee y_{1}, x_{i} \vee y_{2} \mid x_{i} \in S\right\}$ are included into the outcome and $\left|\left\{x_{i} \vee y_{1}, x_{i} \vee y_{2} \mid x_{i} \in S\right\}\right|=2|S|$. On the other hand, all variables in $S$ and at least $|S|$ conclusions in $\left\{x_{1} \vee x_{j} \mid 1 \leq j \leq n\right\}$ (that is, $\left\{x_{1} \vee x_{j} \mid x_{j} \in S\right\}$ if $x_{1} \notin S$ and $\left\{x_{1} \vee x_{j} \mid 2 \leq\right.$ $j \leq n\}$ if $x_{1} \in S$ ) will also be included into the outcome. Thus, the Hamming distance will be

Table 10: Instance of UPQR-Microbribery with three judges and positive monotone clauses of length $s=2$ in the proof of Theorem 9 .

| Judgment Set | $x_{i}$ | $x^{*}$ | $y_{i}$ | $y^{*}$ |  | $x_{i} \vee x_{j}$ | $x_{i} \vee x^{*}$ | $x_{i} \vee y_{i}$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $J^{*} \vee y^{*}$ |  |  |  |  |  |  |  |  |
| $J_{i}(1 \leq i \leq m+1)$ | 1 | 1 | 0 | 0 |  | 1 | 1 | 1 |
| $J_{i}(m+2 \leq i \leq 2 m+1)$ | 0 | 0 | 0 | 0 |  | 0 | 0 | 0 |
| $\mathrm{UPQR}_{1 / 2}$ | 1 | 1 | 0 | 0 | $\Rightarrow$ | 1 | 1 | 1 |
| $J$ | 0 | 0 | 1 | 1 |  | 0 | 0 | 1 |

increased by at least $2|S|-2|S|=0$ after this bribery, which contradicts with that the bribery is successful.

### 5.2 UPQR-Microbribery

Finally, we consider UPQR-Microbribery, where the budget $k$ is on the number of changes in premises. Different from UPQR-Bribery, if we restrict the budget $k$ to be a fixed constant, then UPQR-MICROBRIBERY in general is solvable in polynomial time since we can simply try all possible $k$ premises to change. When $k$ is not fixed, we show that UPQR-Microbribery is NP-hard even for the most basic case when all conclusions are positive monotone clauses of length $\ell=2$, which is the same as UPQR-Bribery (see Theorem [8).

Theorem 9. UPQR-MICrobribery with conclusions being positive monotone clauses of length $\ell=2$ is NP-hard and W[1]-hard with respect to budget $k$, even if the desired set is complete.

Proof. We present a reduction from Clique. Given an instance $(G=(V, E), s)$ of Clique, we construct an instance of UPQR-Microbribery with three judges and $q=\frac{1}{2}$ as follows. Denote $n=|V|$. The agenda contains $x_{i}(1 \leq i \leq n), y_{i}(1 \leq i \leq n), x^{*}, y^{*}$ and their negations as the premise set. The conclusion set consists of the following clauses and their negations (Without loss of generality we assume $s$ is an even number).

- $x_{i} \vee x_{j}$ for every edge $\left\{v_{i}, v_{j}\right\} \in E$.
- $x_{i} \vee x^{*}$ for each $1 \leq i \leq n$
- $\frac{s}{2}$ copies of $x_{i} \vee y_{j}$ for each $1 \leq i \leq n$.
- $\frac{s}{2}-1$ copies of $x^{*} \vee y^{*}$.

Note that the number of copies of $x^{*} \vee y^{*}$ is one smaller than the number of copies of $x_{i} \vee y_{j}$. This will be essential to create a feasible bribery such that the Hamming distance is decreased by one. The judgment sets of all $2 m+1(m=s+1)$ judges and the desired set $J$ of the briber are shown in Table 10. Finally, we set the budget of the briber to be $k=s+1$. Note that the briber cannot change the value of $y_{i}$ or $y^{*}$, since the briber needs to change $m+1=s+2>k$ entries to change any one of them. To change the value of $x_{i}$ or $x^{*}$, the briber needs to change 1 entry. We show that there is a clique of size $s$ in $G$ if and only if there is a successful bribery with at most $k=s+1$ microbribes.
$\Rightarrow$ Assume there is a clique of size $s$ in $G$, then the manipulator can change the values of the corresponding $s$ variables from $\left\{x_{1}, \ldots, x_{n}\right\}$ and $x^{*}$. One one hand, $\binom{s}{2}$ of $\neg\left(x_{i} \vee x_{j}\right)$ and $s$ of $\neg\left(x_{i} \vee x^{*}\right)$ will be included in the outcome. So the number of newly included desired formulas in the outcome is

$$
n_{1}=\binom{s}{2}+s=\binom{s+1}{2} .
$$

One the other hand, $\frac{s}{2} s$ of $x_{i} \vee y_{j}$ and $\frac{s}{2}-1$ of $x^{*} \vee y^{*}$ will be excluded from the outcome. So the number of lost desired formulas in the outcome is

$$
n_{2}=\frac{s}{2} s+\frac{s}{2}-1=\binom{s+1}{2}-1
$$

Since $n_{1}>n_{2}$, this is a successful bribery.
$\Leftarrow$ Assume there is a successful bribery with at most $k$ microbribes. Denote $n_{1}$ the number of newly included desired formulas, and $n_{2}$ the number of excluded desired formulas after a successful bribery. Suppose $s^{\prime} \leq k$ variables from $\left\{x_{1}, \ldots, x_{n}\right\}$ are changed after this bribery. Then $p \leq\binom{ s^{\prime}}{2}$ of $x_{i} \vee x_{j}$ are changed.

If $x^{*}$ is not changed, then

$$
n_{1}=p \leq\binom{ s^{\prime}}{2}
$$

On the other hand, $\frac{s}{2} s^{\prime}$ of $x_{i} \vee y_{j}$ are changed. So

$$
n_{2}=\frac{s}{2} s^{\prime} .
$$

Since $n_{1}>n_{2}$, we have $s^{\prime}-1>s=k-1$, which contradicts with $s^{\prime} \leq k$.
Therefore, $x^{*}$ has to be changed and hence $s^{\prime} \leq s$. Now $p \leq\binom{ s^{\prime}}{2}$ of $x_{i} \vee x_{j}$ and $s^{\prime}$ of $x_{i} \vee x^{*}$ are changed, so

$$
n_{1}=p+s^{\prime} .
$$

On the other hand, $\frac{s}{2} s^{\prime}$ of $x_{i} \vee y_{j}$ and $\frac{s}{2}-1$ of $x^{*} \vee y^{*}$ are changed. So

$$
n_{2}=\frac{s}{2} s^{\prime}+\frac{s}{2}-1 \geq \frac{s\left(s^{\prime}+1\right)}{2}-1 .
$$

Since $n_{1}>n_{2}$, we have

$$
p>\frac{s\left(s^{\prime}+1\right)}{2}-s^{\prime}-1 \Rightarrow p \geq \frac{s\left(s^{\prime}+1\right)}{2}-s^{\prime} .
$$

Since $s^{\prime} \leq s$ and $p \leq\binom{ s^{\prime}}{2}$, we have $s^{\prime}=s$ and $p=\binom{s^{\prime}}{2}$, which means the corresponding $s^{\prime}=s$ vertices form a clique in $G$.

Notice that in both Theorem 8 for UPQR-Bribery and Theorem 9 for UPQR-Microbribery, the number of judges in the reductions is not bounded. However, different from Theorem 8 , where the complexity comes from the choice of $k$ judges to bribe, the complexity in Theorem 9 comes from the choices of variables to change. In particular, in Theorem 9 the main role of the unbounded number of judges is to enforce that some variables ( $y_{i}, z_{i}$ ) cannot be changed by the briber. Then a natural question is what happens if we restrict the number of judges to be constant in UPQR-Microbribery.

Baumeister et al. [4, Theorem 15] claims that UPQR-Microbribery is NP-hard when the number of judges is a fixed constant. However, their proof seems to be wrong. In their proof they present a reduction from Dominate Set to UPQR-Microbribery with all conclusions being positive monotone clauses, and the correctness of this reduction relies on a wrong assumption that in the constructed instance of UPQR-Microbribery the budget (corresponding to the size of the dominate set) is smaller than the number of judges (which is constant).

In the remaining part of this section, we consider UPQR-MICrobribery with constant number of judges. It turns out that the reduction in Theorem 9 can be easily adapted to this case if the desired set of the briber is incomplete.
Theorem 10. UPQR-MIcrobribery with conclusions being positive monotone clauses of length $\ell=2$ is NP-hard and W[1]-hard with respect to budget $k$, even if there are only three judges.

Table 11: Instance of UPQR-Microbribery with three judges and positive monotone clauses of length $s=2$ in the proof of Theorem (10.

| Judgment Set | $x_{i}$ | $x^{*}$ | $y_{i}$ | $y^{*}$ |  | $x_{i} \vee x_{j}$ | $x_{i} \vee x^{*}$ | $x_{i} \vee y_{i}$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :--- | :--- | :--- |
| $x^{*} \vee y^{*}$ |  |  |  |  |  |  |  |  |
| $J_{1}$ | 1 | 1 | 0 | 0 |  | 1 | 1 | 1 |
| $J_{2}$ | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 1 |
| $J_{3}$ | 0 | 0 | 0 | 0 |  | 0 | 0 | 0 |
| $\mathrm{UPQR}_{1 / 2}$ | 1 | 1 | 0 | 0 | $\Rightarrow$ | 1 | 1 | 1 |
| $J$ |  |  |  |  |  | 0 | 0 | 1 |

Proof. We present a reduction from Clique. The construction is the same as the proof of Theorem 9 except that now we have only three judges and the profile is given in Table 11, where $J$ is incomplete and consistent (as it can be satisfied by setting $y^{*}=0$ and $y_{i}=0$ ). We still set the budget of the briber to be $k=s+1$. Note that to change the value of $x^{*}$ or $x_{i}(1 \leq i \leq n)$ the briber need to change one entry, while for $y^{*}$ or $y_{i}(1 \leq i \leq n)$ the briber need to change two entries. We show that there is a clique of size $s$ in $G$ if and only if there is a successful bribery with at most $k=s+1$ changes. The "only if" direction is the same as before and we just consider the "if" direction.
$\Leftarrow$ Assume there is a successful bribery with at most $k$ microbribes. Denote $n_{1}$ the number of newly included desired formulas, and $n_{2}$ the number of excluded desired formulas after a successful bribery. Suppose $s^{\prime} \leq k$ variables from $\left\{x_{1}, \ldots, x_{n}\right\}$ are changed after this bribery. Then $p \leq\binom{ s^{\prime}}{2}$ of $x_{i} \vee x_{j}$ are changed. Note that $s^{\prime}>0$ since changing $x^{*}$ alone cannot change the outcome of $x_{i} \vee x_{j}$ or $x_{i} \vee x^{*}$.

If $x^{*}$ is not changed, then

$$
n_{1}=p \leq\binom{ s^{\prime}}{2}
$$

On the other hand, with the remaining budget $k-s^{\prime}$ the briber can change at most $\left\lfloor\frac{k-s^{\prime}}{2}\right\rfloor$ of $y_{i}$, then at least $\frac{s}{2} s^{\prime}-\left\lfloor\frac{k-s^{\prime}}{2}\right\rfloor$ of $x_{i} \vee y_{j}$ are changed. So

$$
n_{2} \geq \frac{s}{2} s^{\prime}-\left\lfloor\frac{k-s^{\prime}}{2}\right\rfloor .
$$

Since $n_{1}>n_{2}$, we have $\binom{s^{\prime}}{2}>\frac{s}{2} s^{\prime}-\left\lfloor\frac{k-s^{\prime}}{2}\right\rfloor$. By some computation we get $\left(s^{\prime}-s-1\right)\left(\frac{s^{\prime}-1}{2}\right)>0$, which contradicts with $1 \leq s^{\prime} \leq k=s+1$.

Therefore, $x^{*}$ has to be changed and hence $s^{\prime} \leq s$. Now $p \leq\binom{ s^{\prime}}{2}$ of $x_{i} \vee x_{j}$ and $s^{\prime}$ of $x_{i} \vee x^{*}$ are changed, so

$$
n_{1}=p+s^{\prime} .
$$

On the other hand, with the remaining budget $k-s^{\prime}-1=s-s^{\prime}$ the briber can change at most $\left\lfloor\frac{s-s^{\prime}}{2}\right\rfloor$ of $y_{i}$, then at least $\frac{s}{2} s^{\prime}-\left\lfloor\frac{s-s^{\prime}}{2}\right\rfloor$ of $x_{i} \vee y_{j}$ and $\frac{s}{2}-1$ of $x^{*} \vee y^{*}$ are changed. So

$$
n_{2} \geq \frac{s}{2} s^{\prime}-\left\lfloor\frac{s-s^{\prime}}{2}\right\rfloor+\frac{s}{2}-1 \geq \frac{s^{\prime}(s+1)}{2}-1 .
$$

Since $n_{1}>n_{2}$, we have

$$
p>\frac{s^{\prime}(s+1)}{2}-s^{\prime}-1 \Rightarrow p \geq \frac{s^{\prime}(s-1)}{2} .
$$

Since $p \leq\binom{ s^{\prime}}{2}$, we have $s=s^{\prime}$ and $p=\binom{s^{\prime}}{2}$, which means the corresponding $s^{\prime}=s$ vertices form a clique in $G$.

Theorems 9 and 10 show that for positive monotone clauses of length $\ell=2$, UPQRMicrobribery is NP-hard even if the desired set is complete or there are only three judges. We complement these two results by showing that when the desired set is complete and there are only constant number of judges, UPQR-Microbribery becomes solvable in polynomial time for positive monotone clauses (of unbounded length).

Proposition 2. UPQR-Microbribery with conclusions being positive monotone clauses is solvable in polynomial time when the number of judges is a fixed constant and the desired set is complete.

Proof. Let $m$ be the number of judges that is fixed a constant. Without loss of generality, we can assume $k \geq m$, since otherwise we can simply try all possible $k$ premises in polynomial time. Consequently, with budget $k$ the briber is able to change the outcome of every variable. Let $J$ be the desired set of the briber. Denote $J_{c}^{+} \subseteq J$ the set of desired positive conclusions and $J_{c}^{-} \subseteq J$ the set of desired negative conclusions. If there is a variable $x \in J$ that is not in the outcome, then changing the value of $x$ from 0 to 1 is a successful bribery since then $x$ will be included into the outcome and all conclusions containing $x$ are from $J_{c}^{+}$. Thus, we can assume all variables in $J$ are already in the outcome, which then implies that all conclusions in $J_{c}^{+}$are already in the outcome. Now if there is a variable $x$ such that $\neg x \in J$ and $x$ is in the outcome, then changing the value of $x$ from 1 to 0 is a successful bribery since then $\neg x$ will be included into the outcome, all conclusions in $J_{c}^{+}$are still in the outcome, and conclusions from $J_{c}^{-}$that are in the original outcome are still in the outcome. The remaining case is that all premises in $J$ are already included in the outcome, for which there is no feasible bribery, because the Hamming distance between the outcome and desired judgement set is already zero.

## 6 Conclusion and Discussion

This paper provides a refined picture in terms of the computational complexity of different variants of Manipulation and Bribery in judgment aggregation. Our results for basic variants of Manipulation are summarized in Table 12, UPQR-Robustness-Manipulation and UPQR-Possible-Manipulation are easy to be manipulated as long as all conclusions are clauses, while the computational complexity of UPQR-Necessary-Manipulation and UPQR-Exact-Manipulation with conclusions chosen from a standard-form clause set $\mathcal{C}$ is the same as the computational complexity of the corresponding $\mathcal{C}$-SAT.

The results for UPQR-HD-Manipulation, UPQR-Bribery and UPQR-Microbribery are summarized in Table 13. For UPQR-HD-Manipulation, we show that NP-hardness holds even if all conclusions are positive monotone clauses with length $\ell=3$ but that the problem becomes solvable in polynomial time when $\ell=2$. For monotone or Horn clauses with $\ell=2$, the problem is also NP-hard which is in stark contrast to all basic variants of Manipulation that remain polynomial-time solvable for Horn and positive monotone clauses of arbitrary length. For Bribery, we show that both UPQR-Bribery and UPQR-Microbribery remain NPhard even when all conclusions are positive monotone clauses of length 2. Specifically, UPQRBribery with conclusions being positive monotone clauses of length 2 is W [2]-hard with respect to the number $k$ of judges that can be bribed, and is solvable in polynomial time when $k$ is fixed.

All variants of Manipulation and Bribery we considered were known to be generally NPhard, which was seen and sold as "barrier against manipulative behavior" (4). The main message of this work is that several basic variants of Manipulation can be solved efficiently for simple but well-motivated restrictions of conclusions (e.g. Horn clauses and generalizations thereof) whereas other variants remain computationally intractable for most restrictions. Hence, our results question whether there really is a barrier against strategic behavior in case of realistically simple formulas.

Table 12: Computational complexity of basic variants of Manipulation.

| UPQR- $M$-MANIPULATIon $M=$ | Possible /Robustness | Necessary / Exact |
| :---: | :---: | :---: |
| no restriction | NP-h [4] | NP-h [4] |
| standard-form clause set $\mathcal{C}$ | P (Lem. 3) | $\mathcal{C - S a t ~ ( T h m . ~} 1$ \& 2) |
| clauses with length $\ell \leq 3$ | P (Lem. 3) | NP-h (Cor. 2 ) |
| monotone clauses | P (Lem. 3) | NP-h (Cor. 2 ) |
| Horn clauses | P (Lem. 3) | P (Cor. 1) |
| positive monotone clauses | P (Lem. 3) | P (Cor. 1) |

Table 13: Computational complexity of UPQR-HD-MANIPULATION, UPQR-Bribery and UPQR-Microbribery. UPQR-Bribery with unbounded $k$ is NP-hard even for positive monotone clauses with $\ell=2$.

|  | Manipulation | Bribery (fixed $k$ ) | Microbribery |
| :--- | :--- | :--- | :--- |
| positive monotone with $\ell=2$ | P (Thm. (3) | P (Thm. [7) | NP-h (Thm. (9) |
| positive monotone with $\ell=3$ | NP-h (Thm. 44) | NP-h (Cor. [3) | NP-h (Thm. (9) |
| Horn or monotone with $\ell=2$ | NP-h (Thm. [5 \& 6) | NP-h (Cor. 3) | NP-h (Thm. [9) |
| positive monotone | NP-h [4] | NP-h [4] | NP-h [4] |

We see our results as an important step and expect further effects decreasing the computational complexity by considering other realistic structural properties. Possible next steps include a systematic investigation of the parameterized complexity for both judgment aggregationspecific parameters (e.g. "number of judges" or "size of the desired set") and formula specific parameters (e.g. "number of clauses" or "variable frequency"). We note that considering the parameter "number of judges" alone, however, will not lead to tractable cases because this parameter is fixed to three in most of our reductions.

Another direction is to extend our polynomial-time solvable results to more expressive formulas. In this paper, we restrict conclusions to be clauses (disjunctions of literals), under which some basic variants of Manipulation become solvable in polynomial time. Recall that all of our results can be directly translated to the case where we restrict conclusions to be conjunctions of literals (see Section 2.3). Based on these results, one can consider more expressive restrictions, like Horn formulas (conjunctions of Horn clauses) or Krom formulas (conjunctions of clauses of length 2), to explore the boundary of tractability.

In this paper we consider the two Hamming distance based variants of Bribery introduced by Baumeister et al. [4]. One can also define other variants of Bribery (or Microbribery) similar to basic variants of Manipulation, e.g., possible or necessary, and study their computational complexity using our approach. Note that the reduction from UPQR-HD-Manipulation to UPQR-Bribery in Lemma 13 can be directly generalized to other variants of Manipulation and Bribery, thus our hardness results for different variants of Manipulation under different clause restrictions can be easily adapted to the corresponding variants of Bribery. On the other hand, for clause restrictions under which Manipulation is polynomial-time solvable, Bribery could still be NP-hard when the budget is not fixed since the complexity could also come from the choice of different judges to bribe, similar to the case in Theorem 8. We leave this for future work.

Finally, it is interesting to apply our refined approach to Control in judgment aggregation based on the results of Baumeister et al. 7] or to other judgment aggregation procedures (e.g.,

Kemeny procedure [14]). Furthermore, it seems natural to extend the study to strategic behavior of groups of judges instead of a single judge [9]. Note that most NP-hard variants of Control studied in Baumeister et al. 7] rely on complex formulas, similar to the case in Baumeister et al. [4]. We believe many of our NP-hardness results can be extended to different variants of Control. While for the Kemeny procedure, under which even the outcome determination is usually at least NP-hard, it seems difficult to transfer our results there.

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## A Missing Proofs

Lemma 14. For any $k \geq 3$, $\left(\mathcal{M}_{k}^{+} \cup \mathcal{M}_{2}^{-}\right)$-SAT and $\left(\mathcal{M}_{2}^{+} \cup \mathcal{M}_{k}^{-}\right)$-SAT are $N P$-hard.
Proof. Since $\left(\mathcal{M}_{k}^{+} \cup \mathcal{M}_{2}^{-}\right)$-SAT and $\left(\mathcal{M}_{2}^{+} \cup \mathcal{M}_{k}^{-}\right)$-SAT are equivalent under linear-time reductions, it suffices to show one of them is NP-hard. We present a polynomial-time reduction from $k$-Coloring to $\left(\mathcal{M}_{k}^{+} \cup \mathcal{M}_{2}^{-}\right)$-SAT. Given a graph $G=(V, E)$, we construct a formula of $\left(\mathcal{M}_{k}^{+} \cup \mathcal{M}_{2}^{-}\right)$-SAT as follows. For each vertex $v_{i} \in V$, we add vertex clauses $x_{i}^{1} \vee \cdots \vee x_{i}^{k}$ and $\neg x_{i}^{s} \vee \neg x_{i}^{t}$ for each pair of $s$ and $t$ with $1 \leq s<t \leq k$ into the formula. In addition, for each edge $\left\{v_{i}, v_{j}\right\} \in E$, we add edge clauses $\neg x_{i}^{s} \vee \neg x_{j}^{s}$ for every $1 \leq s \leq k$ into the formula.

Suppose the graph $G$ is $k$-colorable, then we construct an assignment of variables by setting $x_{i}^{s}=1$ if vertex $v_{i}$ is colored $s$. This assignment makes all vertex clauses satisfied since for each vertex $v_{i}$ exactly one variable from $\left\{x_{i}^{1}, x_{i}^{2}, \ldots, x_{i}^{k}\right\}$ is set True. This assignment also makes all edge clauses satisfied since for each edge $\left\{v_{i}, v_{j}\right\} \in E, v_{i}$ and $v_{j}$ have different colors, which means the corresponding two variables $x_{i}^{s}$ and $x_{j}^{t}$ that are set True must satisfy that $s \neq t$.

Suppose there is a satisfying assignment for the formula. For each vertex $v_{i}$, since $x_{i}^{1} \vee \cdots \vee x_{i}^{k}$ is satisfied, at least one variable from $\left\{x_{i}^{1}, x_{i}^{2}, \ldots, x_{i}^{k}\right\}$ is set True. In addition, since all vertex clauses $\neg x_{i}^{s} \vee \neg x_{i}^{t}$ with $1 \leq s<t \leq k$ are satisfied, exactly one variable $x_{i}^{t}$ from $\left\{x_{i}^{1}, x_{i}^{2}, \ldots, x_{i}^{k}\right\}$ is set True. Accordingly, we color $v_{i}$ by $t$. Since all edge clauses are satisfied, we have that for each edge, its two end points have different colors. Therefore, we get a proper $k$-coloring for the graph $G$.

## B Symmetric Version of Lemma 7

Recall that Lemma $\mathbb{1}$ shows that our hardness reductions can be adapted to work for any rational quota $q \in[0,1)$, but from the perspective of parameterized complexity analysis it leaves the following two special cases: (1) instances with high quotas and small numbers of judges where a variable is accepted if all judges accept it and (2) instances with low quotas and small numbers of judges where a variable is accepted if at least one judge accepts it. We now discuss how to adapt our hardness reductions for case (1), and the method for case (2) is analogous.

For case (1), we can still create decision variables with $x \in J_{n} \cap \operatorname{UPQR}_{q}(\boldsymbol{J})$ or $\neg x \in J_{n} \cap$ $\mathrm{UPQR}_{q}(\boldsymbol{J})$ (by setting $x \in J_{i}$ for every $i \in\{1,2, \ldots, n-1\}$ ), while for non-decision variables, $x \in J_{n} \backslash \operatorname{UPQR}_{q}(\boldsymbol{J})$ is possible but $x \in \operatorname{UPQR}_{q}(\boldsymbol{J}) \backslash J_{n}$ is impossible. In other words, the only difference is that, in this case, we cannot create variables that are included in the truthful outcome and cannot be changed by the manipulator. Therefore, all of our hardness reductions that do not use this kind of non-decision variables still work for case (1). The only exceptions are Lemma 6 and 8 , for which we can rename each variable by its negation such that for nondecision variables only $x \in J_{n} \backslash \mathrm{UPQR}_{q}(\boldsymbol{J})$ is used in the reductions. Note that this negation will also change the type of clauses, so after this change we actually show the symmetric versions of Lemma 6 and 8 . Then to finish the proof of Theorem 1, we also need to show the following symmetric version of Lemma 7 for the special case where a variable is accepted if all judges accept it.

Lemma 15. When $\lfloor q n+1\rfloor=n$, UPQR-Necessary-Manipulation with conclusions chosen from $\left(\mathcal{M}_{3}^{+} \cup \mathcal{M}_{2}^{-}\right)$is $N P$-hard.

Proof. We adapt the original reduction in the proof of Lemma 7 to prove the claimed result. The constructed instance is shown in Table 14. We first replace each variable by its negation such that all conclusions are from $\mathcal{M}_{3}^{+} \cup \mathcal{M}_{2}^{-}$. In addition, we make all $x_{i}, y_{i}, z_{i}, w, v$ decided by the manipulator by making them accepted by all other judges. Note that since $\lfloor q n+1\rfloor=n$, a variable is accepted if all judges accept it. Recall that the main idea of the original reduction for Lemma 7 is that since variable $v$ is not decided by the manipulator and $v$ is not included

Table 14: Instance of UPQR-NECESSARY-MANIPULATION with conclusion set $\mathcal{C}=\mathcal{M}_{3}^{+} \cup \mathcal{M}_{2}^{-}$ for the proof of Lemma 15.

| Judgment Set | $x_{i}$ | $y_{i}$ | $z_{i}$ | $w$ | $v$ | $u_{1}$ | $u_{2}$ | $\neg w \vee \neg v$ | $v \vee u_{1} \vee u_{2}$ | $x_{i_{1}} \vee x_{i_{2}} \vee x_{i_{3}}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $J_{i}(i \neq n)$ | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 |
| $J_{n}$ | 1 | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 1 | 1 |
| $\mathrm{UPQR}_{1 / 2}$ | 1 | 0 | 0 | 1 | 0 | 0 | 0 | $\Rightarrow$ | 1 | 0 |
| Judgment Set | $\neg z_{i_{1}} \vee \neg z_{i_{2}}$ | $x_{i} \vee y_{i} \vee w$ | $y_{i} \vee z_{i} \vee w$ | $\neg x_{i} \vee \neg y_{i}$ | $\neg y_{i} \vee \neg z_{i}$ |  |  |  |  |  |
| $J_{i}(i \neq n)$ | 0 |  | 1 |  | 1 | 0 | 0 |  |  |  |
| $J_{n}$ | 1 |  | 1 |  | 1 | 1 | 1 |  |  |  |
| $\mathrm{UPQR}_{1 / 2}$ | 1 |  | 1 |  | 1 | 1 | 1 |  |  |  |

in the truthful outcome, the manipulator has to make $w$ included such that the outcome will include conclusion $w \vee v$. For the new instance with $\lfloor q n+1\rfloor=n$ it is impossible to create a variable that is included in the truthful outcome but not decided by the manipulator. However, we can achieve a similar effect in the new instance by adding two new variables $u_{1}, u_{2}$ that are not decided by the manipulator and a new conclusion $v \vee u_{1} \vee u_{2}$ (as shown in Table 14). The desired set $J$ of the manipulator consists of all positive conclusions. Now $v \vee u_{1} \vee u_{2}$ is the only positive conclusion in $J$ that is not included into the outcome, and to make $v \vee u_{1} \vee u_{2}$ included, the manipulator has to make $v$ included, which then enforces the manipulator to make $\neg w$ included due to $\neg w \vee \neg v$. Once $\neg w$ is included into the outcome, the manipulator has to solve an instance of $\left(\mathcal{M}_{3}^{+} \cup \mathcal{M}_{2}^{-}\right)$-SAT (which is NP-hard according to Lemma 14), and the analysis is analogous to the proof of Lemma 7.


[^0]:    ${ }^{1}$ This implies that the agenda is closed under propositional variables, that is, if $\phi$ is a formula in the agenda, then so is every propositional variable occurring within $\phi$.

[^1]:    ${ }^{2}$ For convenience, when we say conclusions are from a clause set $\mathcal{C}$, we mean every conclusion is either a clause in $\mathcal{C}$ or the negation of a clause in $\mathcal{C}$.

[^2]:    ${ }^{3}$ Similar to before, we put the threshold value $q$ as part of the input and this has no influence on the computational complexity due to Lemma 1

[^3]:    ${ }^{4}$ A very formal interpretation of our model definition with the agenda being a set (not a multiset) may not allow copies. In this case, and later in Theorem 9, instead of copies, we can also introduce a separate fresh $y_{i}$ or $y^{*}$ variable for each clause. Since these variables cannot be changed, the proof does not change.

