Residuated implications derived from quasi-overlap functions on lattices

Rui Paiva^a, Benjamín Bedregal^b, Regivan Santiago^b

^aInstituto Federal de Educação, Ciência e Tecnologia do Ceará Maracanaú, Brazil Email: rui.brasileiro@ifce.edu.br ^bDepartamento de Informática e Matemática Aplicada Universidade Federal do Rio Grande do Norte Natal, Brazil Email: {bedregal,regivan}@dimap.ufrn.br

Abstract

In this paper, we introduce the concept of residuated implications derived from quasi-overlap functions on lattices and prove some related properties. In addition, we formalized the residuation principle for the case of quasi-overlap functions on lattices and their respective induced implications, as well as revealing that the class of quasi-overlap functions that fulfill the residuation principle is the same class of continuous functions according to topology of Scott. Also, Scott's continuity and the notion of densely ordered posets are used to generalize a classification theorem for residuated quasi-overlap functions. Finally, the concept of automorphisms are extended to the context of quasi-overlap functions over lattices, taking these lattices into account as topological spaces, with a view to obtaining quasi-overlap functions conjugated by the action of automorphisms.

Keywords: Quasi-overlap functions, Scott topology, Residuated implications, Residuation principle, Lattices

1. Introduction

Overlap functions were introduced by Bustince et al. [4] as a class of aggregate functions with two entries over the interval [0, 1] to be applied to the image processing field. Basically, these functions transform pixel images with values at [0, 1]. Many researchers have began to develop the theory of overlap functions to explore their potentialities in different scenarios, such as problems involving classification or decision making [12, 6, 19, 22, 23, 24, 26] and from the theoretical

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point of view [1, 8, 9, 21, 27]. However, when you consider that pixels (or signs) may contain uncertainties, for example noise, this noise information can be captured on objects that extend real numbers, for example intervals, fuzzy numbers or interval-valued fuzzy sets, intuitionistic fuzzy sets and soft sets, which offer different perspectives for the structures containing the uncertainties. In this case, the notion overlap needed to be extended to handle this types of objects. In this perspective, in [21] the authors generalized the notion of overlap to the context of lattices and introduced a weaker definition, called a quasi-overlap, that arises from the removal of the continuity condition. To this end, the main properties of (quasi-) overlaps over bounded lattices, namely: convex sum, migrativity, homogeneity, idempotency, and cancellation law was investigated, as well as an overlap characterization of Archimedian overlap functions was presented.

In this paper, we propose a theoretical framework of order theory and topology with a view to establishing a connection between the notion of convergence in terms of order and Scott's topology, to obtain a pair of residuated applications, namely: (O, I_O) , where O is a quasi-overlap function and I_O is an induced implication of O. We proved that the class of quasi-overlap functions that fulfill the residuation principle is the same class of Scott-continuous quasi-overlap functions. Also, Scott's continuity and the notion of densely ordered posets are used to generalize a classification theorem for residuated quasi-overlap functions. Finally, the concept of automorphisms are extended to the context of quasi-overlap functions over lattices, taking these lattices into account as topological spaces, with a view to obtaining quasi-overlap functions conjugated by the action of automorphisms.

To this end, the Section 2 presents an interaction between order theory and topology. The directed complete posets class (DCPO's) and the filtered complete posets class (FCPO's) as well as the lattice class are briefly explored. This section also shows the notion of convergence via nets (a generalization of sequences for general topological spaces.). Finally, an overview of Galois connections and residuated mappings is recalled. In Section 3, we present some results investigated on residuated implications induced by fuzzy conjunctions that extend overlap functions to any lattice and some properties that these implications satisfy are presented. This section also shows how the notion of dense order coincides with the concept of density of topological spaces in Scott's topology. In section 4, presents a definition that generalizes automorphisms of bounded lattices, taking these lattices as topological spaces and the class of quasi-overlap functions is closed under Ω -automorphisms, where Ω represents, in this context, Scott's topology. In addition, some immediate properties of the action of Scott-automorphisms

on quasi-overlap functions are explored. Finally, Section 5 gives some final remarks.

2. Preliminaries

2.1. Partial orders

In this subsection we will review some results of order theory, the branch of mathematics that deals among other things with order relations. For more details we recommend [7, 11, 13, 17, 25].

Definition 2.1. Let $\langle X, \leq \rangle$ be a poset. A subset D of X is called directed if D is not empty and $\forall u, v \in D$, $\exists w \in D$ such that $u \leq w$ and $v \leq w$. On the other hand, a subset F of X is called dual directed or co-directed or filtered if F is not empty and $\forall u, v \in F$, $\exists w \in F$ such that $w \leq u$ and $w \leq v$.

Remark 2.1. Since one usually can work on the dual order explicitly, notions of directed set and filtered set satisfy the principle of duality.

In what follows, it is easy to prove the

Lemma 2.1. Let $\langle X, \leq \rangle$ be a poset. The following are valid:

- (*i*) A non-empty chain in X is directed and filtered;
- (ii) For any $x \in X$, the set $\downarrow x = \{y \in X \mid y \leq x\}$ is directed and $\sup \downarrow x = x$;
- (iii) For any $x \in X$, the set $\uparrow x = \{y \in X \mid x \leq y\}$ is filtered and $\inf \uparrow x = x$;
- (iv) In a finite poset X, a subset of X has maximal element \top if, and only if, it *is directed;*
- (v) In a finite poset X, a subset of X has minimal element \perp if, and only if, it is filtered.

Remark 2.2. The sets $\downarrow x$ and $\uparrow x$ are known in the literature by principal ideal generated by x and principal filter generated by x respectively.

Definition 2.2 ([7], p. 587). A poset $\langle X, \leq \rangle$ is called a complete partial order with respect to directed sets (DCPO), if any directed subset of X has supremum in X. Dually, a poset $\langle X, \leq \rangle$ is called a complete partial order with respect to filtered sets (FCPO), if any filtered subset of X has infimum in X. **Remark 2.3.** We remember that every poset $\langle X, \leq \rangle$ in which any two elements $x, y \in X$ have infimum and supremum, denoted respectively by $x \wedge y$ and $x \vee y$, is called lattice. We also remember that a lattice is said to be complete, if for every non-empty subset Y of X, Y has infimum and supremum in X. Thus, every complete lattice is a DCPO and a FCPO.

Definition 2.3 (Order-density of posets). Let $\langle X, \leq \rangle$ be a poset. A subset Y of X is order dense in X if for any elements $x, y \in X$ satisfying the condition x < y, there exists an element z in Y such that x < z < y. If X himself has order dense then $\langle X, \leq \rangle$ has order dense.

2.2. Scott topology

This subsection will discuss important issues for the development of this paper. It is assumed that the reader is familiar with some elementary notions of general topology, such as the notions of topological spaces, open and closed sets, the basis of a topology, as well as the separation axioms. Some of the results presented are well known in the literature, however, For more details we suggest [11, 15, 18].

Definition 2.4 (Scott's open sets). Let $\langle X, \leq \rangle$ be a DCPO and $A \subseteq X$. The set A says a Scott open if it satisfies:

- (i) If $x \in A$ and $x \leq y$ then $y \in A$;
- (ii) If $D \subseteq X$ is a set directed and $\sup D \in A$ then $D \cap A \neq \emptyset$.

Proposition 2.1 ([11], Remark II-1.4). Let $\langle X, \leq \rangle$ be a DCPO and consider the set

$$\sigma(X) = \{ A \subseteq X \mid A \text{ is a Scott open set} \}.$$

Under these conditions, $\sigma(X)$ is a topology on X. Moreover, $\langle X, \sigma(X) \rangle$ is a topological T_0^{-1} space.

Remark 2.4. $\sigma(X)$ it's well-known Scott's topology on X.

We recall that the notation $\langle X, \mathcal{T}, \leq \rangle$ is used to denote a set X endowed with a topology \mathcal{T} and a order relation " \leq " on X. Such a structure is called *ordered topolo-gical space*. From now on, every DCPO will be considered an ordered topological space, in which the topology considered is Scott's topology.

¹ A topological space X is a T_0 space or Kolmogorov space if, for any two different points x and y there is an open set which contains one of these points and not the other.

Proposition 2.2 ([11], Proposition II-2.1). Given a function $f : X \to Y$, where X and Y are DCPO's. The following conditions are equivalents:

- (i) f is continuous with respect to Scott's topology: $f^{-1}(V) \in \sigma(X)$, for all $V \in \sigma(Y)$;
- (ii) f preserves supremum of directed sets, that is, f preserves order and satisfies: $f(\sup D) = \sup f(D)$, for every directed set D of X.

Similarly, the dual theorem can be demonstrated.

Proposition 2.3. Given a function $f : X \to Y$, where X and Y are FCPO's. The following conditions are equivalents:

- (i) f is continuous with respect to Scott's topology: $f^{-1}(V) \in \sigma(X)$, for all $V \in \sigma(Y)$;
- (ii) f preserves infimum of filtered sets, that is, $f(\inf F) = \inf f(F)$, for every set filtered F of X.

Therefore, considering the Remark 2.3 and the Propositions 2.2 and 2.3, the next result is quickly obtained.

Theorem 2.1. Given a function $f : X \to Y$, where X and Y are complete lattices. The following conditions are equivalents:

- (i) f is continuous with respect to Scott's topology: $f^{-1}(V) \in \sigma(X)$, for all $V \in \sigma(Y)$;
- (ii) f preserves supremum of directed sets and infimum of filtered sets: f preserves order and satisfies: $f(\sup D) = \sup f(D)$ and $f(\inf F) = \inf f(F)$, for every directed set D and every set filtered F, both subsets of X.

2.3. Scott-convergence

An interesting concept in topological spaces is that of convergence, as well as the properties of boundary points. In this section we will discuss these and other subject in terms of nets. For more details we suggest [11, 15]. **Definition 2.5.** A net in a set X is a function $j \mapsto x_j : J \to X$ whose domain is a join-semilattice². Nets are also denoted by $(x_j)_{j \in J}$, by (x_j) or x_j , whenever the context is clear. If the set X is provided with an order, then the net x_j is called monotonic if $i \leq j$ implies $x_i \leq x_j$. A subnet of (x_j) is any net of the form $(x_{\psi(i)})_{i \in I}$, where I is a directed set and there is an application $\psi : I \to J$ such that for each $j \in J$ there is eventually $\psi(i) \geq j$ in I.

In the following definition, it is verified that the convergence of nets is a natural generalization of the convergence of sequences.

Definition 2.6. A net $(x_j)_{j \in J}$ in a topological space X converges to $x \in X$ (notation $x_j \to x$) if, whenever $U \subseteq X$ is open and $x \in U$, so there is a $i \in J$ such that $x_j \in U$ for all $j \ge i$.

Remark 2.5. Each subnet of a net that converges to a point (relative to a topological space) converges to the same limit. [15, Affirmation b, p. 74]

We still recall that given a topological space X, a collection $\mathcal{A} = \{A_{\lambda}\}_{\lambda \in I}$ of subsets of X is called a covering of X, when $X \subseteq \bigcup_{\lambda \in I} A_{\lambda}$. We say \mathcal{A} is an open (closed) cover of X when all elements of the cover are opened (closed). A topological space X is called *compact* when all open covering of X has a finite subcollection that covers it. We say that Y is a compact subset of X if Y, with the topology induced by X, is a compact topological space.

We now present a well-known characterization of topological space compactness via nets.

Proposition 2.4 ([15], Theorem 2, p. 136). A topological space X is compact if and only if each net in X admits a subnet converging to a point of X.

In general, a net in a topological space X can converge to several different points. For example, consider the two element set $\{a, b\}$ with topology $\{\emptyset, \{b\}, \{a, b\}\}$. Then every net, that converges to a also converges to b and the net, which is constant b converges only to b. However, the following proposition points out spaces in which the convergence is unique in the sense that if a net s_n converges to s and also to a point t, then s = t. Before, we recall some useful

²We recall that J is a join-semilattice if J is a partially ordered set that has a join (a least upper bound) for any nonempty finite subset of J.

notions. By a neighbourhood of a subset A (in particular of a singleton and therefore of a point) in a topological space X, we mean a subset of X that contains an open set containing A. We say that a topological space X is *Hausdorff* if any two distinct points of X have disjoint neighbourhoods.

Proposition 2.5 ([15], Theorem 3, p. 67). A topological space X is a Hausdorff space if and only if each net in X converges to at most one point.

In view of proposition 2.5 above, the notion of lower bound and upper bound for nets is given below. This is a particular case of Definition II-1.1 in [11], for the case where X is a complete lattice.

Definition 2.7. Given a complete lattice X and a net $(x_i)_{i \in J}$ in X, the lower limit of $(x_i)_{i \in J}$ is:

$$\underline{\lim}_{i \in J} x_i = \sup_{i \in J} \inf_{j \ge i} x_j \tag{1}$$

and its upper limit is:

$$\overline{\lim}_{i\in J} x_i = \inf_{i\in J} \sup_{j\geq i} x_j \tag{2}$$

Let S be the class of those elements $u \in X$ such that $u \leq \underline{\lim}_{i \in J} x_i$ and T be the class of those elements $w \in X$ such that $\overline{\lim}_{i \in J} x_i \leq w$. For each such elements we say that u is a lower S-limit and w is a upper T-limit of $(x_i)_{i \in J}$. In this case we write respectively $u \equiv_S \underline{\lim}_{i \in J} x_i$ and $w \equiv_T \overline{\lim}_{i \in J} x_i$.

Proposition 2.6 ([11], prop. II-2.1). Let X and Y be DCPO's and $f : X \to Y$ a function. The following conditions are equivalent:

1. f preserves suprema of directed sets, i.e. f is order preserving and

$$f(\sup \Delta) = \sup\{f(x) \mid x \in \Delta\}$$
(3)

for all directed subset Δ of X,

2. f is order preserving and

$$f(\underline{\lim}_{i\in J}x_i) \le \underline{\lim}_{i\in J}f(x_i) \tag{4}$$

for any net $(x_i)_{i \in J}$ on X such that $\underline{\lim}_{i \in J} x_i$ and $\underline{\lim}_{i \in J} f(x_i)$ both exist.

Similarly, the dual proposition can be demonstrated.

Proposition 2.7. Let X and Y be FCPO's and $f : X \to Y$ a function. The following conditions are equivalent:

1. f preserves infimum of filtered sets, i.e. f is order preserving and

$$f(\inf \Delta) = \inf\{f(x) \mid x \in \Delta\}$$
(5)

for all filtered subset Δ of X;

2. f is order preserving and

$$f(\overline{\lim}_{i\in J}x_i) \ge \overline{\lim}_{i\in J}f(x_i) \tag{6}$$

for any net $(x_i)_{i \in J}$ on X such that $\overline{\lim}_{i \in J} x_i$ and $\overline{\lim}_{i \in J} f(x_i)$ both exist.

Notice that all complete lattice is a DCPO (FCPO) in which $\underline{\lim}_{i \in J} x_i$ and $\underline{\lim}_{i \in J} f(x_i)$ ($\overline{\lim}_{i \in J} x_i$ and $\overline{\lim}_{i \in J} f(x_i)$) always exist [11]. Hence Propositions 2.6 and 2.7 hold for complete lattices.

Theorem 2.1 establishes a connection between convergence given in terms of lower bound order theory, or liminfs, and Scott's topology. In this perspective, Equations (3) and (5) generalize the notion of continuity of functions on lattices. These facts motivate the following definition.

Definition 2.8. Let X and Y be two complete lattices. A function $f : X \to Y$ is Scott-continuous (simply continuous, if the context is clear) if and only if it satisfies any of the Equations (3) or (5).

Remark 2.6. Note that if X is finite so any function $f : X \to Y$ is continuous because for each directed set Δ of X, $\sup \Delta \in \Delta$ and for each filtered set Δ , $\inf \Delta \in \Delta$.

2.4. Galois connections and the residuation principle

Galois connections generalize the correspondence between subgroups and fields investigated in Galois theory. In order theory, a Galois connection is a particular correspondence between posets and is closely related to the concept of residuated functions. In turn, the residuated functions, besides being important in themselves, have a very relevant role in the characterization of partial orders. In this section we provide the basics necessary for the development of this paper. More details at [2, 3, 5, 10, 20].

Definition 2.9. A monotonic Galois connection from a poset X to a poset Y is a pair (α, β) of monotonic applications $X \xrightarrow{\alpha} Y \xrightarrow{\beta} X$ such that for all $x \in X$ and $y \in Y$, one has that

$$\alpha(x) \leq_Y y \iff x \leq_X \beta(y).$$

The α application is called the lower adjunct while the β application is called the upper adjunct connection:



Figure 1: Galois monotonic connection between the posets X and Y

In the family of monotonic functions between partial orders there is a very important class of functions, called residuated functions.

Definition 2.10. A function $f : X \to Y$ between posets X and Y is said to be residuated if it is monotonic, and in addition there is a monotonic function $g: Y \to X$ such that

$$g \circ f \ge id_X \quad and \quad f \circ g \le id_Y.$$
 (7)

If f is a residuated function, the monotonic function g that satisfies the inequalities in (7) is called *residue* of f and is denoted by f^R . It is easy to see that the residue of a residuated function, when it exists, is unique ([3], p. 7).

Residuated functions, besides being important in themselves, play a fundamental role in the preservation of ideals.

Theorem 2.2 ([3], Theorem 1.3). Let $f : X \to Y$ be an application between posets. The following statements are equivalents:

- (*i*) *f* is residuated;
- (ii) For each principal ideal ↓ w of Y, the set f⁻¹ (↓ w) is a principal ideal of X.

Theorem 2.3 ([10], Lemma 3.2). Let X and Y be posets. An application $f : X \to Y$ is residuated if, and only if, the pair (f, f^R) forms a monotonic Galois connection.

Therefore, whenever f is a residuated function, it is established that the pair (f, f^R) satisfies the residuation principle, or adjunct principle, or that it forms a Galois connection. The next result can be found in [5], page 162, in topic 7.33. It is a characterization theorem for residuated functions.

Theorem 2.4 (Characterization of residuated functions). Let $f : X \to Y$ and $g : Y \to X$ be functions any between the posets X and Y. The following statements are equivalents:

- (i) f is residuated and $g = f^R$;
- (ii) For all $x \in X$ and $y \in Y$ one has $x \leq g(y) \Leftrightarrow f(x) \leq y$;
- (iii) f is monotonic and for each $y \in Y$, $g(y) = \max\{x \in X \mid f(x) \le y\}$;
- (iv) g is monotonic and for each $x \in X$, $f(x) = \min\{y \in Y \mid x \le g(y)\}$.

Galois connections are used to describe classes of functions for modeling fuzzy logic connectors. In the next section, the notions of quasi-overlap on lattices, as well as their derived implications, will be investigated.

3. Residuated implications derived from quasi-overlap on lattices

In this section, we present some results investigated on residuated implications induced by conjunctions that extend overlap functions to any lattice. Overlap functions were proposed by Bustince et al. [4] in order to solve the problem of fuzziness on the process of image classification. Initially, overlap functions were defined as continuous functions. Bustince et al. in [4] justify the requirement of continuity by saying that it is considered in order to avoid *O* to be a uninorm, however it is easy to see that if a uninorm is an overlap function, then it is necessarily a t-norm. However, in some contexts, continuity is not an indispensable property, especially when we consider finite lattices. This situation appears in some situations in the field of digital image processing. Considering this, in [21] the authors proposed a more general definition, called of *quasi-overlap*, which arises from the removal of the continuity condition. **Definition 3.1 ([21], Definition 3.2).** Let X be a bounded lattice. A function $O : X^2 \to X$ is called a quasi-overlap function on X (simply quasi-overlap, if the context is clear) if all of following properties hold:

- (OL1) O(x, y) = O(y, x) for all $x, y \in X$;
- (*OL2*) O(x, y) = 0 if and only if x = 0 or y = 0;
- (**OL3**) O(x, y) = 1 if and only if x = y = 1;
- (OL4) O is non-decreasing in each variable, that is

$$x_1 \le x_2 \Rightarrow O(x_1, y) \le O(x_2, y)$$

$$y_1 \le y_2 \Rightarrow O(x, y_1) \le O(x, y_2).$$

Proposition 3.1. A quasi-overlap O is associative if and only if, for any $x, y, z \in X$, it satisfies the exchange principle: O(x, O(y, z)) = O(y, O(x, z)).

Proof. The necessary and the sufficient conditions follows easily from the commutativity of O.

In order to extend the notion of continuity presented in the Definition 2.8 for the context of quasi-overlap functions, the following Definition is considered.

Definition 3.2 ([21], Definition 3.1). Let X be a complete lattice. An Overlap function on X is a quasi-overlap that is Scott-continuous.

In the following, the concept of residuated implications derived from quasioverlap functions on lattices will be introduced.

Definition 3.3. Let X be a bounded lattice. A binary operation $I : X^2 \to X$ is called a implication if it is descending on the first variable, and nondecreasing with respect to the second variable. Moreover, I(0,0) = I(0,1) = I(1,1) = 1 and I(1,0) = 0.

In the following, some properties that implications satisfy are presented.

Definition 3.4. An implication I is said to fulfill:

(NP) Neutral Property: I(1, y) = y, where $y \in X$;

(EP) Exchange Principle: I(x, I(y, z)) = I(y, I(x, z)) where $x, y, z \in X$;

- (IP) Identity Principle: I(x, x) = 1 where $x \in X$;
- (**OP**) Ordering Property: $x \le y \Leftrightarrow I(x, y) = 1$ where $x, y \in X$.

Lemma 3.1. Let X be a complete lattice. Given a quasi-overlap $O : X^2 \to X$, the function defined by formula

$$I_O(x,y) = \sup\{t \in X \mid O(x,t) \le y\}, \, \forall x, y \in X$$
(8)

is nondecreasing with respect to second variable and decreasing with respect to first variable. Moreover, $I_O(0,0) = I_O(0,1) = I_O(1,1) = 1$ and $I_O(1,0) = 0$.

Proof. In fact, first note that the function I_O is well defined. Fix $x, y \in X$ and denote

$$R(x,y) \coloneqq \{t \in X \mid O(x,t) \le y\}.$$
(9)

Since $0 \le O(x, 0) = 0$, it follows that $0 \in R(x, y)$. This means that $R(x, y) \ne \emptyset$ and once X is complete, there is $\sup R(x, y)$ in X. Let $x, y, v \in X$, with $y \le v$. Then

$$\{t\in X\,|\,O(x,t)\leq y\}\subseteq\{t\in X\,|\,O(x,t)\leq v\}$$

and therefore, $\sup\{t \in X | O(x,t) \leq y\} \leq \sup\{t \in X | O(x,t) \leq v\}$, that is, $I_O(x,y) \leq I_O(x,v)$. This means that the function I_O is nondecreasing on the second variable. Now, let $x, u, y \in X$, com $x \leq u$. From the monotonicity of O with respect to the first variable one has that $O(x,t) \leq O(u,t)$, for all $t \in X$. Therefore,

$$\{t \in X \mid O(u,t) \le y\} \subseteq \{t \in X \mid O(x,t) \le y\},\$$

then $\sup\{t \in X | O(u,t) \le y\} \le \sup\{t \in X | O(x,t) \le y\}$ and therefore, $I_O(u,y) \le I_O(x,y)$. Thus I_O is decreasing on the first variable. Moreover, since every quasi-overlap satisfies (**OL2**) and (**OL3**) from Definition 3.1, one has that

- $I_O(0,0) = \sup\{t \in X \mid O(0,t) \le 0\} = 1;$
- $I_O(0,1) = \sup\{t \in X \mid O(0,t) \le 1\} = 1;$
- $I_O(1,1) = \sup\{t \in X \mid O(1,t) \le 1\} = 1;$
- $I_O(1,0) = \sup\{t \in X \mid O(1,t) \le 0\} = 0.$

Definition 3.5. Let O be a quasi-over a complete lattice X. The function I_O defined by (8) is called the implication induced by O.

In order to develop the notion of residuation for quasi-overlap and its induced implications, it is necessary to explore some important facts about a particular class of lattices, namely, the class of dense complete lattices. For this, we present below a sequence of useful results.

First we remember that given a subset $B \subseteq X$ of a topological space X, the interior of B, denoted by int(B), is the largest open set contained in B.

Lemma 3.2. Le $\langle X, \leq \rangle$ be a poset. If $B \subseteq X$, then $int(B) = \{x \in B \mid \uparrow x \subseteq B\}$.

Proof. Let $w \in \{x \in B \mid \uparrow x \subseteq B\}$. Then $\uparrow w \subseteq B$. But $\uparrow w$ is open, hence $\uparrow w \subseteq int(B)$. Thus, $w \in int(B)$. On the other hand, let $y \in int(B) \subseteq B$. Since $int(B) = \uparrow int(B)$, one has that $int(B) \subseteq \uparrow B$. Then, $y \in \{x \in B \mid \uparrow x \subseteq B\}$. \Box

We recall also that a subset S of a topological space X is *dense* in X when its closure \overline{S} coincides with the whole space X. This is equivalent to say that every open non-empty in X contains some point of S, or else, that the complement of S does not have interior points. In order to establish a connection between this topological definition for density and that given by the Definition 2.3 and which is given in terms of order, we present the following assertion.

Theorem 3.1. Let $\langle X, \leq \rangle$ be a DCPO. A set $S \subseteq X$ is order dense in X if, and only if, S is dense in X in the Scott's topology.

Proof. (\Rightarrow) Suppose that the set $S \subseteq X$ is order dense in X and let $x, y \in X$ be such that x < y. Then, the set $D = \{z \in S \mid x < z < y\}$ is non-empty and directed, with $\sup D = y$. On the other hand, $\uparrow x = \{r \in X \mid x \le r\}$ is an open set of X in Scott's topology, since it satisfies the properties (i) and (ii) of Definition 2.4. Therefore, since $\uparrow x \cap D \neq \emptyset$ and $D \subseteq S$, it follows that $\uparrow x \cap S \neq \emptyset$. Therefore, S is dense in X in the Scott's topology.

In that follows, if neither $x \le y$ nor $y \le x$, then x and y are said to be incomparable, which is denoted here as $x \parallel y$.

(\Leftarrow) For each $x \in X$ which satisfies the condition: x < y or $x \parallel y$, for some $y \in X$, define the set $P_x = \{w \in X \mid x < w \text{ or } x \parallel y\}$. It is clear that $P_x \neq \emptyset$. Let's show that P_x is a Scott's open. In fact, the condition (*i*) of Definition 2.4 is trivially satisfied. As for condition (*ii*), suppose there is a directed set $D \subseteq X$

such that $\sup D \in P_x$ and $D \cap P_x = \emptyset$. So, for all $w \in D$ one has that $w \leq x$. Therefore, x is an upper bound for w and $\sup D \leq x$, which contradicts the fact that $\sup D \in P_x$. Therefore, $D \cap P_x \neq \emptyset$ and hence P_x is a Scott's open. Moreover, since S is topologically dense in X, one has that $P_x \cap S \neq \emptyset$ and hence exists $z \in S$ such that x < z < y for all $x, y \in X$ that satisfy the condition x < y. \Box

Proposition 3.2. Let $\langle X, \leq \rangle$ be a poset provided with Scott's topology $\sigma(X)$. For all $y \in X$, the set $\uparrow y$ is compact.

Proof. In fact, let $\mathcal{A} = \{A_{\lambda}\}_{\lambda \in I}$ be an open covering of X. Then, for all $y \in X$, $y \in \bigcup_{\lambda \in I} A_{\lambda}$. Hence, $y \in A_{\lambda_0}$, for some $\lambda_0 \in I$. But A_{λ_0} is Scott's open, then $A_{\lambda_0} = \uparrow A_{\lambda_0}$. Also, as $\uparrow y \subseteq \uparrow A_{\lambda_0}$, it follows that $\uparrow A_{\lambda_0}$ is a finite subcollection of \mathcal{A} . Therefore, since $\uparrow y \subseteq \uparrow A_{\lambda_0}$, it follows that $\uparrow y$ is compact. \Box

In Scott's topology, for complete lattices, compactness is a trivial feature.

Proposition 3.3. If $\langle X, \sigma(X), \leq \rangle$ is a complete lattice equipped with Scott topology, then X is compact.

Proof. let $\mathcal{A} = \{A_{\lambda}\}_{\lambda \in I}$ be an open covering of X. Since X is complete it follows that it is bounded. Let 0 be its bottom element. Then, $0 \in \bigcup_{\lambda \in I} A_{\lambda}$. Hence, $0 \in A_{\lambda_0}$, for some $\lambda_0 \in I$. But A_{λ_0} is Scott's open, then $A_{\lambda_0} = \uparrow A_{\lambda_0}$. Also, as $\uparrow 0 \subseteq \uparrow A_{\lambda_0}$, it follows that $\uparrow A_{\lambda_0}$ is a finite subcollection of \mathcal{A} . Also, since $\uparrow 0 = \{x \in X \mid 0 \leq x\}$, it follows that $X \subseteq \uparrow 0$. Therefore, X is compact. \Box

Therefore, follows from Theorem 3.1 and Proposition 3.3, which every complete dense order lattice is compact and dense in Scott's topology.

Another point that deserves attention is the fact that these spaces must not necessarily have a definite total order about them. Consider the set of subintervals of [0,1] defined as $\mathbb{I}([0,1]) = \{[a,b] \mid a \leq b \text{ and } a, b \in [0,1]\}$, provided with the product order " \leq " defined as follows:

$$[u, v] \preceq [p, q]$$
 if, and only if, $u \leq p$ and $v \leq q$

where " \leq " is the usual order of \mathbb{R} . The next lemma shows that $\langle \mathbb{I}([0,1]), \preceq \rangle$ is a partially ordered complete lattice.

Lemma 3.3. The structure $\langle \mathbb{I}([0,1]), \Box, \bigcup, \mathbf{0}, \mathbf{1} \rangle$, where $[a,b] \Box [c,d] = [a \land c, b \land d]$ and $[a,b] \sqcup [c,d] = [a \lor c, b \lor d]$, for all $a, b, c, d \in [0,1]$, it is a complete lattice with the top element $\mathbf{1} = [1,1]$ and the bottom element $\mathbf{0} = [0,0]$. **Proof.** According to the definition of \sqcap and \sqcup operators, just consider for each $[a, b], [c, d] \in \mathbb{I}([0, 1]), \inf\{[a, b], [c, d]\} = [a, b] \sqcap [c, d]$ and $\sup\{[a, b], [c, d]\} = [a, b] \sqcup [c, d]$. Thus $\langle \mathbb{I}([0, 1]), \sqcap, \sqcup, \mathbf{0}, \mathbf{1} \rangle$ is a lattice. Now consider the non-empty set $X \subseteq \mathbb{I}([0, 1])$. It is obvious that [0, 0] is a lower bound of X, then the set:

$$X^{\ell} = \{J \in \mathbb{I}([0,1]) \mid J \text{ is lower bound of } X\}$$

it is not empty. Define

$$v = \sup_{[p,q] \in X^{\ell}}(p)$$
 and $w = \sup_{[p,q] \in X^{\ell}}(q).$

This implies that [v, w] is lower bound of X. We affirm that [v, w] it is the largest of the lower bounds of X. Indeed, suppose there exists $[r, s] \in X^{\ell}$ such that $[v, w] \sqsubseteq [r, s]$, then $v \le r$ and $w \le s$. On the other hand, by way v and w are defined, we have $v \ge r$ and $w \ge s$. Therefore v = r, w = s and hence $\inf X = [v, w]$. Similarly, since [1, 1] is upper bound for X, define

$$X^u = \{T \in \mathbb{I}([0,1]) \mid T \text{ is upper bound of } X\} \neq \emptyset.$$

and call

$$m = \inf_{[a,b]\in X^u}(a)$$
 and $n = \inf_{[a,b]\in X^u}(b).$

This leads us to conclude that $\sup X = [m, n]$.

In addition, since [0, 1] is dense, it follows that $\mathbb{I}([0, 1])$ is also dense. On the other hand, the interval [0, 1] provided with the usual order of reais is also complete lattice of dense order. Finally, a final point to be discussed on this topic is linked to the issue of convergence, which is clarified in the next lemma.

Lemma 3.4. If X is a complete lattice of dense order, then X is Hausdorff.

Proof. Let $x, y \in X$, so we have the following possibilities:

(i) x and y incomparable: In this case, x, y < sup{x, y}. Then, of the density of X, there are x₀, y₀ ∈ X such that x < x₀ < sup{x, y} and y < y₀ < sup{x, y}. Let's show that the sets B = {w ∈ X | x ≤ w < x₀} and C = {u ∈ X | y ≤ u < y₀} are open non-empty of X in Scott's topology and that B∩C = Ø. Indeed, first note that by definition int(B) ⊆ B always worth it. On the other hand, if b ∈ B ⊆ X, then x ≤ b < x₀ and more, the set ↑ x ∩ B = {r ∈ B | x ≤ r} is an

open of B in Scott's topology induced of X. Therefore, by Lemma 3.2, $b \in int(B) = \{r \in B \mid \uparrow r \subseteq B\}$. That is, B is an open of X that contains x. Similarly it is shown that C = int(C). That is, C is a open of X that contains y. Moreover, if $B \cap C \neq \emptyset$ so there is $\alpha \in X$ such that $x \leq \alpha$ and $y \leq \alpha$, which leads to a contradiction with $\sup\{x, y\}$.

(ii) x and y are comparables: In this case, assume without loss of generality x < y. From the density of X there is $z \in X$ such that x < z < y. Define the sets $M = \{p \in X \mid x \le p < z\}$ and $\uparrow z = \{q \in X \mid z \le q\}$. Notice that M is a open of X that contains x and $\uparrow z$ is a open of X that contains y. Moreover, $M \cap \uparrow z = \emptyset$.

Therefore, it follows that the lattice X is a Hausdorff space.

Remark 3.1. The Lemma 3.4 together with Proposition 2.5 guarantee the uniqueness of convergence of converging nets in a complete lattice of dense order.

Let X be the complete lattice of dense order. For each $x \in X$, define the functions $O_x, I_{O_x} : X \to X$ por $O_x(z) = O(x, z)$ and $I_{O_x}(y) = I_O(x, y)$, for all $y, z \in X$, where O is a quasi-overlap and I_O its induced implication. In what follows, it will be presented under what conditions O_x and I_{O_x} represent a family of residuated functions and their respective family of residues.

Definition 3.6. The pair (O, I_O) is said to satisfy the residuation principle whenever

$$O(x,z) \le y \iff z \le I_O(x,y), \ \forall x,y,z \in X.$$
(10)

The next theorem reveals that the class of quasi-overlap functions that fulfill the residuation principle is the class of continuous functions according to Scott's topology.

Theorem 3.2. Let X be a complete lattice of dense order and O a quasi-overlap over X. So the following items are equivalent:

- (i) O is Scott-continuous;
- (ii) O and I_O satisfy the residuation principle;
- (*iii*) $I_O(x, y) = \max\{t \in X \mid O(x, t) \le y\}.$

Proof. ((i) \Rightarrow (ii)): For any $x, y, z \in X$ suppose that $O(x, z) \leq y$. Then $z \in R(x, y)$ (cf. Equation (9)). Hence, $z \leq \sup R(x, y) = I_O(x, y)$. Now assume that for $x, y, z \in X$ one has $z \leq I_O(x, y)$. If $z < I_O(x, y)$ then, since X is dense order, there is $t_0 \in X$ such that $z < t_0 < I_O(x, y)$ and $O(x, t_0) \leq y$. From the monotonicity of O in each variable one has thate $O(x, z) \leq y$. On the other hand, if $z = I_O(x, y)$, so we have two possibilities:

- (P1) $z \in R(x, y)$: In this case obviously that $O(x, z) \le y$;
- (P2) $z \notin R(x, y)$: In this case, since X is complete and of dense order, then X It is compact and dense in Scott's topology. Thus, by Proposition 2.4, There is a non-decreasing net $(z_j)_{j\in J}$ in X such that $z_j < z$ and, since O is monotonic in the second variable, from the residuation principle it follows that $O(x, z_j) \leq O(x, z) \leq y$, for all $j \in J$. Let's show that $z = \underline{\lim}_{j\in J} z_j$. In fact, let A be a Scott open containing z. Since $\{z_j \in X \mid O(x, z_j) \leq y\}$ is directed (since (z_j) is non-decreasing) and $z = \sup\{z_j \in X \mid O(x, z_j) \leq y\}$, then by item (ii) from Definition 2.4 (Scott's open), it follows $\{z_j \in X \mid O(x, z_j) \leq y\} \cap A \neq \emptyset$. Therefore, for some $i \in J$, we have $x_j \in A$ for all $j \geq i$. Thus, by Definition 2.6, we have $z_j \to z$. That is, $z = \underline{\lim}_{j\in J} z_j$. Finally, since O is Scott-continuous, by Proposition 2.6

$$O(x,z) = O(x,\underline{\lim}_{j\in J} z_j) \le \underline{\lim}_{j\in J} O(x,z_j) \le y.$$
(11)

Therefore, anyway, one has that $O(x, z) \le y$.

((ii) \Rightarrow (iii)): Assume that pair (O, I_O) satisfies the residuation principle. So since $I_O(x, y) \leq I_O(x, y)$ for all $x, y \in X$, it follows that $O(x, I_O(x, y)) \leq y$. This means that $I_O(x, y) \in R(x, y)$ and $\sup R(x, y) = \max R(x, y)$. ((iii) \Rightarrow (i)): Suppose that $I_O(x, y) = \max\{t \in X \mid O(x, t) \leq y\}$ for all $x, y \in Q(x, t) = \max\{t \in X \mid O(x, t) \leq y\}$ for all $x, y \in Q(x, t) = \max\{t \in X \mid O(x, t) \leq y\}$ for all $x, y \in Q(x, t) = \max\{t \in X \mid O(x, t) \leq y\}$ for all $x, y \in Q(x, t) = \max\{t \in X \mid Q(x, t) \leq y\}$ for all $x, y \in Q(x, t) = \max\{t \in X \mid Q(x, t) \leq y\}$ for all $x, y \in Q(x, t) = \max\{t \in X \mid Q(x, t) \leq y\}$ for all $x, y \in Q(x, t) = \max\{t \in X \mid Q(x, t) \leq y\}$ for all $x, y \in Q(x, t) = \max\{t \in X \mid Q(x, t) \leq y\}$ for all $x, y \in Q(x, t) = \max\{t \in X \mid Q(x, t) \leq y\}$ for all $x, y \in Q(x, t) = \max\{t \in X \mid Q(x, t) \leq y\}$ for all $x, y \in Q(x, t) = \max\{t \in X \mid Q(x, t) \leq y\}$ for all $x, y \in Q(x, t) = \max\{t \in X \mid Q(x, t) \leq y\}$ for all $x, y \in Q(x, t) = \max\{t \in X \mid Q(x, t) \leq y\}$ for all $x, y \in Q(x, t) = \max\{t \in X \mid Q(x, t) \leq y\}$ for all $x, y \in Q(x, t) = \max\{t \in X \mid Q(x, t) \leq y\}$ for all $x, y \in Q(x, t) = \max\{t \in X \mid Q(x, t) \leq y\}$ for all $x, y \in Q(x, t) = \max\{t \in Q(x, t) \in Q(x, t) \in Q(x, t) \leq y\}$ for all $x, y \in Q(x, t) = \max\{t \in Q(x, t) \in Q(x, t) \in Q(x, t)\}$ for all $x, y \in Q(x, t) \in Q(x, t) \in Q(x, t)$ for all $x, y \in Q(x, t) \in Q(x, t)$ for all $x, y \in Q(x, t) \in Q(x, t)$ for all $x, y \in Q(x, t) \in Q(x, t)$ for all $x, y \in Q(x, t) \in Q(x, t)$ for all $x, y \in Q(x, t) \in Q(x, t)$ for all $x, y \in Q(x, t)$ for all x, y

X. We must show that $O(x, \sup\{z_j | j \in J\}) = \sup\{O(x, z_j) | j \in J\}$, for each $x \in X$ and for any non-decreasing net $(z_j)_{j \in J}$ in X. On the one hand, the monotonicity of O and by definition of supremum it follows that

$$\sup\{O(x, z_j) \mid j \in J\} \le O\left(x, \sup\{z_j \mid j \in J\}\right)$$
(12)

On the other hand, let $w = \sup\{O(x, z_j) \mid j \in J\}$. Then $O(x, z_j) \leq w$ and so for all $j \in J$, $z_j \in \{t \in X \mid O(x, t) \leq w\}$ and consequently, $z_j \leq I_O(x, w)$ for all

 $j \in J$. Therefore, by monotonicity of O one has

$$O\left(x, \sup\{z_j \mid j \in J\}\right) \le O(x, I_O(x, w)) \le w = \sup\{O(x, z_j) \mid j \in J\}.$$
 (13)

Therefore, from inequalities (12) and (13), it is concluded that O is continuous. \Box

Corollary 3.1. If O is a Quasi-overlap over X and X is order dense, then O and I_O satisfy residuation principle and $I_O(x, y) = \max\{z \mid O(x, z) \le y\}$.

Definition 3.7. The functions O and I_O are respectively called residuated quasioverlap and residuated implication (or R_O -implication), if any of the items in Theorem 3.2 are checked.

In the following, properties that R_O -implications and their residuated quasioverlap satisfy are presented.

Proposition 3.4. Let X be a complete lattice of order dense and O a residuated quasi-overlap over X. Then:

- (i) I_O satisfies (NP) if, and only if, 1 is neutral element of O;
- (ii) I_O satisfies (IP) if, and only if, O is deflationary:

$$O(x,1) \le x, x \in X;$$

(iii) I_O satisfies (**OP**) if, and only if, O have neutral element 1.

Proof. The proof is based on considerations similar to [16]. But adapted to the lattice context. Indeed,

(i) (\Rightarrow) Suppose that for all $y \in X$

$$I_O(1, y) = \max\{t \in X \mid O(1, t) \le y\} = y.$$
(14)

So for an arbitrary $y \in X$ one has $O(1, y) \leq y$. If for some y_0 in X, one has $O(1, y_0) < y_0$, then by density of X exists z such that $z < y_0$ and $O(1, y_0) < z$. According to the residuation principle, $z < y_0 \leq I_O(1, z)$, which contradicts the equation (14).

(\Leftarrow) Suppose that O(1, r) = r, for all $r \in X$. Then

$$I_{O}(1, y) = \max\{t \in X \mid O(1, t) \le y\} = \max\{t \in X \mid t \le y\} = y.$$
(15)

(ii) Just note that for an arbitrary $x \in X$, we have

$$I_O(x,x) = \max\{t \in X \mid O(x,t) \le_X x\} = 1 \Leftrightarrow O(x,1) \le x.$$

(iii) (\Rightarrow) Suppose for each $x, y \in X$, such that x and y are comparables, one has $x \leq y \Leftrightarrow I_O(x, y) = 1$. Then $I_O(x, x) = \max\{t \in X \mid O(x, t) \leq x\} = 1$. This means that $O(x, 1) \leq x$, for all $x \in X$. Moreover, by the monotonicity of O,

$$I_O(x, O(x, 1)) = \max\{t \in X \mid O(x, t) \le O(x, 1)\} = 1.$$

Thus, by (**OP**), $x \le O(x, 1)$. So for an arbitrary $x \in X$, O(x, 1) = x. (\Leftarrow) Suppose *O* has neutral element 1. If for $x, y \in X$,

$$I_O(x, y) = \max\{t \in X \mid O(x, t) \le y\} = 1,$$

then we have $x = O(x, 1) \le y$. On the other hand, if for each $x, y \in X$, if $x \le y$, so since 1 is neutral element of O, one has $O(x, 1) = x \le y$. Therefore, by residuation, it follows that $I_O(x, y) = 1$.

Proposition 3.5. Let X be a complete lattice of order dense and O a residuated quasi-overlap over X. Under these conditions:

- (i) If I_O satisfies (**EP**), O(x, O(y, z)) and O(y, O(x, z)) are comparables for all $x, y, z \in X$, then O is associative;
- (ii) If O is associative, then I_O satisfies (**EP**).

Proof. (i) Assume that I_O fulfills the property of the exchange principle (**EP**). Suppose that there are $x, y, z \in X$ such that $O(x, O(y, z)) \neq O(O(x, y), z)$. Then, by Proposition 3.1 it follows that $O(x, O(y, z)) \neq O(y, O(x, z))$. Thus, by hipoteses, we can assume without loss of generality that O(x, O(y, z)) < O(y, O(x, z)). Applying two times the residuation principle we get

$$z < I_O(y, I_O(x, O(y, O(x, z)))).$$

Using the exchange principle we have $z < I_O(x, I_O(y, O(x, z))))$. Applying the residuation principle again two times we go back to

$$O(y, O(x, z)) < O(y, O(x, z)),$$

which is trivially a contradiction.

(ii) Assume that O is associative. From residuation principle we have

$$I_{O}(x, I_{O}(y, z)) = \max\{t \in X \mid O(x, t) \leq I_{O}(y, z)\}\$$

= $\max\{t \in X \mid O(y, O(x, t)) \leq z\}\$
= $\max\{t \in X \mid O(O(y, x), t) \leq z\}\$
= $\max\{t \in X \mid O(O(x, y), t) \leq z\}\$
= $\max\{t \in X \mid O(x, O(y, t)) \leq z\}\$
= $\max\{t \in X \mid O(y, t) \leq I_{O}(x, z)\}\$
= $I_{O}(y, I_{O}(x, z)).$

4. Quasi-overlap conjugated and their induced implications

We begin this section by presenting a definition that generalizes automorphisms of bounded lattices, taking these lattices as topological spaces.

Definition 4.1. Let X be a bounded lattice and Ω a topology on X. A function $\rho: X \to X$ is a Ω -automorphism if:

- (i) ρ is bijective;
- (ii) ρ is continuous according to topology Ω ;
- (iii) $x \leq y$ if, and only if, $\rho(x) \leq \rho(y)$.

Remark 4.1. The set of all Ω -automorphism of X is denoted by $Aut_{\Omega}(X)$.

Lemma 4.1. The set $Aut_{\Omega}(X)$ of all Ω -automorphism of a bounded lattice X is a group under composition of mappings.

Proof. It is routine to check this proof.

Note that Definition 4.1 generalizes the definition of automorphisms over [0, 1], which implies continuity in the Euclidean topology. In addition, since ρ is a continuous bijection whose inverse ρ^{-1} is also continuous, it follows that ρ is an application known in topology as homeomorphism. It is an application that preserves the topological structure of your space. It should also be noted that ρ (as well as its inverse) can be seen as an order isomorphism.

Furthermore, the next result shows that the class of quasi-overlap functions is closed under Ω -automorphisms, where Ω represents, in this context, Scott's topology, and for this reason, instead of Ω -automorphism the term Scott-automorphism is used.

Proposition 4.1. Let O be a quasi-overlap function and ρ a Scott-automorphism, both defined over a complete lattice X. Then, the conjugated of O, denoted by O^{ρ} , is also quasi-overlap function. Moreover, if O is Scott-continuous, O^{ρ} is also Scott-continuous.

Proof. (**OL1**): It follows directly from the fact that composed of non-decreasing functions is a non-decreasing function.;

(OL2): Immediately follows from the commutativity of O; **(OL3)**: (\Rightarrow) Suppose that $O^{\rho}(x, y) = 0$. Then we have

$$\rho^{-1} \left(O\left(\rho(x), \rho(y)\right) \right) = 0 \quad \Leftrightarrow \quad O\left(\rho(x), \rho(y)\right) = 0$$
$$\Leftrightarrow \quad \rho(x) = 0 \text{ or } \rho(y) = 0$$
$$\Leftrightarrow \quad x = 0 \text{ or } y = 0$$

(\Leftarrow) If x = 0 or y = 0 then, suppose without loss of generality that x = 0. Then, $\rho(x) = 0$ and so,

$$O^{\rho}(x,y) = \rho^{-1} (O (\rho(x), \rho(y)))$$

= $\rho^{-1} (O (0, \rho(y)))$
= $\rho^{-1}(0)$
= 0.

8: (\Rightarrow) Suppose that $O^{\rho}(x, y) = 1$. Then one has that

$$\rho^{-1} \left(O\left(\rho(x), \rho(y)\right) \right) = 1 \quad \Leftrightarrow \quad O\left(\rho(x), \rho(y)\right) = 1$$
$$\Leftrightarrow \quad \rho(x) = \rho(y) = 1$$
$$\Leftrightarrow \quad x = y = 1$$

(\Leftarrow) Suppose that x = y = 1, $\rho(x) = \rho(y) = 1$. Then one has that

$$O^{\rho}(x, y) = \rho^{-1} (O (1, 1))$$

= $\rho^{-1}(1)$
= 1.

The continuity of O^{ρ} follows from the composition of continuous functions. \Box

A first application of the action of Scott-automorphisms on quasi-overlap functions is that the conjugated of an induced implication of a quasi-overlap O coincides with the induced implication of the conjugated O^{ρ} .

Proposition 4.2. I_O^{ρ} coincides with $I_{O^{\rho}}$.

Proof. Indeed,

$$\begin{split} I_{O}^{\rho}(x,y) &= \rho^{-1} \left(I_{O}(\rho(x),\rho(y)) \right) \\ &= \rho^{-1} \left(\sup\{\rho(z) \in X | O(\rho(x),\rho(z)) \leq \rho(y) \} \right) \\ &= \max\{z \in X | \rho^{-1} \left(O(\rho(x),\rho(z)) \right) \leq y \} \\ &= \max\{z \in X | O^{\rho} \left(x,z \right) \leq y \} \\ &= I_{O^{\rho}}(x,y). \end{split}$$

Remark 4.2. The above proposition states that the processes for obtaining conjugated adjunct or adjunct conjugated are invariant, as shown in Figure 2.



Figure 2: Diagram of Adjunct and Conjugation

Another interesting application of quasi-overlap conjugated is linked to the notion of closed operators³. The following is a theoretical framework for obtaining the closure of the conjugated of O and I_O .

Proposition 4.3. Let X be a complete lattice of dense order and O^{ρ} a conjugated of quasi-overlap function O set over X. The following conditions are equivalent:

³Remember that a function $f : X \to X$ over a poset $\langle X, \leq \rangle$ is a closed operator on X if f is non-descreasing, idempotent (f(f(x)) = f(x)), and inflationary $(x \leq f(x))$.

- (i) O^{ρ} is residuated if, and only if, O_{y}^{ρ} and O_{x}^{ρ} are both monotonic and *Scott-continuous;*
- (ii) O^{ρ} is residuated if, and only if, O^{ρ}_{u} and O^{ρ}_{x} are residuated.

Proof. ((i) \Rightarrow (ii)) On the space X define the following partial order relation:

$$(a,b) \leq (u,v) \Leftrightarrow a \leq u \text{ and } b \leq v.$$

It is routine to check that this provides two natural topologies X^2 , namely: Scott's topology in space $\langle X, \leq \rangle$ and Scott's topology product in $\langle X^2, \leq \rangle$. So if a function defined on X^2 is Scott-continuous, its projections on the X factor are clearly Scott-continuous. Therefore, since O is commutative, non-decreasing at each variable and residuated (particularly Scott-continuous), so for each $x \in X$ fixed, set $O_x : X \to X$ by $O_x(z) = O(z, y)$ for all $y, z \in X$, and for each $y \in X$ fixed, set $O_y : X \to X$ by $O_y(z) = O(x, z)$ for all $x, z \in X$. Hence, by the Theorem 3.2, follow the result.

 $((ii) \Rightarrow (i))$ It is an immediate consequence of (i).

A pictorial representation of the Proposition 4.3 can be seen in Figure 3.



Figure 3: Diagram of residuated functions

Therefore, given a residuated application $O^{\rho} : X^2 \to X$ and $z \in X$ the function $\varphi_z : X \to X$ defined by $\varphi_z(x) = I_{O_x^{\rho}}(z)$, where $I_{O_y^{\rho}}$ is the residue of $O_x^{\rho} : X \to X$. Similarly define $\psi_z : X \to X$ by $\psi_z(y) = I_{O_y^{\rho}}(z)$, where $I_{O_y^{\rho}}$ is the residue of $O_y^{\rho} : X \to X$.

Corollary 4.1. Let X a complete lattice of dense order. For all $z \in X$ the following items are worth:

- (i) The pair (ψ_z, φ_z) forms an adjunction;
- (ii) The applications ψ_z and φ_z are closed operators on X.

5. Final remarks

In this paper we propose was the residuation principle for the case of quasioverlap functions on lattices and their respective induced implications. It has been found that the class of quasi-overlap functions that fulfill the residuation principle is the same class of continuous functions according to Scott's topology. Get a generalization of the residuation principle for quasi-orverlap functions was one of the motivations that led to the writing of this paper, which demonstrated the need for a topology on lattice. This topology is Scott's topology. In fact, given any lattice, it is always possible to know how each element behaves relative to the other elements, but it is difficult to know what the overall structure looks like. However, by defining Scott's topology, the topological properties related to the order that this lattice contains allowed to develop its own visualization for this lattice. Thus, a large number of properties that occur in the closed real interval [0, 1] (eg density, connectivity, as well as being a Hausdorff space) could be generalized to general lattices with specific topological properties. Thus, concepts such as density were expressed both in topological terms and in terms of the defined order relation over the set. Another example was the concept of compactness, which allowed generalizations of the well-known extreme value theorem⁴. The results for any lattice adjunctions using overlap functions allow these operators to be used in tools such as *Mathematical Morphology*, which is applied to the field of signal and image processing through dilation operators, erosion, and others [14].

In the field of logic, an important point is that residuation is an essential algebraic property that must be required to have good semantics for fuzzy logic systems based on the modus ponens rule, the necessary and sufficient condition for a conjunction fuzzy have a residue is not continuity but left-continuity. Since Scott-continuous quasi-overlap functions are actually a generalization of left-continuous overlap functions to lattices, it is definitely interesting, from a logical point of view, to focus on the study of properties related to left-continuous overlap functions, as well as investigating how these properties are interpreted for Scott-continuous quasi-overlap functions. It is noteworthy that knowledge of

⁴The extreme value theorem ensures that a continuous function defined in a compact set reaches its maximum and minimum somewhere in the set.

left-continuous overlap functions is drastically limited compared to the good description in the continuous case literature.

References

- Bedregal, B., Bustince, H., Palmeira, E., Dimuro, G., Fernandez, J., 2017. Generalized interval-valued OWA operators with interval weights derived from interval-valued overlap functions. International Journal of Approximate Reasoning 90, 1 – 16. doi:10.1016/j.ijar.2017.07.001.
- [2] Birkhoff, G., 1967. Lattice Theory. 3rd ed., American Mathematical Society, Providence.
- [3] Blyth, T., 2005. Lattices and Ordered Algebraic Structures. Universitext, Springer London.
- [4] Bustince, H., Fernandez, J., Mesiar, R., Montero, J., Orduna, R., 2010. Overlap functions. Nonlinear Analysis: Theory, Methods & Applications 72, 1488 – 1499.
- [5] Davey, B.A., Priestley, H.A., 2002. Introduction to Lattices and Order. 2 ed., Cambridge University Press. doi:10.1017/CB09780511809088.
- [6] De Miguel, L., Gómez, D., Rodríguez, J.T., Montero, J., Bustince, H., Dimuro, G.P., Sanz, J.A., 2018. General overlap functions. Fuzzy Sets and Systems doi:10.1016/j.fss.2018.08.003.
- [7] Dickmann, M., Schwartz, N., Tressl, M., 2019. Spectral Spaces. New Mathematical Monographs, Cambridge University Press. doi:10.1017/9781316543870.
- [8] Dimuro, G.P., Bedregal, B., 2015. On residual implications derived from overlap functions. Information Sciences 312, 78 - 88. doi:10.1016/j.ins.2015.03.049.
- [9] Dimuro, G.P., Bedregal, B., Bustince, H., Asiáin, M.J., Mesiar, R., 2016. On additive generators of overlap functions. Fuzzy Sets and Systems 287, 76 96. doi:10.1016/j.fss.2015.02.008.
- [10] Galatos, N., Jipsen, P., Kowalski, T., Ono, H., 2007. Residuated Lattices: An Algebraic Glimpse at Substructural Logics, Volume 151. 1st ed., Elsevier Science, San Diego, USA.

- [11] Gierz, G., Hofmann, K.H., Keimel, K., Lawson, J.D., Mislove, M., Scott, D.S., 2003. Continuous Lattices and Domains. Encyclopedia of Mathematics and its Applications, Cambridge University Press.
- [12] Gómez, D., Rodríguez, J.T., Montero, J., Bustince, H., Barrenechea, E., 2016. n-dimensional overlap functions. Fuzzy Sets and Systems 287, 57 75. doi:10.1016/j.fss.2014.11.023.
- [13] Goubault-Larrecq, J., 2016. Non-Hausdorff Topology and Domain Theory: Selected Topics in Point-Set Topology. Cambridge University Press, New York, NY, USA.
- Haralick, R.M., Sternberg, S.R., Zhuang, X., 1987. Image analysis using mathematical morphology. IEEE Transactions on Pattern Analysis and Machine Intelligence PAMI-9, 532–550. doi:10.1109/TPAMI.1987.4767941.
- [15] Kelley, J.L., 1975. General Topology. Graduate texts in Mathematics, 27, Springer, New York.
- [16] Król, A., 2011. Dependencies between fuzzy conjunctions and implications, in: Proceedings of the 7th conference of the European Society for Fuzzy Logic and Technology, EUSFLAT 2011, Aix-Les-Bains, France, July 18-22, 2011, pp. 230–237. doi:10.2991/eusflat.2011.155.
- [17] Mislove, M.W., 1998. Topology, domain theory and theoretical computer science. Topology and its Applications 89, 3 59. doi:10.1016/S0166-8641(97)00222-8.
- [18] Munkres, J., 2000. Topology. Topology, Prentice-Hall.
- [19] Nolasco, D.H., Costa, F.B., Palmeira, E.S., Alves, D.K., Bedregal, B.R., Rocha, T.O., Ribeiro, R.L., Silva, J.C., 2019. Wavelet-fuzzy power quality diagnosis system with inference method based on overlap functions: Case study in an AC microgrid. Engineering Applications of Artificial Intelligence 85, 284 – 294. doi:10.1016/j.engappai.2019.05.016.
- [20] Ore, O., 1944. Galois connexions. Transactions of the American Mathematical Society 55, 493–513.

- [21] Paiva, R., Palmeira, E.S., Santiago, R.H.N., Bedregal, B.R.C., 2019. Latticevalued overlap and quasi-overlap functions. CoRR abs/1902.00133. URL: http://arxiv.org/abs/1902.00133.
- [22] Qiao, J., Hu, B.Q., 2018a. The distributive laws of fuzzy implications over overlap and grouping functions. Information Sciences 438, 107 – 126. doi:10.1016/j.ins.2018.01.047.
- [23] Qiao, J., Hu, B.Q., 2018b. On generalized migrativity property for overlap functions. Fuzzy Sets and Systems doi:10.1016/j.fss.2018.01.007.
- [24] Qiao, J., Hu, B.Q., 2018c. On the migrativity of uninorms and nullnorms over overlap and grouping functions. Fuzzy Sets and Systems 346, 1 – 54. doi:10.1016/j.fss.2017.11.012.
- [25] Scott, D., 1972. Continuous lattices, in: Lawvere, F.W. (Ed.), Toposes, Algebraic Geometry and Logic, Springer Berlin Heidelberg, Berlin, Heidelberg. pp. 97–136.
- [26] Wang, Y.M., Liu, H.W., 2019. The modularity condition for overlap and grouping functions. Fuzzy Sets and Systems 372, 97 - 110. doi:10.1016/j.fss.2018.09.015.
- [27] Zhang, T., Qin, F., 2020. On distributive laws between 2-uninorms and overlap (grouping) functions. International Journal of Approximate Reasoning 119, 353 – 372. doi:10.1016/j.ijar.2020.01.008.