Constructing 2-uninorms on bounded lattices by using additive generators^{*}

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Abstract

In this article, we present two methods to construct 2-uninorms on bounded lattices by using additive generators, which are further used for inducing uninorms, nullnorms, uni-nullnorms and null-uninorms, respectively. We also provide some examples for illustrating the constructing methods of 2-uninorms.

Keywords: Bounded lattice; Additive generator; Uninorm; 2-uninorm

1. Introduction

Triangular norms (t-norms for short) and triangular conorms (t-conorms for short) are associative, commutative and monotone binary operations with the neutral element 1 and 0, respectively. Schweizer and Sklar [21] studied t-norms and t-conorms on [0, 1] based on Menger's notion [20]. T-norms and t-conorms have been proved to be useful in many fields like fuzzy set theory [18], fuzzy logic [2], fuzzy systems modeling [34], and probabilistic metric spaces [22, 23].

As important generalizations of t-norms and t-conorms, the concepts of uninorms and nullnorms were introduced on the unit interval [0, 1] by Yager and Rybalov [33] and Calvo et al. [5], respectively. Uninorms allow for a neutral element anywhere in the unit interval, whereas nullnorms allow for a zero element kanywhere in the unit interval, while keeping 0 as neutral element on [0, k] and 1 as neutral element on [k, 1]. A series of works have been done for uninorms [7, 9, 14, 17] and nullnorms [6, 8, 25, 31, 36], respectively.

To unify uninorms and nullnorms, 2-uninorms were first investigated by Akella [1]. Since then, some properties of 2-uninorms on the unit interval are studied. For example, Drygaś and Rak [10] solved the functional equations of distributivity between 2-uninorms. Wang and Qin [28] studied the distributivity equations for 2-uninorms over semi-uninorms. Zong et al. [37] described the structures of 2-uninorms. Then Sun and Liu [24] investigated the left (resp. right) distributivity of semi-t-operators over 2-uninorms. Almost at the same time, Zhang and Qin [35] obtained some sufficient and necessary conditions of the distributivity equations between five classes of basic 2-uninorms and overlap (resp. grouping) functions. Wang et al. [30] introduced the discrete 2-uninorms. Huang and Qin [13] made a deep study on the migrativity of uninorms over 2-uninorms.

As a bounded lattice is more general than [0, 1], the study of 2-uninorms on the unit interval has already been extended to bounded lattices. For instance, Ertuğrul [11] and Xie and Yi [32] gave the constructions of 2-uninorms. Recently, Sun and Liu [26] explored the additive generators of t-norms and t-conorms. Also, He and Wang [12] studied the additive generators of uninorms, and they even extended the classical additive generators to partially ordered cases by adding some conditions. As 2-uninorms are generalizations

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of uninorms, this leads to a natural question: Could we construct 2-uninorms on bounded lattices by using additive generators? This article will focus on this question.

The remainder of this article is organized as follows. In Section 2, we provide the necessary background material. Section 3 is devoted to the constructions of 2-uninorms on bounded lattices based on additive generators. Finally, conclusions are drawn in Section 4.

2. Preliminaries

This section presents some basic definitions and results that are used latter.

A *lattice* [4] is a nonempty set L equipped with a partial order \leq such that any two elements x and y have a greatest lower bound (called meet or infimum), denoted by $x \wedge y$, as well as a smallest upper bound (called join or supremum), denoted by $x \vee y$. A lattice is called *bounded* if it has a top element 1_L and a bottom element 0_L . For short, we use the notation L instead of a bounded lattice $(L, \leq, 0_L, 1_L)$ throughout this article. Let $x, y \in L$. The elements x and y are called *comparable* if $x \leq y$ or $y \leq x$. Otherwise, x and y are called *incomparable*. The symbol $x \parallel y$ is used when x and y are incomparable. If x and y are comparable, then we use the symbol $x \not\parallel y$. In what follows, I_a denotes the set of all incomparable elements with $a \in L$, i.e., $I_a = \{x \in L : x \parallel a\}$. Let $a, b \in L$ with $a \leq b$. An interval [a, b] is defined as $[a, b] = \{x \in L \mid a \leq x \leq b\}$, other intervals can be defined similarly, $(a, b] = \{x \in L \mid a < x \leq b\}$, $[a, b) = \{x \in L \mid a \leq x < b\}$, $(a, b) = \{x \in L \mid a < x < b\}$.

Definition 2.1 ([3, 19]).

- (i) A binary operation $T: L^2 \to L$ is called a *t-norm* if it is commutative, associative, and increasing with respect to both variables and it satisfies $T(x, 1_L) = x$ for all $x \in L$.
- (ii) A binary operation $S: L^2 \to L$ is called a *t-conorm* if it is commutative, associative, and increasing with respect to both variables and it satisfies $S(x, 0_L) = x$ for all $x \in L$.

Definition 2.2 ([16, 33]). A binary operation $U : L^2 \to L$ is called a *uninorm* if it has commutativity, associativity, and increasing with respect to both variables and a neutral element $e \in L$.

Obviously, a t-norm (resp. t-conorm) on L is exactly a uninorm with the neutral element $e = 1_L$ (resp. $e = 0_L$).

Definition 2.3 ([5, 15]). A binary operation $V : L^2 \to L$ is called a *nullnorm* if it is commutative, associative, increasing with respect to both variables, and there exists an element $a \in L$, which is called a *zero element* for V, such that $V(x, 0_L) = x$ for all $x \in [0_L, a]$ and $V(x, 1_L) = x$ for all $x \in [a, 1_L]$.

Obviously, a t-norm (resp. t-conorm) on L is exactly a nullnorm with the zero element $a = 0_L$ (resp. $a = 1_L$).

Definition 2.4 ([26]). Let P, Q be two partially ordered sets and $f: P \to Q$ be a non-constant injective monotone function. If f is increasing, then a pseudo-inverse $f^{(-1)}: Q \to P$ is given by (2.1) when it exists.

$$f^{(-1)}(y) = \begin{cases} \inf\{x \in P | f(x) > y\}, & \text{if } \operatorname{card}\{f^{-1}(y)\} = 0, \\ f^{-1}(y), & \text{if } \operatorname{card}\{f^{-1}(y)\} = 1. \end{cases}$$
(2.1)

If f is decreasing then a pseudo-inverse $f^{(-1)}: Q \to P$ is given by (2.2) when it exists.

$$f^{(-1)}(y) = \begin{cases} \sup\{x \in P | f(x) > y\}, & \text{if } \operatorname{card}\{f^{-1}(y)\} = 0, \\ f^{-1}(y), & \text{if } \operatorname{card}\{f^{-1}(y)\} = 1. \end{cases}$$
(2.2)

Definition 2.5 ([12]). Let $0 \in A \subseteq [-\infty, +\infty]$. For two non-zero elements $x, a \in A$, if there exists $b \in A$ fulfilling x = a + b and ab > 0, then we call a a summand of x in A.

Remark 2.1 ([12]). Let $0 \in A \subseteq [-\infty, +\infty]$ and $x \in A$. If x = 0 then x has no summands. If $x \neq 0$, then x is always a summand of itself and each summand a of x satisfies xa > 0

Definition 2.6 ([27, 29]). A binary operation $F : L^2 \to L$ is called a *uni-nullnorm* if it satisfies the commutativity, associativity, monotonicity with respect to both variables, and there exist some elements $e, a \in L$ with $0_L \leq e < a \leq 1_L$ such that F(e, x) = x for all $x \in [0_L, a]$ and $F(x, 1_L) = x$ for all $x \in [a, 1_L]$.

Definition 2.7 ([1, 11]). Let $e_1, a, e_2 \in L$ with $0_L \leq e_1 \leq a \leq e_2 \leq 1_L$. A binary operation $\mathbb{U} : L^2 \to L$ is called a *2-uninorm* if it is commutative, associative, increasing with respect to both variables and fulfilling $\mathbb{U}(e_1, x) = x$ for all $x \leq a$ and $\mathbb{U}(e_2, x) = x$ for all $x \geq a$.

From Definitions 2.2, 2.3, 2.6 and 2.7, one can easily check the following remark.

Remark 2.2.

- (i) A 2-uninorm with $e_2 = 1_L$ is a uni-nullnorm.
- (ii) A 2-uninorm with $e_1 = 0_L$ is a null-uninorm.
- (iii) A 2-uninorm with $a = 1_L$ or $a = 0_L$ is a uninorm.
- (iv) A 2-uninorm with $e_1 = 0_L$ and $e_2 = 1_L$ is a nullnorm.

3. Constructions of 2-uninorms on bounded lattices

In this section, we first give two new methods for constructing 2-uninorms on bounded lattices L, which are further used for inducing uninorms, nullnorms, uni-nullnorms and null-uninorms, respectively. We then provide some examples to illustrate the new methods.

Let $a \in L$. Denoted by $\mathcal{F}_a = \{f(x)|0 < f(x) < f(a) \text{ and } x \in I_a\}$ when $f: L \to [-\infty, +\infty]$ is injective increasing, and by $\mathcal{G}_a = \{f(x)|f(a) < f(x) < 0 \text{ and } x \in I_a\}$ when $f: L \to [-\infty, +\infty]$ is injective decreasing.

Lemma 3.1. Let $f : L \to [-\infty, +\infty]$ be an injective increasing function, and $x, y \in L$ with $x \leq y$. If $f(x) \leq f(y) \leq f(a)$ and $x \in I_a$, then $y \in I_a$.

Proof. Assume that $y \notin I_a$. Then $y \leq a$ since $f(y) \leq f(a)$ and f is injective increasing. Thus from $x \leq y$ we obtain $x \leq y \leq a$, contrary to the fact that $x \in I_a$.

Theorem 3.1. Let $e_1, a, e_2 \in L$ with $0_L \leq e_1 \leq a \leq e_2 \leq 1_L$, and $f : L \to [-\infty, +\infty]$ be an injective increasing function with $f(e_1) = 0$. If f satisfies the following five conditions: for all $x, y \in L$,

(i) if $(f(x), f(y)) \in [0, f(a)]^2 \cup [-\infty, 0]^2$ then

$$\min\{f(x) + f(y), f(a)\} \in Ran(f) \cup [-\infty, f(0_L)),$$

- (ii) for all $(f(x), f(y)) \in [0, f(a)]^2 \cup [-\infty, 0]^2$, if f(x) and f(y) have at least one same summand $f(z) \in Ran(f)$ then $x \not\parallel y$,
- (iii) if $f(x) < 0 < f(y) \le f(a)$ and $x < e_1$, then $x \not\parallel y$,
- (iv) if $f(a) \leq f(x) < f(y)$ or $f(x) < 0 < f(a) \leq f(y)$, then $x \not\parallel y$,
- (v) for all $f(x), f(y) \in (0, f(a))$, if $f(x) + f(y) \in Ran(f)$ and $0 < f(x) + f(y) \le f(a)$ then $f^{-1}(f(x) + f(y)) \not\parallel a$,

then the following function $\mathbb{U}: L^2 \to L$ is a 2-uninorm, and we call f an additive generator of \mathbb{U} . For all $x, y \in L$,

$$\mathbb{U}(x,y) = \begin{cases} f^{(-1)}(f(x) + f(y)), & (f(x), f(y)) \in [-\infty, 0]^2, \\ f^{(-1)}(\min\{f(x) + f(y), f(a)\}), & (f(x), f(y)) \in [0, f(a)]^2 \text{ and } x \notin I_a \text{ and } y \notin I_a, \\ a, & (f(x), f(y)) \in (f(a), +\infty] \times [0, f(a)] \\ & \cup [0, f(a)] \times (f(a), +\infty] \cup [0, f(a)] \times \mathcal{F}_a \\ & \cup \mathcal{F}_a \times [0, f(a)], \\ f^{(-1)}(\max\{f(x), f(y)\}), & (f(x), f(y)) \in (f(e_2), +\infty]^2 \cup (f(e_2), +\infty] \\ & \times (f(a), f(e_2)] \cup (f(a), f(e_2)] \times (f(e_2), +\infty], \\ f^{(-1)}(\min\{f(x), f(y)\}), & otherwise. \end{cases}$$
(3.1)

Proof. First it is necessary to check that \mathbb{U} is well-defined. The proof is made in five cases:

- (I) If $(f(x), f(y)) \in [f(a), +\infty) \times [0, f(a)] \cup [0, f(a)] \times [f(a), +\infty) \cup [0, f(a)] \times \mathcal{F}_a \cup \mathcal{F}_a \times [0, f(a)]$, then $\mathbb{U}(x, y) = a$. Thus \mathbb{U} is well-defined by (3.1).
- (II) If $(f(x), f(y)) \in [-\infty, 0]^2$, then $f(x) + f(y) \le f(a)$, subsequently,

$$f^{(-1)}(\min\{f(x) + f(y), f(a)\}) = f^{(-1)}(f(x) + f(y)).$$

There are two subcases as follows.

Subcase (II-a). If $f(x) + f(y) \in Ran(f)$ then clearly $card\{f^{-1}(f(x) + f(y))\} = 1$ since f is injective. Thus $\mathbb{U}(x, y)$ is well-defined by (2.1).

Subcase (II-b). If $f(x) + f(y) \in [-\infty, f(0_L))$ then $\operatorname{card} \{ f^{-1}(f(x) + f(y)) \} = 0$. By (3.1), we know that

$$f^{(-1)}(f(x) + f(y)) = \inf\{z \in L : f(z) > f(x) + f(y)\} = \wedge L = 0_L.$$

Hence $\mathbb{U}(x, y)$ is well-defined by (2.1).

(III) If $(f(x), f(y)) \in [0, f(a)]^2$, $x, y \notin I_a$ and $f(x) + f(y) \leq f(a)$, then $f(x) + f(y) \in Ran(f)$, subsequently,

$$f^{(-1)}(\min\{f(x) + f(y), f(a)\}) = f^{(-1)}(f(x) + f(y)).$$

Thus, similar to (II-a), $\mathbb{U}(x, y)$ is well-defined.

(IV) If $(f(x), f(y)) \in [0, f(a)]^2$, $x, y \notin I_a$ and $f(x) + f(y) \ge f(a)$, then

$$f^{(-1)}(\min\{f(x) + f(y), f(a)\}) = f^{(-1)}(f(a)) = a.$$

Thus \mathbb{U} is well-defined by (3.1).

(V) For the other cases, without loss of generality, we suppose that $f(x) \leq f(y)$. Then

$$f^{(-1)}(\min\{f(x), f(y)\}) = f^{(-1)}(f(x)) = x,$$

and

$$f^{(-1)}(\max\{f(x), f(y)\}) = f^{(-1)}(f(y)) = y.$$

Thus $\mathbb{U}(x, y)$ is well-defined by (3.1).

Therefore, from (I)-(V), we know that $\mathbb{U}(x, y)$ is well-defined for any $x, y \in L$.

Next, we prove that U is a 2-uninorm. First, from (3.1), we have the following three statements.

(a) \mathbb{U} is a commutative binary operation on L.

- (b) If $x \in [0_L, a]$, i.e., $f(x) \leq f(a)$ and $x \notin I_a$, then $\mathbb{U}(e_1, x) = f^{(-1)}(\min\{f(e_1) + f(x), f(a)\}) = f^{(-1)}((f(x)) = x$ since f is injective increasing.
- (c) If $x \in [a, 1_L]$, i.e., $f(x) \ge f(a)$, then $\mathbb{U}(e_2, x) = f^{(-1)}(\min\{f(e_2), f(x)\}) = f^{(-1)}(f(x)) = x$ for $f(a) \le f(x) \le f(e_2)$ and $\mathbb{U}(e_2, x) = f^{(-1)}(\max\{f(e_2), f(x)\}) = f^{(-1)}(f(x)) = x$ for $f(x) \ge f(e_2)$ since f is injective increasing.

Then it remains to show the monotonicity and associativity of the binary operation \mathbb{U} .

- (1) **Monotonicity.** Let $x, y, z \in L$ with $x \leq y$. We need to verify $\mathbb{U}(x, z) \leq \mathbb{U}(y, z)$. Noticing that $f(x) \leq f(y)$ since $x \leq y$ and f is injective increasing. Therefore, the proof is split into all possible cases as follows.
 - 1. f(z) = 0.
 - 1.1 If $f(x) \leq f(y) < 0$, or $f(x) < 0 \leq f(y) \leq f(a)$ and $y \notin I_a$, then

$$\mathbb{U}(x,z) = f^{(-1)}(f(x) + f(z)) = x \le y = f^{(-1)}(f(y) + f(z)) = \mathbb{U}(y,z)$$

1.2 If $f(x) < 0 \le f(y) \le f(a)$ and $y \in I_a$, or f(x) < 0 < f(a) < f(y), then from (iv), we have

$$\mathbb{U}(x,z) = f^{(-1)}(f(x) + f(z)) = f^{(-1)}(f(x)) = x \le a = \mathbb{U}(y,z).$$

 $1.3 \ 0 \le f(x) \le f(y) \le f(a).$

1.3.1 If $x, y \notin I_a$, then $\mathbb{U}(x, z) = x \leq y = \mathbb{U}(y, z)$.

1.3.2 If $x \in I_a$, then $y \in I_a$ by Lemma 3.1, subsequently, $\mathbb{U}(x, z) = a = \mathbb{U}(y, z)$.

1.3.3 If $x \notin I_a$ and $y \in I_a$, then $\mathbb{U}(x, z) = x \leq a = \mathbb{U}(y, z)$.

1.4 If $0 \le f(x) \le f(a) < f(y)$ and $x \notin I_a$, then $\mathbb{U}(x,z) = f^{(-1)}(f(x) + f(z)) = x \le a = \mathbb{U}(y,z)$.

1.5 If $0 \le f(x) \le f(a) < f(y)$ and $x \in I_a$, then $\mathbb{U}(x, z) = a = \mathbb{U}(y, z)$.

1.6 If $f(a) < f(x) \le f(y)$, then $\mathbb{U}(x, z) = a = \mathbb{U}(y, z)$.

2. f(z) < 0.

2.1. If $f(x) \leq f(y) \leq 0$, then we have $f(x) + f(z) \leq 0 < f(a)$, subsequently,

$$\mathbb{U}(x,z) = f^{(-1)}(\min\{f(x) + f(z), f(a)\}) = f^{(-1)}(f(x) + f(z)).$$

Analogously, we obtain $\mathbb{U}(y, z) = f^{(-1)}(\min\{f(y) + f(z), f(a)\}) = f^{(-1)}(f(y) + f(z)).$ 2.1.1. If $f(x) + f(z) \in [-\infty, f(0_L))$, then $\mathbb{U}(x, z) = f^{(-1)}(f(x) + f(z)) = \wedge L = 0_L < \mathbb{U}(y, z).$

2.1.2. If $f(x) + f(z) \in Ran(f)$, then $f(y) + f(z) \in Ran(f)$. Noticing that f(z) is the same summand of both f(x) + f(z) and f(y) + f(z), thus $\mathbb{U}(x, z) \notin \mathbb{U}(y, z)$ by (ii). Therefore, $\mathbb{U}(x, z) \leq \mathbb{U}(y, z)$ since $f(x) + f(z) \leq f(y) + f(z)$ and f is injective increasing.

2.2. If f(x) < 0 < f(y), then $\mathbb{U}(x, z) = f^{(-1)}(f(x) + f(z))$, and

$$\mathbb{U}(y,z) = f^{(-1)}(\min\{f(x), f(z)\}) = f^{(-1)}(f(z)) = z.$$

2.2.1. If $f(x) + f(z) \in [-\infty, f(0_L))$, then $\mathbb{U}(x, z) = 0_L \leq \mathbb{U}(y, z)$ from 2.1.1.

2.2.2. If $f(x) + f(z) \in Ran(f)$, then f(z) is the same summand of both f(x) + f(z) and f(z) by Remark 2.1. Thus $\mathbb{U}(x,z) \not\models f^{-1}(f(z))$ by (ii). Therefore, from $f(x) \leq 0$ we have $\mathbb{U}(x,z) \leq f^{-1}(f(z)) = z = \mathbb{U}(y,z)$ since $f(x) + f(z) \leq f(z)$ and f is injective increasing.

2.3. If $0 \le f(x) \le f(y)$, then $\mathbb{U}(x, z) = f^{(-1)}(\min\{f(x), f(z)\}) = f^{(-1)}(f(z)) = z$ and

$$\mathbb{U}(y,z) = f^{(-1)}(\min\{f(y), f(z)\}) = f^{(-1)}(f(z)) = z,$$

therefore, $\mathbb{U}(x, z) = z = \mathbb{U}(y, z)$. 3. f(z) > 0. 3.1 If $f(x) \le f(y) < 0$, then

$$\mathbb{U}(x,z) = f^{(-1)}(\min\{f(x), f(z)\}) = x \le y = f^{(-1)}(\min\{f(y), f(z)\}) = \mathbb{U}(y,z).$$

 $3.2 \ f(x) < 0 = f(y).$

3.2.1 If $0 < f(z) \leq f(a)$ and $z \notin I_a$, then from (iii) we have

$$\mathbb{U}(x,z) = f^{(-1)}(\min\{f(x), f(z)\}) = x \le z = f^{(-1)}(\min\{f(y) + f(z), f(a)\}) = \mathbb{U}(y,z).$$

3.2.2 If $0 < f(z) \le f(a)$ and $z \in I_a$, then from (iii) we have

$$\mathbb{U}(x,z) = f^{(-1)}(\min\{f(x), f(z)\}) = x \le a = \mathbb{U}(y,z).$$

3.2.3 If f(z) > f(a), then from (iii), we have

$$\mathbb{U}(x,z) = f^{(-1)}(\min\{f(x), f(z)\}) = x \le a = \mathbb{U}(y,z).$$

3.3 $f(x) = 0 < f(y) \le f(a)$. 3.3.1 If $0 < f(z) \le f(a)$, and $z \in I_a$, then $\mathbb{U}(x, z) = a = \mathbb{U}(y, z)$. 3.3.2 $0 < f(z) \le f(a)$ and $y, z \notin I_a$. 3.3.2.1 If $f(y) + f(z) \le f(a)$, then from (ii), we have

$$\mathbb{U}(y,z) = f^{(-1)}(\min\{f(y) + f(z), f(a)\}) = f^{(-1)}(f(y) + f(z)) \ge f^{(-1)}(z) = z = \mathbb{U}(x,z).$$

3.3.2.2 If $f(y) + f(z) \ge f(a)$, then from $z \notin I_a$ and $f(z) \le f(a)$, we have

$$\mathbb{U}(y,z) = f^{(-1)}(\min\{f(y) + f(z), f(a)\}) = a \ge z = f^{(-1)}(\min\{f(x) + f(z), f(a)\}) = \mathbb{U}(x,z).$$

3.3.3 If $0 < f(z) \le f(a), y \in I_a$ and $z \notin I_a$, then

$$\mathbb{U}(y,z) = a \ge z = f^{(-1)}(\min\{f(x) + f(z), f(a)\}) = \mathbb{U}(x,z).$$

3.3.4 If f(z) > f(a), then $\mathbb{U}(y, z) = a = \mathbb{U}(x, z)$. 3.4 If f(x) < 0 < f(y), then

$$\mathbb{U}(x,z) = f^{(-1)}(\min\{f(x), f(z)\}) = x.$$
(3.2)

3.4.1 $(f(y), f(z)) \in (0, f(a)]^2$. 3.4.1.1 $y, z \notin I_a$. 3.4.1.1 If $f(y) + f(z) \leq f(a)$ then, because of (3.2) and (ii), we have

$$\mathbb{U}(y,z) = f^{(-1)}(\min\{f(y) + f(z), f(a)\}) = f^{(-1)}(f(y) + f(z)) \ge f^{(-1)}(y) = y \ge x = \mathbb{U}(x,z).$$

3.4.1.1.2 If $f(y) + f(z) \ge f(a)$ then, because of (iv), we have

$$\mathbb{U}(y,z) = f^{(-1)}(\min\{f(y) + f(z), f(a)\}) = f^{(-1)}(f(a)) = a \ge x = \mathbb{U}(x,z).$$

3.4.1.2 If $z \in I_a$ or $y \in I_a$, then from (3.2) and (iv) we have $\mathbb{U}(y, z) = a \ge x = \mathbb{U}(x, z)$. 3.4.2 $(f(y), f(z)) \in (f(a), f(e_2)]^2$. 3.4.2.1 If $f(y) \le f(z)$, then

$$\mathbb{U}(y,z) = f^{(-1)}(\min\{f(y), f(z)\}) = f^{(-1)}(f(y)) = y \ge x = \mathbb{U}(x,z)$$

3.4.2.2 If $f(y) \ge f(z)$, then from (iv) we have

$$\mathbb{U}(y,z) = f^{(-1)}(\min\{f(y), f(z)\}) = f^{(-1)}(f(z)) = z \ge x = \mathbb{U}(x,z).$$

 $\begin{aligned} 3.4.3 \ (f(y), f(z)) \in (f(e_2), +\infty]^2 \cup (f(e_2), +\infty] \times (f(a), f(e_2)] \cup (f(a), f(e_2)] \times (f(e_2), +\infty]. \\ 3.4.3.1 \ \text{If} \ f(y) \geq f(z), \text{ then} \end{aligned}$

$$\mathbb{U}(y,z) = f^{(-1)}(\max\{f(y), f(z)\}) = f^{(-1)}(y) = y \ge x = \mathbb{U}(x,z).$$

3.4.3.2 If $f(y) \leq f(z)$, then from (iv), we have

$$\mathbb{U}(y,z) = f^{(-1)}(\max\{f(y),f(z)\}) = f^{(-1)}(z) = z \ge x = \mathbb{U}(x,z).$$

3.4.4 If $(f(y), f(z)) \in [f(a), +\infty] \times [0, f(a)] \cup [0, f(a)] \times [f(a), +\infty]$, then from (iv), we have

$$\mathbb{U}(y,z) = a \ge x = \mathbb{U}(x,z).$$

$$\begin{split} &3.5\ 0 < f(x) \leq f(y) \leq f(a). \\ &3.5.1\ 0 < f(z) \leq f(a). \\ &3.5.1.1\ \text{If}\ z \in I_a, \ \text{then}\ \mathbb{U}(x,z) = a = \mathbb{U}(y,z). \\ &3.5.1.2\ x, y, z \notin I_a. \\ &3.5.1.2.1\ \text{If}\ f(a) \leq f(x) + f(z) \leq f(y) + f(z), \ \text{then}\ \mathbb{U}(x,z) = a = \mathbb{U}(y,z). \\ &3.5.1.2.2\ \text{If}\ f(x) + f(z) \leq f(a) \leq f(y) + f(z) \ \text{then}, \ \text{because of}\ (v), \ \text{we have} \end{split}$$

$$\mathbb{U}(x,z) = f^{(-1)}(\min\{f(x) + f(z), f(a)\}) = f^{(-1)}(f(x) + f(z)) \le a = \mathbb{U}(y,z).$$

3.5.1.2.3 If $f(x) + f(z) \le f(y) + f(z) \le f(a)$ then, from (ii), we have

$$\mathbb{U}(x,z) = f^{(-1)}(f(x) + f(z)) \le f^{(-1)}(f(y) + f(z)) = \mathbb{U}(y,z)$$

3.5.1.3 If $z \notin I_a$ and $x \in I_a$, then $y \in I_a$ by Lemma 3.1, subsequently, $\mathbb{U}(x, z) = a = \mathbb{U}(y, z)$. 3.5.1.4 If $z, x \notin I_a$ and $y \in I_a$, then from (v), we have $\mathbb{U}(x, z) = f^{(-1)}(f(x) + f(z)) \leq a = \mathbb{U}(y, z)$. 3.5.2 If f(z) > f(a), then $\mathbb{U}(x, z) = a = \mathbb{U}(y, z)$. 3.6 $0 \leq f(x) \leq f(a) < f(y)$. 3.6.1 $0 < f(z) \le f(a)$. 3.6.1.1 $x, z \notin I_a$. 3.6.1.1.1 If $f(x) \ne 0$ and $f(x) + f(z) \le f(a)$ then, because of (v), we have

$$\mathbb{U}(x,z) = f^{(-1)}(\min\{f(x) + f(z), f(a)\}) = f^{(-1)}(f(x) + f(z)) \le a = \mathbb{U}(y,z).$$

3.6.1.1.2 If f(x) = 0, then $\mathbb{U}(x, z) = z \le a = \mathbb{U}(y, z)$. 3.6.1.1.3 If $f(x) + f(z) \ge f(a)$, then $\mathbb{U}(x, z) = f^{(-1)}(\min\{f(x) + f(z), f(a)\}) = a = \mathbb{U}(y, z)$. 3.6.1.2 If $x \in I_a$ or $z \in I_a$, then $\mathbb{U}(x, z) = a = \mathbb{U}(y, z)$. 3.6.2 If $(f(y), f(z)) \in (f(a), f(e_2)]^2$, then from (iv) and $f(y), f(z) \ge f(a)$, we have

$$\mathbb{U}(x,z) = a \le f^{(-1)}(\min\{f(y), f(z)\}) = \mathbb{U}(y,z).$$

3.6.3 If $(f(y), f(z)) \in (f(e_2), +\infty]^2 \cup (f(e_2), +\infty] \times (f(a), f(e_2)] \cup (f(a), f(e_2)] \times (f(e_2), +\infty]$, then from (iv) and $f(y), f(z) \ge f(a)$, we have

$$\mathbb{U}(x,z) = a \le f^{(-1)}(\max\{f(y), f(z)\}) = \mathbb{U}(y,z).$$

3.7
$$f(a) < f(x) \le f(y)$$
.
3.7.1 If $0 < f(z) \le f(a)$, then $\mathbb{U}(x, z) = a = \mathbb{U}(y, z)$.
3.7.2 $f(a) < f(z) \le f(e_2)$.
3.7.2.1 If $f(a) < f(x) \le f(y) \le f(e_2)$, then $\mathbb{U}(x, z) = f^{(-1)}(\min\{f(x), f(z)\})$ and $\mathbb{U}(y, z) = f^{(-1)}(\min\{f(y), f(z)\})$.

3.7.2.1.1 If $f(x) \leq f(y) \leq f(z)$, then $\mathbb{U}(x, z) = x \leq y = \mathbb{U}(y, z)$. 3.7.2.1.2 If $f(z) \leq f(x) \leq f(y)$, then $\mathbb{U}(x, z) = z = \mathbb{U}(y, z)$. 3.7.2.1.3 If $f(x) \leq f(z) \leq f(y)$ then, because of (iv), we have $\mathbb{U}(x, z) = x \leq z = \mathbb{U}(y, z)$. 3.7.2.2 If $f(a) < f(x) \leq f(e_2) < f(y)$, then $\mathbb{U}(x, z) = f^{(-1)}(\min\{f(x), f(z)\})$ and

$$\mathbb{U}(y,z) = f^{(-1)}(\max\{f(y), f(z)\}) = f^{(-1)}(f(y)) = y.$$

3.7.2.2.1 If $f(x) \leq f(z)$, then $\mathbb{U}(x, z) = x \leq y = \mathbb{U}(y, z)$. 3.7.2.2.2 If $f(x) \geq f(z)$ then, because of (iv), we have $\mathbb{U}(x, z) = z \leq y = \mathbb{U}(y, z)$. 3.7.2.3 If $f(e_2) < f(x) \leq f(y)$, then $\mathbb{U}(x, z) = x \leq y = \mathbb{U}(y, z)$. 3.7.3 If $f(z) > f(e_2)$, then $\mathbb{U}(x, z) = f^{(-1)}(\max\{f(x), f(z)\})$ and $\mathbb{U}(y, z) = f^{(-1)}(\max\{f(y), f(z)\})$. 3.7.3.1 If $f(x) \leq f(y) \leq f(z)$, then $\mathbb{U}(x, z) = z = \mathbb{U}(y, z)$. 3.7.3.2 If $f(x) \leq f(z) \leq f(y)$ then, because of (iv), we obtain $\mathbb{U}(x, z) = z \leq y = \mathbb{U}(y, z)$. 3.7.3.3 If $f(z) \leq f(x) \leq f(y)$, then $\mathbb{U}(x, z) = x \leq y = \mathbb{U}(y, z)$.

- (2) Associativity. Let $x, y, z \in L$. We need to verify $\mathbb{U}(x, \mathbb{U}(y, z)) = \mathbb{U}(\mathbb{U}(x, y), z)$. The proof is split into all possible cases as follows.
 - 1. f(x) < 0. 1.1. f(y) < 0. 1.1.1. f(z) < 0.

$$\begin{aligned} 1.1.1.1 & \text{If } f(x) + f(y) \in Ran(f) \text{ and } f(y) + f(z) \in Ran(f), \text{ then} \\ \mathbb{U}(\mathbb{U}(x,y),z) &= \mathbb{U}(f^{(-1)}(\min\{(f(x) + f(y)), f(a)\}), z) \\ &= f^{(-1)}(f \circ f^{(-1)}(f(x) + f(y)) + f(z)) \quad (\text{since } f(x) + f(y) \leq 0 \leq f(a)) \\ &= f^{(-1)}(f(x) + f(y) + f(z)). \\ \mathbb{U}(x,\mathbb{U}(y,z)) &= \mathbb{U}(x, f^{(-1)}(\min\{(f(y) + f(z)), f(a)\}) \\ &= f^{(-1)}(f(x) + f(y) + f(z)). \end{aligned}$$

1.1.1.2 If $f(x) + f(y) \in [-\infty, f(0_L))$ and $f(y) + f(z) \in [-\infty, f(0_L))$, then

$$\begin{split} \mathbb{U}(\mathbb{U}(x,y),z) &= & \mathbb{U}(f^{(-1)}(\min\{(f(x)+f(y)),f(a)\}),z) \\ &= & \mathbb{U}(\wedge L,z) \\ &= & \mathbb{U}(0_L,z). \\ &= & f^{(-1)}(f(z)+f(0_L)) \\ &= & 0_L. \\ \mathbb{U}(x,\mathbb{U}(y,z)) &= & \mathbb{U}(x,f^{(-1)}(\min\{(f(y)+f(z)),f(a)\}) \\ &= & \mathbb{U}(x,\wedge L) \\ &= & \mathbb{U}(x,0_L) \\ &= & f^{(-1)}(f(x)+f(0_L)) \\ &= & 0_L. \end{split}$$

1.1.1.3 If $f(x) + f(y) \in [-\infty, f(0_L))$ and $f(y) + f(z) \in Ran(f)$, then

$$\begin{split} \mathbb{U}(\mathbb{U}(x,y),z) &= \mathbb{U}(f^{(-1)}(\min\{(f(x)+f(y)),f(a)\}),z) \\ &= \mathbb{U}(\wedge L,z) \\ &= \mathbb{U}(0_L,z) \\ &= f^{(-1)}(f(z)+f(0_L)) \\ &= 0_L. \\ \mathbb{U}(x,\mathbb{U}(y,z)) &= \mathbb{U}(x,f^{(-1)}(\min\{(f(y)+f(z)),f(a)\}) \\ &= f^{(-1)}(f(x)+f(y)+f(z)) \\ &= \wedge L \\ &= 0_L. \end{split}$$

1.1.1.4 If $f(x) + f(y) \in Ran(f)$ and $f(y) + f(z) \in [-\infty, f(0_L))$, then

$$\begin{split} \mathbb{U}(\mathbb{U}(x,y),z) &= & \mathbb{U}(f^{(-1)}(\min\{(f(x)+f(y)),f(a)\}),z) \\ &= & f^{(-1)}(f(x)+f(y)+f(z)) \\ &= & \wedge L \\ &= & 0_L. \\ \mathbb{U}(x,\mathbb{U}(y,z)) &= & \mathbb{U}(x,f^{(-1)}(\min\{(f(y)+f(z)),f(a)\}) \\ &= & \mathbb{U}(x,\wedge L) \\ &= & \mathbb{U}(x,0_L) \\ &= & 0_L. \end{split}$$

1.1.2. $0 \le f(z)$.

1.1.2.1 If $f(x) + f(y) \in Ran(f)$, then

$$\begin{split} \mathbb{U}(\mathbb{U}(x,y),z) &= \mathbb{U}(f^{(-1)}(\min\{(f(x)+f(y)),f(a)\}),z) \\ &= f^{(-1)}(\min\{f\circ f^{(-1)}(f(x)+f(y)),f(z)\}) \quad (\text{since } f(x)+f(y) \leq 0 \leq f(a)) \\ &= f^{(-1)}(f(x)+f(y)). \\ \mathbb{U}(x,\mathbb{U}(y,z)) &= \mathbb{U}(x,f^{(-1)}(\min\{f(y),f(z)\}) \\ &= f^{(-1)}(\min\{f(x)+f(y),f(a)\}) \\ &= f^{(-1)}(\min\{f(x)+f(y)). \end{split}$$

1.1.2.2 If $f(x) + f(y) \in [-\infty, f(0_L))$, then

$$\begin{split} \mathbb{U}(\mathbb{U}(x,y),z) &= \mathbb{U}(f^{(-1)}(\min\{(f(x)+f(y)),f(a)\}),z) \\ &= \mathbb{U}(\wedge L,z) \\ &= 0_L. \\ \mathbb{U}(x,\mathbb{U}(y,z)) &= \mathbb{U}(x,f^{(-1)}(\min\{f(y),f(z)\}) \\ &= f^{(-1)}(\min\{f(x)+f(y),f(a)\}) \\ &= \wedge L \\ &= 0_L. \end{split}$$

1.2. $0 \le f(y) \le f(a)$. 1.2.1. f(z) < 0. 1.2.1.1 If $f(x) + f(z) \in Ran(f)$, then

$$\begin{aligned} \mathbb{U}(\mathbb{U}(x,y),z) &= & \mathbb{U}(z,\mathbb{U}(x,y)) \\ &= & f^{(-1)}((f(x)+f(z)). & \text{(by 1.1.2.1)} \\ \mathbb{U}(x,\mathbb{U}(y,z)) &= & \mathbb{U}(x,f^{(-1)}(\min\{f(y),f(z)\})) \\ &= & f^{(-1)}((f(x)+f(z)). \end{aligned}$$

1.2.1.2 If $f(x) + f(z) \in [-\infty, f(0_L))$, then by 1.1.2.2,

1.2.2. If $0 \le f(z) \le f(a)$, then $\min\{f(y) + f(z), f(a)\} \in Ran(f)$. 1.2.2.1 If $y, z \notin I_a$, then

$$\begin{split} \mathbb{U}(\mathbb{U}(x,y),z) &= & \mathbb{U}(f^{(-1)}(\min\{f(x),f(y)\}),z) \\ &= & f^{(-1)}(\min\{f(x),f(z)\}) \\ &= & f^{(-1)}(f(x)). \\ \mathbb{U}(x,\mathbb{U}(y,z)) &= & \mathbb{U}(x,f^{(-1)}(\min\{(f(y)+f(z)),f(a)\})) \\ &= & f^{(-1)}(f(x)). \end{split}$$

 $1.2.2.2 \text{ If } y \in I_a \text{ or } z \in I_a, \text{ then } \mathbb{U}(\mathbb{U}(x,y),z) = \mathbb{U}(x,z) = x = \mathbb{U}(x,a) = \mathbb{U}(x,\mathbb{U}(y,z)).$

1.2.3. If f(a) < f(z), then

$$U(U(x, y), z) = U(f^{(-1)}(\min\{f(x), f(y)\}, z))$$

= $f^{(-1)}(\min\{f(x), f(z)\})$
= $f^{(-1)}((f(x)).$
$$U(x, U(y, z)) = U(x, a)$$

= $f^{(-1)}((f(x)).$

1.3. f(a) < f(y). 1.3.1. If f(z) < 0, then

$$\begin{aligned} \mathbb{U}(\mathbb{U}(x,y),z) &= & \mathbb{U}(f^{(-1)}(\min\{f(x),f(y)\},z) = \mathbb{U}(x,z). \\ \mathbb{U}(x,\mathbb{U}(y,z)) &= & \mathbb{U}(x,f^{(-1)}(\min\{f(y),f(z)\})) = \mathbb{U}(x,z). \end{aligned}$$

1.3.2. If $0 \le f(z) \le f(a)$, then

$$\begin{split} \mathbb{U}(\mathbb{U}(x,y),z) &= \mathbb{U}(f^{(-1)}(\min\{f(x),f(y)\})),z) \\ &= \mathbb{U}(x,z) \\ &= f^{(-1)}(\min\{f(x),f(z)\}) \\ &= x. \\ \mathbb{U}(x,\mathbb{U}(y,z) &= \mathbb{U}(x,a) = f^{(-1)}(\min\{f(x),f(a)\})\} = x. \end{split}$$

1.3.3 f(z) > f(a).

$$\begin{split} 1.3.3.1 \ \mathrm{If} \ (f(y), f(z)) &\in (f(a), f(e_2)]^2, \ \mathrm{then} \\ &\mathbb{U}(\mathbb{U}(x, y), z) \ = \ \mathbb{U}(f^{(-1)}(\min\{f(x), f(y)\})), z) \\ &= \ \mathbb{U}(x, z) \\ &= \ f^{(-1)}(\min\{f(x), f(z)\}) \\ &= \ x. \\ &\mathbb{U}(x, \mathbb{U}(y, z)) \ = \ f^{(-1)}(\min\{f(x), f \circ f^{(-1)}(\min\{f(y), f(z)\})\}) = x. \\ 1.3.3.2 \ \mathrm{If} \ (f(y), f(z)) &\in [f(e_2), +\infty]^2 \cup [f(e_2), +\infty] \times (f(a), f(e_2)] \cup (f(a), f(e_2)] \times [f(e_2), +\infty], \ \mathrm{then} \\ &\mathbb{U}(\mathbb{U}(x, y), z) \ = \ \mathbb{U}(f^{(-1)}(\min\{f(x), f(y)\})), z) \\ &= \ \mathbb{U}(x, z) \\ &= \ f^{(-1)}(\min\{f(x), f(z)\}) \\ &= \ x. \\ &\mathbb{U}(x, \mathbb{U}(y, z) \ = \ \mathbb{U}(x, f^{(-1)}(\{\max\{f(y), f(z)\}))) \\ &= \ f^{(-1)}(\min\{f(x), f \circ f^{(-1)}(\{\max\{f(y), f(z)\})\})\} \\ &= \ x. \end{split}$$

2. $0 \le f(x) \le f(a)$. 2.1. f(y) < 0. 2.1.1. f(z) < 0. 2.1.1.1 If $f(y) + f(z) \in Ran(f)$ and $f(x) + f(y) \in Ran(f)$, then $\mathbb{U}(\mathbb{U}(x,y),z) = \mathbb{U}(z,\mathbb{U}(x,y))$ (by the commutativity of \mathbb{U}) $= f^{(-1)}(f(z) + f(y)). \quad (by \ 1.2.1.1)$ $\mathbb{U}(x,\mathbb{U}(y,z)) = \mathbb{U}(\mathbb{U}(y,z),x)$ (by the commutativity of \mathbb{U}) $= f^{(-1)}(f(z) + f(y)). \quad (by \ 1.1.2.1)$ 11

2.1.1.2 If $f(y) + f(z) \in [-\infty, f(0_L))$, then

$$\begin{split} \mathbb{U}(\mathbb{U}(x,y),z) &= \mathbb{U}(z,\mathbb{U}(x,y)) & \text{(by the commutativity of } \mathbb{U}) \\ &= 0_L. & \text{(by 1.2.1.2)} \\ \mathbb{U}(x,\mathbb{U}(y,z)) &= \mathbb{U}(\mathbb{U}(y,z),x) & \text{(by the commutativity of } \mathbb{U}) \\ &= f^{(-1)}(f(z) + f(y)) & \text{(by 1.1.2.1)} \\ &= \wedge L \\ &= 0_L. \end{split}$$

2.1.2. If $0 \le f(z) \le f(a)$, then

$$\begin{split} \mathbb{U}(\mathbb{U}(x,y),z) &= \mathbb{U}(f^{(-1)}(\min\{f(x),f(y)\}),z), \\ &= \mathbb{U}(f^{(-1)}(f(y)),z) \\ &= \mathbb{U}(y,z) \\ &= f^{(-1)}(\min\{f(y),f(z)\}) \\ &= y. \\ \mathbb{U}(x,\mathbb{U}(y,z)) &= \mathbb{U}(x,f^{(-1)}(\min\{f(y),f(z)\})) \\ &= \mathbb{U}(x,y) \\ &= f^{(-1)}(\min\{f(x),f(y)\}) \\ &= y. \end{split}$$

2.1.3. If f(z) > f(a), then

$$\begin{split} \mathbb{U}(\mathbb{U}(x,y),z) &= \mathbb{U}(f^{(-1)}(\min\{f(x),f(y)\}),z), \\ &= \mathbb{U}(f^{(-1)}(f(y)),z) \\ &= \mathbb{U}(y,z) \\ &= f^{(-1)}(\min\{f(y),f(z)\}) \\ &= y. \\ \mathbb{U}(x,\mathbb{U}(y,z)) &= \mathbb{U}(x,f^{(-1)}(\min\{f(y),f(z)\})) \\ &= \mathbb{U}(x,y) \\ &= f^{(-1)}(\min\{f(x),f(y)\}) \\ &= y. \end{split}$$

2.2. $0 \le f(y) \le f(a)$. 2.2.1. If f(z) < 0, then

$$\begin{split} \mathbb{U}(\mathbb{U}(x,y),z) &= \mathbb{U}(z,\mathbb{U}(x,y)) & \text{(by the commutativity of } \mathbb{U}) \\ &= f^{(-1)}(f(z)). & \text{(by 1.2.2.1-1.2.2.2)} \\ \mathbb{U}(x,\mathbb{U}(y,z)) &= \mathbb{U}(x,f^{(-1)}(\min\{f(y),f(z)\})) \\ &= \mathbb{U}(x,z) \\ &= f^{(-1)}(\min\{f(x),f(z)\}) \\ &= f^{(-1)}(f(z)). \end{split}$$

2.2.2. If $0 \le f(z) \le f(a)$, then both $\min\{f(x) + f(y), f(a)\} \in Ran(f)$ and $\min\{f(y) + f(z), f(a)\}$ belong to Ran(f).

2.2.2.1 If $x, y, z \notin I_a$, then

$$\begin{split} \mathbb{U}(\mathbb{U}(x,y),z) &= \mathbb{U}(f^{(-1)}(\min\{f(x)+f(y),f(a)\}),z) \\ &= f^{(-1)}(\min\{f\circ f^{(-1)}(\min\{f(x)+f(y),f(a)\})+f(z),f(a)\}) \\ &= f^{(-1)}(\min\{(\min\{f(x)+f(y),f(a)\})+f(z),f(a)\}) \\ &= f^{(-1)}(\min\{(f(x)+f(y)+f(z),f(a)+f(z)\},f(a)\}) \\ &= f^{(-1)}(\min\{f(x)+f(y)+f(z),f(a)+f(z),f(a)\}) \\ &= f^{(-1)}(\min\{f(x)+f(y)+f(z),f(a)\}). \quad (\text{since } f(a)+f(z) \ge f(a)). \\ \mathbb{U}(x,\mathbb{U}(y,z)) &= \mathbb{U}(\mathbb{U}(y,z),x) \qquad (\text{by the commutativity of } \mathbb{U}) \\ &= f^{(-1)}(\min\{f(x)+f(y)+f(z),f(a)\}). \quad (\text{by } 2.2.2.) \end{split}$$

 $\begin{array}{l} 2.2.2.2 \text{ If } x,y,z \in I_a \text{ or } x,z \in I_a, \text{ then } \mathbb{U}(\mathbb{U}(x,y),z) = a = \mathbb{U}(x,\mathbb{U}(y,z)).\\ 2.2.2.3 \text{ If } y,z \in I_a \text{ and } x \notin I_a, \text{ then } \end{array}$

$$\mathbb{U}(\mathbb{U}(x,y),z) = a = f^{(-1)}(\min\{f(a) + f(x), f(a)\}) = \mathbb{U}(x,\mathbb{U}(y,z)).$$

2.2.2.4 The case $x, y \in I_a$ and $z \notin I_a$ is analogous to 2.2.2.3. 2.2.2.5 If $x, y \notin I_a$ and $z \in I_a$, then

$$\mathbb{U}(\mathbb{U}(x,y),z) = \mathbb{U}(f^{(-1)}(\min\{f(x) + f(y), f(a)\}, z) = a = \mathbb{U}(x,a) = \mathbb{U}(x,\mathbb{U}(y,z)).$$

2.2.2.6 Both the case $x \in I_a$ and $y, z \notin I_a$ and the case $y \in I_a$ and $x, z \notin I_a$ are analogous to 2.2.2.5. 2.2.3. f(a) < f(z).

2.2.3.1 If $x, y \notin I_a$, then from $f^{(-1)}(\min\{f(x) + f(y), f(a)\}) \in [0, f(a)]$, we have

$$\mathbb{U}(\mathbb{U}(x,y),z) = \mathbb{U}(f^{(-1)}(\min\{f(x) + f(y), f(a)\}), z) = a = \mathbb{U}(x,a) = \mathbb{U}(x,\mathbb{U}(y,z)).$$

2.2.3.2 If $x, y \in I_a$, then $\mathbb{U}(\mathbb{U}(x, y), z) = \mathbb{U}(a, z) = a = \mathbb{U}(x, a) = \mathbb{U}(x, \mathbb{U}(y, z))$. 2.2.3.3 If $x \notin I_a$ and $y \in I_a$, then $\mathbb{U}(\mathbb{U}(x, y), z) = \mathbb{U}(a, z) = a = \mathbb{U}(x, a) = \mathbb{U}(x, \mathbb{U}(y, z))$. 2.2.3.4 If $x \in I_a$ and $y \notin I_a$, then

$$\mathbb{U}(\mathbb{U}(x,y),z) = \mathbb{U}(a,z) = a = \mathbb{U}(x,a) = \mathbb{U}(x,\mathbb{U}(y,z)).$$

2.3 f(a) < f(y). 2.3.1 If f(z) < 0, then

$$\mathbb{U}(\mathbb{U}(x,y),z) = \mathbb{U}(a,z) = z = \mathbb{U}(x,f^{(-1)}(\min\{f(y),f(z))\}) = \mathbb{U}(x,\mathbb{U}(y,z)).$$

2.3.2. If $0 \le f(z) \le f(a)$, then from 2.2.3.1-2.2.3.4, we have

$$\mathbb{U}(\mathbb{U}(x,y),z)=\mathbb{U}(z,\mathbb{U}(x,y))=a=\mathbb{U}(x,a)=\mathbb{U}(x,\mathbb{U}(y,z)).$$

2.3.3. f(a) < f(z). 2.3.3.1 If $(f(y), f(z)) \in (f(a), f(e_2)]^2$, then

$$\mathbb{U}(\mathbb{U}(x,y),z) = \mathbb{U}(a,z) = a = \mathbb{U}(x,f^{(-1)}(\min\{f(y),f(z)\})) = \mathbb{U}(x,\mathbb{U}(y,z)).$$

2.3.3.2 If $(f(y), f(z)) \in [f(e_2), +\infty]^2 \cup [f(e_2), +\infty] \times (f(a), f(e_2)] \cup (f(a), f(e_2)] \times [f(e_2), +\infty]$, then $\mathbb{U}(\mathbb{U}(x, y), z) = \mathbb{U}(a, z) = a = \mathbb{U}(x, f^{(-1)}(\max\{f(y), f(z)\})) = \mathbb{U}(x, \mathbb{U}(y, z)).$ 3. f(a) < f(x). 3.1. f(y) < 0. 3.1.1. If f(z) < 0, then

$$\begin{split} \mathbb{U}(\mathbb{U}(x,y),z) &= \mathbb{U}(z,\mathbb{U}(x,y)) & \text{(by the commutativity of } \mathbb{U}) \\ &= \mathbb{U}(z,y) & \text{(by 1.3.1.)} \\ &= f^{(-1)}(f(y) + f(z)). \\ \mathbb{U}(x,\mathbb{U}(y,z)) &= \mathbb{U}(\mathbb{U}(y,z),x) & \text{(by the commutativity of } \mathbb{U}) \\ &= f^{(-1)}(f(y) + f(z)). & \text{(by 1.1.2.)} \end{split}$$

3.1.2. If $0 \le f(z) \le f(a)$, then

$$\begin{split} \mathbb{U}(\mathbb{U}(x,y),z) &= \mathbb{U}(z,\mathbb{U}(x,y)) & \text{(by the commutativity of }\mathbb{U}) \\ &= \mathbb{U}(z,y) & \text{(by 2.3.1.)} \\ &= f^{(-1)}(\min\{f(z),f(y)\}) \\ &= y. \\ \mathbb{U}(x,\mathbb{U}(y,z)) &= \mathbb{U}(\mathbb{U}(y,z),x) & \text{(by the commutativity of }\mathbb{U}) \\ &= y. & \text{(by 1.2.3.)} \end{split}$$

3.1.3 If f(a) < f(z), then

$$\begin{split} \mathbb{U}(\mathbb{U}(x,y),z) &= \mathbb{U}(y,z) \\ &= f^{(-1)}(\min\{f(y),f(z)\}) \\ &= y. \\ \mathbb{U}(x,\mathbb{U}(y,z)) &= \mathbb{U}(\mathbb{U}(y,z),x) \quad \text{(by the commutativity of }\mathbb{U}) \\ &= y. \quad \text{(by 1.3.3.)} \end{split}$$

3.2. $0 \le f(y) \le f(a)$.

3.2.1. If f(z) < 0, then $\mathbb{U}(\mathbb{U}(x, y), z) = \mathbb{U}(a, z) = z = \mathbb{U}(x, z) = \mathbb{U}(x, \mathbb{U}(y, z))$. 3.2.2. If $0 \le f(z) \le f(a)$, then by 2.3.2, 2.2.3 and the commutativity of \mathbb{U} , we have

$$\mathbb{U}(\mathbb{U}(x,y),z)=\mathbb{U}(z,\mathbb{U}(x,y))=a=\mathbb{U}(\mathbb{U}(y,z),x)=\mathbb{U}(x,\mathbb{U}(y,z)).$$

3.2.3. If f(a) < f(z), then $\mathbb{U}(\mathbb{U}(x, y), z) = \mathbb{U}(a, z) = a = \mathbb{U}(x, a) = \mathbb{U}(x, \mathbb{U}(y, z))$. 3.3. f(a) < f(y). 3.3.1. If f(z) < 0, then

$$\begin{split} \mathbb{U}(\mathbb{U}(x,y),z) &= \mathbb{U}(z,\mathbb{U}(x,y) & \text{(by the commutativity of }\mathbb{U}) \\ &= z. & \text{(by 1.3.3.)} \\ \mathbb{U}(x,\mathbb{U}(y,z)) &= \mathbb{U}(\mathbb{U}(y,z),x) & \text{(by the commutativity of }\mathbb{U}) \\ &= z. & \text{(by 3.1.3.)} \end{split}$$

3.3.2. If $0 \le f(z) \le f(a)$, then

$$\begin{split} \mathbb{U}(\mathbb{U}(x,y),z) &= \mathbb{U}(z,\mathbb{U}(x,y) & \text{(by the commutativity of }\mathbb{U}) \\ &= a. & \text{(by 2.3.3.)} \\ \mathbb{U}(x,\mathbb{U}(y,z)) &= \mathbb{U}(\mathbb{U}(y,z),x) & \text{(by the commutativity of }\mathbb{U}) \\ &= a. & \text{(by 3.2.3.)} \end{split}$$

3.3.3.2 If exactly one of f(x), f(y), f(z) belongs to $(f(e_2), +\infty]$, say $f(x) \in (f(e_2), +\infty]$, then

$$\begin{split} \mathbb{U}(\mathbb{U}(x,y),z) &= \mathbb{U}(f^{(-1)}(\max\{f(x),f(y)\}),z) \\ &= U(x,z) \\ &= f^{(-1)}(\max\{f(x),f(z)\}) \\ &= x. \\ \mathbb{U}(x,\mathbb{U}(y,z)) &= \mathbb{U}(x,f^{(-1)}(\min\{f(y),f(z)\})) \\ &= f^{(-1)}(\max\{f(x),f\circ f^{(-1)}(\min\{f(y),f(z)\})\}) \\ &= x. \end{split}$$

3.3.3.3 If at least two of f(x), f(y), f(z) belong to $(f(e_2), +\infty]$, then

$$\begin{split} \mathbb{U}(\mathbb{U}(x,y),z) &= \mathbb{U}(f^{(-1)}(\max\{f(x),f(y)\}),z) \\ &= f^{(-1)}(\max\{f\circ f^{(-1)}(\max\{f(x),f(y)\}),f(z)\}) \\ &= f^{(-1)}(\max\{f(x),f(y),f(z)\}. \\ \mathbb{U}(x,\mathbb{U}(y,z)) &= \mathbb{U}(x,f^{(-1)}(\max\{f(y),f(z)\})) \\ &= f^{(-1)}(\max\{f(x),f\circ f^{(-1)}(\max\{f(y),f(z)\})) \\ &= f^{(-1)}(\max\{f(x),f(y),f(z)\}. \end{split}$$

Hence, \mathbb{U} is associative.

Consequently, \mathbb{U} is a 2-uninorm on L.

The following theorem is a dual consequence of Theorem 3.1.

Theorem 3.2. Let $e_1, a, e_2 \in L$ with $0_L \leq e_1 \leq a \leq e_2 \leq 1_L$, and $f : L \to [-\infty, +\infty]$ be an injective decreasing function with $f(e_1) = 0$. If f satisfies the following five conditions: for all $x, y \in L$,

(i) if $(f(x), f(y)) \in [f(a), 0]^2 \cup [0, +\infty]^2$ then

$$\max\{f(x) + f(y), f(a)\} \in Ran(f) \cup (f(0_L), +\infty],$$

- (ii) for all $(f(x), f(y)) \in [f(a), 0]^2 \cup [0, +\infty]^2$, if f(x) and f(y) have at least one same summand $f(z) \in Ran(f)$, then $x \not\parallel y$,
- (iii) if $f(a) \leq f(x) < 0 < f(y)$ and $e_1 < y$, then $x \not\parallel y$,
- (iv) if $f(a) \ge f(x) > f(y)$ or $f(x) \le f(a) < 0 < f(y)$, then $x \not\parallel y$,
- (v) for all $f(x), f(y) \in (f(a), 0)$, if $f(x) + f(y) \in Ran(f)$ and $f(a) \leq f(x) + f(y) < 0$ then $f^{-1}(f(x) + f(y)) \not| a$,

then the following function $\mathbb{U}_d: L^2 \to L$ is a 2-uninorm, and we call f an additive generator of \mathbb{U}_d . For all $x, y \in L$,

$$\mathbb{U}_{d}(x,y) = \begin{cases} f^{(-1)}(f(x) + f(y)), & (f(x), f(y)) \in [0, +\infty]^{2}, \\ f^{(-1)}(\max\{f(x) + f(y), f(a)\}), & (f(x), f(y)) \in [f(a), 0]^{2} \text{ and } x, y \notin I_{a}, \\ a, & (f(x), f(y)) \in [-\infty, f(a)) \times [f(a), 0] \\ & \cup [f(a), 0] \times [-\infty, f(a)) \\ & \cup [f(a), 0] \times \mathcal{G}_{a} \cup \mathcal{G}_{a} \times [f(a), 0], \\ f^{(-1)}(\min\{f(x), f(y)\}), & (f(x), f(y)) \in [-\infty, f(e_{2}))^{2} \cup [-\infty, f(e_{2})) \\ & \times [f(e_{2}), f(a)) \cup [f(e_{2}), f(a)) \times [-\infty, f(e_{2})), \\ f^{(-1)}(\max\{f(x), f(y)\}), & otherwise. \end{cases}$$
(3.3)

From Remark 2.2, we have the following corollary.

Corollary 3.1. (i) Taking $e_2 = 1_L$ in Theorem 3.1, we obtain the uni-nullnorm U_N as follows. For all $x, y \in L$,

$$U_N(x,y) = \begin{cases} f^{(-1)}(f(x) + f(y)), & (f(x), f(y)) \in [-\infty, 0]^2, \\ f^{(-1)}(\min\{f(x) + f(y), f(a)\}), & (f(x), f(y)) \in [0, f(a)]^2 \text{ and } x, y \notin I_a, \\ a, & (f(x), f(y)) \in [f(a), +\infty) \times [0, f(a)] \\ & \cup [0, f(a)] \times [f(a), +\infty) \\ & \cup [0, f(a)] \times \mathcal{F}_a \cup \mathcal{F}_a \times [0, f(a)], \\ f^{(-1)}(\min\{f(x), f(y)\}), & otherwise. \end{cases}$$

(ii) Taking $e_2 = 1_L$ in Theorem 3.2, we have the uni-nullnorm U_N as follows. For all $x, y \in L$,

$$U_N(x,y) = \begin{cases} f^{(-1)}(f(x) + f(y)), & (f(x), f(y)) \in (0, +\infty]^2, \\ f^{(-1)}(\max\{f(x) + f(y), f(a)\}), & (f(x), f(y)) \in [f(a), 0]^2 \text{ and } x, y \notin I_a, \\ a, & (f(x), f(y)) \in [-\infty, f(a)) \times [f(a), 0] \\ & \cup [f(a), 0] \times [-\infty, f(a)) \\ & \cup [f(a), 0] \times \mathcal{G}_a \cup \mathcal{G}_a \times [f(a), 0], \\ f^{(-1)}(\max\{f(x), f(y)\}), & otherwise. \end{cases}$$

(iii) Taking $e_1 = 0_L$ in Theorem 3.1, we get the null-uninorm as below. For all $x, y \in L$,

$$N_{U}(x,y) = \begin{cases} f^{(-1)}(\min\{f(x) + f(y), f(a)\}), & (f(x), f(y)) \in [0, f(a)]^{2} \text{ and } x, y \notin I_{a}, \\ a, & (f(x), f(y)) \in [f(a), +\infty) \times [0, f(a)] \\ & \cup [0, f(a)] \times [f(a), +\infty) \\ & \cup [0, f(a)] \times \mathcal{F}_{a} \cup \mathcal{F}_{a} \times [0, f(a)], \end{cases}$$
$$f^{(-1)}(\min\{f(x), f(y)\}), & (f(x), f(y)) \in [f(a), f(e_{2})]^{2}, \\ f^{(-1)}(\max\{f(x), f(y)\}), & otherwise. \end{cases}$$

In this case, f is an injective increasing function from L to $[0, +\infty]$ with $f(0_L) = 0$ and it satisfies the conditions (i), (ii), (v) in Theorem 3.1 and (iv') if $f(a) \leq f(x) < f(y)$, then $x \not\parallel y$.

(iv) Taking $e_1 = 0_L$ in Theorem 3.2, we have the null-uninorm as below. For all $x, y \in L$,

$$N_{U}(x,y) = \begin{cases} f^{(-1)}(\max\{f(x) + f(y), f(a)\}), & (f(x), f(y)) \in [f(a), 0]^{2} \text{ and } x, y \notin I_{a}, \\ a, & (f(x), f(y)) \in [-\infty, f(a)) \times [f(a), 0] \\ & \cup [f(a), 0] \times [-\infty, f(a)) \\ & \cup [f(a), 0] \times \mathcal{G}_{a} \cup \mathcal{G}_{a} \times [f(a), 0] \\ f^{(-1)}(\max\{f(x), f(y)\}), & (f(x), f(y)) \in [f(e_{2}), f(a)]^{2}, \\ f^{(-1)}(\min\{f(x), f(y)\}), & otherwise. \end{cases}$$

In this case, f is an injective decreasing function from L to $[-\infty, 0]$ with $f(0_L) = 0$ and it satisfies the conditions (i), (ii), (v) in Theorem 3.2 and (iv") if $f(x) < f(y) \le f(a)$, then $x \not\parallel y$.

(v) Taking $e_1 = 0_L$ and $e_2 = 1_L$ in Theorem 3.1, we get the nullnorm as follows. For all $x, y \in L$,

$$N(x,y) = \begin{cases} f^{(-1)}(\min\{f(x) + f(y), f(a)\}), & (f(x), f(y)) \in [0, f(a)]^2 \text{ and } x, y \notin I_a, \\ a, & (f(x), f(y)) \in [f(a), +\infty) \times [0, f(a)] \\ & \cup [0, f(a)] \times [f(a), +\infty) \\ & \cup [0, f(a)] \times \mathcal{F}_a \cup \mathcal{F}_a \times [0, f(a)], \\ f^{(-1)}(\min\{f(x), f(y)\}), & otherwise. \end{cases}$$

In this case, f is an injective increasing function from L to $[0, +\infty]$ with $f(0_L) = 0$ and it satisfies the conditions (i), (ii), (v) in Theorem 3.1 and (iv').

(vi) Taking $e_1 = 0_L$ and $e_2 = 1_L$ in Theorem 3.2, we obtain the nullnorm as below. For all $x, y \in L$,

$$N(x,y) = \begin{cases} f^{(-1)}(\max\{f(x) + f(y), f(a)\}), & (f(x), f(y)) \in [f(a), 0]^2 \text{ and } x, y \notin I_a, \\ a, & (f(x), f(y)) \in [-\infty, f(a)) \times [f(a), 0] \\ & \cup [f(a), 0] \times [-\infty, f(a)) \\ & \cup [f(a), 0] \times \mathcal{G}_a \cup \mathcal{G}_a \times [f(a), 0], \\ f^{(-1)}(\max\{f(x), f(y)\}), & otherwise. \end{cases}$$

In this case, f is an injective decreasing function from L to $[-\infty, 0]$ with $f(0_L) = 0$ and it satisfies the conditions (i), (ii), (v) in Theorem 3.2 and (iv").

(vii) Taking $a = 1_L$ in Theorem 3.1, we have the uninorm U in [12] as follows. For all $x, y \in L$,

$$U(x,y) = \begin{cases} f^{(-1)}(f(x) + f(y)), & (f(x), f(y)) \in [-\infty, 0]^2 \cup [0, +\infty]^2, \\ f^{(-1)}(\min\{f(x), f(y)\}), & otherwise. \end{cases}$$
(3.4)

In this case, f just needs to satisfy the conditions (i), (ii) and (iii). Subsequently, we further obtain the t-norm and t-conorm given in [26] by taking $e_1 = 1_L$ and $e_1 = 0_L$ in Formula (3.4), respectively.

(viii) Taking $a = 1_L$ in Theorem 3.2, we deduce the uninorm U as below. For all $x, y \in L$,

$$U(x,y) = \begin{cases} f^{(-1)}(f(x) + f(y)), & (f(x), f(y)) \in [-\infty, 0]^2 \cup [0, +\infty]^2, \\ f^{(-1)}(\max\{f(x), f(y)\}), & otherwise. \end{cases}$$

In this case, f just needs to satisfy the conditions (i), (ii) and (iii).

Remark 3.1.

- (i) Since we require the functions f in both Theorems 3.1 and 3.2 to be injective, it is obviously impossible to choose a suitable $f: L \to [-\infty, +\infty]$ when the cardinality of L is strictly greater than \aleph_1 .
- (ii) The following are two alternative forms of (3.1) and (3.3), respectively. For all $x, y \in L$,

$$\mathbb{U}(x,y) = \begin{cases} f^{-1}(f(x) + f(y)), & \text{either } (f(x), f(y)) \in [-\infty, 0]^2 \text{ and } f(x) + f(y) \in Ran(f) \\ & \text{or } (f(x), f(y)) \in [0, f(a)]^2, f(x) + f(y) \leq f(a), f(x) + f(y) \in Ran(f) \\ & \text{and } x, y \notin I_a; \end{cases} \\ 0_L, & (f(x), f(y)) \in [-\infty, 0]^2 \text{ and } f(x) + f(y) \in [-\infty, f(0_L)); \\ a, & \text{either } (f(x), f(y)) \in [0, f(a)]^2 \text{ and } f(x) + f(y) \geq f(a), \\ & \text{or } (f(x), f(y)) \in [f(a), +\infty] \times [0, f(a)] \cup [0, f(a)] \times [f(a), +\infty] \\ & \text{or } (f(x), f(y)) \in [0, f(a)] \times \mathcal{F}_a \cup \mathcal{F}_a \times [0, f(a)]; \\ x, & \text{either } (f(x), f(y)) \in [-\infty, 0] \times [0, +\infty], \\ & \text{or } (f(x), f(y)) \in [f(e_2), +\infty] \times [f(a), f(e_2)], \\ & \text{or } (f(x), f(y)) \in [f(e_2), +\infty] \times [f(a), f(e_2)], \\ & \text{or } (f(x), f(y)) \in [f(e_2), +\infty]^2 \text{ and } f(x) \leq f(y) \\ & \text{or } (f(x), f(y)) \in [f(a), f(e_2)] \times [f(e_2), +\infty], \\ & \text{or } (f(x), f(y)) \in [f(a), f(e_2)] \times [f(e_2), +\infty], \\ & \text{or } (f(x), f(y)) \in [f(a), f(e_2)]^2 \text{ and } f(x) \geq f(y), \\ & \text{or } (f(x), f(y)) \in [f(a), f(e_2)] \times [f(e_2), +\infty], \\ & \text{or } (f(x), f(y)) \in [f(a), f(e_2)]^2 \text{ and } f(x) \geq f(y) \\ & \text{or } (f(x), f(y)) \in [f(a), f(e_2)]^2 \text{ and } f(x) \geq f(y), \\ & \text{or } (f(x), f(y)) \in [f(a), f(e_2)]^2 \text{ and } f(x) \geq f(y) \\ & \text{or } (f(x), f(y)) \in [f(a), f(e_2)]^2 \text{ and } f(x) \geq f(y) \\ & \text{or } (f(x), f(y)) \in [f(e_2), +\infty]^2 \text{ and } f(x) \leq f(y). \end{cases}$$

For all $x, y \in L$,

$$\mathbb{U}_{d}(x,y) = \begin{cases} f^{-1}(f(x) + f(y)), & \text{either } (f(x), f(y)) \in [0, +\infty]^{2} \text{ and } f(x) + f(y) \in Ran(f) \\ & \text{or } (f(x), f(y)) \in [f(a), 0]^{2} \text{ and } f(x) + f(y) \geq f(a) \text{ and } \\ & f(x) + f(y) \in Ran(f) \text{ and } x, y \notin I_{a}; \\ 0_{L}, & (f(x), f(y)) \in [0, +\infty]^{2} \text{ and } f(x) + f(y) \in [f(0_{L}), +\infty); \\ a, & \text{either } (f(x), f(y)) \in [f(a), f(e_{1})] \times [-\infty, f(a)] \\ & \cup [-\infty, f(a)] \times [f(a), f(e_{1})], \\ & \text{or } (f(x), f(y)) \in [0, f(a)]^{2} \text{ and } x \notin I_{a} \text{ and } y \notin I_{a} \\ & \text{or } (f(x), f(y)) \in [f(a), 0] \times \mathcal{F}_{a} \cup \mathcal{F}_{a} \times [f(a), 0]; \\ x, & \text{either } (f(x), f(y)) \in [f(a), 0] \times \mathcal{F}_{a} \cup \mathcal{F}_{a} \times [f(a), 0]; \\ & \text{or } (f(x), f(y)) \in [f(e_{2}), f(a)] \times [-\infty, f(e_{2})], \\ & \text{or } (f(x), f(y)) \in [f(e_{2}), f(a)]^{2} \text{ and } f(x) \geq f(y) \\ & \text{or } (f(x), f(y)) \in [-\infty, f(e_{2})]^{2} \text{ and } f(x) \geq f(y); \\ & \text{or } (f(x), f(y)) \in [-\infty, f(e_{2})] \times [f(e_{2}), f(a)], \\ & \text{or } (f(x), f(y)) \in [-\infty, f(e_{2})] \times [f(e_{2}), f(a)], \\ & \text{or } (f(x), f(y)) \in [f(e_{2}), f(a)]^{2} \text{ and } f(x) \leq f(y) \\ & \text{or } (f(x), f(y)) \in [-\infty, f(e_{2})] \times [f(e_{2}), f(a)], \\ & \text{or } (f(x), f(y)) \in [-\infty, f(e_{2})]^{2} \text{ and } f(x) \leq f(y) \\ & \text{or } (f(x), f(y)) \in [-\infty, f(e_{2})]^{2} \text{ and } f(x) \leq f(y). \\ \end{array}$$

(iii) If L = [0, 1], then f in Theorem 3.1 only needs to satisfy the condition (i) since $x \not\parallel y$ for any $x, y \in [0, 1]$,

and the following function $\mathbb{U}(x, y)$ is a 2-uninorm on [0, 1]. For all $x, y \in [0, 1]$,

$$\mathbb{U}(x,y) = \begin{cases} f^{(-1)}(\min\{f(x) + f(y), f(a)\}), & (f(x), f(y)) \in [-\infty, 0]^2 \cup [0, f(a)]^2; \\ a, & (f(x), f(y)) \in (f(a), +\infty] \times [0, f(a)] \\ & \cup [0, f(a)] \times (f(a), +\infty]; \\ f^{(-1)}(\max\{f(x), f(y)\}, & (f(x), f(y)) \in (f(e_2), +\infty]^2 \cup (f(e_2), +\infty] \\ & \times (f(a), f(e_2)] \cup (f(a), f(e_2)] \times (f(e_2), +\infty]; \\ f^{(-1)}(\min\{f(x), f(y)\}), & \text{otherwise.} \end{cases}$$
(3.5)

The following two examples illustrate Theorem 3.1.

Example 3.1. Consider the lattice $L_1 = \{0_{L_1}, x_1, x_2, x_3, x_4, e_1, x_5, x_6, a, x_7, x_8, x_9, e_2, x_{10}, x_{11}, 1_{L_1}\}$ given in Fig. 1 and the injective increasing function f defined by Table 1. One can check that the function f satisfies Theorem 3.1, and the 2-uninorm \mathbb{U} is shown by Table 2.

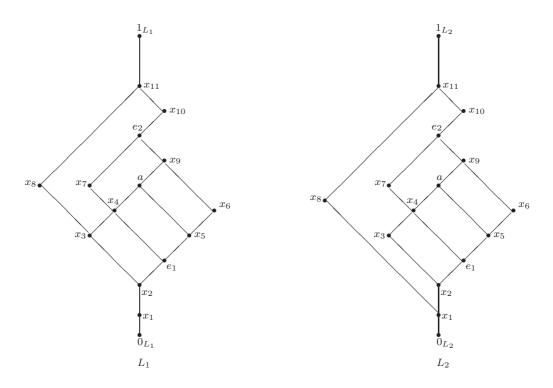


Figure 1. Two bounded lattices L_1 and L_2

Table 1 The gener	rator	: <i>f</i> .															
	x	0_{L_1}	x_1	x_2	x_3	x_4	e_1	x_5	x_6	a	x_7	x_8	x_9	e_2	x_{10}	x_{11}	1_{L_1}
	f	-11	-8	-4	6	12	0	9	14	15	13	11	17	18	20	22	24

The 2-uninorm \mathbb{U} .	Tab	le 2	
	The	2-uninorm	$\mathbb{U}.$

\mathbb{U}	0_{L_1}	x_1	x_2	x_3	x_4	e_1	x_5	x_6	a	x_7	x_8	x_9	e_2	x_{10}	x_{11}	1_{L_1}
0_{L_1}	0_{L_1}	$0_{L_{1}}$	0_{L_1}													
x_1	0_{L_1}	0_L	0_L	x_1	x_1											
x_2	0_{L_1}	0_L	x_1	x_2	x_2											
x_3	0_{L_1}	x_1	x_2	x_4	a	x_3	a	a	a	a	a	a	a	a	a	a
x_4	0_{L_1}	x_1	x_2	a	a	x_4	a	a	a	a	a	a	a	a	a	a
e_1	0_{L_1}	x_1	x_2	x_3	x_4	e_1	x_5	a	a	a	a	a	a	a	a	a
x_5	0_{L_1}	x_1	x_2	a	a	x_5	a	a	a	a	a	a	a	a	a	a
x_6	0_{L_1}	x_1	x_2	a	a	a	a	a	a	a	a	a	a	a	a	a
a	0_{L_1}	x_1	x_2	a	a	a	a	a	a	a	a	a	a	a	a	a
x_7	0_{L_1}	x_1	x_2	a	a	a	a	a	a	a	a	a	a	a	a	a
x_8	0_{L_1}	x_1	x_2	a	a	a	a	a	a	a	a	a	a	a	a	a
x_9	0_{L_1}	x_1	x_2	a	a	a	a	a	a	a	a	x_9	x_9	x_{10}	x_{11}	1_{L_1}
e_2	0_{L_1}	x_1	x_2	a	a	a	a	a	a	a	a	x_9	e_2	x_{10}	x_{11}	1_{L_1}
x_{10}	0_{L_1}	x_1	x_2	a	a	a	a	a	a	a	a	x_{10}	x_{10}	x_{10}	x_{11}	1_{L_1}
x_{11}	0_{L_1}	x_1	x_2	a	a	a	a	a	a	a	a	x_{11}	x_{11}	x_{11}	x_{11}	1_{L_1}
1_L	0_{L_1}	x_1	x_2	a	a	a	a	a	a	a	a	1_{L_1}	1_{L_1}	1_{L_1}	1_{L_1}	1_{L_1}

Example 3.2. Consider the bounded lattice (L, \leq) as shown in Fig. 2. Let $e_1 = x_4$, $e_2 = x_9$ and $a = x_7$. Then the function $f : L \to [-\infty, +\infty]$ defined by $f(x_i) = i - 4$ is an injective increasing function with $f(x_4) = 0$. Further, we have:

- (i) If $i \in [0,3]$, then $f(x_i) \in [-4,-1]$ and we have $f(x_i) + f(x_j) \in [-\infty, f(x_0)) \cup [-4,-2]$ for all $i, j \in [0,3] \cup \{4\}$. If $i \in [5,7]$, then $f(x_i) \in [1,3]$ and we have $\min\{f(x_i) + f(x_j), f(x_7)\} \in [2,3]$ for all $i, j \in [5,7] \cup \{4\}$. Hence the condition (i) in Theorem 3.1 is satisfied since $[-4,-2] \cup [2,3] \subseteq Ran(f)$.
- (ii) For all $(f(x_i), f(x_j)) \in [0, f(x_7)]^2 \cup [-\infty, 0]^2$, if $f(x_i)$ and $f(x_j)$ have at least one same summand $f(x_k) \in Ran(f)$ then we have $i, j \in [0, 2]$, or $i, j \in [6, 7]$, or $i \in [0, 2]$ and $j \in [2, 3]$, or $i \in [5, 5.5]$ and $j \in [6, 7]$, subsequently, $x_i \not\models x_j$ by Fig. 2. Therefore, the condition (ii) in Theorem 3.1 is satisfied.
- (iii) For all $f(x_i) < 0$ and $0 < f(x_j) \le f(x_7)$, we have $i \in [0,3]$ and $j \in [5,7]$. Then $x_i \not| x_j$ by Fig. 2 and the condition (iii) in Theorem 3.1 is satisfied.
- (iv) For all $f(a) \leq f(x_i) < f(x_j)$, we have $i, j \in [7, 10]$. Then $x_i \not| x_j$ by Fig. 2. For all $f(x_i) < 0 < f(a) \leq f(x_j)$, we have $i \in [0, 3]$ and $j \in [7, 10]$. Thus $x_i \not| x_j$ by Fig. 2. Therefore, the condition (iv) in Theorem 3.1 is satisfied.
- (v) For all $f(x_i), f(x_j) \in (0, f(x_7))$, if $f(x_k) = f(x_i) + f(x_j) \in Ran(f)$ and $0 < f(x_i) + f(x_j) \le f(x_7)$, then $f(x_k) = f(x_i) + f(x_j) \in [2, 3]$. Subsequently, $k \in [6, 7]$. Then the condition (v) in Theorem 3.1 is easily checked since $x_k \not| x_7$ by Fig. 2, i.e., $f^{-1}(f(x_i) + f(x_j)) \not| x_8$.

So that by Theorem 3.1, the function f is an additive generator of the 2-uninorm \mathbb{U} given by: for all $x_i, x_j \in L$,

$$\mathbb{U}(x_i, x_j) = \begin{cases} x_{\max\{i+j-4,0\}}, & (x_i, x_j) \in ([x_0, x_4] \cup I_{x_4})^2, \\ x_{\min\{i+j-4,7\}}, & (x_i, x_j) \in [x_4, x_7]^2, \\ \min\{x_i, x_j\}, & (x_i, x_j) \in [x_7, x_9]^2 \cup [x_0, x_4) \times [x_4, x_{10}] \cup [x_4, x_{10}] \times [x_0, x_4), \\ \max\{x_i, x_j\}, & (x_i, x_j) \in [x_9, x_{10}]^2 \cup (x_9, x_{10}] \times (x_7, x_9] \cup (x_7, x_9] \times (x_9, x_{10}], \\ x_7, & \text{otherwise.} \end{cases}$$

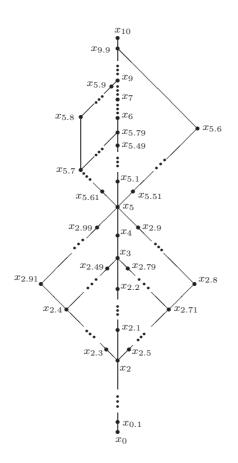


Figure 2. A bounded lattice L

The following example shows that each of the conditions (i) - (v) in Theorems 3.1 can't be dropped, respectively.

Example 3.3. Consider the two bounded lattices L_1 and L_2 in Fig. 1 and the five injective increasing functions f_i , i = 1, 2, 3, 4, 5, with $f_i(e_1) = 0$ defined by Tables 3, 4, 5, 6 and 7, respectively.

Table 3

The generator f_1

The gene	erator	f_1 .															
	x	0_{L_1}	x_1	x_2	x_3	x_4	e_1	x_5	x_6	a	x_7	x_8	x_9	e_2	x_{10}	x_{11}	1_{L_1}
	f_1	-11	-8	-4	6	10	0	9	14	15	13	11	17	18	20	22	24
Table 4																	
The generator f_2 .																	
	x	0_{L_1}	x_1	x_2	x_3	x_4	e_1	x_5	x_6	a	x_7	x_8	x_9	e_2	x_{10}	x_{11}	1_{L_1}
	f_2	-11	-8	-4	6	9	0	12	14	15	13	11	17	18	20	22	24
Table 5																	
The gene	erator	f_3 .															
	x	0_{L_2}	x_1	x_2	x_3	x_4	e_1	x_5	x_6	a	x_7	x_8	x_9	e_2	x_{10}	x_{11}	1_{L_2}
	f_3	-11	-8	-4	6	12	0	9	14	15	13	11	17	18	20	22	24

Table 6

The generator f_4 .

x	0_{L_1}	x_1	x_2	x_3	x_4	e_1	x_5	x_6	a	x_7	x_8	x_9	e_2	x_{10}	x_{11}	1_{L_1}
f_4	-11	-8	-4	6	12	0	9	14	15	13	19	17	18	20	22	24

Table 7

The gene	erator	f_5 .															
	x	0_{L_1}	x_1	x_2	x_3	x_4	e_1	x_5	x_6	a	x_7	x_8	x_9	e_2	x_{10}	x_{11}	1_{L_1}
	f_5	-11	-8	-4	9	13	0	6	12	15	14	11	17	18	20	22	24

- (i) One can check that $f_1 : L_1 \to [-11, 24]$ satisfies (ii), (iii), (iv) and (v), but it doesn't satisfy (i) in Theorem 3.1 since $f_1(x_3) + f_1(x_3) = 12 \notin Ran(f_1) \cup [-\infty, f_1(0_L)]$. Applying Formula (3.1), we know that $\mathbb{U}(\mathbb{U}(x_3, x_3), x_4) = \mathbb{U}(e_1, x_4) = x_4 \neq a = \mathbb{U}(x_3, a) = \mathbb{U}(x_3, \mathbb{U}(x_3, x_4))$. Thus \mathbb{U} isn't a 2-uninorm.
- (ii) Also, one can easily verify that $f_2: L_1 \to [-11, 24]$ satisfies (i), (iii), (iv) and (v), but it doesn't satisfy (ii) of Theorem 3.1 since both $f_2(x_5)$ and $f_2(x_3)$ have a same summand $f_2(x_3)$ but $x_3 \parallel x_5$. By using Formula (3.1), we have $\mathbb{U}(x_3, e_1) = x_3 \parallel x_5 = \mathbb{U}(x_3, x_3)$ while $e_1 \leq x_3$. It follows that \mathbb{U} isn't a 2-uninorm.
- (iii) One can clarify that $f_3: L_2 \rightarrow [-11, 24]$ satisfies (i),(ii),(iv) and (v), but f_3 doesn't satisfy (iii) in Theorem 3.1 since $f_3(x_2) < 0, x_2 < e_1$ but $x_2 \parallel x_8$. By using Formula (3.1), we obtain $\mathbb{U}(x_2, x_8) = x_2$ and $\mathbb{U}(e_1, x_8) = x_8$. Thus $\mathbb{U}(x_2, x_8) \parallel \mathbb{U}(e_1, x_8)$ while $x_2 < e_1$. It follows that \mathbb{U} isn't a 2-uninorm.
- (iv) One can prove that $f_4 : L_1 \to [-11, 24]$ satisfies (i), (ii), (iii) and (v), but f_3 doesn't satisfy (iv) in Theorem 3.1 since $f_4(a) < f_4(x_9) < f_4(x_{10})$, but $x_8 \parallel x_{10}$. By using Formula (3.1), one has $\mathbb{U}(x_9, x_8) = x_8$ and $\mathbb{U}(x_{10}, x_8) = x_{10}$. Thus $\mathbb{U}(x_9, x_8) \parallel \mathbb{U}(x_{10}, x_8)$ while $x_9 < x_{10}$. It follows that \mathbb{U} isn't a 2-uninorm.
- (v) One can check that $f_5: L_1 \to [-11, 24]$ satisfies (i), (ii), (iii) and (iv), but f_5 doesn't satisfy (v) in Theorem 3.1 since $0 < f_5(x_5) < f_5(a)$, but $x_6 \not\parallel a$. By using Formula (3.1), one knows $\mathbb{U}(x_5, x_5) = x_6$ and $\mathbb{U}(x_5, x_6) = a$. Thus $\mathbb{U}(x_5, x_5) = f_5^{-1}(f_5(x_5) + f_5(x_5)) = x_6 \parallel a = \mathbb{U}(x_5, x_6)$ while $x_5 < x_6$. It follows that \mathbb{U} isn't a 2-uninorm.

4. Conclusions

In this article, we mainly presented two construction methods of 2-uninorms on bounded lattices by using additive generators, and also supplied two examples to show that the existence of such additive generators on bounded lattices. It is worth pointing out that uninorms, nullnorms, uni-nullnorms, null-uninorms are all special 2-uninorms, respectively. Consequently, we can obtain their additive generators. Dually, one may discuss the multiplicative generators of these binary operations. It is an interesting problem to further find some lax conditions in both Theorems 3.1 and 3.2 by modifying Formulas (3.1) and (3.3), respectively.

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