# Constructing 2-uninorms on bounded lattices by using additive generators ${ }^{\star}$ 

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#### Abstract

In this article, we present two methods to construct 2 -uninorms on bounded lattices by using additive generators, which are further used for inducing uninorms, nullnorms, uni-nullnorms and null-uninorms, respectively. We also provide some examples for illustrating the constructing methods of 2 -uninorms.


Keywords: Bounded lattice; Additive generator; Uninorm; 2-uninorm

## 1. Introduction

Triangular norms (t-norms for short) and triangular conorms (t-conorms for short) are associative, commutative and monotone binary operations with the neutral element 1 and 0 , respectively. Schweizer and Sklar [21] studied t-norms and t-conorms on [0, 1] based on Menger's notion 20. T-norms and t-conorms have been proved to be useful in many fields like fuzzy set theory [18], fuzzy logic [2], fuzzy systems modeling [34], and probabilistic metric spaces [22, 23].

As important generalizations of t -norms and t -conorms, the concepts of uninorms and nullnorms were introduced on the unit interval [0, 1] by Yager and Rybalov [33] and Calvo et al. 5], respectively. Uninorms allow for a neutral element anywhere in the unit interval, whereas nullnorms allow for a zero element $k$ anywhere in the unit interval, while keeping 0 as neutral element on $[0, k]$ and 1 as neutral element on $[k, 1]$. A series of works have been done for uninorms [7, 9, 14, 17] and nullnorms [6, 8, 25, 31, 36], respectively.

To unify uninorms and nullnorms, 2-uninorms were first investigated by Akella [1]. Since then, some properties of 2-uninorms on the unit interval are studied. For example, Drygaś and Rak [10] solved the functional equations of distributivity between 2-uninorms. Wang and Qin 28] studied the distributivity equations for 2 -uninorms over semi-uninorms. Zong et al. 37] described the structures of 2 -uninorms. Then Sun and Liu [24] investigated the left (resp. right) distributivity of semi-t-operators over 2-uninorms. Almost at the same time, Zhang and Qin [35] obtained some sufficient and necessary conditions of the distributivity equations between five classes of basic 2 -uninorms and overlap (resp. grouping) functions. Wang et al. [30] introduced the discrete 2-uninorms. Huang and Qin [13] made a deep study on the migrativity of uninorms over 2-uninorms.

As a bounded lattice is more general than $[0,1]$, the study of 2 -uninorms on the unit interval has already been extended to bounded lattices. For instance, Ertuğrul 11] and Xie and Yi 32] gave the constructions of 2-uninorms. Recently, Sun and Liu 26] explored the additive generators of t-norms and t-conorms. Also, He and Wang [12] studied the additive generators of uninorms, and they even extended the classical additive generators to partially ordered cases by adding some conditions. As 2-uninorms are generalizations

[^0]of uninorms, this leads to a natural question: Could we construct 2-uninorms on bounded lattices by using additive generators? This article will focus on this question.

The remainder of this article is organized as follows. In Section 2 we provide the necessary background material. Section 3 is devoted to the constructions of 2 -uninorms on bounded lattices based on additive generators. Finally, conclusions are drawn in Section 4

## 2. Preliminaries

This section presents some basic definitions and results that are used latter.
A lattice [4] is a nonempty set $L$ equipped with a partial order $\leq$ such that any two elements $x$ and $y$ have a greatest lower bound (called meet or infimum), denoted by $x \wedge y$, as well as a smallest upper bound (called join or supremum), denoted by $x \vee y$. A lattice is called bounded if it has a top element $1_{L}$ and a bottom element $0_{L}$. For short, we use the notation $L$ instead of a bounded lattice $\left(L, \leq, 0_{L}, 1_{L}\right)$ throughout this article. Let $x, y \in L$. The elements $x$ and $y$ are called comparable if $x \leq y$ or $y \leq x$. Otherwise, $x$ and $y$ are called incomparable. The symbol $x \| y$ is used when $x$ and $y$ are incomparable. If $x$ and $y$ are comparable, then we use the symbol $x \nVdash y$. In what follows, $I_{a}$ denotes the set of all incomparable elements with $a \in L$, i.e., $I_{a}=\{x \in L: x \| a\}$. Let $a, b \in L$ with $a \leq b$. An interval $[a, b]$ is defined as $[a, b]=\{x \in L \mid a \leq x \leq b\}$, other intervals can be defined similarly, $(a, b]=\{x \in L \mid a<x \leq b\},[a, b)=\{x \in L \mid a \leq x<b\}$, $(a, b)=\{x \in L \mid a<x<b\}$.

Definition 2.1 ([3, [19] ).
(i) A binary operation $T: L^{2} \rightarrow L$ is called a $t$-norm if it is commutative, associative, and increasing with respect to both variables and it satisfies $T\left(x, 1_{L}\right)=x$ for all $x \in L$.
(ii) A binary operation $S: L^{2} \rightarrow L$ is called a $t$-conorm if it is commutative, associative, and increasing with respect to both variables and it satisfies $S\left(x, 0_{L}\right)=x$ for all $x \in L$.

Definition 2.2 ( $[\mathbf{1 6}, \boxed{33}])$. A binary operation $U: L^{2} \rightarrow L$ is called a uninorm if it has commutativity, associativity, and increasing with respect to both variables and a neutral element $e \in L$.

Obviously, a t-norm (resp. t-conorm) on $L$ is exactly a uninorm with the neutral element $e=1_{L}$ (resp. $e=0_{L}$ ).

Definition 2.3 ([5, [15]). A binary operation $V: L^{2} \rightarrow L$ is called a nullnorm if it is commutative, associative, increasing with respect to both variables, and there exists an element $a \in L$, which is called a zero element for $V$, such that $V\left(x, 0_{L}\right)=x$ for all $x \in\left[0_{L}, a\right]$ and $V\left(x, 1_{L}\right)=x$ for all $x \in\left[a, 1_{L}\right]$.

Obviously, a t-norm (resp. t-conorm) on $L$ is exactly a nullnorm with the zero element $a=0_{L}$ (resp. $a=1_{L}$ )

Definition $2.4([\mathbf{2 6}])$. Let $P, Q$ be two partially ordered sets and $f: P \rightarrow Q$ be a non-constant injective monotone function. If $f$ is increasing, then a pseudo-inverse $f^{(-1)}: Q \rightarrow P$ is given by (2.1) when it exists.

$$
f^{(-1)}(y)= \begin{cases}\inf \{x \in P \mid f(x)>y\}, & \text { if } \operatorname{card}\left\{f^{-1}(y)\right\}=0  \tag{2.1}\\ f^{-1}(y), & \text { if } \operatorname{card}\left\{f^{-1}(y)\right\}=1\end{cases}
$$

If $f$ is decreasing then a pseudo-inverse $f^{(-1)}: Q \rightarrow P$ is given by (2.2) when it exists.

$$
f^{(-1)}(y)= \begin{cases}\sup \{x \in P \mid f(x)>y\}, & \text { if } \operatorname{card}\left\{f^{-1}(y)\right\}=0  \tag{2.2}\\ f^{-1}(y), & \text { if } \operatorname{card}\left\{f^{-1}(y)\right\}=1\end{cases}
$$

Definition 2.5 ([12]). Let $0 \in A \subseteq[-\infty,+\infty]$. For two non-zero elements $x, a \in A$, if there exists $b \in A$ fulfilling $x=a+b$ and $a b>0$, then we call $a$ a summand of $x$ in $A$.

Remark 2.1 ([12] $)$. Let $0 \in A \subseteq[-\infty,+\infty]$ and $x \in A$. If $x=0$ then $x$ has no summands. If $x \neq 0$, then $x$ is always a summand of itself and each summand $a$ of $x$ satisfies $x a>0$

Definition 2.6 ( $[\mathbf{2 7},[\mathbf{2 9}])$. A binary operation $F: L^{2} \rightarrow L$ is called a uni-nullnorm if it satisfies the commutativity, associativity, monotonicity with respect to both variables, and there exist some elements $e, a \in L$ with $0_{L} \leq e<a \leq 1_{L}$ such that $F(e, x)=x$ for all $x \in\left[0_{L}, a\right]$ and $F\left(x, 1_{L}\right)=x$ for all $x \in\left[a, 1_{L}\right]$.

Definition $2.7([1,11])$. Let $e_{1}, a, e_{2} \in L$ with $0_{L} \leq e_{1} \leq a \leq e_{2} \leq 1_{L}$. A binary operation $\mathbb{U}: L^{2} \rightarrow L$ is called a 2-uninorm if it is commutative, associative, increasing with respect to both variables and fulfilling $\mathbb{U}\left(e_{1}, x\right)=x$ for all $x \leq a$ and $\mathbb{U}\left(e_{2}, x\right)=x$ for all $x \geq a$.

From Definitions 2.2 2.3, 2.6 and 2.7, one can easily check the following remark.

## Remark 2.2.

(i) A 2-uninorm with $e_{2}=1_{L}$ is a uni-nullnorm.
(ii) A 2-uninorm with $e_{1}=0_{L}$ is a null-uninorm.
(iii) A 2-uninorm with $a=1_{L}$ or $a=0_{L}$ is a uninorm.
(iv) A 2-uninorm with $e_{1}=0_{L}$ and $e_{2}=1_{L}$ is a nullnorm.

## 3. Constructions of 2-uninorms on bounded lattices

In this section, we first give two new methods for constructing 2 -uninorms on bounded lattices $L$, which are further used for inducing uninorms, nullnorms, uni-nullnorms and null-uninorms, respectively. We then provide some examples to illustrate the new methods.

Let $a \in L$. Denoted by $\mathcal{F}_{a}=\left\{f(x) \mid 0<f(x)<f(a)\right.$ and $\left.x \in I_{a}\right\}$ when $f: L \rightarrow[-\infty,+\infty]$ is injective increasing, and by $\mathcal{G}_{a}=\left\{f(x) \mid f(a)<f(x)<0\right.$ and $\left.x \in I_{a}\right\}$ when $f: L \rightarrow[-\infty,+\infty]$ is injective decreasing.

Lemma 3.1. Let $f: L \rightarrow[-\infty,+\infty]$ be an injective increasing function, and $x, y \in L$ with $x \leq y$. If $f(x) \leq f(y) \leq f(a)$ and $x \in I_{a}$, then $y \in I_{a}$.

Proof. Assume that $y \notin I_{a}$. Then $y \leq a$ since $f(y) \leq f(a)$ and $f$ is injective increasing. Thus from $x \leq y$ we obtain $x \leq y \leq a$, contrary to the fact that $x \in I_{a}$.

Theorem 3.1. Let $e_{1}, a, e_{2} \in L$ with $0_{L} \leq e_{1} \leq a \leq e_{2} \leq 1_{L}$, and $f: L \rightarrow[-\infty,+\infty]$ be an injective increasing function with $f\left(e_{1}\right)=0$. If $f$ satisfies the following five conditions: for all $x, y \in L$,
(i) if $(f(x), f(y)) \in[0, f(a)]^{2} \cup[-\infty, 0]^{2}$ then

$$
\min \{f(x)+f(y), f(a)\} \in \operatorname{Ran}(f) \cup\left[-\infty, f\left(0_{L}\right)\right)
$$

(ii) for all $(f(x), f(y)) \in[0, f(a)]^{2} \cup[-\infty, 0]^{2}$, if $f(x)$ and $f(y)$ have at least one same summand $f(z) \in$ $\operatorname{Ran}(f)$ then $x \nVdash y$,
(iii) if $f(x)<0<f(y) \leq f(a)$ and $x<e_{1}$, then $x \nVdash y$,
(iv) if $f(a) \leq f(x)<f(y)$ or $f(x)<0<f(a) \leq f(y)$, then $x \nVdash y$,
(v) for all $f(x), f(y) \in(0, f(a))$, if $f(x)+f(y) \in \operatorname{Ran}(f)$ and $0<f(x)+f(y) \leq f(a)$ then $f^{-1}(f(x)+$ $f(y)) \nVdash a$,
then the following function $\mathbb{U}: L^{2} \rightarrow L$ is a 2-uninorm, and we call $f$ an additive generator of $\mathbb{U}$. For all $x, y \in L$,

$$
\mathbb{U}(x, y)= \begin{cases}f^{(-1)}(f(x)+f(y)), & (f(x), f(y)) \in[-\infty, 0]^{2},  \tag{3.1}\\ f^{(-1)}(\min \{f(x)+f(y), f(a)\}), & (f(x), f(y)) \in[0, f(a)]^{2} \text { and } x \notin I_{a} \text { and } y \notin I_{a}, \\ a, & (f(x), f(y)) \in(f(a),+\infty] \times[0, f(a)] \\ & \cup[0, f(a)] \times(f(a),+\infty] \cup[0, f(a)] \times \mathcal{F}_{a} \\ & \cup \mathcal{F}_{a} \times[0, f(a)], \\ & (f(x), f(y)) \in\left(f\left(e_{2}\right),+\infty\right]^{2} \cup\left(f\left(e_{2}\right),+\infty\right] \\ f^{(-1)}(\max \{f(x), f(y)\}), & \times\left(f(a), f\left(e_{2}\right)\right] \cup\left(f(a), f\left(e_{2}\right)\right] \times\left(f\left(e_{2}\right),+\infty\right], \\ & \text { otherwise. }\end{cases}
$$

Proof. First it is necessary to check that $\mathbb{U}$ is well-defined. The proof is made in five cases:
(I) If $(f(x), f(y)) \in[f(a),+\infty) \times[0, f(a)] \cup[0, f(a)] \times[f(a),+\infty) \cup[0, f(a)] \times \mathcal{F}_{a} \cup \mathcal{F}_{a} \times[0, f(a)]$, then $\mathbb{U}(x, y)=a$. Thus $\mathbb{U}$ is well-defined by (3.1).
(II) If $(f(x), f(y)) \in[-\infty, 0]^{2}$, then $f(x)+f(y) \leq f(a)$, subsequently,

$$
f^{(-1)}(\min \{f(x)+f(y), f(a)\})=f^{(-1)}(f(x)+f(y))
$$

There are two subcases as follows.
Subcase (II-a). If $f(x)+f(y) \in \operatorname{Ran}(f)$ then clearly card $\left\{f^{-1}(f(x)+f(y))\right\}=1$ since $f$ is injective. Thus $\mathbb{U}(x, y)$ is well-defined by (2.1).
Subcase (II-b). If $f(x)+f(y) \in\left[-\infty, f\left(0_{L}\right)\right)$ then $\operatorname{card}\left\{f^{-1}(f(x)+f(y))\right\}=0$. By (3.1), we know that

$$
f^{(-1)}(f(x)+f(y))=\inf \{z \in L: f(z)>f(x)+f(y)\}=\wedge L=0_{L}
$$

Hence $\mathbb{U}(x, y)$ is well-defined by (2.1).
(III) If $(f(x), f(y)) \in[0, f(a)]^{2}, x, y \notin I_{a}$ and $f(x)+f(y) \leq f(a)$, then $f(x)+f(y) \in \operatorname{Ran}(f)$, subsequently,

$$
f^{(-1)}(\min \{f(x)+f(y), f(a)\})=f^{(-1)}(f(x)+f(y))
$$

Thus, similar to (II-a), $\mathbb{U}(x, y)$ is well-defined.
(IV) If $(f(x), f(y)) \in[0, f(a)]^{2}, x, y \notin I_{a}$ and $f(x)+f(y) \geq f(a)$, then

$$
f^{(-1)}(\min \{f(x)+f(y), f(a)\})=f^{(-1)}(f(a))=a
$$

Thus $\mathbb{U}$ is well-defined by (3.1).
(V) For the other cases, without loss of generality, we suppose that $f(x) \leq f(y)$. Then

$$
f^{(-1)}(\min \{f(x), f(y)\})=f^{(-1)}(f(x))=x
$$

and

$$
f^{(-1)}(\max \{f(x), f(y)\})=f^{(-1)}(f(y))=y
$$

Thus $\mathbb{U}(x, y)$ is well-defined by (3.1).
Therefore, from (I)-(V), we know that $\mathbb{U}(x, y)$ is well-defined for any $x, y \in L$.
Next, we prove that $\mathbb{U}$ is a 2 -uninorm. First, from (3.1), we have the following three statements.
(a) $\mathbb{U}$ is a commutative binary operation on $L$.
(b) If $x \in\left[0_{L}, a\right]$, i.e., $f(x) \leq f(a)$ and $x \notin I_{a}$, then $\mathbb{U}\left(e_{1}, x\right)=f^{(-1)}\left(\min \left\{f\left(e_{1}\right)+f(x), f(a)\right\}\right)=$ $f^{(-1)}((f(x))=x$ since $f$ is injective increasing.
(c) If $x \in\left[a, 1_{L}\right]$, i.e., $f(x) \geq f(a)$, then $\mathbb{U}\left(e_{2}, x\right)=f^{(-1)}\left(\min \left\{f\left(e_{2}\right), f(x)\right\}\right)=f^{(-1)}(f(x))=x$ for $f(a) \leq$ $f(x) \leq f\left(e_{2}\right)$ and $\mathbb{U}\left(e_{2}, x\right)=f^{(-1)}\left(\max \left\{f\left(e_{2}\right), f(x)\right\}\right)=f^{(-1)}(f(x))=x$ for $f(x) \geq f\left(e_{2}\right)$ since $f$ is injective increasing.

Then it remains to show the monotonicity and associativity of the binary operation $\mathbb{U}$.
(1) Monotonicity. Let $x, y, z \in L$ with $x \leq y$. We need to verify $\mathbb{U}(x, z) \leq \mathbb{U}(y, z)$. Noticing that $f(x) \leq f(y)$ since $x \leq y$ and $f$ is injective increasing. Therefore, the proof is split into all possible cases as follows.

1. $f(z)=0$.
1.1 If $f(x) \leq f(y)<0$, or $f(x)<0 \leq f(y) \leq f(a)$ and $y \notin I_{a}$, then

$$
\mathbb{U}(x, z)=f^{(-1)}(f(x)+f(z))=x \leq y=f^{(-1)}(f(y)+f(z))=\mathbb{U}(y, z)
$$

1.2 If $f(x)<0 \leq f(y) \leq f(a)$ and $y \in I_{a}$, or $f(x)<0<f(a)<f(y)$, then from (iv), we have

$$
\mathbb{U}(x, z)=f^{(-1)}(f(x)+f(z))=f^{(-1)}(f(x))=x \leq a=\mathbb{U}(y, z) .
$$

$1.30 \leq f(x) \leq f(y) \leq f(a)$.
1.3.1 If $x, y \notin I_{a}$, then $\mathbb{U}(x, z)=x \leq y=\mathbb{U}(y, z)$.
1.3.2 If $x \in I_{a}$, then $y \in I_{a}$ by Lemma 3.1, subsequently, $\mathbb{U}(x, z)=a=\mathbb{U}(y, z)$.
1.3.3 If $x \notin I_{a}$ and $y \in I_{a}$, then $\mathbb{U}(x, z)=x \leq a=\mathbb{U}(y, z)$.
1.4 If $0 \leq f(x) \leq f(a)<f(y)$ and $x \notin I_{a}$, then $\mathbb{U}(x, z)=f^{(-1)}(f(x)+f(z))=x \leq a=\mathbb{U}(y, z)$.
1.5 If $0 \leq f(x) \leq f(a)<f(y)$ and $x \in I_{a}$, then $\mathbb{U}(x, z)=a=\mathbb{U}(y, z)$.
1.6 If $f(a)<f(x) \leq f(y)$, then $\mathbb{U}(x, z)=a=\mathbb{U}(y, z)$.
2. $f(z)<0$.
2.1. If $f(x) \leq f(y) \leq 0$, then we have $f(x)+f(z) \leq 0<f(a)$, subsequently,

$$
\mathbb{U}(x, z)=f^{(-1)}(\min \{f(x)+f(z), f(a)\})=f^{(-1)}(f(x)+f(z))
$$

Analogously, we obtain $\mathbb{U}(y, z)=f^{(-1)}(\min \{f(y)+f(z), f(a)\})=f^{(-1)}(f(y)+f(z))$.
2.1.1. If $f(x)+f(z) \in\left[-\infty, f\left(0_{L}\right)\right)$, then $\mathbb{U}(x, z)=f^{(-1)}(f(x)+f(z))=\wedge L=0_{L} \leq \mathbb{U}(y, z)$.
2.1.2. If $f(x)+f(z) \in \operatorname{Ran}(f)$, then $f(y)+f(z) \in \operatorname{Ran}(f)$. Noticing that $f(z)$ is the same summand of both $f(x)+f(z)$ and $f(y)+f(z)$, thus $\mathbb{U}(x, z) \nVdash \mathbb{U}(y, z)$ by (ii). Therefore, $\mathbb{U}(x, z) \leq \mathbb{U}(y, z)$ since $f(x)+f(z) \leq f(y)+f(z)$ and $f$ is injective increasing.
2.2. If $f(x)<0<f(y)$, then $\mathbb{U}(x, z)=f^{(-1)}(f(x)+f(z))$, and

$$
\mathbb{U}(y, z)=f^{(-1)}(\min \{f(x), f(z)\})=f^{(-1)}(f(z))=z
$$

2.2.1. If $f(x)+f(z) \in\left[-\infty, f\left(0_{L}\right)\right)$, then $\mathbb{U}(x, z)=0_{L} \leq \mathbb{U}(y, z)$ from 2.1.1.
2.2.2. If $f(x)+f(z) \in \operatorname{Ran}(f)$, then $f(z)$ is the same summand of both $f(x)+f(z)$ and $f(z)$ by Remark 2.1 Thus $\mathbb{U}(x, z) \nVdash f^{-1}(f(z))$ by (ii). Therefore, from $f(x) \leq 0$ we have $\mathbb{U}(x, z) \leq f^{-1}(f(z))=z=$ $\mathbb{U}(y, z)$ since $f(x)+f(z) \leq f(z)$ and $f$ is injective increasing.
2.3. If $0 \leq f(x) \leq f(y)$, then $\mathbb{U}(x, z)=f^{(-1)}(\min \{f(x), f(z)\})=f^{(-1)}(f(z))=z$ and

$$
\mathbb{U}(y, z)=f^{(-1)}(\min \{f(y), f(z)\})=f^{(-1)}(f(z))=z,
$$

therefore, $\mathbb{U}(x, z)=z=\mathbb{U}(y, z)$.
3. $f(z)>0$.
3.1 If $f(x) \leq f(y)<0$, then

$$
\mathbb{U}(x, z)=f^{(-1)}(\min \{f(x), f(z)\})=x \leq y=f^{(-1)}(\min \{f(y), f(z)\})=\mathbb{U}(y, z)
$$

$3.2 f(x)<0=f(y)$.
3.2.1 If $0<f(z) \leq f(a)$ and $z \notin I_{a}$, then from (iii) we have

$$
\mathbb{U}(x, z)=f^{(-1)}(\min \{f(x), f(z)\})=x \leq z=f^{(-1)}(\min \{f(y)+f(z), f(a)\})=\mathbb{U}(y, z) .
$$

3.2.2 If $0<f(z) \leq f(a)$ and $z \in I_{a}$, then from (iii) we have

$$
\mathbb{U}(x, z)=f^{(-1)}(\min \{f(x), f(z)\})=x \leq a=\mathbb{U}(y, z) .
$$

3.2.3 If $f(z)>f(a)$, then from (iii), we have

$$
\mathbb{U}(x, z)=f^{(-1)}(\min \{f(x), f(z)\})=x \leq a=\mathbb{U}(y, z) .
$$

$3.3 f(x)=0<f(y) \leq f(a)$.
3.3.1 If $0<f(z) \leq f(a)$, and $z \in I_{a}$, then $\mathbb{U}(x, z)=a=\mathbb{U}(y, z)$.
3.3.2 $0<f(z) \leq f(a)$ and $y, z \notin I_{a}$.
3.3.2.1 If $f(y)+f(z) \leq f(a)$, then from (ii), we have

$$
\mathbb{U}(y, z)=f^{(-1)}(\min \{f(y)+f(z), f(a)\})=f^{(-1)}(f(y)+f(z)) \geq f^{(-1)}(z)=z=\mathbb{U}(x, z)
$$

3.3.2.2 If $f(y)+f(z) \geq f(a)$, then from $z \notin I_{a}$ and $f(z) \leq f(a)$, we have

$$
\mathbb{U}(y, z)=f^{(-1)}(\min \{f(y)+f(z), f(a)\})=a \geq z=f^{(-1)}(\min \{f(x)+f(z), f(a)\})=\mathbb{U}(x, z) .
$$

3.3.3 If $0<f(z) \leq f(a), y \in I_{a}$ and $z \notin I_{a}$, then

$$
\mathbb{U}(y, z)=a \geq z=f^{(-1)}(\min \{f(x)+f(z), f(a)\})=\mathbb{U}(x, z) .
$$

3.3.4 If $f(z)>f(a)$, then $\mathbb{U}(y, z)=a=\mathbb{U}(x, z)$.
3.4 If $f(x)<0<f(y)$, then

$$
\begin{equation*}
\mathbb{U}(x, z)=f^{(-1)}(\min \{f(x), f(z)\})=x . \tag{3.2}
\end{equation*}
$$

3.4.1 $(f(y), f(z)) \in(0, f(a)]^{2}$.
3.4.1.1 $y, z \notin I_{a}$.
3.4.1.1.1 If $f(y)+f(z) \leq f(a)$ then, because of (3.2) and (ii), we have

$$
\mathbb{U}(y, z)=f^{(-1)}(\min \{f(y)+f(z), f(a)\})=f^{(-1)}(f(y)+f(z)) \geq f^{(-1)}(y)=y \geq x=\mathbb{U}(x, z)
$$

3.4.1.1.2 If $f(y)+f(z) \geq f(a)$ then, because of (iv), we have

$$
\mathbb{U}(y, z)=f^{(-1)}(\min \{f(y)+f(z), f(a)\})=f^{(-1)}(f(a))=a \geq x=\mathbb{U}(x, z) .
$$

3.4.1.2 If $z \in I_{a}$ or $y \in I_{a}$, then from (3.2) and (iv) we have $\mathbb{U}(y, z)=a \geq x=\mathbb{U}(x, z)$.
3.4.2 $(f(y), f(z)) \in\left(f(a), f\left(e_{2}\right)\right]^{2}$.
3.4.2.1 If $f(y) \leq f(z)$, then

$$
\mathbb{U}(y, z)=f^{(-1)}(\min \{f(y), f(z)\})=f^{(-1)}(f(y))=y \geq x=\mathbb{U}(x, z)
$$

3.4.2.2 If $f(y) \geq f(z)$, then from (iv) we have

$$
\mathbb{U}(y, z)=f^{(-1)}(\min \{f(y), f(z)\})=f^{(-1)}(f(z))=z \geq x=\mathbb{U}(x, z)
$$

3.4.3 $(f(y), f(z)) \in\left(f\left(e_{2}\right),+\infty\right]^{2} \cup\left(f\left(e_{2}\right),+\infty\right] \times\left(f(a), f\left(e_{2}\right)\right] \cup\left(f(a), f\left(e_{2}\right)\right] \times\left(f\left(e_{2}\right),+\infty\right]$.
3.4.3.1 If $f(y) \geq f(z)$, then

$$
\mathbb{U}(y, z)=f^{(-1)}(\max \{f(y), f(z)\})=f^{(-1)}(y)=y \geq x=\mathbb{U}(x, z)
$$

3.4.3.2 If $f(y) \leq f(z)$, then from (iv), we have

$$
\mathbb{U}(y, z)=f^{(-1)}(\max \{f(y), f(z)\})=f^{(-1)}(z)=z \geq x=\mathbb{U}(x, z) .
$$

3.4.4 If $(f(y), f(z)) \in[f(a),+\infty] \times[0, f(a)] \cup[0, f(a)] \times[f(a),+\infty]$, then from (iv), we have

$$
\mathbb{U}(y, z)=a \geq x=\mathbb{U}(x, z)
$$

$3.50<f(x) \leq f(y) \leq f(a)$.
3.5.1 $0<f(z) \leq f(a)$.
3.5.1.1 If $z \in I_{a}$, then $\mathbb{U}(x, z)=a=\mathbb{U}(y, z)$.
3.5.1.2 $x, y, z \notin I_{a}$.
3.5.1.2.1 If $f(a) \leq f(x)+f(z) \leq f(y)+f(z)$, then $\mathbb{U}(x, z)=a=\mathbb{U}(y, z)$.
3.5.1.2.2 If $f(x)+f(z) \leq f(a) \leq f(y)+f(z)$ then, because of (v), we have

$$
\mathbb{U}(x, z)=f^{(-1)}(\min \{f(x)+f(z), f(a)\})=f^{(-1)}(f(x)+f(z)) \leq a=\mathbb{U}(y, z) .
$$

3.5.1.2.3 If $f(x)+f(z) \leq f(y)+f(z) \leq f(a)$ then, from (ii), we have

$$
\mathbb{U}(x, z)=f^{(-1)}(f(x)+f(z)) \leq f^{(-1)}(f(y)+f(z))=\mathbb{U}(y, z)
$$

3.5.1.3 If $z \notin I_{a}$ and $x \in I_{a}$, then $y \in I_{a}$ by Lemma 3.1] subsequently, $\mathbb{U}(x, z)=a=\mathbb{U}(y, z)$.
3.5.1.4 If $z, x \notin I_{a}$ and $y \in I_{a}$, then from (v), we have $\mathbb{U}(x, z)=f^{(-1)}(f(x)+f(z)) \leq a=\mathbb{U}(y, z)$.
3.5.2 If $f(z)>f(a)$, then $\mathbb{U}(x, z)=a=\mathbb{U}(y, z)$.
$3.60 \leq f(x) \leq f(a)<f(y)$.
3.6.1 $0<f(z) \leq f(a)$.
3.6.1.1 $x, z \notin I_{a}$.
3.6.1.1.1 If $f(x) \neq 0$ and $f(x)+f(z) \leq f(a)$ then, because of (v), we have

$$
\mathbb{U}(x, z)=f^{(-1)}(\min \{f(x)+f(z), f(a)\})=f^{(-1)}(f(x)+f(z)) \leq a=\mathbb{U}(y, z) .
$$

3.6.1.1.2 If $f(x)=0$, then $\mathbb{U}(x, z)=z \leq a=\mathbb{U}(y, z)$.
3.6.1.1.3 If $f(x)+f(z) \geq f(a)$, then $\mathbb{U}(x, z)=f^{(-1)}(\min \{f(x)+f(z), f(a)\})=a=\mathbb{U}(y, z)$.
3.6.1.2 If $x \in I_{a}$ or $z \in I_{a}$, then $\mathbb{U}(x, z)=a=\mathbb{U}(y, z)$.
3.6.2 If $(f(y), f(z)) \in\left(f(a), f\left(e_{2}\right)\right]^{2}$, then from (iv) and $f(y), f(z) \geq f(a)$, we have

$$
\mathbb{U}(x, z)=a \leq f^{(-1)}(\min \{f(y), f(z)\})=\mathbb{U}(y, z) .
$$

3.6.3 If $(f(y), f(z)) \in\left(f\left(e_{2}\right),+\infty\right]^{2} \cup\left(f\left(e_{2}\right),+\infty\right] \times\left(f(a), f\left(e_{2}\right)\right] \cup\left(f(a), f\left(e_{2}\right)\right] \times\left(f\left(e_{2}\right),+\infty\right]$, then from (iv) and $f(y), f(z) \geq f(a)$, we have

$$
\mathbb{U}(x, z)=a \leq f^{(-1)}(\max \{f(y), f(z)\})=\mathbb{U}(y, z)
$$

$3.7 f(a)<f(x) \leq f(y)$.
3.7.1 If $0<f(z) \leq f(a)$, then $\mathbb{U}(x, z)=a=\mathbb{U}(y, z)$.
3.7.2 $f(a)<f(z) \leq f\left(e_{2}\right)$.
3.7.2.1 If $f(a)<f(x) \leq f(y) \leq f\left(e_{2}\right)$, then $\mathbb{U}(x, z)=f^{(-1)}(\min \{f(x), f(z)\})$ and

$$
\mathbb{U}(y, z)=f^{(-1)}(\min \{f(y), f(z)\})
$$

3.7.2.1.1 If $f(x) \leq f(y) \leq f(z)$, then $\mathbb{U}(x, z)=x \leq y=\mathbb{U}(y, z)$.
3.7.2.1.2 If $f(z) \leq f(x) \leq f(y)$, then $\mathbb{U}(x, z)=z=\mathbb{U}(y, z)$.
3.7.2.1.3 If $f(x) \leq f(z) \leq f(y)$ then, because of (iv), we have $\mathbb{U}(x, z)=x \leq z=\mathbb{U}(y, z)$.
3.7.2.2 If $f(a)<f(x) \leq f\left(e_{2}\right)<f(y)$, then $\mathbb{U}(x, z)=f^{(-1)}(\min \{f(x), f(z)\})$ and

$$
\mathbb{U}(y, z)=f^{(-1)}(\max \{f(y), f(z)\})=f^{(-1)}(f(y))=y .
$$

3.7.2.2.1 If $f(x) \leq f(z)$, then $\mathbb{U}(x, z)=x \leq y=\mathbb{U}(y, z)$.
3.7.2.2.2 If $f(x) \geq f(z)$ then, because of (iv), we have $\mathbb{U}(x, z)=z \leq y=\mathbb{U}(y, z)$.
3.7.2.3 If $f\left(e_{2}\right)<f(x) \leq f(y)$, then $\mathbb{U}(x, z)=x \leq y=\mathbb{U}(y, z)$.
3.7.3 If $f(z)>f\left(e_{2}\right)$, then $\mathbb{U}(x, z)=f^{(-1)}(\max \{f(x), f(z)\})$ and $\mathbb{U}(y, z)=f^{(-1)}(\max \{f(y), f(z)\})$.
3.7.3.1 If $f(x) \leq f(y) \leq f(z)$, then $\mathbb{U}(x, z)=z=\mathbb{U}(y, z)$.
3.7.3.2 If $f(x) \leq f(z) \leq f(y)$ then, because of (iv), we obtain $\mathbb{U}(x, z)=z \leq y=\mathbb{U}(y, z)$.
3.7.3.3 If $f(z) \leq f(x) \leq f(y)$, then $\mathbb{U}(x, z)=x \leq y=\mathbb{U}(y, z)$.
(2) Associativity. Let $x, y, z \in L$. We need to verify $\mathbb{U}(x, \mathbb{U}(y, z))=\mathbb{U}(\mathbb{U}(x, y), z)$. The proof is split into all possible cases as follows.

1. $f(x)<0$.
1.1. $f(y)<0$.
1.1.1. $f(z)<0$.
1.1.1.1 If $f(x)+f(y) \in \operatorname{Ran}(f)$ and $f(y)+f(z) \in \operatorname{Ran}(f)$, then

$$
\begin{aligned}
\mathbb{U}(\mathbb{U}(x, y), z) & =\mathbb{U}\left(f^{(-1)}(\min \{(f(x)+f(y)), f(a)\}), z\right) \\
& =f^{(-1)}\left(f \circ f^{(-1)}(f(x)+f(y))+f(z)\right) \quad(\text { since } f(x)+f(y) \leq 0 \leq f(a)) \\
& =f^{(-1)}(f(x)+f(y)+f(z)) . \\
\mathbb{U}(x, \mathbb{U}(y, z)) & =\mathbb{U}\left(x, f^{(-1)}(\min \{(f(y)+f(z)), f(a)\})\right. \\
& =f^{(-1)}(f(x)+f(y)+f(z)) .
\end{aligned}
$$

1.1.1.2 If $f(x)+f(y) \in\left[-\infty, f\left(0_{L}\right)\right)$ and $f(y)+f(z) \in\left[-\infty, f\left(0_{L}\right)\right)$, then

$$
\begin{aligned}
\mathbb{U}(\mathbb{U}(x, y), z) & =\mathbb{U}\left(f^{(-1)}(\min \{(f(x)+f(y)), f(a)\}), z\right) \\
& =\mathbb{U}(\wedge L, z) \\
& =\mathbb{U}\left(0_{L}, z\right) . \\
& =f^{(-1)}\left(f(z)+f\left(0_{L}\right)\right) \\
& =0_{L} \\
\mathbb{U}(x, \mathbb{U}(y, z)) & =\mathbb{U}\left(x, f^{(-1)}(\min \{(f(y)+f(z)), f(a)\})\right. \\
& =\mathbb{U}(x, \wedge L) \\
& =\mathbb{U}\left(x, 0_{L}\right) \\
& =f^{(-1)}\left(f(x)+f\left(0_{L}\right)\right) \\
& =0_{L}
\end{aligned}
$$

1.1.1.3 If $f(x)+f(y) \in\left[-\infty, f\left(0_{L}\right)\right)$ and $f(y)+f(z) \in \operatorname{Ran}(f)$, then

$$
\begin{aligned}
\mathbb{U}(\mathbb{U}(x, y), z) & =\mathbb{U}\left(f^{(-1)}(\min \{(f(x)+f(y)), f(a)\}), z\right) \\
& =\mathbb{U}(\wedge L, z) \\
& =\mathbb{U}\left(0_{L}, z\right) \\
& =f^{(-1)}\left(f(z)+f\left(0_{L}\right)\right) \\
& =0_{L} \\
\mathbb{U}(x, \mathbb{U}(y, z)) & =\mathbb{U}\left(x, f^{(-1)}(\min \{(f(y)+f(z)), f(a)\})\right. \\
& =f^{(-1)}(f(x)+f(y)+f(z)) \\
& =\wedge L \\
& =0_{L} .
\end{aligned}
$$

1.1.1.4 If $f(x)+f(y) \in \operatorname{Ran}(f)$ and $f(y)+f(z) \in\left[-\infty, f\left(0_{L}\right)\right)$, then

$$
\begin{aligned}
\mathbb{U}(\mathbb{U}(x, y), z) & =\mathbb{U}\left(f^{(-1)}(\min \{(f(x)+f(y)), f(a)\}), z\right) \\
& =f^{(-1)}(f(x)+f(y)+f(z)) \\
& =\wedge L \\
& =0_{L} \\
\mathbb{U}(x, \mathbb{U}(y, z)) & =\mathbb{U}\left(x, f^{(-1)}(\min \{(f(y)+f(z)), f(a)\})\right. \\
& =\mathbb{U}(x, \wedge L) \\
& =\mathbb{U}\left(x, 0_{L}\right) \\
& =0_{L} .
\end{aligned}
$$

1.1.2. $0 \leq f(z)$.
1.1.2.1 If $f(x)+f(y) \in \operatorname{Ran}(f)$, then

$$
\begin{aligned}
\mathbb{U}(\mathbb{U}(x, y), z) & =\mathbb{U}\left(f^{(-1)}(\min \{(f(x)+f(y)), f(a)\}), z\right) \\
& =f^{(-1)}\left(\min \left\{f \circ f^{(-1)}(f(x)+f(y)), f(z)\right\}\right) \quad(\text { since } f(x)+f(y) \leq 0 \leq f(a)) \\
& =f^{(-1)}(f(x)+f(y)) . \\
\mathbb{U}(x, \mathbb{U}(y, z)) & =\mathbb{U}\left(x, f^{(-1)}(\min \{f(y), f(z)\})\right. \\
& =f^{(-1)}(\min \{f(x)+f(y), f(a)\}) \\
& =f^{(-1)}(f(x)+f(y)) .
\end{aligned}
$$

1.1.2.2 If $f(x)+f(y) \in\left[-\infty, f\left(0_{L}\right)\right)$, then

$$
\begin{aligned}
\mathbb{U}(\mathbb{U}(x, y), z) & =\mathbb{U}\left(f^{(-1)}(\min \{(f(x)+f(y)), f(a)\}), z\right) \\
& =\mathbb{U}(\wedge L, z) \\
& =0_{L} \\
\mathbb{U}(x, \mathbb{U}(y, z)) & =\mathbb{U}\left(x, f^{(-1)}(\min \{f(y), f(z)\})\right. \\
& =f^{(-1)}(\min \{f(x)+f(y), f(a)\}) \\
& =\wedge L \\
& =0_{L}
\end{aligned}
$$

1.2. $0 \leq f(y) \leq f(a)$.
1.2.1. $f(z)<0$.
1.2.1.1 If $f(x)+f(z) \in \operatorname{Ran}(f)$, then

$$
\begin{aligned}
\mathbb{U}(\mathbb{U}(x, y), z) & =\mathbb{U}(z, \mathbb{U}(x, y)) \\
& =f^{(-1)}((f(x)+f(z)) . \quad(\text { by 1.1.2.1) } \\
\mathbb{U}(x, \mathbb{U}(y, z)) & =\mathbb{U}\left(x, f^{(-1)}(\min \{f(y), f(z)\})\right) \\
& =f^{(-1)}((f(x)+f(z)) .
\end{aligned}
$$

1.2.1.2 If $f(x)+f(z) \in\left[-\infty, f\left(0_{L}\right)\right)$, then by 1.1.2.2,

$$
\begin{aligned}
\mathbb{U}(\mathbb{U}(x, y), z) & =\mathbb{U}(z, \mathbb{U}(x, y))=0_{L} \\
\mathbb{U}(x, \mathbb{U}(y, z)) & =\mathbb{U}\left(x, f^{(-1)}(\min \{f(y), f(z)\})\right) \\
& =f^{(-1)}((f(x)+f(z)) \\
& =\wedge L \\
& =0_{L}
\end{aligned}
$$

1.2.2. If $0 \leq f(z) \leq f(a)$, then $\min \{f(y)+f(z), f(a)\} \in \operatorname{Ran}(f)$.
1.2.2.1 If $y, z \notin I_{a}$, then

$$
\begin{aligned}
\mathbb{U}(\mathbb{U}(x, y), z) & =\mathbb{U}\left(f^{(-1)}(\min \{f(x), f(y)\}), z\right) \\
& =f^{(-1)}(\min \{f(x), f(z)\}) \\
& =f^{(-1)}(f(x)) \\
\mathbb{U}(x, \mathbb{U}(y, z)) & =\mathbb{U}\left(x, f^{(-1)}(\min \{(f(y)+f(z)), f(a)\})\right) \\
& =f^{(-1)}(f(x)) .
\end{aligned}
$$

1.2.2.2 If $y \in I_{a}$ or $z \in I_{a}$, then $\mathbb{U}(\mathbb{U}(x, y), z)=\mathbb{U}(x, z)=x=\mathbb{U}(x, a)=\mathbb{U}(x, \mathbb{U}(y, z))$.
1.2.3. If $f(a)<f(z)$, then

$$
\begin{aligned}
\mathbb{U}(\mathbb{U}(x, y), z) & =\mathbb{U}\left(f^{(-1)}(\min \{f(x), f(y)\}, z)\right. \\
& =f^{(-1)}(\min \{f(x), f(z)\}) \\
& =f^{(-1)}((f(x)) . \\
\mathbb{U}(x, \mathbb{U}(y, z)) & =\mathbb{U}(x, a) \\
& =f^{(-1)}((f(x)) .
\end{aligned}
$$

1.3. $f(a)<f(y)$.
1.3.1. If $f(z)<0$, then

$$
\begin{aligned}
\mathbb{U}(\mathbb{U}(x, y), z) & =\mathbb{U}\left(f^{(-1)}(\min \{f(x), f(y)\}, z)=\mathbb{U}(x, z)\right. \\
\mathbb{U}(x, \mathbb{U}(y, z)) & =\mathbb{U}\left(x, f^{(-1)}(\min \{f(y), f(z)\})\right)=\mathbb{U}(x, z) .
\end{aligned}
$$

1.3.2. If $0 \leq f(z) \leq f(a)$, then

$$
\begin{aligned}
\mathbb{U}(\mathbb{U}(x, y), z) & \left.=\mathbb{U}\left(f^{(-1)}(\min \{f(x), f(y)\})\right), z\right) \\
& =\mathbb{U}(x, z) \\
& =f^{(-1)}(\min \{f(x), f(z)\}) \\
& =x . \\
\mathbb{U}(x, \mathbb{U}(y, z) & \left.=\mathbb{U}(x, a)=f^{(-1)}(\min \{f(x), f(a)\})\right\}=x .
\end{aligned}
$$

1.3.3 $f(z)>f(a)$.
1.3.3.1 If $(f(y), f(z)) \in\left(f(a), f\left(e_{2}\right)\right]^{2}$, then

$$
\begin{aligned}
\mathbb{U}(\mathbb{U}(x, y), z) & \left.=\mathbb{U}\left(f^{(-1)}(\min \{f(x), f(y)\})\right), z\right) \\
& =\mathbb{U}(x, z) \\
& =f^{(-1)}(\min \{f(x), f(z)\}) \\
& =x \\
\mathbb{U}(x, \mathbb{U}(y, z)) & =f^{(-1)}\left(\min \left\{f(x), f \circ f^{(-1)}(\min \{f(y), f(z)\})\right\}\right)=x
\end{aligned}
$$

1.3.3.2 If $(f(y), f(z)) \in\left[f\left(e_{2}\right),+\infty\right]^{2} \cup\left[f\left(e_{2}\right),+\infty\right] \times\left(f(a), f\left(e_{2}\right)\right] \cup\left(f(a), f\left(e_{2}\right)\right] \times\left[f\left(e_{2}\right),+\infty\right]$, then

$$
\begin{aligned}
\mathbb{U}(\mathbb{U}(x, y), z) & \left.=\mathbb{U}\left(f^{(-1)}(\min \{f(x), f(y)\})\right), z\right) \\
& =\mathbb{U}(x, z) \\
& =f^{(-1)}(\min \{f(x), f(z)\}) \\
& =x . \\
\mathbb{U}(x, \mathbb{U}(y, z) & =\mathbb{U}\left(x, f^{(-1)}(\{\max \{f(y), f(z)\}))\right) \\
& =f^{(-1)}\left(\min \left\{f(x), f \circ f^{(-1)}(\{\max \{f(y), f(z)\})\}\right)\right\} \\
& =x .
\end{aligned}
$$

2. $0 \leq f(x) \leq f(a)$.
2.1. $f(y)<0$.
2.1.1. $f(z)<0$.
2.1.1.1 If $f(y)+f(z) \in \operatorname{Ran}(f)$ and $f(x)+f(y) \in \operatorname{Ran}(f)$, then

$$
\begin{aligned}
\mathbb{U}(\mathbb{U}(x, y), z) & =\mathbb{U}(z, \mathbb{U}(x, y)) & & \text { (by the commutativity of } \mathbb{U}) \\
& =f^{(-1)}(f(z)+f(y)) . & & \text { (by 1.2.1.1) } \\
\mathbb{U}(x, \mathbb{U}(y, z)) & =\mathbb{U}(\mathbb{U}(y, z), x) & & \text { (by the commutativity of } \mathbb{U}) \\
& =f^{(-1)}(f(z)+f(y)) . & & \text { (by 1.1.2.1) }
\end{aligned}
$$

2.1.1.2 If $f(y)+f(z) \in\left[-\infty, f\left(0_{L}\right)\right)$, then

$$
\begin{array}{rlrl}
\mathbb{U}(\mathbb{U}(x, y), z) & =\mathbb{U}(z, \mathbb{U}(x, y)) & & \text { (by the commutativity of } \mathbb{U} \text { ) } \\
& =0_{L} . & & \text { (by 1.2.1.2) } \\
\mathbb{U}(x, \mathbb{U}(y, z)) & =\mathbb{U}(\mathbb{U}(y, z), x) & & \text { (by the commutativity of } \mathbb{U}) \\
& =f^{(-1)}(f(z)+f(y)) & & \text { (by 1.1.2.1) } \\
& =\wedge L & & \\
& =0_{L} . &
\end{array}
$$

2.1.2. If $0 \leq f(z) \leq f(a)$, then

$$
\begin{aligned}
\mathbb{U}(\mathbb{U}(x, y), z) & =\mathbb{U}\left(f^{(-1)}(\min \{f(x), f(y)\}), z\right) \\
& =\mathbb{U}\left(f^{(-1)}(f(y)), z\right) \\
& =\mathbb{U}(y, z) \\
& =f^{(-1)}(\min \{f(y), f(z)\}) \\
& =y . \\
\mathbb{U}(x, \mathbb{U}(y, z)) & =\mathbb{U}\left(x, f^{(-1)}(\min \{f(y), f(z)\})\right) \\
& =\mathbb{U}(x, y) \\
& =f^{(-1)}(\min \{f(x), f(y)\}) \\
& =y .
\end{aligned}
$$

2.1.3. If $f(z)>f(a)$, then

$$
\begin{aligned}
\mathbb{U}(\mathbb{U}(x, y), z) & =\mathbb{U}\left(f^{(-1)}(\min \{f(x), f(y)\}), z\right) \\
& =\mathbb{U}\left(f^{(-1)}(f(y)), z\right) \\
& =\mathbb{U}(y, z) \\
& =f^{(-1)}(\min \{f(y), f(z)\}) \\
& =y \\
\mathbb{U}(x, \mathbb{U}(y, z)) & =\mathbb{U}\left(x, f^{(-1)}(\min \{f(y), f(z)\})\right) \\
& =\mathbb{U}(x, y) \\
& =f^{(-1)}(\min \{f(x), f(y)\}) \\
& =y
\end{aligned}
$$

2.2. $0 \leq f(y) \leq f(a)$.
2.2.1. If $f(z)<0$, then

$$
\begin{array}{rlr}
\mathbb{U}(\mathbb{U}(x, y), z) & =\mathbb{U}(z, \mathbb{U}(x, y)) \quad \text { (by the commutativity of } \mathbb{U}) \\
& =f^{(-1)}(f(z)) . & \quad \text { by 1.2.2.1-1.2.2.2) } \\
\mathbb{U}(x, \mathbb{U}(y, z)) & =\mathbb{U}\left(x, f^{(-1)}(\min \{f(y), f(z)\})\right) \\
& =\mathbb{U}(x, z) \\
& =f^{(-1)}(\min \{f(x), f(z)\}) \\
& =f^{(-1)}(f(z)) .
\end{array}
$$

2.2.2. If $0 \leq f(z) \leq f(a)$, then both $\min \{f(x)+f(y), f(a)\} \in \operatorname{Ran}(f)$ and $\min \{f(y)+f(z), f(a)\}$ belong to $\operatorname{Ran}(f)$.
2.2.2.1 If $x, y, z \notin I_{a}$, then

$$
\begin{aligned}
\mathbb{U}(\mathbb{U}(x, y), z) & =\mathbb{U}\left(f^{(-1)}(\min \{f(x)+f(y), f(a)\}), z\right) \\
& =f^{(-1)}\left(\min \left\{f \circ f^{(-1)}(\min \{f(x)+f(y), f(a)\})+f(z), f(a)\right\}\right) \\
& =f^{(-1)}(\min \{(\min \{f(x)+f(y), f(a)\})+f(z), f(a)\}) \\
& =f^{(-1)}(\min \{\min \{(f(x)+f(y)+f(z), f(a)+f(z)\}, f(a)\}) \\
& =f^{(-1)}(\min \{f(x)+f(y)+f(z), f(a)+f(z), f(a)\}) \\
& \left.=f^{(-1)}(\min \{f(x)+f(y)+f(z), f(a)\}) . \quad \text { (since } f(a)+f(z) \geq f(a)\right) . \\
\mathbb{U}(x, \mathbb{U}(y, z)) & =\mathbb{U}(\mathbb{U}(y, z), x) \\
& \left.=f^{(-1)}(\min \{f(x)+f(y)+f(z), f(a)\}) . \quad \text { (by the commutativity of } \mathbb{U}\right) \\
& \text { (by 2.2.2.) }
\end{aligned}
$$

2.2.2.2 If $x, y, z \in I_{a}$ or $x, z \in I_{a}$, then $\mathbb{U}(\mathbb{U}(x, y), z)=a=\mathbb{U}(x, \mathbb{U}(y, z))$.
2.2.2.3 If $y, z \in I_{a}$ and $x \notin I_{a}$, then

$$
\mathbb{U}(\mathbb{U}(x, y), z)=a=f^{(-1)}(\min \{f(a)+f(x), f(a)\})=\mathbb{U}(x, \mathbb{U}(y, z))
$$

2.2.2.4 The case $x, y \in I_{a}$ and $z \notin I_{a}$ is analogous to 2.2.2.3.
2.2.2.5 If $x, y \notin I_{a}$ and $z \in I_{a}$, then

$$
\mathbb{U}(\mathbb{U}(x, y), z)=\mathbb{U}\left(f^{(-1)}(\min \{f(x)+f(y), f(a)\}, z)=a=\mathbb{U}(x, a)=\mathbb{U}(x, \mathbb{U}(y, z)) .\right.
$$

2.2.2.6 Both the case $x \in I_{a}$ and $y, z \notin I_{a}$ and the case $y \in I_{a}$ and $x, z \notin I_{a}$ are analogous to 2.2.2.5. 2.2.3. $f(a)<f(z)$.
2.2.3.1 If $x, y \notin I_{a}$, then from $f^{(-1)}(\min \{f(x)+f(y), f(a)\}) \in[0, f(a)]$, we have

$$
\mathbb{U}(\mathbb{U}(x, y), z)=\mathbb{U}\left(f^{(-1)}(\min \{f(x)+f(y), f(a)\}), z\right)=a=\mathbb{U}(x, a)=\mathbb{U}(x, \mathbb{U}(y, z)) .
$$

2.2.3.2 If $x, y \in I_{a}$, then $\mathbb{U}(\mathbb{U}(x, y), z)=\mathbb{U}(a, z)=a=\mathbb{U}(x, a)=\mathbb{U}(x, \mathbb{U}(y, z))$.
2.2.3.3 If $x \notin I_{a}$ and $y \in I_{a}$, then $\mathbb{U}(\mathbb{U}(x, y), z)=\mathbb{U}(a, z)=a=\mathbb{U}(x, a)=\mathbb{U}(x, \mathbb{U}(y, z))$.
2.2.3.4 If $x \in I_{a}$ and $y \notin I_{a}$, then

$$
\mathbb{U}(\mathbb{U}(x, y), z)=\mathbb{U}(a, z)=a=\mathbb{U}(x, a)=\mathbb{U}(x, \mathbb{U}(y, z))
$$

$2.3 f(a)<f(y)$.
2.3.1 If $f(z)<0$, then

$$
\mathbb{U}(\mathbb{U}(x, y), z)=\mathbb{U}(a, z)=z=\mathbb{U}\left(x, f^{(-1)}(\min \{f(y), f(z))\}\right)=\mathbb{U}(x, \mathbb{U}(y, z)) .
$$

2.3.2. If $0 \leq f(z) \leq f(a)$, then from 2.2.3.1-2.2.3.4, we have

$$
\mathbb{U}(\mathbb{U}(x, y), z)=\mathbb{U}(z, \mathbb{U}(x, y))=a=\mathbb{U}(x, a)=\mathbb{U}(x, \mathbb{U}(y, z)) .
$$

2.3.3. $f(a)<f(z)$.
2.3.3.1 If $(f(y), f(z)) \in\left(f(a), f\left(e_{2}\right)\right]^{2}$, then

$$
\mathbb{U}(\mathbb{U}(x, y), z)=\mathbb{U}(a, z)=a=\mathbb{U}\left(x, f^{(-1)}(\min \{f(y), f(z)\})\right)=\mathbb{U}(x, \mathbb{U}(y, z))
$$

2.3.3.2 If $(f(y), f(z)) \in\left[f\left(e_{2}\right),+\infty\right]^{2} \cup\left[f\left(e_{2}\right),+\infty\right] \times\left(f(a), f\left(e_{2}\right)\right] \cup\left(f(a), f\left(e_{2}\right)\right] \times\left[f\left(e_{2}\right),+\infty\right]$, then

$$
\mathbb{U}(\mathbb{U}(x, y), z)=\mathbb{U}(a, z)=a=\mathbb{U}\left(x, f^{(-1)}(\max \{f(y), f(z)\})\right)=\mathbb{U}(x, \mathbb{U}(y, z)) .
$$

3. $f(a)<f(x)$.
3.1. $f(y)<0$.
3.1.1. If $f(z)<0$, then

$$
\begin{array}{rlrl}
\mathbb{U}(\mathbb{U}(x, y), z) & =\mathbb{U}(z, \mathbb{U}(x, y)) & & \text { (by the commutativity of } \mathbb{U}) \\
& =\mathbb{U}(z, y) & \text { (by 1.3.1.) } \\
& =f^{(-1)}(f(y)+f(z)) . & \\
\mathbb{U}(x, \mathbb{U}(y, z)) & =\mathbb{U}(\mathbb{U}(y, z), x) & & \text { (by the commutativity of } \mathbb{U}) \\
& =f^{(-1)}(f(y)+f(z)) . & \text { (by 1.1.2.) }
\end{array}
$$

3.1.2. If $0 \leq f(z) \leq f(a)$, then

$$
\begin{aligned}
\mathbb{U}(\mathbb{U}(x, y), z) & =\mathbb{U}(z, \mathbb{U}(x, y)) \quad \text { (by the commutativity of } \mathbb{U} \text { ) } \\
& =\mathbb{U}(z, y) \quad \text { (by 2.3.1.) } \\
& =f^{(-1)}(\min \{f(z), f(y)\}) \\
& =y . \\
& =\mathbb{U}(x, \mathbb{U}(y, z), x) \quad \text { (by the commutativity of } \mathbb{U}) \\
& =y . \quad(\text { by } 1.2 .3 .)
\end{aligned}
$$

3.1.3 If $f(a)<f(z)$, then

$$
\begin{aligned}
\mathbb{U}(\mathbb{U}(x, y), z) & =\mathbb{U}(y, z) \\
& =f^{(-1)}(\min \{f(y), f(z)\}) \\
& =y . \\
\mathbb{U}(x, \mathbb{U}(y, z)) & =\mathbb{U}(\mathbb{U}(y, z), x) \quad \text { (by the commutativity of } \mathbb{U}) \\
& =y . \quad(\text { by } 1.3 .3 .)
\end{aligned}
$$

3.2. $0 \leq f(y) \leq f(a)$.
3.2.1. If $f(z)<0$, then $\mathbb{U}(\mathbb{U}(x, y), z)=\mathbb{U}(a, z)=z=\mathbb{U}(x, z)=\mathbb{U}(x, \mathbb{U}(y, z))$.
3.2.2. If $0 \leq f(z) \leq f(a)$, then by 2.3.2, 2.2.3 and the commutativity of $\mathbb{U}$, we have

$$
\mathbb{U}(\mathbb{U}(x, y), z)=\mathbb{U}(z, \mathbb{U}(x, y))=a=\mathbb{U}(\mathbb{U}(y, z), x)=\mathbb{U}(x, \mathbb{U}(y, z)) .
$$

3.2.3. If $f(a)<f(z)$, then $\mathbb{U}(\mathbb{U}(x, y), z)=\mathbb{U}(a, z)=a=\mathbb{U}(x, a)=\mathbb{U}(x, \mathbb{U}(y, z))$.
3.3. $f(a)<f(y)$.
3.3.1. If $f(z)<0$, then

$$
\begin{aligned}
\mathbb{U}(\mathbb{U}(x, y), z) & =\mathbb{U}(z, \mathbb{U}(x, y) & & \text { (by the commutativity of } \mathbb{U} \text { ) } \\
& =z . & & \text { (by 1.3.3.) } \\
\mathbb{U}(x, \mathbb{U}(y, z)) & =\mathbb{U}(\mathbb{U}(y, z), x) & & \text { (by the commutativity of } \mathbb{U}) \\
& =z . & & \text { (by 3.1.3.) }
\end{aligned}
$$

3.3.2. If $0 \leq f(z) \leq f(a)$, then

$$
\begin{aligned}
\mathbb{U}(\mathbb{U}(x, y), z) & =\mathbb{U}(z, \mathbb{U}(x, y) & & \text { (by the commutativity of } \mathbb{U}) \\
& =a . & & \text { (by 2.3.3.) } \\
\mathbb{U}(x, \mathbb{U}(y, z)) & =\mathbb{U}(\mathbb{U}(y, z), x) & & \text { (by the commutativity of } \mathbb{U}) \\
& =a . & & \text { (by 3.2.3.) }
\end{aligned}
$$

3.3.3. $f(a)<f(z)$.
3.3.3.1 If $f(x), f(y), f(z) \notin\left(f\left(e_{2}\right),+\infty\right]$, then

$$
\begin{aligned}
\mathbb{U}(\mathbb{U}(x, y), z) & =\mathbb{U}\left(f^{(-1)}(\min \{f(x), f(y)\}), z\right) \\
& =f^{(-1)}\left(\min \left\{f \circ f^{(-1)}(\min \{f(x), f(y)\}), f(z)\right\}\right) \\
& =f^{(-1)}(\min \{\min \{f(x), f(y)\}, f(z)\}) \\
& =f^{(-1)}(\min \{f(x), f(y), f(z)\}) . \\
\mathbb{U}(x, \mathbb{U}(y, z)) & =\mathbb{U}(\mathbb{U}(y, z), x) \quad \text { (by the commutativity of } \mathbb{U}) \\
& =f^{(-1)}(\min \{f(x), f(y), f(z)\}) . \quad \text { (by 3.3.3.) }
\end{aligned}
$$

3.3.3.2 If exactly one of $f(x), f(y), f(z)$ belongs to $\left(f\left(e_{2}\right),+\infty\right]$, say $f(x) \in\left(f\left(e_{2}\right),+\infty\right]$, then

$$
\begin{aligned}
\mathbb{U}(\mathbb{U}(x, y), z) & =\mathbb{U}\left(f^{(-1)}(\max \{f(x), f(y)\}), z\right) \\
& =U(x, z) \\
& =f^{(-1)}(\max \{f(x), f(z)\}) \\
& =x . \\
\mathbb{U}(x, \mathbb{U}(y, z)) & =\mathbb{U}\left(x, f^{(-1)}(\min \{f(y), f(z)\})\right) \\
& =f^{(-1)}\left(\max \left\{f(x), f \circ f^{(-1)}(\min \{f(y), f(z)\})\right\}\right) \\
& =x .
\end{aligned}
$$

3.3.3.3 If at least two of $f(x), f(y), f(z)$ belong to $\left(f\left(e_{2}\right),+\infty\right]$, then

$$
\begin{aligned}
\mathbb{U}(\mathbb{U}(x, y), z) & =\mathbb{U}\left(f^{(-1)}(\max \{f(x), f(y)\}), z\right) \\
& =f^{(-1)}\left(\max \left\{f \circ f^{(-1)}(\max \{f(x), f(y)\}), f(z)\right\}\right) \\
& =f^{(-1)}(\max \{f(x), f(y), f(z)\} . \\
\mathbb{U}(x, \mathbb{U}(y, z)) & =\mathbb{U}\left(x, f^{(-1)}(\max \{f(y), f(z)\})\right) \\
& =f^{(-1)}\left(\max \left\{f(x), f \circ f^{(-1)}(\max \{f(y), f(z)\})\right\}\right) \\
& =f^{(-1)}(\max \{f(x), f(y), f(z)\} .
\end{aligned}
$$

Hence, $\mathbb{U}$ is associative.
Consequently, $\mathbb{U}$ is a 2 -uninorm on $L$.
The following theorem is a dual consequence of Theorem 3.1
Theorem 3.2. Let $e_{1}$, a, $e_{2} \in L$ with $0_{L} \leq e_{1} \leq a \leq e_{2} \leq 1_{L}$, and $f: L \rightarrow[-\infty,+\infty]$ be an injective decreasing function with $f\left(e_{1}\right)=0$. If $f$ satisfies the following five conditions: for all $x, y \in L$,
(i) if $(f(x), f(y)) \in[f(a), 0]^{2} \cup[0,+\infty]^{2}$ then

$$
\max \{f(x)+f(y), f(a)\} \in \operatorname{Ran}(f) \cup\left(f\left(0_{L}\right),+\infty\right]
$$

(ii) for all $(f(x), f(y)) \in[f(a), 0]^{2} \cup[0,+\infty]^{2}$, if $f(x)$ and $f(y)$ have at least one same summand $f(z) \in$ Ran $(f)$, then $x \nVdash y$,
(iii) if $f(a) \leq f(x)<0<f(y)$ and $e_{1}<y$, then $x \nVdash y$,
(iv) if $f(a) \geq f(x)>f(y)$ or $f(x) \leq f(a)<0<f(y)$, then $x \nVdash y$,
(v) for all $f(x), f(y) \in(f(a), 0)$, if $f(x)+f(y) \in \operatorname{Ran}(f)$ and $f(a) \leq f(x)+f(y)<0$ then $f^{-1}(f(x)+$ $f(y)) \nVdash a$,
then the following function $\mathbb{U}_{d}: L^{2} \rightarrow L$ is a 2-uninorm, and we call $f$ an additive generator of $\mathbb{U}_{d}$. For all $x, y \in L$,

$$
\mathbb{U}_{d}(x, y)=\left\{\begin{array}{lc}
f^{(-1)}(f(x)+f(y)), & (f(x), f(y)) \in[0,+\infty]^{2},  \tag{3.3}\\
f^{(-1)}(\max \{f(x)+f(y), f(a)\}), & (f(x), f(y)) \in[f(a), 0]^{2} \text { and } x, y \notin I_{a}, \\
a, & (f(x), f(y)) \in[-\infty, f(a)) \times[f(a), 0] \\
& \cup[f(a), 0] \times[-\infty, f(a)) \\
& \cup[f(a), 0] \times \mathcal{G}_{a} \cup \mathcal{G}_{a} \times[f(a), 0], \\
f^{(-1)}(\min \{f(x), f(y)\}), & (f(x), f(y)) \in\left[-\infty, f\left(e_{2}\right)\right)^{2} \cup\left[-\infty, f\left(e_{2}\right)\right) \\
& \times\left[f\left(e_{2}\right), f(a)\right) \cup\left[f\left(e_{2}\right), f(a)\right) \times\left[-\infty, f\left(e_{2}\right)\right), \\
& \text { otherwise. }
\end{array}\right.
$$

From Remark [2.2, we have the following corollary.
Corollary 3.1. (i) Taking $e_{2}=1_{L}$ in Theorem 3.1, we obtain the uni-nullnorm $U_{N}$ as follows. For all $x, y \in L$,

$$
U_{N}(x, y)= \begin{cases}f^{(-1)}(f(x)+f(y)), & (f(x), f(y)) \in[-\infty, 0]^{2}, \\
f^{(-1)}(\min \{f(x)+f(y), f(a)\}), & (f(x), f(y)) \in[0, f(a)]^{2} \text { and } x, y \notin I_{a}, \\
a, & (f(x), f(y)) \in[f(a),+\infty) \times[0, f(a)] \\
& \multicolumn{1}{l}{\begin{array}{ll} 
& \cup[0, f(a)] \times[f(a),+\infty) \\
& \cup[0, f(a)] \times \mathcal{F}_{a} \cup \mathcal{F}_{a} \times[0, f(a)], \\
& \\
f^{(-1)}(\min \{f(x), f(y)\}), & \text { otherwise. }
\end{array}}\end{cases}
$$

(ii) Taking $e_{2}=1_{L}$ in Theorem 3.2, we have the uni-nullnorm $U_{N}$ as follows. For all $x, y \in L$,

$$
U_{N}(x, y)= \begin{cases}f^{(-1)}(f(x)+f(y)), & (f(x), f(y)) \in(0,+\infty]^{2}, \\
f^{(-1)}(\max \{f(x)+f(y), f(a)\}), & (f(x), f(y)) \in[f(a), 0]^{2} \text { and } x, y \notin I_{a}, \\
a, & (f(x), f(y)) \in[-\infty, f(a)) \times[f(a), 0] \\
& \cup[f(a), 0] \times[-\infty, f(a)) \\
& \cup[f(a), 0] \times \mathcal{G}_{a} \cup \mathcal{G}_{a} \times[f(a), 0], \\
& \multicolumn{1}{c}{\left(\begin{array}{l}
\text { otherwise. }
\end{array}\right.}\end{cases}
$$

(iii) Taking $e_{1}=0_{L}$ in Theorem 3.1, we get the null-uninorm as below. For all $x, y \in L$,

$$
N_{U}(x, y)= \begin{cases}f^{(-1)}(\min \{f(x)+f(y), f(a)\}), & (f(x), f(y)) \in[0, f(a)]^{2} \text { and } x, y \notin I_{a}, \\ a, & (f(x), f(y)) \in[f(a),+\infty) \times[0, f(a)] \\ & \cup[0, f(a)] \times[f(a),+\infty) \\ & \cup[0, f(a)] \times \mathcal{F}_{a} \cup \mathcal{F}_{a} \times[0, f(a)], \\ & (f(x), f(y)) \in\left[f(a), f\left(e_{2}\right)\right]^{2}, \\ f^{(-1)}(\min \{f(x), f(y)\}), & \text { otherwise. }\end{cases}
$$

In this case, $f$ is an injective increasing function from $L$ to $[0,+\infty]$ with $f\left(0_{L}\right)=0$ and it satisfies the conditions (i), (ii), (v) in Theorem 3.1 and
(iv') if $f(a) \leq f(x)<f(y)$, then $x \nVdash y$.
(iv) Taking $e_{1}=0_{L}$ in Theorem 3.2, we have the null-uninorm as below. For all $x, y \in L$,

$$
N_{U}(x, y)= \begin{cases}f^{(-1)}(\max \{f(x)+f(y), f(a)\}), & (f(x), f(y)) \in[f(a), 0]^{2} \text { and } x, y \notin I_{a}, \\ a, & (f(x), f(y)) \in[-\infty, f(a)) \times[f(a), 0] \\ & \cup[f(a), 0] \times[-\infty, f(a)) \\ & \cup[f(a), 0] \times \mathcal{G}_{a} \cup \mathcal{G}_{a} \times[f(a), 0], \\ f^{(-1)}(\max \{f(x), f(y)\}), & (f(x), f(y)) \in\left[f\left(e_{2}\right), f(a)\right]^{2}, \\ f^{(-1)}(\min \{f(x), f(y)\}), & \text { otherwise. }\end{cases}
$$

In this case, $f$ is an injective decreasing function from $L$ to $[-\infty, 0]$ with $f\left(0_{L}\right)=0$ and it satisfies the conditions (i), (ii), (v) in Theorem 3.2 and (iv") if $f(x)<f(y) \leq f(a)$, then $x \nVdash y$.
(v) Taking $e_{1}=0_{L}$ and $e_{2}=1_{L}$ in Theorem 3.1, we get the nullnorm as follows. For all $x, y \in L$,

$$
N(x, y)= \begin{cases}f^{(-1)}(\min \{f(x)+f(y), f(a)\}), & (f(x), f(y)) \in[0, f(a)]^{2} \text { and } x, y \notin I_{a}, \\
a, & (f(x), f(y)) \in[f(a),+\infty) \times[0, f(a)] \\
& \cup[0, f(a)] \times[f(a),+\infty) \\
& \cup[0, f(a)] \times \mathcal{F}_{a} \cup \mathcal{F}_{a} \times[0, f(a)], \\
& \multicolumn{1}{l}{\begin{array}{l}
\text { otherwise } .
\end{array}}\end{cases}
$$

In this case, $f$ is an injective increasing function from $L$ to $[0,+\infty]$ with $f\left(0_{L}\right)=0$ and it satisfies the conditions (i), (ii), (v) in Theorem 3.1 and (iv').
(vi) Taking $e_{1}=0_{L}$ and $e_{2}=1_{L}$ in Theorem 3.2, we obtain the nullnorm as below. For all $x, y \in L$,

$$
N(x, y)= \begin{cases}f^{(-1)}(\max \{f(x)+f(y), f(a)\}), & (f(x), f(y)) \in[f(a), 0]^{2} \text { and } x, y \notin I_{a}, \\ a, & (f(x), f(y)) \in[-\infty, f(a)) \times[f(a), 0] \\ & \cup[f(a), 0] \times[-\infty, f(a)) \\ & \cup[f(a), 0] \times \mathcal{G}_{a} \cup \mathcal{G}_{a} \times[f(a), 0], \\ & \\ f^{(-1)}(\max \{f(x), f(y)\}), & \text { otherwise. }\end{cases}
$$

In this case, $f$ is an injective decreasing function from $L$ to $[-\infty, 0]$ with $f\left(0_{L}\right)=0$ and it satisfies the conditions (i), (ii), (v) in Theorem 3.2 and (iv").
(vii) Taking $a=1_{L}$ in Theorem [3.1, we have the uninorm $U$ in [12] as follows. For all $x, y \in L$,

$$
U(x, y)= \begin{cases}f^{(-1)}(f(x)+f(y)), & (f(x), f(y)) \in[-\infty, 0]^{2} \cup[0,+\infty]^{2}  \tag{3.4}\\ f^{(-1)}(\min \{f(x), f(y)\}), & \text { otherwise }\end{cases}
$$

In this case, $f$ just needs to satisfy the conditions (i), (ii) and (iii). Subsequently, we further obtain the $t$-norm and $t$-conorm given in [26] by taking $e_{1}=1_{L}$ and $e_{1}=0_{L}$ in Formula (3.4), respectively.
(viii) Taking $a=1_{L}$ in Theorem 3.2, we deduce the uninorm $U$ as below. For all $x, y \in L$,

$$
U(x, y)= \begin{cases}f^{(-1)}(f(x)+f(y)), & (f(x), f(y)) \in[-\infty, 0]^{2} \cup[0,+\infty]^{2} \\ f^{(-1)}(\max \{f(x), f(y)\}), & \text { otherwise }\end{cases}
$$

In this case, $f$ just needs to satisfy the conditions (i), (ii) and (iii).

## Remark 3.1.

(i) Since we require the functions $f$ in both Theorems 3.1 and 3.2 to be injective, it is obviously impossible to choose a suitable $f: L \rightarrow[-\infty,+\infty]$ when the cardinality of $L$ is strictly greater than $\aleph_{1}$.
(ii) The following are two alternative forms of (3.1) and (3.3), respectively. For all $x, y \in L$,

$$
\mathbb{U}(x, y)= \begin{cases}f^{-1}(f(x)+f(y)), & \text { either }(f(x), f(y)) \in[-\infty, 0]^{2} \text { and } f(x)+f(y) \in \operatorname{Ran}(f) \\ & \text { or }(f(x), f(y)) \in[0, f(a)]^{2}, f(x)+f(y) \leq f(a), f(x)+f(y) \in \operatorname{Ran}(f) \\ & \text { and } x, y \notin I_{a} ; \\ & (f(x), f(y)) \in[-\infty, 0]^{2} \text { and } f(x)+f(y) \in\left[-\infty, f\left(0_{L}\right)\right) ; \\ 0_{L}, & \text { either }(f(x), f(y)) \in[0, f(a)]^{2} \text { and } f(x)+f(y) \geq f(a), \\ a, & \text { or }(f(x), f(y)) \in[f(a),+\infty] \times[0, f(a)] \cup[0, f(a)] \times[f(a),+\infty] \\ & \text { or }(f(x), f(y)) \in[0, f(a)] \times \mathcal{F}_{a} \cup \mathcal{F}_{a} \times[0, f(a)] ; \\ & \text { either }(f(x), f(y)) \in[-\infty, 0] \times[0,+\infty], \\ & \text { or }(f(x), f(y)) \in\left[f\left(e_{2}\right),+\infty\right] \times\left[f(a), f\left(e_{2}\right)\right], \\ & \text { or }(f(x), f(y)) \in\left[f(a), f\left(e_{2}\right)\right]^{2} \text { and } f(x) \leq f(y) \\ & \text { or }(f(x), f(y)) \in\left[f\left(e_{2}\right),+\infty\right]^{2} \text { and } f(x) \geq f(y) ; \\ & \text { either }(f(x), f(y)) \in[0,+\infty] \times[-\infty, 0], \\ & \text { or }(f(x), f(y)) \in\left[f(a), f\left(e_{2}\right)\right] \times\left[f\left(e_{2}\right),+\infty\right], \\ & \text { or }(f(x), f(y)) \in\left[f(a), f\left(e_{2}\right)\right]^{2} \text { and } f(x) \geq f(y) \\ & \text { or }(f(x), f(y)) \in\left[f\left(e_{2}\right),+\infty\right]^{2} \text { and } f(x) \leq f(y) .\end{cases}
$$

For all $x, y \in L$,

$$
\mathbb{U}_{d}(x, y)= \begin{cases}f^{-1}(f(x)+f(y)), & \text { either }(f(x), f(y)) \in[0,+\infty]^{2} \text { and } f(x)+f(y) \in \operatorname{Ran}(f) \\ & \text { or }(f(x), f(y)) \in[f(a), 0]^{2} \text { and } f(x)+f(y) \geq f(a) \text { and } \\ & f(x)+f(y) \in \operatorname{Ran}(f) \text { and } x, y \notin I_{a} ; \\ 0_{L}, & (f(x), f(y)) \in[0,+\infty]^{2} \text { and } f(x)+f(y) \in\left[f\left(0_{L}\right),+\infty\right) ; \\ a, & \text { either }(f(x), f(y)) \in\left[f(a), f\left(e_{1}\right)\right] \times[-\infty, f(a)] \\ \quad \cup[-\infty, f(a)] \times\left[f(a), f\left(e_{1}\right)\right], \\ & \text { or }(f(x), f(y)) \in[0, f(a)]^{2} \text { and } x \notin I_{a} \text { and } y \notin I_{a} \\ & \text { or }(f(x), f(y)) \in[f(a), 0] \times \mathcal{F}_{a} \cup \mathcal{F}_{a} \times[f(a), 0] ; \\ & \text { either }(f(x), f(y)) \in[0,+\infty] \times[-\infty, 0], \\ & \text { or }(f(x), f(y)) \in\left[f\left(e_{2}\right), f(a)\right] \times\left[-\infty, f\left(e_{2}\right)\right], \\ & \text { or }(f(x), f(y)) \in\left[f\left(e_{2}\right), f(a)\right]^{2} \text { and } f(x) \geq f(y) \\ & \text { or }(f(x), f(y)) \in\left[-\infty, f\left(e_{2}\right)\right]^{2} \text { and } f(x) \geq f(y) ; \\ & \text { either }(f(x), f(y)) \in[-\infty, 0] \times[0,+\infty], \\ & \text { or }(f(x), f(y)) \in\left[-\infty, f\left(e_{2}\right)\right] \times\left[f\left(e_{2}\right), f(a)\right], \\ & \text { or }(f(x), f(y)) \in\left[f\left(e_{2}\right), f(a)\right]^{2} \text { and } f(x) \leq f(y) \\ & \text { or }(f(x), f(y)) \in\left[-\infty, f\left(e_{2}\right)\right]^{2} \text { and } f(x) \leq f(y) .\end{cases}
$$

(iii) If $L=[0,1]$, then $f$ in Theorem 3.1]only needs to satisfy the condition (i) since $x \nVdash y$ for any $x, y \in[0,1]$,
and the following function $\mathbb{U}(x, y)$ is a 2 -uninorm on $[0,1]$. For all $x, y \in[0,1]$,

$$
\mathbb{U}(x, y)= \begin{cases}f^{(-1)}(\min \{f(x)+f(y), f(a)\}), & (f(x), f(y)) \in[-\infty, 0]^{2} \cup[0, f(a)]^{2} ;  \tag{3.5}\\ a, & (f(x), f(y)) \in(f(a),+\infty] \times[0, f(a)] \\ & \cup[0, f(a)] \times(f(a),+\infty] ; \\ f^{(-1)}(\max \{f(x), f(y)\}, & (f(x), f(y)) \in\left(f\left(e_{2}\right),+\infty\right]^{2} \cup\left(f\left(e_{2}\right),+\infty\right] \\ & \quad \times\left(f(a), f\left(e_{2}\right)\right] \cup\left(f(a), f\left(e_{2}\right)\right] \times\left(f\left(e_{2}\right),+\infty\right] ; \\ & \text { otherwise. }\end{cases}
$$

The following two examples illustrate Theorem 3.1.
Example 3.1. Consider the lattice $L_{1}=\left\{0_{L_{1}}, x_{1}, x_{2}, x_{3}, x_{4}, e_{1}, x_{5}, x_{6}, a, x_{7}, x_{8}, x_{9}, e_{2}, x_{10}, x_{11}, 1_{L_{1}}\right\}$ given in Fig. 1 and the injective increasing function $f$ defined by Table 1 One can check that the function $f$ satisfies Theorem 3.1] and the 2 -uninorm $\mathbb{U}$ is shown by Table 2

$L_{1}$

$L_{2}$

Figure 1. Two bounded lattices $L_{1}$ and $L_{2}$

## Table 1

The generator $f$.

| $x$ | $0_{L_{1}}$ | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ | $e_{1}$ | $x_{5}$ | $x_{6}$ | $a$ | $x_{7}$ | $x_{8}$ | $x_{9}$ | $e_{2}$ | $x_{10}$ | $x_{11}$ | $1_{L_{1}}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $f$ | -11 | -8 | -4 | 6 | 12 | 0 | 9 | 14 | 15 | 13 | 11 | 17 | 18 | 20 | 22 | 24 |

Table 2
The 2 -uninorm $\mathbb{U}$.

| $U$ | $0_{L_{1}}$ | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ | $e_{1}$ | $x_{5}$ | $x_{6}$ | $a$ | $x_{7}$ | $x_{8}$ | $x_{9}$ | $e_{2}$ | $x_{10}$ | $x_{11}$ | $1_{L_{1}}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $0_{L_{1}}$ | $0_{L_{1}}$ | $0_{L_{1}}$ | $0_{L_{1}}$ | $0_{L_{1}}$ | $0_{L_{1}}$ | $0_{L_{1}}$ | $0_{L_{1}}$ | $0_{L_{1}}$ | $0_{L_{1}}$ | $0_{L_{1}}$ | $0_{L_{1}}$ | $0_{L_{1}}$ | $0_{L_{1}}$ | $0_{L_{1}}$ | $0_{L_{1}}$ | $0_{L_{1}}$ |
| $x_{1}$ | $0_{L_{1}}$ | $0_{L}$ | $0_{L}$ | $x_{1}$ | $x_{1}$ | $x_{1}$ | $x_{1}$ | $x_{1}$ | $x_{1}$ | $x_{1}$ | $x_{1}$ | $x_{1}$ | $x_{1}$ | $x_{1}$ | $x_{1}$ | $x_{1}$ |
| $x_{2}$ | $0_{L_{1}}$ | $0_{L}$ | $x_{1}$ | $x_{2}$ | $x_{2}$ | $x_{2}$ | $x_{2}$ | $x_{2}$ | $x_{2}$ | $x_{2}$ | $x_{2}$ | $x_{2}$ | $x_{2}$ | $x_{2}$ | $x_{2}$ | $x_{2}$ |
| $x_{3}$ | $0_{L_{1}}$ | $x_{1}$ | $x_{2}$ | $x_{4}$ | $a$ | $x_{3}$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ |
| $x_{4}$ | $0_{L_{1}}$ | $x_{1}$ | $x_{2}$ | $a$ | $a$ | $x_{4}$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ |
| $e_{1}$ | $0_{L_{1}}$ | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ | $e_{1}$ | $x_{5}$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ |
| $x_{5}$ | $0_{L_{1}}$ | $x_{1}$ | $x_{2}$ | $a$ | $a$ | $x_{5}$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ |  |  |
| $x_{6}$ | $0_{L_{1}}$ | $x_{1}$ | $x_{2}$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ |  |
| $a$ | $0_{L_{1}}$ | $x_{1}$ | $x_{2}$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ |
| $x_{7}$ | $0_{L_{1}}$ | $x_{1}$ | $x_{2}$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ |
| $x_{8}$ | $0_{L_{1}}$ | $x_{1}$ | $x_{2}$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ |
| $x_{9}$ | $0_{L_{1}}$ | $x_{1}$ | $x_{2}$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $x_{9}$ | $x_{9}$ | $x_{10}$ | $x_{11}$ | $1_{L_{1}}$ |
| $e_{2}$ | $0_{L_{1}}$ | $x_{1}$ | $x_{2}$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $x_{9}$ | $e_{2}$ | $x_{10}$ | $x_{11}$ | $1_{L_{1}}$ |
| $x_{10}$ | $0_{L_{1}}$ | $x_{1}$ | $x_{2}$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $x_{10}$ | $x_{10}$ | $x_{10}$ | $x_{11}$ | $1_{L_{1}}$ |
| $x_{11}$ | $0_{L_{1}}$ | $x_{1}$ | $x_{2}$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $x_{11}$ | $x_{11}$ | $x_{11}$ | $x_{11}$ | $1_{L_{1}}$ |
| $1_{L}$ | $0_{L_{1}}$ | $x_{1}$ | $x_{2}$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $1_{L_{1}}$ | $1_{L_{1}}$ | $1_{L_{1}}$ | $1_{L_{1}}$ | $1_{L_{1}}$ |

Example 3.2. Consider the bounded lattice $(L, \leq)$ as shown in Fig. 2, Let $e_{1}=x_{4}, e_{2}=x_{9}$ and $a=x_{7}$. Then the function $f: L \rightarrow[-\infty,+\infty]$ defined by $f\left(x_{i}\right)=i-4$ is an injective increasing function with $f\left(x_{4}\right)=0$. Further, we have:
(i) If $i \in[0,3]$, then $f\left(x_{i}\right) \in[-4,-1]$ and we have $f\left(x_{i}\right)+f\left(x_{j}\right) \in\left[-\infty, f\left(x_{0}\right)\right) \cup[-4,-2]$ for all $i, j \in$ $[0,3] \cup\{4\}$. If $i \in[5,7]$, then $f\left(x_{i}\right) \in[1,3]$ and we have $\min \left\{f\left(x_{i}\right)+f\left(x_{j}\right), f\left(x_{7}\right)\right\} \in[2,3]$ for all $i, j \in[5,7] \cup\{4\}$. Hence the condition (i) in Theorem[3.1] is satisfied since $[-4,-2] \cup[2,3] \subseteq \operatorname{Ran}(f)$.
(ii) For all $\left(f\left(x_{i}\right), f\left(x_{j}\right)\right) \in\left[0, f\left(x_{7}\right)\right]^{2} \cup[-\infty, 0]^{2}$, if $f\left(x_{i}\right)$ and $f\left(x_{j}\right)$ have at least one same summand $f\left(x_{k}\right) \in \operatorname{Ran}(f)$ then we have $i, j \in[0,2]$, or $i, j \in[6,7]$, or $i \in[0,2]$ and $j \in[2,3]$, or $i \in[5,5.5]$ and $j \in[6,7]$, subsequently, $x_{i} \nVdash x_{j}$ by Fig. 2. Therefore, the condition (ii) in Theorem[3.1 is satisfied.
(iii) For all $f\left(x_{i}\right)<0$ and $0<f\left(x_{j}\right) \leq f\left(x_{7}\right)$, we have $i \in[0,3]$ and $j \in[5,7]$. Then $x_{i} \nVdash x_{j}$ by Fig. 2 and the condition (iii) in Theorem 3.1 is satisfied.
(iv) For all $f(a) \leq f\left(x_{i}\right)<f\left(x_{j}\right)$, we have $i, j \in[7,10]$. Then $x_{i} \nVdash x_{j}$ by Fig. 2, For all $f\left(x_{i}\right)<0<$ $f(a) \leq f\left(x_{j}\right)$, we have $i \in[0,3]$ and $j \in[7,10]$. Thus $x_{i} \nVdash x_{j}$ by Fig. 2, Therefore, the condition (iv) in Theorem 3.1 is satisfied.
(v) For all $f\left(x_{i}\right), f\left(x_{j}\right) \in\left(0, f\left(x_{7}\right)\right)$, if $f\left(x_{k}\right)=f\left(x_{i}\right)+f\left(x_{j}\right) \in \operatorname{Ran}(f)$ and $0<f\left(x_{i}\right)+f\left(x_{j}\right) \leq f\left(x_{7}\right)$, then $f\left(x_{k}\right)=f\left(x_{i}\right)+f\left(x_{j}\right) \in[2,3]$. Subsequently, $k \in[6,7]$. Then the condition (v) in Theorem 3.1] is easily checked since $x_{k} \nVdash x_{7}$ by Fig. 2 i.e., $f^{-1}\left(f\left(x_{i}\right)+f\left(x_{j}\right)\right) \nVdash x_{8}$.

So that by Theorem 3.1 the function $f$ is an additive generator of the 2 -uninorm $\mathbb{U}$ given by: for all $x_{i}, x_{j} \in L$,

$$
\mathbb{U}\left(x_{i}, x_{j}\right)= \begin{cases}x_{\max \{i+j-4,0\}}, & \left(x_{i}, x_{j}\right) \in\left(\left[x_{0}, x_{4}\right] \cup I_{x_{4}}\right)^{2}, \\ x_{\min \{i+j-4,7\},}, & \left(x_{i}, x_{j}\right) \in\left[x_{4}, x_{7}\right]^{2}, \\ \min \left\{x_{i}, x_{j}\right\}, & \left(x_{i}, x_{j}\right) \in\left[x_{7}, x_{9}\right]^{2} \cup\left[x_{0}, x_{4}\right) \times\left[x_{4}, x_{10}\right] \cup\left[x_{4}, x_{10}\right] \times\left[x_{0}, x_{4}\right), \\ \max \left\{x_{i}, x_{j}\right\}, & \left(x_{i}, x_{j}\right) \in\left[x_{9}, x_{10}\right]^{2} \cup\left(x_{9}, x_{10}\right] \times\left(x_{7}, x_{9}\right] \cup\left(x_{7}, x_{9}\right] \times\left(x_{9}, x_{10}\right], \\ x_{7}, & \text { otherwise. }\end{cases}
$$



Figure 2. A bounded lattice $L$

The following example shows that each of the conditions (i) - (v) in Theorems 3.1 can't be dropped, respectively.

Example 3.3. Consider the two bounded lattices $L_{1}$ and $L_{2}$ in Fig. 1 and the five injective increasing functions $f_{i}, i=1,2,3,4,5$, with $f_{i}\left(e_{1}\right)=0$ defined by Tables 34 , 5 and 7 respectively.

Table 3
The generator $f_{1}$.

| $x$ | $0_{L_{1}}$ | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ | $e_{1}$ | $x_{5}$ | $x_{6}$ | $a$ | $x_{7}$ | $x_{8}$ | $x_{9}$ | $e_{2}$ | $x_{10}$ | $x_{11}$ | $1_{L_{1}}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $f_{1}$ | -11 | -8 | -4 | 6 | 10 | 0 | 9 | 14 | 15 | 13 | 11 | 17 | 18 | 20 | 22 | 24 |

Table 4
The generator $f_{2}$.

| $x$ | $0_{L_{1}}$ | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ | $e_{1}$ | $x_{5}$ | $x_{6}$ | $a$ | $x_{7}$ | $x_{8}$ | $x_{9}$ | $e_{2}$ | $x_{10}$ | $x_{11}$ | $1_{L_{1}}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $f_{2}$ | -11 | -8 | -4 | 6 | 9 | 0 | 12 | 14 | 15 | 13 | 11 | 17 | 18 | 20 | 22 | 24 |

Table 5
The generator $f_{3}$.

| $x$ | $0_{L_{2}}$ | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ | $e_{1}$ | $x_{5}$ | $x_{6}$ | $a$ | $x_{7}$ | $x_{8}$ | $x_{9}$ | $e_{2}$ | $x_{10}$ | $x_{11}$ | $1_{L_{2}}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $f_{3}$ | -11 | -8 | -4 | 6 | 12 | 0 | 9 | 14 | 15 | 13 | 11 | 17 | 18 | 20 | 22 | 24 |

## Table 6

The generator $f_{4}$.

| $x$ | $0_{L_{1}}$ | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ | $e_{1}$ | $x_{5}$ | $x_{6}$ | $a$ | $x_{7}$ | $x_{8}$ | $x_{9}$ | $e_{2}$ | $x_{10}$ | $x_{11}$ | $1_{L_{1}}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $f_{4}$ | -11 | -8 | -4 | 6 | 12 | 0 | 9 | 14 | 15 | 13 | 19 | 17 | 18 | 20 | 22 | 24 |

Table 7
The generator $f_{5}$.

| $x$ | $0_{L_{1}}$ | $x_{1}$ | $x_{2}$ | $x_{3}$ | $x_{4}$ | $e_{1}$ | $x_{5}$ | $x_{6}$ | $a$ | $x_{7}$ | $x_{8}$ | $x_{9}$ | $e_{2}$ | $x_{10}$ | $x_{11}$ | $1_{L_{1}}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $f_{5}$ | -11 | -8 | -4 | 9 | 13 | 0 | 6 | 12 | 15 | 14 | 11 | 17 | 18 | 20 | 22 | 24 |

(i) One can check that $f_{1}: L_{1} \rightarrow[-11,24]$ satisfies (ii), (iii), (iv) and (v), but it doesn't satisfy (i) in Theorem 3.1 since $f_{1}\left(x_{3}\right)+f_{1}\left(x_{3}\right)=12 \notin \operatorname{Ran}\left(f_{1}\right) \cup\left[-\infty, f_{1}\left(0_{L}\right)\right]$. Applying Formula (3.1), we know that $\mathbb{U}\left(\mathbb{U}\left(x_{3}, x_{3}\right), x_{4}\right)=\mathbb{U}\left(e_{1}, x_{4}\right)=x_{4} \neq a=\mathbb{U}\left(x_{3}, a\right)=\mathbb{U}\left(x_{3}, \mathbb{U}\left(x_{3}, x_{4}\right)\right)$. Thus $\mathbb{U}$ isn't a 2-uninorm.
(ii) Also, one can easily verify that $f_{2}: L_{1} \rightarrow[-11,24]$ satisfies (i), (iii), (iv) and (v), but it doesn't satisfy (ii) of Theorem 3.1 since both $f_{2}\left(x_{5}\right)$ and $f_{2}\left(x_{3}\right)$ have a same summand $f_{2}\left(x_{3}\right)$ but $x_{3} \| x_{5}$. By using Formula (3.1), we have $\mathbb{U}\left(x_{3}, e_{1}\right)=x_{3} \| x_{5}=\mathbb{U}\left(x_{3}, x_{3}\right)$ while $e_{1} \leq x_{3}$. It follows that $\mathbb{U}$ isn't a 2-uninorm.
(iii) One can clarify that $f_{3}: L_{2} \rightarrow[-11,24]$ satisfies (i),(ii),(iv) and (v), but $f_{3}$ doesn't satisfy (iii) in Theorem 3.1 since $f_{3}\left(x_{2}\right)<0, x_{2}<e_{1}$ but $x_{2} \| x_{8}$. By using Formula (3.1), we obtain $\mathbb{U}\left(x_{2}, x_{8}\right)=x_{2}$ and $\mathbb{U}\left(e_{1}, x_{8}\right)=x_{8}$. Thus $\mathbb{U}\left(x_{2}, x_{8}\right) \| \mathbb{U}\left(e_{1}, x_{8}\right)$ while $x_{2}<e_{1}$. It follows that $\mathbb{U}$ isn't a 2-uninorm.
(iv) One can prove that $f_{4}: L_{1} \rightarrow[-11,24]$ satisfies (i), (ii), (iii) and (v), but $f_{3}$ doesn't satisfy (iv) in Theorem 3.1 since $f_{4}(a)<f_{4}\left(x_{9}\right)<f_{4}\left(x_{10}\right)$, but $x_{8} \| x_{10}$. By using Formula (3.1), one has $\mathbb{U}\left(x_{9}, x_{8}\right)=x_{8}$ and $\mathbb{U}\left(x_{10}, x_{8}\right)=x_{10}$. Thus $\mathbb{U}\left(x_{9}, x_{8}\right) \| \mathbb{U}\left(x_{10}, x_{8}\right)$ while $x_{9}<x_{10}$. It follows that $\mathbb{U}$ isn't a 2 -uninorm.
(v) One can check that $f_{5}: L_{1} \rightarrow[-11,24]$ satisfies (i), (ii), (iii) and (iv), but $f_{5}$ doesn't satisfy (v) in Theorem 3.1 since $0<f_{5}\left(x_{5}\right)<f_{5}(a)$, but $x_{6} \nVdash a$. By using Formula (3.1), one knows $\mathbb{U}\left(x_{5}, x_{5}\right)=x_{6}$ and $\mathbb{U}\left(x_{5}, x_{6}\right)=a$. Thus $\mathbb{U}\left(x_{5}, x_{5}\right)=f_{5}^{-1}\left(f_{5}\left(x_{5}\right)+f_{5}\left(x_{5}\right)\right)=x_{6} \| a=\mathbb{U}\left(x_{5}, x_{6}\right)$ while $x_{5}<x_{6}$. It follows that $\mathbb{U}$ isn't a 2 -uninorm.

## 4. Conclusions

In this article, we mainly presented two construction methods of 2 -uninorms on bounded lattices by using additive generators, and also supplied two examples to show that the existence of such additive generators on bounded lattices. It is worth pointing out that uninorms, nullnorms, uni-nullnorms, null-uninorms are all special 2-uninorms, respectively. Consequently, we can obtain their additive generators. Dually, one may discuss the multiplicative generators of these binary operations. It is an interesting problem to further find some lax conditions in both Theorems 3.1 and 3.2 by modifying Formulas (3.1) and (3.3), respectively.

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