When a combination of convexity and continuity forces monotonicity of preferences*

Hirbod Assa† Alexander Zimper‡

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Abstract

We consider arbitrary subsets L of random variables defined on an arbitrary non-additive probability space $(\Omega, \mathcal{F}, \nu)$. A topology τ on L satisfies Condition BU if every open set in this topology which contains $X \in L$ as a member also contains as a subset some (c, ϵ)–*ball* around X, defined as $B_{c,\epsilon}(X) = \{Y \in L | \nu(|X - Y| \geq c) < \epsilon\}$. Condition BU is satisfied by any topology of convergence in non-additive measure ν (Ouyang and Zhang 2011; Li 2012) but also by all coarser topologies. Next we consider preference relations that are continuous with respect to any topology τ satisfying Condition BU. For non-atomic ν we prove that any convex and τ-continuous preference relation over the random variables in a given local cone L must satisfy monotonicity in the local cone order. This monotonicity result comes with surprisingly strong decision theoretic implications: (i) The only τ-continuous and convex preference relation defined over all random variables is the indifference relation; (ii) Any τ-continuous and convex preference relation defined over all positive random variables must satisfy payoff-monotonicity; (iii) Any convex and payoff-monotone preference relation defined over all loss random variables must violate τ-continuity.

Keywords: Non-additive probability measure; Non-atomicity; Weak base topologies; Convergence in non-additive measure; Choquet expected utility; Convex risk measures

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†Kent Business School; University of Kent; Kent CT2 7FS; UK; E-mail: h.assa@kent.ac.uk

‡Department of Economics; University of Pretoria; Hatfield 0028; South Africa; E-mail: alexander.zimper@up.ac.za

1 Introduction

Preferences over random variables are typically required to satisfy the three decision-theoretic principles of payoff-monotonicity, continuity, and convexity. Payoff-monotonicity translates the 'more-is-better' principle into a probabilistic context by stipulating that the decision maker should prefer random variables that payoff-dominate other random variables on non-null events. According to the behavioral principle of continuity, the decision maker should evaluate random variables in a similar manner whenever she perceives these random variables as similar. Similarity perceptions are thereby formally pinned down by the modeler's choice of a topological or/and metric space. The principle of convexity is used to characterize risk, uncertainty, and/or ambiguity aversion (cf., e.g., Maccheroni et al. 2006; Cerreia-Vioglio et al. 2011) as well as the desire for portfolio diversification (Dekel 1989; Föllmer and Schied 2002). For example, risk-averse expected utility decision makers are characterized through convex preferences which are represented by concave Bernoulli utility functions. Similarly, convex Choquet expected utility preferences–which combine risk and ambiguity aversion–are represented through a combination of concave Bernoulli utility functions and convex non-additive probability measures.

Given that these fundamental decision-theoretic principles have quite different behavioral meanings, one might intuitively expect that they are also mathematically independent of each other, i.e., that the modeler might impose them as behavioral axioms in arbitrary combinations. Contrary to this intuition we show that preferences which are convex and continuous with respect to topologies from a specific class–including topologies of convergence in a non-additive measure–force monotonicity properties whenever these preferences are defined over a subdomain of random variables that comes with a local cone structure.[1]

Our analysis is organized in two parts. The first part concerns our topological framework which introduces the relevant topological Condition BU and discusses convergence in a non-additive measure. The second part derives new results about mathematical and decision theoretic implications under the assumption that local cones of random variable are endowed with any topology satisfying Condition BU such as, e.g., the topology of convergence in non-additive measure. In what follows, we briefly sketch the main insights from both parts of our analysis.

[1] To work with a non-additive rather than just an additive probability space is motivated by the descriptively successful decision theoretic models of Choquet expected utility theory (Schmeidler 1989; Gilboa 1987) and of (cumulative) prospect theory (Tversky and Kahneman 1992; Wakker and Tversky 1993). These generalizations of expected utility theory use non-additive probability measures to capture, e.g., behavioral attitudes towards ambiguity or/and risk but also cognitive features like likelihood insensitivity (cf. Wakker 2004; 2010).

1.1 Our topological framework

Denote by L^0 the set of all random variables (i.e., measurable functions) defined on a non-additive (=not necessarily additive) probability space $(\Omega, \mathcal{F}, \nu)$. Fix an arbitrary $L \subseteq L^0$. We introduce through a *weak-base* construction the class of τ_C-topologies which all satisfy the following condition–referred to as *Condition BU*: For every open set $\mathcal{U} \in \tau_C$ containing $X \in L$ we can find some (c, ϵ)-ball $B_{c,\epsilon}(X)$ such that

$$B_{c,\epsilon}(X) \subseteq \mathcal{U}$$

where

$$B_{c,\epsilon}(X) = \{Y \in L | \nu(|X - Y| \geq c) < \epsilon\}$$

for $c, \epsilon \in \mathbb{Q}_{>0}$.

A sequence of random variables $\{X_n\} \subseteq L$ converges in the non-additive measure ν to $X \in L$, denoted $X_n \to_\nu X$, if and only if (=iff)

$$\lim_{n \to \infty} \nu(|X_n - X| \geq c) = 0 \text{ for all } c > 0.$$

We have that

$$X_n \to_\nu X \text{ implies } X_n \to_\tau X \tag{1}$$

iff τ satisfies Condition BU. We say that a topology τ is a 'topology of convergence in the non-additive measure ν' iff it satisfies in addition to (1) also

$$X_n \to_\tau X \text{ implies } X_n \to_\nu X.$$

Denote by τ^* the finest τ_C-topology, which is induced by the weak base

$$\left\{B_{\frac{1}{m},\frac{1}{m}}(X) \mid m \in \mathbb{N}^+\right\}, X \in L.$$

Although our subsequent analysis will apply to any topology satisfying Condition BU, we ask for interpretational reasons the following question: Under which conditions on ν is τ^* a topology of convergence in ν? To answer this question we prove two different Theorems in the Appendix.

Theorem 1. *Suppose that ν satisfies*

(i) *'uniform autocontinuity from above' and*

(ii) *'order continuity'.*

Then all $\left(\frac{1}{m}, \frac{1}{m}\right)$-balls are open sets in the finest topology $\tau^* \in T_{\mathcal{C}}$.

Theorem 2. *We have that* $\tau^* = \tau_{d_0}$ *such that the topology* τ_{d_0} *is induced by the distance function*

$$d_0(X,Y) = (C)\int \frac{|X-Y|}{1+|X-Y|}d\nu$$

where the integral is the Choquet integral.

Theorem 1 implies that τ^* is a 'topology of convergence in ν' whenever ν satisfies the two structural properties (i) 'uniform autocontinuity from above' and (ii) 'order continuity'. The topology τ_{d_0} had been introduced by Ouyang and Zhang (2011) who derive the seminal result that 'uniform autocontinuity of ν from above' (as defined in Wang 1984) is a sufficient condition for τ_{d_0} being a topology of convergence in ν. Combined with our Theorem 2, the analysis in Ouyang and Zhang (2011) implies that 'uniform autocontinuity from above' is already sufficient to guarantee that τ^* is a 'topology of convergence in ν' (cf. Remark 4 in the Appendix).

1.2 New results: Combining our topological framework with a local cone structure and non-atomicity

A *local cone* is a set $L \subseteq L^0$ of random variables defined on $(\Omega, \mathcal{F}, \nu)$ that combines a convex cone structure with the locality property that $1_A Z \in L$ whenever $Z \in L$ and $A \in \mathcal{F}$ (where 1_A denotes the indicator function). The space $(\Omega, \mathcal{F}, \nu)$ is *non-atomic* if we can find for every $\epsilon > 0$ some finite partition $\Pi = \{\Omega_1, ..., \Omega_n\} \subseteq F$ such that

$$\nu(\Omega_i) \leq \epsilon \text{ for all } i \in \{1, ..., n\}.$$

As our main mathematical finding we derive the following result:

Theorem 3. *Consider the topological space* (L, τ) *such that (i)* $(\Omega, \mathcal{F}, \nu)$ *is an arbitrary non-atomic, non-additive probability space, (ii) L is a local cone, and (iii)* τ *satisfies Condition BU. We have for any* $X, Z \in L$ *and any open set* $\mathcal{U} \in \tau$ *that*

$$X \in \mathcal{U} \text{ implies } X + Z \in co(\mathcal{U})$$

where $co(\mathcal{U})$ *denotes the convex hull of* $\mathcal{U}$*.*

As one implication of Theorem 3, the only non-empty, convex and open set in the topological space (L^0, τ) is the set of all random variables L^0 itself whenever $(\Omega, \mathcal{F}, \nu)$ is non-atomic and τ satisfies Condition BU.

Turn now to our decision theoretic analysis. A complete preference relation over the random variables in L is *τ-continuous* iff all strictly better and strictly worse sets are open sets in the topological space (L, τ). As our main decision theoretic result we prove:

Theorem 4. *Consider the topological space (L, τ) such that (i) $(\Omega, \mathcal{F}, \nu)$ is an arbitrary non-atomic, non-additive probability space, (ii) L is a local cone, and (iii) τ satisfies Condition BU. Any τ-continuous and convex preference relation on L must be monotone in the local cone order $\leq_L$ defined as follows:*

$$X \leq_L Y \Leftrightarrow Y - X \in L.$$

This monotonicity result comes with surprisingly powerful implications for subdomains of random variables that are relevant to applications in economics and finance. To see this, fix some topology τ that satisfies Condition BU and consider the following three relevant examples of subdomains which all have a local cone structure: (i) the subdomain of all random variables L^0 (e.g., all random monetary gains and losses), (ii) the subdomain of all non-negative random variables L^0_+ (e.g., all random consumption levels in economic applications), and (iii) the subdomain of all non-positive random variables L^0_- (e.g., all random losses in portfolio risk models).[2]

For $L = L^0$ the local cone order is the equivalence order so that the only τ-continuous and convex preference relation on L^0 must be the indifference relation. As a consequence, there cannot exist any non-trivial (i.e., non-constant) Choquet expected utility representation of preferences on L^0 that expresses risk and ambiguity aversion as well as τ-continuity.

For $L = L^0_+$ the local cone order coincides with the payoff dominance order. Any τ-continuous and convex preference relation on L^0_+ must therefore satisfy the fundamental

[2] In addition, we establish conditions on ν such that the subdomains

$$L^{\nu 0}_+ = \left\{ Y \in L^0 \mid \nu\left(Y < 0\right) = 0 \right\}$$

and

$$L^{\nu 1}_+ = \left\{ Y \in L^0 \mid \nu\left(Y \geq 0\right) = 1 \right\}$$

become local cones, respectively. Our mathematical and decision theoretic results will also apply to the corresponding (ν-dependent) local cone orders.

behavioral principle of payoff-monotonicity. Moreover, there exist non-trivial τ-continuous and convex utility representations of preference relation on L_+^0 such as, e.g., the Choquet expected utility with respect to a convex non-additive probability measure and a strictly increasing, concave, and bounded Bernoulli utility function.

Finally, for $L = L_-^0$, the local cone order becomes the loss-dominance order (i.e., the opposite of the payoff-dominance order). Consequently, any τ-continuous and convex preference relation on L_-^0 must violate the behavioral principle of payoff-monotonicity. This implies that preferences which are induced by convex and monotone risk measures defined on the subdomain of loss random variables–as axiomatized, e.g., in Artzner et al. (1997; 1999) and Föllmer and Schied (2002)–cannot be τ-continuous.

In Assa and Zimper (2018) we had shown that any non-trivial preference relation which is complete on the space of all random variables L^0 cannot be simultaneously convex and continuous in a non-atomic, additive measure. On the other hand, there exist non-trivial preference relations that are only complete for the non-negative random variables in L_+^0 which are convex as well as continuous in measure (think, e.g., of expected utility preferences with a strictly concave and bounded Bernoulli utility function defined on $\mathbb{R}_+$ only). As both subdomains of random variables L^0 and L_+^0 are large spaces, this difference had been puzzling to us. This paper establishes that the difference in the local-cone order of both subdomains L^0 and L_+^0 is the underlying mathematical structure that drives this fundamental difference.

The remainder of our analysis proceeds as follows. Section 2 recalls non-additive probability measures and defines non-atomicity. Section 3 develops our topological framework. Section 4 introduces the notions of local cones and local cone orders. In Section 5 we derive new mathematical results that combine our topological framework with the local cone structure of relevant subdomains of random variables. Section 6 discusses decision theoretic implications. Section 7 concludes.

2 Set-up, preliminaries

2.1 Non-additive probability measures

Throughout the paper we consider arbitrary measurable spaces $(\Omega, \mathcal{F})$ where $\mathcal{F}$ denotes a sigma-algebra on the state space Ω. A non-additive probability measure $\nu : \mathcal{F} \to [0,1]$ defined on $(\Omega, \mathcal{F})$ satisfies

- $\nu(\varnothing) = 0$, $\nu(\Omega) = 1$ (normalization);
- For all $A, B \in \mathcal{F}$, $A \subseteq B$ implies $\nu(A) \leq \nu(B)$ (monotonicity).

The following properties may or may not be satisfied by a non-additive probability measure ν whereby these properties become (strictly) weaker in ascending order. That is, if ν satisfies property i, it also satisfies property $i+1$ whereas the converse statement does not hold for all ν.

1. ν is *additive* iff, for all $A, B \in \mathcal{F}$,

$$\nu(A) + \nu(B) = \nu(A \cup B) + \nu(A \cap B).$$

2. ν is *concave* (i.e., *submodular*) iff, for all $A, B \in \mathcal{F}$,

$$\nu(A) + \nu(B) \geq \nu(A \cup B) + \nu(A \cap B).$$

3. ν is *subadditive* iff, for all $A, B \in \mathcal{F}$ with $A \cap B = \emptyset$

$$\nu(A) + \nu(B) \geq \nu(A \cup B).$$

4. ν is *uniformly autocontinuous from above* iff for any $\delta > 0$ there is an $n \in \mathbb{N}^+$ such that, for all $A, B \in \mathcal{F}$,

$$\nu(B) < \frac{1}{n} \text{ implies } \nu(A \cup B) < \nu(A) + \delta.$$

5. ν is *autocontinuous from above* iff, for all $\{A_n\} \subseteq \mathcal{F}$, $A \in \mathcal{F}$,

$$\lim_{n \to \infty} \nu(A_n) = 0 \text{ implies } \lim_{n \to \infty} \nu(A \cup A_n) = \nu(A)$$

The definitions of 'autocontinuity from above' and 'uniform autocontinuity from above' had been introduced by Wang (1984). This author also shows that 'uniform autocontinuity from above' is weaker than 'subadditivity' (Proposition 8 in Wang 1984) but stronger than 'autocontinuity from above' (cf. Example 3 in Wang 1984). Next write

$$A_n \searrow A$$

for

$$A_1 \supseteq A_2 \supseteq \ldots \text{ and } \bigcap_{n \in \mathbb{N}} A_n = A.$$

1. ν is *continuous from above* iff

$$A_n \searrow A \text{ implies } \lim_{n \to \infty} \nu(A_n) = A.$$

2. ν is *order continuous* (i.e., *continuous from above at zero*) iff

$$A_n \searrow \emptyset \text{ implies } \lim_{n\to\infty} \nu(A_n) = 0.$$

As pointed out in the seminal contribution by Wang (1984), 'autocontinuity from above' is a very different structural property than 'continuity from above' (cf. Example 2 in Wang 1984 and Example 4.2 in Ouyang and Zhang 2011). Note that 'order continuity' and 'autocontinuity from above' together imply 'continuity from above' (Proposition 5 in Wang 1984).

2.2 Non-atomicity

We introduce the following definition of non-atomicity for non-additive probability measures.

Definition 1. *We call the non-additive probability space $(\Omega, \mathcal{F}, \nu)$ non-atomic iff there exists for every $\epsilon > 0$ some finite partition $\Pi = \{\Omega_1, ..., \Omega_n\} \subseteq \mathcal{F}$, depending on ϵ, such that*

$$\nu(\Omega_i) \leq \epsilon \text{ for all } i \in \{1, ..., n\}.$$

The standard example of a non-atomic probability space for an additive measure is $\left((0,1), \mathcal{F}^B, \mu^L\right)$ where $\mathcal{F}^B$ is the Borel sigma-algebra on the open unit interval and μ^L is the Lebesgue measure. In particular, this space is sufficient to capture all possible (additive probability) distributions of random variables (cf. the Second Proof of Theorem 14.1 in Billingsley 1995).

Example 1. Consider the non-additive space $\left((0,1), \mathcal{F}^B, \mu^L\right)$ such that the non-additive measure ν is given as

$$\nu(A) = w\left(\mu^L(A)\right) \text{ for all } A \in \mathcal{F} \tag{2}$$

for some probability weighting function $w : [0,1] \to [0,1]$ that is increasing and satisfies $w(0) = 0$ and $w(1) = 1$. One can show the following: If $w(\cdot)$ is strictly increasing and continuous on some open neighborhood around zero, this non-additive probability space must also be non-atomic.

The above definition of a non-atomic space is, however, much more general than the generated ν considered in Example 1. The following example implies that non-atomicity for non-additive measures can even hold for finite state spaces Ω, which would be impossible for additive probability measures.[3]

Example 2. Suppose that there exists some finite partition $\Pi = \{\Omega_1, ..., \Omega_n\} \subseteq \mathcal{F}$ such that $\nu(\Omega_i) = 0$ for $i = 1, .., n$. Non-atomicity is trivially satisfied so that $(\Omega, \mathcal{F}, \nu)$ is non-atomic in our sense.

3 Topological framework

3.1 Condition BU

Denote by L^0 the set of all $\mathcal{F}$-measurable real-valued functions, i.e., all random variables, defined on an arbitrary non-additive probability space $(\Omega, \mathcal{F}, \nu)$. Fix an arbitrary subset of random variables $L \subseteq L^0$. For given $X \in L$ and $c, \epsilon \in \mathbb{Q}_{>0}$ introduce the following (c, ϵ)-ball at X

$$B_{c,\epsilon}(X) = \{Y \in L | \nu(|X - Y| \geq c) < \epsilon\}$$

whereby $B_{c,\epsilon}(X) = L$ for $\epsilon > 1$. Denote by

$$\mathcal{B}(X) = \{B_{c,\epsilon}(X) \mid c, \epsilon \in \mathbb{Q}_{>0}\}$$

the collection of all these (c, ϵ)-balls. $B_{c,\epsilon}(X)$ contains all $Y \in L$ which are close to X in the sense that any difference between X and Y weakly greater than $c > 0$ only happens on an event with ν-probability less than ϵ. Observe that

$$B_{c,\epsilon}(X) \subseteq B_{c',\epsilon'}(X) \text{ for } c \leq c' \text{ and } \epsilon \leq \epsilon'. \tag{3}$$

Consider now any topology τ defined on $L \subseteq L^0$ for a given ν. Central to our analysis will be the following property, which may or may not be satisfied by any given topology.

Definition 2. *We say that a topology τ satisfies the Condition BU iff, for every open set $\mathcal{U} \in \tau$,*

$$X \in \mathcal{U} \text{ implies } B_{c,\epsilon}(X) \subseteq \mathcal{U}$$

for some $B_{c,\epsilon}(X) \in \mathcal{B}(X)$.

[3] We owe this example to an anonymous referee.

Fix any topology τ on L. Recall that the *neighborhood system* at X that corresponds to τ is defined as

$$\mathcal{N}_X = \{\mathcal{V} \subseteq L \mid \mathcal{U} \subseteq \mathcal{V} \text{ for some } \mathcal{U} \in \tau \text{ with } X \in \mathcal{U}\}. \tag{4}$$

A sequence $\{X_n\} \subseteq L$ converges in the topology τ to $X \in L$, denoted $X_n \to_\tau X$, iff it is eventually in every neighborhood $\mathcal{V} \in \mathcal{N}_X$ of X.

A sequence $\{X_n\} \subseteq L$ converges in the non-additive measure ν to X, denoted $X_n \to_\nu X$, iff for every $c > 0$

$$\lim_{n\to\infty} \nu\left(|X_n - X| \geq c\right) = 0.$$

Or equivalently: $X_n \to_\nu X$ iff, for every fixed $c > 0$, the sequence $\{X_n\}$ is eventually in every (c, ϵ)-ball $B_{c,\epsilon}(X)$ with $\epsilon > 0$. By set-inclusion (3), $X_n \to_\nu X$ iff there exists for every $m \in \mathbb{N}^+$ some M such that

$$X_n \in B_{\frac{1}{m},\frac{1}{m}}(X) \text{ for all } n \geq M. \tag{5}$$

Proposition 1. *A topology τ comes with the convergence behavior*

$$X_n \to_\nu X \text{ implies } X_n \to_\tau X \tag{6}$$

iff τ satisfies Condition BU.

Proof. The if-part. If τ satisfies Condition BU, there exists some $B_{c,\epsilon}(X) \subseteq \mathcal{U}$ whenever $X \in \mathcal{U} \in \tau$. Fix this $B_{c,\epsilon}(X)$ and $\mathcal{U}$ and suppose that $X_n \to_\nu X$. If $\{X_n\}$ is eventually in every $B_{\frac{1}{m},\frac{1}{m}}(X)$, it must eventually be in $B_{c,\epsilon}(X)$ and therefore in every neighborhood $\mathcal{V}$ of X such that $\mathcal{U} \subseteq \mathcal{V}$. Since this argument applies to every $\mathcal{U} \in \tau$ with $X \in \mathcal{U}$, $\{X_n\}$ will be eventually in every neighborhood of X which gives us the convergence behavior (6).

The only-if part. Any $\mathcal{U} \in \tau$ with $X \in \mathcal{U}$ is itself a neighborhood of X. Suppose that $X_n \to_\nu X$ so that (5) must hold. If Condition BU is violated, there are no $\left(\frac{1}{m}, \frac{1}{m}\right)$-balls such that $B_{\frac{1}{m},\frac{1}{m}}(X) \subseteq \mathcal{U}$. Consequently, $\{X_n\}$ will never be in the neighborhood $\mathcal{U} \in \mathcal{N}_X$ of X.□□

3.2 The class of $\tau_{\mathcal{C}}$-topologies

Fix ν and $L \subseteq L^0$. We introduce the class of $\tau_{\mathcal{C}}$-topologies which all satisfy, by a *weak base* construction, Condition BU. To this purpose, define the following countable collection of all finite intersections of (c, ϵ)-balls in $\mathcal{B}(X)$

$$\mathcal{C}^*(X) = \{B_{c_1,\epsilon_1}(X) \cap \cdots \cap B_{c_n,\epsilon_n}(X) \,|\, n \in \mathbb{N}, c_i, \epsilon_i \in \mathbb{Q}_{>0}\}. \tag{7}$$

We write $\mathcal{C}(X)$ as the generic expression for any subset of $\mathcal{C}^*(X)$ that is closed under finite intersections, i.e., $\mathcal{C}(X) \subseteq \mathcal{C}^*(X)$ and

$$y, z \in \mathcal{C}(X) \text{ implies } y \cap z \in \mathcal{C}(X).$$

Definition 3. *Fix some collection of $\mathcal{C}(X)$ for all $X \in L$:*

$$\mathcal{C} = \{\mathcal{C}(X) \mid X \in L\}. \tag{8}$$

(i) *We say that the following topology is induced by $\mathcal{C}$:*

$$\tau_{\mathcal{C}} = \{\mathcal{U} \subseteq L \mid \text{ for all } X \in \mathcal{U}, \mathcal{U} \in \mathcal{W}_X\} \cup \{\emptyset, L\}$$

such that

$$\mathcal{W}_X = \{\mathcal{V} \subseteq L \mid y \subseteq \mathcal{V} \text{ for some } y \in \mathcal{C}(X)\}.$$

(ii) *We denote by $T_{\mathcal{C}}$ the class of all such $\tau_{\mathcal{C}}$-topologies induced by all possible collections (8).*

The family $\mathcal{C}$ forms a *weak base* for the $\tau_{\mathcal{C}}$-topology so that $\tau_{\mathcal{C}}$ is *weakly first-countable* (or *g-first countable*) (cf. Arhangel'skiĭ 1966; Siwiec 1974; Hong 1999; Yang and Shi 2011). To see that any $\tau_{\mathcal{C}} \in T_{\mathcal{C}}$ satisfies Condition BU, observe that there must be for every $\mathcal{U} \in \tau_{\mathcal{C}}$ with $X \in \mathcal{U}$ some $y \subseteq \mathcal{U}$ such that $y \in \mathcal{C}(X)$ has the following form

$$y = B_{c_1,\epsilon_1}(X) \cap \cdots \cap B_{c_n,\epsilon_n}(X).$$

Let now $\frac{1}{m} \leq \min\{c_1, \epsilon_1, ..., c_n, \epsilon_n\}$ so that we obtain, by set-inclusion (3),

$$B_{\frac{1}{m},\frac{1}{m}}(X) \subseteq y \subseteq \mathcal{U}.$$

Since $B_{\frac{1}{m},\frac{1}{m}}(X) \in \mathcal{B}(X)$, Condition BU is satisfied.

Denote by $\mathcal{C}$ and $\mathcal{C}'$ two weak bases with corresponding topologies $\tau_{\mathcal{C}}$ and $\tau_{\mathcal{C}'}$. Suppose that we have for all $X \in L$: for every $y \in \mathcal{C}(X) \in \mathcal{C}$ there exists some $y' \in \mathcal{C}'(X) \in \mathcal{C}'$ such that

$$y' \subseteq y.$$

In that case, we must have for the corresponding topologies

$$\begin{array}{rcl} \mathcal{U} & \in & \tau_{\mathcal{C}} \text{ implies } \mathcal{U} \in \tau_{\mathcal{C}'} \\ & \Leftrightarrow & \\ \tau_{\mathcal{C}} & \subseteq & \tau_{\mathcal{C}'} \end{array}$$

so that $\tau_{\mathcal{C}'}$ is *finer* than $\tau_{\mathcal{C}}$ (or: $\tau_{\mathcal{C}}$ is *coarser* than $\tau_{\mathcal{C}'}$). Both topologies are *equivalent*, i.e., $\tau_{\mathcal{C}} = \tau_{\mathcal{C}'}$, iff $\tau_{\mathcal{C}'}$ is finer than $\tau_{\mathcal{C}}$ while $\tau_{\mathcal{C}}$ is also finer than $\tau_{\mathcal{C}'}$.

The *coarsest* topology in $T_{\mathcal{C}}$ is the trivial topology

$$\tau_{\mathcal{C}} = \{L, \emptyset\}$$

induced by

$$\mathcal{C} = \{\mathcal{C}(X) = \{L\} \mid X \in L\}.$$

The *finest* topology in $T_{\mathcal{C}}$ is induced by (7) itself for all $X \in L$:

$$\mathcal{C} = \{\mathcal{C}^*(X) \mid X \in L\}.$$

For later reference we denote the finest topology in $T_{\mathcal{C}}$ by τ^*.

Proposition 2. *The finest topology τ^* in $T_{\mathcal{C}}$ is equivalent to the topology $\tau_{\mathcal{C}'} \in T_{\mathcal{C}}$ induced by*

$$\mathcal{C}' = \left\{\mathcal{C}'(X) = \left\{B_{\frac{1}{m},\frac{1}{m}}(X) \mid m \in \mathbb{N}^+\right\} \mid X \in L\right\}.$$

Proof. We have, by set inclusion (3), for every

$$y = B_{c_1,\epsilon_1}(X) \cap \cdots \cap B_{c_n,\epsilon_n}(X) \in \mathcal{C}^*(X)$$

some

$$y' = B_{\frac{1}{m},\frac{1}{m}}(X) \in \mathcal{C}'(X)$$

with $\frac{1}{m} \leq \min\{c_1, \epsilon_1, ..., c_n, \epsilon_n\}$ such that

$$y' \subseteq y.$$

This shows that $\tau_{\mathcal{C}'}$ is finer than τ^*. On the other hand, τ^* is trivially finer than $\tau_{\mathcal{C}'}$ because all $B_{\frac{1}{m},\frac{1}{m}}(X) \in \mathcal{C}^*(X)$.□□

By Proposition 1, we thus obtain the following equivalent characterization of τ^*.

Observation 1. *The finest topology τ^* in $T_{\mathcal{C}}$ is equivalently given as*

$$\tau^* = \{\mathcal{U} \subseteq L \mid \text{ for all } X \in \mathcal{U}, \mathcal{U} \in \mathcal{W}_X\} \cup \{\emptyset, L\}$$

such that

$$\mathcal{W}_X = \left\{\mathcal{V} \subseteq L \mid B_{\frac{1}{m},\frac{1}{m}}(X) \subseteq \mathcal{V} \text{ for some } m \in \mathbb{N}^+\right\}.$$

3.3 Convergence in the non-additive measure ν

We call a topology τ with convergence behavior

$$X_n \to_\tau X \text{ iff } X_n \to_\nu X$$

a 'topology of convergence in (the non-additive measure) ν'. By Proposition 1, we know that any 'topology of convergence in ν' must satisfy Condition BU. Although our subsequent mathematical and decision-theoretic results will be derived for all topologies satisfying Condition BU–regardless of whether they are 'topologies of convergence in ν' or not–it is interesting to ask under which conditions does a topology τ become a 'topology of convergence in ν'.

Proposition 3.

(i) *The finest topology $\tau^* \in T_{\mathcal{C}}$ is a topology of convergence in ν, i.e.,*

$$X_n \to_{\tau^*} X \text{ iff } X_n \to_\nu X,$$

whenever all $\left(\frac{1}{m},\frac{1}{m}\right)$-balls at X are neighborhoods of X.

(ii) *This is in particular the case whenever all $\left(\frac{1}{m},\frac{1}{m}\right)$-balls are open sets in τ^*.*

Proof. The if-part. As a $\tau_{\mathcal{C}}$-topology τ^* satisfies, by construction, Condition BU which gives us, by Proposition 1, that $X_n \to_\nu X$ implies $X_n \to_{\tau^*} X$.

The only-if part. Suppose now that $X_n \to_{\tau^*} X$. If $\{X_n\}$ is eventually in every neighborhood of X, it must also be eventually in every $B_{\frac{1}{m},\frac{1}{m}}(X)$ whenever all $\left(\frac{1}{m},\frac{1}{m}\right)$-balls at X are neighborhoods of X, i.e., $X_n \to_{\tau^*} X$ implies $X_n \to_{\nu} X$. □□

Proposition 3 comes with the caveat, however, that not all $\left(\frac{1}{m},\frac{1}{m}\right)$-balls are necessarily neighborhoods in the topology $\tau^* \in T_{\mathcal{C}}$ for arbitrary $(\Omega,\mathcal{F},\nu)$ and L. As it is, there are $(\Omega,\mathcal{F},\nu)$ and L for which there does not exist any topology of convergence in ν whereas the finest topology $\tau^* \in T_{\mathcal{C}}$ always exists by construction.

Example 3. Let $\{A_1, A_2\} \subseteq \mathcal{F}$ be some partition of Ω and assume that

$$\nu(A_1) = \nu(A_2) = 0.$$

Consider $L = \{X, Y, Z\}$ such that

	A_1	A_2
X	1	1
Y	1	0
Z	0	1

Note that we have for any $m > 1$

$$\begin{aligned} B_{\frac{1}{m},\frac{1}{m}}(X) &= \{X, Y, Z\}, \\ B_{\frac{1}{m},\frac{1}{m}}(Y) &= \{X, Y\}, \\ B_{\frac{1}{m},\frac{1}{m}}(Z) &= \{X, Z\}, \end{aligned}$$

resulting for the weak base

$$\mathcal{C} = \left\{\mathcal{C}(X) = \left\{B_{\frac{1}{m},\frac{1}{m}}(X) \mid m \in \mathbb{N}^+\right\} \mid X \in L\right\}$$

in

$$\begin{aligned} \mathcal{W}_X &= \{\{X, Y, Z\}\}, \\ \mathcal{W}_Y &= \{\{X, Y\}, \{X, Y, Z\}\}, \\ \mathcal{W}_Z &= \{\{X, Z\}, \{X, Y, Z\}\}. \end{aligned}$$

This gives us as the finest topology in $T_{\mathcal{C}}$ the trivial topology

$$\tau^* = \{\emptyset, \{X, Y, Z\}\}$$

for which the $B_{\frac{1}{m},\frac{1}{m}}(Y)$ and $B_{\frac{1}{m},\frac{1}{m}}(Z)$ are neither open sets nor neighborhoods of Y and Z, respectively, for $m > 1$. Note that everything converges to everything in the trivial topology τ^*, including $Y \to_{\tau^*} Z$. On the other hand, the Y do not converge in ν to Z because, for every $c \in (0,1]$,

$$\lim_{n\to\infty} \nu\left(|Y - Z| \geq c\right) = 1,$$

implying

$$Y \to_{\tau^*} Z \text{ but not } Y \to_{\nu} X.$$

Consequently, τ^* is not a topology of convergence in ν. To see that there does not exist any topology of convergence in ν for the above example, observe that any topology which is strictly finer than τ^* would violate Condition BU.□

3.4 Excursus: When is the τ^*-topology a topology of convergence in ν?

Although it is not essential to the remaining analysis of this paper–which applies to all topologies satisfying Condition BU–it would be interesting to identify structural properties of ν which ensure that τ^* is a topology of convergence in ν for arbitrary $L \subseteq L^0$. We prove the following result in the Appendix.

Theorem 1. *Suppose that ν satisfies*

(i) *'uniform autocontinuity from above' and*

(ii) *'order continuity'.*

Then all $\left(\frac{1}{m}, \frac{1}{m}\right)$-balls are open sets in the finest topology $\tau^* \in T_{\mathcal{C}}$.

By Proposition 3(ii), we immediately obtain the following corollary to Theorem 1.

Corollary 1. *The finest topology $\tau^* \in T_{\mathcal{C}}$ is a 'topology of convergence in ν' for any ν that satisfies*

(i) *'uniform autocontinuity from above' and*

(ii) *'order continuity'.*

Theorem 1 and Corollary 1 are closely related to the analysis in Ouyang and Zhang (2011)–and to a lesser extend to Li (2012)[4]–who consider topologies of convergence in ν induced by some distance function d, denoted τ_d. These authors' strategy for constructing τ_d-topologies that are also topologies of convergence in ν is as follows:

1. Find a distance function d that comes with the convergence behavior
$$\lim_{n\to\infty} d\left(X_n, X\right) = 0 \text{ iff } X_n \to_\nu X. \tag{9}$$
2. Identify conditions on ν which ensure that all *d-balls* are open sets in τ_d.

To be precise, recall that a *distance function* $d : L^0 \times L^0 \to [0, \infty)$ has to satisfy $d(X, X) = 0$. Fix a distance function $d : L^0 \times L^0 \to [0, \infty)$ and an arbitrary $L \subseteq L^0$. Define the *d-ball* at X of radius $\frac{1}{n}$ as

$$B^d_{\frac{1}{n}}(X) = \left\{ Y \in L \mid d(X, Y) < \frac{1}{n} \right\}$$

and consider the collection of all such d-balls at X

$$\mathcal{B}^d(X) = \left\{ B^d_{\frac{1}{n}}(X) \mid n \in \mathbb{N}^+ \right\}.$$

Definition 4. *For a fixed d and $L \subseteq L^0$ define the d-induced topology*

$$\tau_d = \left\{ \mathcal{U} \subseteq L \mid \text{ for all } X \in \mathcal{U}, \mathcal{U} \in \mathcal{W}^d_X \right\} \cup \{\emptyset, L\}$$

such that

$$\mathcal{W}^d_X = \left\{ \mathcal{V} \subseteq L \mid B^d_{\frac{1}{n}}(X) \subseteq \mathcal{V} \text{ for some } n \in \mathbb{N}^+ \right\}.$$

In our terminology, the family

$$\left\{ \mathcal{B}^d(X) \mid X \in L \right\}$$

[4] Li (2012) considers conditons such that the distance function

$$d_{\inf \varepsilon}(X, Y) = \inf \left\{ \varepsilon \mid \nu\left(|X - Y| > \varepsilon\right) \leq \varepsilon \right\},$$

which comes with convergence behavior (9), induces a pseudometric space.

forms a *weak base* for the τ_d-topology. Ouyang and Zhang (2011) consider the topology τ_{d_0} induced by the distance function $d_0 : L^0 \times L^0 \to [0,1)$ such that[5]

$$d_0\left(X,Y\right) = (C)\int \frac{|X-Y|}{1+|X-Y|}d\nu, \tag{10}$$

which comes with convergence behavior (9). The integral in (10) is the Choquet integral with respect to ν defined for bounded, $\mathcal{F}$-measurable functions f as follows (Choquet 1954; Schmeidler 1986; Wakker 1993; König 2003):

$$(C)\int f d\nu = \int_{-\infty}^{0} \left(\nu\left(f \geq t\right) - 1\right) dt + \int_{0}^{\infty} \nu\left(f \geq t\right) dt.$$

Ouyang and Zhang (2011) show, in a first step, that d_0 becomes a pseudometric for concave ν because concavity of ν ensures–via the triangle inequality–that all d_0-balls are open sets (Theorem 3.1 in Ouyang and Zhang 2011). For non-concave ν, however, d_0 might violate the triangle-inequality so that it is no longer guaranteed that the d_0-balls are open sets. In a second step, Ouyang and Zhang (2011) look therefore for conditions on ν beyond concavity which would ensure that all d_0-balls are open sets. The proof of Theorem 4.1 in Ouyang and Zhang (2011) implies the following results for $L = L^0$.[6]

Sufficiency part. *Suppose that ν is 'uniformly autocontinuous from above'. Then the d_0-balls are open sets in τ_{d_0}.*

Necessity part. *If ν violates 'autocontinuity from above', not all d_0-balls are open sets in τ_{d_0}.*

In the Appendix we prove the following equivalence result.

Theorem 2. *For arbitrary $(\Omega, \mathcal{F}, \nu)$ and $L \subseteq L^0$ we have th cfr345at $\tau^* = \tau_{d_0}$.*

[5]For additive ν, d_0 is the standard pseudometric which captures convergence in ν and which becomes a metric for the standard concept of equivalence classes of random variables (cf. 13.40 Lemma in Aliprantis and Border 2006).

[6]The original formulation of Theorem 4.1 in Ouyang and Zhang (2011) states that (i) 'uniform autocontinuity from above' is sufficient for τ_{d_0} being a topology whereas (ii) 'autocontinuity from above' is necessary for τ_{d_0} being a topology. In our opinion, this formulation is somewhat misleading because, by the weak base construction of Definition 4, τ_{d_0} is a tautology for an arbitrary ν. What these authors actually prove is that the respective autocontinuity conditions are sufficient, resp. necessary, for the d_0-balls to be open sets in τ_{d_0}. Also note that one can construct examples such that $\tau^* = \tau_{d_0}$ is a topology of convergence in ν even if 'autocontinuity from above' is violated so that not all d_0-balls are open sets in τ_{d_0}.

Combined with the sufficiency part of Theorem 4.1 in Ouyang and Zhang (2011), Theorem 2 implies that all d_0-balls are open sets in τ^* whenever ν is 'uniformly autocontinuous from above'. Moreover, the proof of Theorem 2 shows that there exists for every $\left(\frac{1}{m}, \frac{1}{m}\right)$-ball $B_{\frac{1}{m},\frac{1}{m}}(X)$ some d_0-ball $B^{d_0}_{\frac{1}{n}}(X)$ such that

$$B^{d_0}_{\frac{1}{n}}(X) \subseteq B_{\frac{1}{m},\frac{1}{m}}(X).$$

Consequently, any $\left(\frac{1}{m}, \frac{1}{m}\right)$-ball $B_{\frac{1}{m},\frac{1}{m}}(X)$ is a neighborhood of X in the topology τ^* whenever the d_0-balls $B^{d_0}_{\frac{1}{n}}(X)$ are open sets in τ^*. By Proposition 3(i), we therefore obtain the following stronger version of Corollary 1.

Corollary 1*. *The finest topology $\tau^* \in T_{\mathcal{C}}$ is a 'topology of convergence in ν' for any ν that satisfies 'uniform autocontinuity from above'.*

4 Local cones and local cone orders

We are going to use the following concepts and termini.

- A set $L \subseteq L^0$ is a *convex cone* iff (i) $X + Y \in L$ for all $X, Y \in L$ and (ii) $\lambda X \in L$ for all $\lambda \geq 0$, $X \in L$.
- A set $L \subseteq L^0$ is *local* iff, for any $X \in L$ and any $A \in \mathcal{F}$, $X1_A \in L$ where 1_A denotes the indicator function.
- We call $L \subseteq L^0$ a *local cone* iff L is a convex cone and local.

Definition 5. *Let L be a local cone. We define the local cone order $\leq_L$ (or simply the L-order) as follows:*

$$X \leq_L Y \Leftrightarrow Y - X \in L.$$

If L is a local cone such that $X \leq_L Y$ and $X \leq_L Z$, we also have, for $\lambda \in (0,1)$,

$$\begin{aligned} \lambda(Y - X) + (1-\lambda)(Z - X) &= \lambda Y + (1-\lambda) Z - X \in L \\ &\Leftrightarrow \\ X &\leq_L \lambda Y + (1-\lambda) Z. \end{aligned}$$

The remainder of this section introduces examples of local cone orders that will be relevant to our subsequent analysis.

4.1 ν-independent local cone orders

Denote by $\leq$ the payoff-dominance order, i.e.,

$$X \leq Y \text{ iff } X(\omega) \leq Y(\omega) \text{ for all } \omega \in \Omega.$$

If $X \leq Y$, we say that Y dominates X. Relevant subdomains of random variables which are local cones are (i) the set L^0 of all random variables itself, (ii) the set of all non-negative random variables

$$L^0_+ = \left\{Y \in L^0 \mid 0 \leq Y(\omega) \text{ for all } \omega \in \Omega\right\},$$

and (iii) the set of all non-positive random variables

$$L^0_- = \left\{Y \in L^0 \mid Y(\omega) \leq 0 \text{ for all } \omega \in \Omega\right\}.$$

The respective local cone orders for these relevant subdomains of random variables are given as follows.

Observation 2.

(i) *For $L = L^0$ the L-order coincides with the equivalence order, i.e., for all $X, Y \in L^0$,*

$$X \leq_L Y.$$

(ii) *For $L = L^0_+$ the L-order coincides with the payoff-dominance order, i.e., for all $X, Y \in L^0_+$,*

$$X \leq_L Y \Leftrightarrow X \leq Y.$$

(iii) *For $L = L^0_-$ the L-order coincides with the reversed payoff-dominance order, i.e., for all $X, Y \in L^0_-$,*

$$X \leq_L Y \Leftrightarrow Y \leq X.$$

4.2 ν-dependent local cone orders

We are going to refer to the following properties, which may or may not be satisfied by any given ν.

- ν is *convex at one* iff, for all $A, B \in \mathcal{F}$,

$$\nu(A) = 1 \text{ and } \nu(B) = 1 \text{ implies } \nu(A \cap B) = 1. \tag{11}$$

- ν is *concave at zero* iff, for all $A, B \in \mathcal{F}$,

$$\nu(A) = 0 \text{ and } \nu(B) = 0 \text{ implies } \nu(A \cup B) = 0. \tag{12}$$

For a fixed ν define the subdomain $L_+^{\nu 0}$ of random variables that are non-negative except on an ν-zero event, i.e.,

$$L_+^{\nu 0} = \left\{ Y \in L^0 \mid \nu(Y < 0) = 0 \right\}. \tag{13}$$

$L_+^{\nu 0}$ is local because we have, for any $A \in \mathcal{F}$,

$$(X 1_A < 0) \subseteq (X < 0),$$

so that, by monotonicity of ν,

$$\nu(X < 0) = 0 \text{ implies } \nu(X 1_A < 0) = 0.$$

In contrast to the ν-independent definition of L_+^0, we can, in general, not be sure that $L_+^{\nu 0}$ is also a convex cone. Clearly, we have that $\lambda X \in L_+^{\nu 0}$ for all $\lambda \geq 0$ and $X \in L_+^{\nu 0}$. But we might encounter situations where $X, Y \in L_+^{\nu 0}$ does not necessarily imply $X + Y \in L_+^{\nu 0}$.

Example 4. Consider the space $(\Omega, \mathcal{F}, \nu)$ such that for some partition $\{A_1, A_2\} \subset \mathcal{F}$ of Ω

$$\nu(A_1) = \nu(A_2) = 0.$$

Observe for the following random variables

	A_1	A_2
X	-1	0
Y	0	-1
$X+Y$	-1	-1

that

$$\nu(X < 0) = 0 \text{ and } \nu(Y < 0) = 0 \text{ but not } \nu(X + Y < 0) = 0.$$

Consequently, $L_+^{\nu 0}$ is not a convex cone.□

Proposition 4. *If ν is concave at zero, the subdomain $L_+^{\nu 0}$ defined by (13) is a local cone.*

Proof. To show that concavity at zero (12) is sufficient for $L_{+}^{0\nu}$ being a local cone, we have to establish that $X, Y \in L_{+}^{0\nu}$ implies $X + Y \in L_{+}^{0\nu}$ whenever ν is concave at zero. To this purpose, define the following events in $\mathcal{F}$

$$\begin{aligned} A_1 &= (X < 0) \cap (Y \geq 0)\,, \\ A_2 &= (X < 0) \cap (Y < 0)\,, \\ A_3 &= (X \geq 0) \cap (Y < 0)\,, \\ A_4 &= (X \geq 0) \cap (Y \geq 0)\,, \end{aligned}$$

which form a partition of Ω. Next, observe that

$$(X + Y < 0) \subseteq (A_1 \cup A_2 \cup A_3)$$

as well as

$$\begin{aligned} (X < 0) &= A_1 \cup A_2, \\ (Y < 0) &= A_2 \cup A_3. \end{aligned}$$

By concavity at zero (12),

$$\nu\,(A_1 \cup A_2) = 0 \text{ and } \nu\,(A_2 \cup A_3) = 0 \text{ implies } \nu\,(A_1 \cup A_2 \cup A_3) = 0$$

while monotonicity implies

$$\nu\,(X + Y < 0) \subseteq \nu\,(A_1 \cup A_2 \cup A_3)\,.$$

Collecting terms gives us the desired result

$$\begin{aligned} \nu\,(X < 0) &= 0 \text{ and } \nu\,(Y < 0) = 0 \text{ implies } \nu\,(X + Y < 0) = 0 \\ &\Leftrightarrow \\ X, Y &\in L_{+}^{0\nu} \text{ implies } X + Y \in L_{+}^{0\nu}. \end{aligned}$$

□□

As an alternative to (13), define now the subdomain $L_{+}^{\nu 1}$ of random variables that are positive on an ν-one event, i.e.,

$$L_{+}^{\nu 1} = \left\{ Y \in L^0 \mid \nu\,(Y \geq 0) = 1 \right\}. \tag{14}$$

$L_+^{\nu 1}$ is local because, by monotonicity of ν,

$$\begin{array}{rcl} (X \geq 0) & \subseteq & (X 1_A \geq 0) \\ & \Rightarrow & \\ \nu (X \geq 0) & = & 1 \text{ implies } \nu (X 1_A \geq 0) = 1. \end{array}$$

Whereas $\lambda X \in L_+^{\nu 1}$ holds for all $\lambda \geq 0$ and $X \in L_+^{\nu 1}$, we can, in general, not guarantee that $X, Y \in L_+^{\nu 1}$ implies $X + Y \in L_+^{\nu 1}$.

Example 4 (modified). Consider the same set-up as under Example 3 except for the modified non-additive probability measure ν, which now satisfies

$$\nu (A_1) = \nu (A_2) = 1.$$

Observe that $\nu (0 \leq X) = 1$ and $\nu (0 \leq Y) = 1$ but not $\nu (0 \leq X + Y) = 1$, which violates the definition of a convex cone.$\square$

Proposition 5. *If ν is convex at one, the subdomain $L_+^{\nu 1}$ defined by (14) is a local cone.*

Proof. We have to show that

$$\nu (0 \leq X) = 1 \text{ and } \nu (0 \leq Y) = 1 \text{ implies } \nu (0 \leq X + Y) = 1$$

whenever convexity at one (11) is satisfied. Observe that

$$((0 \leq X) \cap (0 \leq Y)) \subseteq (0 \leq X + Y)$$

so that monotonicity implies

$$\nu ((0 \leq X) \cap (0 \leq Y)) \leq \nu (0 \leq X + Y).$$

The result follows because we have, by (11), that

$$\nu (0 \leq X) = 1 \text{ and } \nu (0 \leq Y) = 1 \text{ implies } \nu ((0 \leq X) \cap (0 \leq Y)) = 1.$$

$\square\square$

Note that we have, by (13) and (14),

$$\begin{aligned} Y - X &\in L_+^{\nu 0} \Leftrightarrow \nu\left(Y - X < 0\right) = 0, \\ Y - X &\in L_+^{\nu 1} \Leftrightarrow \nu\left(Y - X \geq 0\right) = 1, \end{aligned}$$

which gives us the following ν-dependent local cone orders for these respective local cones.

Observation 3.

(i) *Suppose that ν is concave at zero. For $L = L_+^{\nu 0}$ the L-order coincides with the 'ν-zero payoff dominance order' according to which Y dominates X except on an ν-zero event, i.e.,*

$$X \leq_L Y \Leftrightarrow \nu\left(Y < X\right) = 0.$$

(ii) *Suppose that ν is convex at one. For $L = L_+^{\nu 1}$ the L-order coincides with the 'ν-one payoff dominance order' according to which Y dominates X on an ν-one event, i.e.,*

$$X \leq_L Y \Leftrightarrow \nu\left(X \leq Y\right) = 1.$$

5 New mathematical results

Our subsequent results will apply to any topology that satisfies Condition BU. The following Lemma combines a local cone structure of L with the topological Condition BU.

Lemma 1. *Consider the topological space (L, τ) such that $(\Omega, \mathcal{F}, \nu)$ is an arbitrary non-additive probability space and L is a local cone. Suppose that τ satisfies Condition BU so that there exists for any given open set $\mathcal{U} \in \tau$ with $X \in \mathcal{U}$ some $c, \epsilon > 0$ such that*

$$B_{c,\epsilon}\left(X\right) \subseteq \mathcal{U}.$$

Then we also have that

$$\nu\left(A\right) < \epsilon \text{ implies } X + Y 1_A \in B_{c,\epsilon}\left(X\right) \subseteq \mathcal{U} \tag{15}$$

for an arbitrary $Y \in L$.

Proof. By the local cone structure, we must have $X + Y1_A \in L$ for any $X, Y \in L$ and any $A \in \mathcal{F}$. Suppose that there is some $c, \epsilon > 0$ such that

$$X \in B_{c,\epsilon}\left(X\right) \subseteq \mathcal{U}.$$

Observe that

$$\begin{aligned}\nu\left(|X+Y1_A-X|\geq c\right) &= \nu\left(|Y1_A|\geq c\right)\\ &\leq \nu\left(A\right)\end{aligned}$$

where Y is arbitrary. Note that $\nu\left(A\right)<\epsilon$ implies

$$\begin{aligned}\nu\left(|X+Y1_A-X|\geq c\right) &< \epsilon\\ &\Rightarrow\\ X+Y1_A &\in B_{c,\epsilon}\left(X\right)\subseteq\mathcal{U},\end{aligned}$$

which gives us the desired result (15).□□

To derive the following Theorem, we add non-atomicity to the assumptions of Lemma 1.

Theorem 3. *Consider the topological space (L,τ) such that (i) $(\Omega,\mathcal{F},\nu)$ is an arbitrary non-atomic, non-additive probability space, (ii) L is a local cone, and (iii) τ satisfies Condition BU. We have for any $X,Z\in L$ and any open set $\mathcal{U}\in\tau$ that*

$$X\in\mathcal{U}\text{ implies }X+Z\in co\left(\mathcal{U}\right)$$

where $co\left(\mathcal{U}\right)$ denotes the convex hull of $\mathcal{U}$.

In words: Under the conditions of Theorem 3, we can add arbitrary $Z\in L$ to $X\in L$ such that the resulting random variables $X+Z$ belong to the convex hull of any open set $\mathcal{U}$ around X.

Proof of Theorem 3. Fix $X,Z\in L$. Consider any partition $\Pi=\{\Omega_1,...,\Omega_n\}\subseteq\mathcal{F}$ for $n\in\mathbb{N}^+$. Because L is a local cone, we have (i) $nZ\in L$ as well as (ii) $X+nZ1_{\Omega_i}\in L$, $i=1,...,n$. Since $(\Omega,\mathcal{F},\nu)$ is non-atomic, there exists for every $\epsilon>0$ some sufficiently fine partition Π such that $\nu\left(\Omega_i\right)<\epsilon$ for every $i=1,...,n$. By Lemma 1, we obtain that $\nu\left(\Omega_i\right)<\epsilon$ implies

$$X+nZ1_{\Omega_i}\in B_{c,\epsilon}\left(X\right)\subseteq\mathcal{U}$$

whenever

$$X\in B_{c,\epsilon}\left(X\right)\subseteq\mathcal{U}.$$

Because of

$$X + Z = \sum_{i=1}^{n} \frac{1}{n} \left(X + nZ1_{\Omega_i} \right),$$

we obtain the desired result $X + Z \in co(\mathcal{U})$ whenever $X \in \mathcal{U}$.□□

Observe that Theorem 3 only states that $X + Z$ belongs to the convex hull of $\mathcal{U}$ and not to $\mathcal{U}$ itself. However, under the assumption that the open set $\mathcal{U}$ around X is convex, i.e., $\mathcal{U} = co(\mathcal{U})$, Theorem 3 comes with the powerful implication that

$$X \in \mathcal{U} \text{ implies } X + Z \in \mathcal{U} \text{ for all } Z \in L. \tag{16}$$

An application of (16) to the local cones considered in Observations 2 and 3 gives us the following Corollary to Theorem 3 for relevant subdomains of random variables.

Corollary 2. *Suppose that the assumptions of Theorem 3 are satisfied.*

(i) *Any open and convex set $\mathcal{U}$ in the topological space (L^0, τ) around $X \in L^0$ contains all random variables $Y \in L^0$.*

(ii) *Any open and convex set $\mathcal{U}$ in the topological space $\left(L_+^0, \tau\right)$ around $X \in L_+^0$ contains all random variables $Y \in L_+$ that payoff-dominate X, i.e., $X \leq Y$.*

(iii) *Any open and convex set $\mathcal{U}$ in the topological space $\left(L_-^0, \tau\right)$ around $X \in L_-^0$ contains all random variables $Y \in L_-$ that are payoff-dominated by X, i.e., $Y \leq X$.*

(iv) *Let ν be concave at zero. Any open and convex set $\mathcal{U}$ in the topological space $\left(L_+^{\nu 0}, \tau\right)$ around $X \in L_+^{\nu 0}$ contains all random variables $Y \in L_+^{\nu 0}$ that ν-zero payoff-dominate X, i.e., $\nu(Y < X) = 0$.*

(v) *Let ν be convex at one. Any open and convex set $\mathcal{U}$ in the topological space $\left(L_+^{\nu 1}, \tau\right)$ around $X \in L_+^{\nu 1}$ contains all random variables $Y \in L_+^{\nu 1}$ that ν-one payoff-dominate X, i.e., $\nu(X \leq Y) = 1$.*

Part (i) of Corollary 2 immediately gives us another result:

Corollary 3. *Suppose that the assumptions of Theorem 3 are satisfied. The only non-empty open and convex set in the topological space (L^0, τ) is the set of all random variables L^0 itself.*

Remark 1. Denote by μ any non-atomic, additive probability measure. In Assa and Zimper (2018, Proposition 1) we had derived the following result: *The only convex subset of the topological space* (L^0, τ_μ) *with non-empty interior is the set* L^0 *itself.* Corollary 3 thus generalizes Proposition 1 in Assa and Zimper (2018) to any non-atomic, non-additive measure. The conclusion of Proposition 1 in Assa and Zimper (2018) has been well-established for topological L^p-spaces with $0 < p < 1$. Topological L^p-spaces with $0 < p < 1$ are defined for all random variables X such that $\int_\Omega |X|^p \, d\mu$ exists and they are induced by the pseudometric

$$d_p(X, Y) = \int_\Omega |X - Y|^p \, d\mu$$

(cf. Theorem 13.30 in Aliprantis and Border 2006). In particular, Rudin (1991, paragraph 1.47) proves: *The only convex subset of the topological space* L^p*-space with* $0 < p < 1$ *with non-empty interior is the set* L^p *itself.* This finding is mathematically equivalent to Theorem 1 in Day (1940) who shows that the null-functional is the only continuous linear functional on a topological space L^p-space with $0 < p < 1$.

6 New decision theoretic results

6.1 Continuous preferences on local cones

We consider a transitive and reflexive preference relation $\preceq$ that is complete over all pairs $(X, Y) \in L \times L$ for a fixed subdomain $L \subseteq L^0$.[7] The usual conventions apply: $X \preceq Y$ means that Y is at least as desirable as X; $X \preceq Y$ AND $Y \preceq X$ stand for 'indifference', abbreviated by $X \sim Y$; $X \preceq Y$ AND NOT $Y \sim X$ mean that Y is strictly preferred to X, denoted $X \prec Y$. Introduce the *strictly better* set at X

$$S^*(X) = \{Z \in L | X \prec Z\}.$$

as well as the *strictly worse* set at X

$$s^*(X) = \{Z \in L | Z \prec X\}.$$

Definitions 6.

[7] See Remark 3 for a possible weakening of this completeness assumption.

(i) *The preference relation* $\preceq$ *is* τ*-continuous on* L *iff* $S^*(X)$ *and* $s^*(X)$*, are open sets in the topological space* (L, τ)*.*

(ii) *The preference relation* $\preceq$ *is convex on* L *iff, for any* $X \in L$, $S^*(X)$ *is a convex set.*

(iii) *The preference relation* $\preceq$ *is* L*-monotone iff*

$$X \leq_L Y \Rightarrow X \preceq Y.$$

The following result–based on Theorem 3–shows that τ-continuity with respect to any topology τ that satisfies Condition BU in combination with convexity forces an ordering property in terms of the local cone order $\leq_L$.

Theorem 4. *Consider the topological space* (L, τ) *such that (i)* $(\Omega, \mathcal{F}, \nu)$ *is an arbitrary non-atomic, non-additive probability space, (ii)* L *is a local cone, and (iii)* τ *satisfies Condition BU. If* $\preceq$ *is* τ*-continuous and convex on* L*, then* $\preceq$ *must also be* L*-monotone.*

Proof. Step 1. Let $X, Y \in L$ such that

$$X \leq_L Y \Leftrightarrow Y - X \in L.$$

Fix an open set $\mathcal{U}$ such that $X \in \mathcal{U}$. By Theorem 3, we obtain

$$X + Y - X = Y \in co(\mathcal{U}).$$

Step 2. It remains to prove the following statement: If $S^*(Y)$ is open in (L, τ) and convex, then $X \preceq Y$. Assume to the contrary that $S^*(Y)$ is open and convex but $Y \prec X$. Let $\mathcal{U} = S^*(Y)$ so that $Y \prec X$ implies $X \in \mathcal{U}$. By Step 1, $Y \in co(\mathcal{U})$. But by convexity of $S^*(Y)$, we get $co(\mathcal{U}) = co(S^*(Y)) = S^*(Y)$. This implies $Y \in S^*(Y)$, i.e., $Y \prec Y$, which is a contradiction.□□

Theorem 4 comes with the following implications for the different L-orders of the relevant subdomains identified in Observations 2 and 3.

Corollary 4. *Suppose that the assumptions of Theorem 4 are satisfied.. Let* $\preceq$ *be* τ*-continuous and convex on* L*.*

(i) *If* $L = L^0$*, then* $\preceq$ *must be the indifference relation i.e., for all* $X, Y \in L^0$*,* $X \sim Y$*.*

(ii) *If* $L = L^0_+$*, then* $\preceq$ *must be payoff-monotone, i.e., if* $X \leq Y$*, then* $X \preceq Y$*.*

(iii) *If* $L = L^0_-$*, then* $\preceq$ *must be reverse payoff-monotone, i.e., if* $X \leq Y$*, then* $Y \preceq X$*.*

(iv) *Let* ν *be concave at zero. If* $L = L^{\nu 0}_+$*, then* $\preceq$ *must be* ν*-zero payoff-monotone, i.e., if* $\nu(Y < X) = 0$*, then* $X \preceq Y$*.*

(v) *Let* ν *be convex at one. If* $L = L^{\nu 1}_+$*, then* $\preceq$ *must be* ν*-one payoff-monotone, i.e., if* $\nu(X \leq Y) = 1$*, then* $X \preceq Y$*.*

Remark 2. Observe that the conclusion (and the proof of) Theorem 4 applies to preferences that are complete for all pairs $(X, Y) \in L \times L$ such that $X \leq_L Y$. Of course, for $L = L^0$ this means that the preference ordering $\preceq$ has to be complete over all pairs of random variables in L^0 because $X \leq_L Y$ for all $(X, Y) \in L^0 \times L^0$. However, for the other subdomains of Corollary 4 the assumption of complete preferences over all pairs in these subdomains can effectively be weakened. For example, for $L = L^0_+$ the preference ordering $\preceq$ needs only to be complete over all $(X, Y) \in L^0_+ \times L^0_+$ for which payoff-monotonicity holds, i.e., $X \leq Y$.

Remark 3. In Assa and Zimper (2018) we consider the topology of convergence in additive measure μ on equivalence classes of all random variables in L^0 induced by the metric

$$d_0(X, Y) = \int \frac{|X - Y|}{1 + |X - Y|} d\mu.$$

We had derived the following result for a non-atomic, additive probability space:

> Theorem 1(a): Consider a binary preference relation $\preceq$ on L^0 that is non-trivial and complete. The preference relation $\preceq$ cannot simultaneously satisfy continuity and convexity (of strictly better sets).

Non-triviality of $\preceq$ means that there is at least one pair $(X, Y) \in L^0 \times L^0$ such that $X \prec Y$. Consequently, for $\nu = \mu$ this result from our previous paper obtains as a special case of Corollary 4(i). Put differently, Corollary 4(i) generalizes Theorem 1(a) (Assa and Zimper 2018) from a non-atomic, additive probability space $(\Omega, \mathcal{F}, \mu)$ to a non-atomic, non-additive probability space $(\Omega, \mathcal{F}, \nu)$. It also generalizes our previous analysis to all topologies that satisfy Condition BU.

6.2 Implications for utility- and risk-measure representations

Fix some topology τ that satisfies Condition BU. Our decision theoretic analysis was originally motivated by the following question: Given that both L^0 and L^0_+ are large spaces of random variables, what is the underlying mathematical structure for the fact that there does not exist a non-trivial Choquet expected utility representation for τ-continuous and convex preferences over all random variables whereas such representation exists for all non-negative random variables? Theorem 4–and, more specifically, Corollary 4(i)-(ii)– answers this question by identifying the different local cone orders for the subdomains L^0 and L^0_+, respectively, as the underlying mathematical reason.

To see this, consider an utility representation of preferences $U : L \to \mathbb{R}$ such that

$$U(X) \leq U(Y) \Leftrightarrow X \preceq Y \text{ for all } X, Y \in L.$$

By part (i) of Corollary 4, the only utility representation of τ-continuous and convex preferences over all random variables in L^0 is the constant utility function

$$U(X) = c \text{ for all } X \in L^0$$

for some $c \in \mathbb{R}$.

In contrast, part (ii) of Corollary 4 only requires payoff-monotonicity for τ–continuous and convex preferences defined over all non-negative random variables only. In particular, observe that the payoff-monotone Choquet expected utility representation $U : L^0_+ \to \mathbb{R}$ such that

$$U(X) = (C) \int_{\Omega} u(X(\omega))\, d\widetilde{\nu}$$

for an arbitrary non-additive measure $\widetilde{\nu}$ defined on $(\Omega, \mathcal{F})$ would represent τ-continuous and convex preferences on L^0_+ whenever (i) the Bernoulli utility function $u : \mathbb{R}_+ \to \mathbb{R}$ is increasing, continuous (in the Euclidean metric), concave, and bounded from above and (ii) the non-additive probability measure $\widetilde{\nu}$ is *convex*, i.e., satisfying for all $A, B \in \mathcal{F}$

$$\widetilde{\nu}(A) + \widetilde{\nu}(B) \leq \widetilde{\nu}(A \cup B) + \widetilde{\nu}(A \cap B).$$

To sum up: Every utility representation of τ-continuous and convex preferences must adhere to monotonicity in the local cone order. This local cone order corresponds to equivalence for the subdomain L^0 and to payoff-dominance for the subdomain L^0_+, respectively.

Turn now to the local cone order on the subdomain L^0_- which corresponds to loss-dominance, i.e., to the opposite of payoff-dominance. Fix some risk measure $\rho : L^0_- \to \mathbb{R}$

with the interpretation that the random variable Y is less risky than the random variable X if and only if $\rho(Y) \leq \rho(X)$ whereby we consider *ρ-induced preferences* in the sense that

$$\rho(Y) \leq \rho(X) \Leftrightarrow X \preceq Y \text{ for all } X, Y \in L_{-}^{0}.$$

A convex risk measure ρ must satisfy (i) monotonicity in the payoff-dominance order, i.e., $X \leq Y$ implies $\rho(Y) \leq \rho(X)$, as well as (ii) convexity, i.e.,

$$\rho(\lambda Y + (1-\lambda) X) \leq \lambda \rho(Y) + (1-\lambda) \rho(X) \text{ for all } \lambda \in [0,1].$$

Convex risk measures induce convex preferences (cf. Föllmer and Schied 2002; 2016).

Convexity of a risk measure expresses the natural idea that a diversified portfolio should be less risky than a non-diversified portfolio. Convexity is, e.g., satisfied by *coherent* risk measures that have been axiomatized in order to address the non-convexity of the value-at-risk criterion (cf. Artzner et al. 1997; 1999; Delbaen 2002; 2009). By part (iii) of Corollary 4, preferences that are induced by a convex risk measure $\rho : L_{-}^{0} \to R$ must violate τ–continuity on the subdomain L_{-}^{0} of all loss random variables. The following example illustrates this fact for the popular convex risk measure *average value-at-risk* (also referred to as *expected shortfall* or *conditional value-at-risk*, cf. Föllmer and Schied 2016, Chapter 4.4)

Example 5: Average value-at-risk. Fix the non-atomic, non-additive probability space $((0,1), \mathcal{F}^{B}, \nu)$ where $\mathcal{F}^{B}$ denotes the Borel-sigma algebra on $(0,1)$. The average value-at-risk of any loss-random variable $Z \in L_{-}^{0}$ for the confidence level interval $(0, \beta)$ is defined as

$$\mathrm{AVaR}_{\beta}(Z) = \frac{1}{\beta} \int_{0}^{\beta} \mathrm{VaR}_{\alpha}(Z)\, d\alpha.$$

such that the value-at-risk of Z at confidence level α with respect to ν is given as follows (cf. Miryana 2014):

$$\mathrm{VaR}_{\alpha}(Z) = -\inf\{x \mid \alpha \leq \nu(Z \leq x)\}$$

Let $Y = 0$ be the loss-random variable that gives a constant "loss" of zero. Introduce the sequence $\{\Pi^{\epsilon}\}$ of partitions of $(0,1)$ such that, for $\epsilon = \frac{1}{n}$ with $n \geq 1$,

$$\Pi^{\epsilon} = \{\Omega_{1}^{\epsilon}, ..., \Omega_{n}^{\epsilon}\} = \left\{ \left(0, \frac{1}{n}\right), \left[\frac{1}{n}, \frac{2}{n}\right), ..., \left[\frac{n-1}{n}, 1\right) \right\}$$

and define the loss-random variables $Y_i^\epsilon \in L_-^0$ such that

$$Y_i^\epsilon(\omega) = \begin{cases} 0 & \text{if } \omega \in \Omega \backslash \Omega_i^\epsilon \\ -\frac{1}{\nu\left(\Omega_i^\epsilon\right)} & \text{if } \omega \in \Omega_i^\epsilon \end{cases}$$

The Y_i^ϵ converge, by construction, in measure to Y, i.e., $Y_i^\epsilon \to_\nu Y$. Next observe that

$$\begin{aligned} \mathrm{VaR}_\alpha(Y) &= 0, \\ \mathrm{VaR}_\alpha(Y_i^\epsilon) &= \begin{cases} 0 & \text{if } \nu\left(\Omega_i^\epsilon\right) < \alpha \\ \frac{1}{\nu\left(\Omega_i^\epsilon\right)} & \text{if } \nu\left(\Omega_i^\epsilon\right) \geq \alpha \end{cases} \end{aligned}$$

implying, for $\nu\left(\Omega_i^\epsilon\right) < \beta$,

$$\begin{aligned} \mathrm{AVaR}_\beta(Y) &= 0, \\ \mathrm{AVaR}_\beta(Y_i^\epsilon) &= \frac{1}{\beta}\left(\int_0^{\nu\left(\Omega_i^\epsilon\right)} \frac{1}{\nu\left(\Omega_i^\epsilon\right)} d\alpha + \int_{\nu\left(\Omega_i^\epsilon\right)}^{\beta} 0 d\alpha \right) = \frac{1}{\beta}. \end{aligned}$$

Consequently,

$$\lim_{\epsilon \to 0} \mathrm{AVaR}_\beta(Y_i^\epsilon) = \frac{1}{\beta} \neq 0 = \mathrm{AVaR}_\beta(Y)$$

which shows that the average value-at-risk criterion AVaR_β induces preferences that violate τ–continuity on the subdomain L_-^0.□

7 Concluding remarks

We introduce the notions of (i) non-atomicity of a non-additive probability space $(\Omega, \mathcal{F}, \nu)$, (ii) a local cone structure of a set of random variables L, and (iii) the Condition BU for a topology τ. Condition BU is always satisfied for the class of $\tau_{\mathcal{C}}$-topologies which includes the topology of convergence in the non-additive measure ν but also coarser topologies.

Consider any open set $\mathcal{U}$ in (L, τ) such that τ satisfies Condition BU, $(\Omega, \mathcal{F}, \nu)$ is non-atomic, and L is a local cone. We have established that $X \in \mathcal{U}$ implies that any random variable $X + Z$ with $Z \in L$ must belong to the convex hull of $\mathcal{U}$. For preferences over the random variables in L we have shown that (i) convexity of preferences combined with (ii) τ-continuity of preferences forces monotonicity in the local cone order. The local cone order for the set of all random variables is the equivalence order whereas it is payoff-dominance for the set of all positive random variables. Our decision theoretic finding thus explains why there exist non-trivial, convex, and τ-continuous Choquet expected utility representations for the subdomain of positive but not for the subdomain of all random variables. Another implication of our analysis is the impossibility of τ-continuous preferences induced by convex risk measures defined on the subdomain of all loss random variables.

Appendix

A Proof of Theorem 1

By Proposition 3(ii), τ^* is a topology of convergence in ν if all $\left(\frac{1}{m}, \frac{1}{m}\right)$-balls are open sets in τ^*.

Lemma 2. *Suppose that the finest topology τ^* in $T_{\mathcal{C}}$ satisfies the following condition for all $m \in \mathbb{N}^+$ and all $X \in L$:*

Condition YX: $Y \in B_{\frac{1}{m},\frac{1}{m}}(X)$ *implies that there exists some* $m_Y \in \mathbb{N}^+$ *such that*

$$B_{\frac{1}{m_Y},\frac{1}{m_Y}}(Y) \subseteq B_{\frac{1}{m},\frac{1}{m}}(X)\,. \tag{17}$$

Then all $\left(\frac{1}{m}, \frac{1}{m}\right)$-balls are open sets in τ^*.

Proof. By definition, $B_{\frac{1}{m},\frac{1}{m}}(X)$ is an open set in τ^* iff for every $Y \in B_{\frac{1}{m},\frac{1}{m}}(X)$

$$B_{\frac{1}{m},\frac{1}{m}}(X) \in \mathcal{W}_Y = \left\{\mathcal{V} \subseteq L \mid B_{\frac{1}{m_Y},\frac{1}{m_Y}}(Y) \subseteq \mathcal{V} \text{ for some } m_Y \in \mathbb{N}^+\right\}.$$

Letting $B_{\frac{1}{m},\frac{1}{m}}(X) = \mathcal{V}$ shows that $B_{\frac{1}{m},\frac{1}{m}}(X)$ is an open set in τ^* if (17) is satisfied for all $Y \in B_{\frac{1}{m},\frac{1}{m}}(X)$.□□

Proof of Theorem 1. Step 1. Let

$$\begin{array}{rcl} Y & \in & B_{\frac{1}{m},\frac{1}{m}}(X) \\ & \Leftrightarrow & \\ \nu\left(|X-Y| \geq \dfrac{1}{m}\right) & < & \dfrac{1}{m} \end{array}$$

and fix

$$\delta^* = \frac{1}{m} - \ \nu\left(|X-Y| \geq \frac{1}{m}\right) > 0. \tag{18}$$

Observe that, for any $n > m$,

$$\begin{aligned} \left(|X-Z| \geq \frac{1}{m}\right) &\subseteq \left(|X-Y| + |Y-Z| \geq \frac{1}{m}\right) \\ &\subseteq \left(|X-Y| \geq \frac{1}{m} - \frac{1}{n}\right) \cup \left(|Y-Z| \geq \frac{1}{n}\right) \end{aligned}$$

so that

$$\nu\left(|X-Z|\geq\frac{1}{m}\right)\leq\nu\left(\left(|X-Y|\geq\frac{1}{m}-\frac{1}{n}\right)\cup\left(|Y-Z|\geq\frac{1}{n}\right)\right).$$

Pick some $\delta>0$ such that $\delta<\delta^*$ and write $\delta^*=\delta+\varepsilon$. If 'uniform autocontinuity from above' holds, there exists some n_δ such that

$$\nu\left(|Y-Z|\geq\frac{1}{n}\right)<\frac{1}{n_\delta}$$

implies

$$\nu\left(|X-Z|\geq\frac{1}{m}\right)<\nu\left(|X-Y|\geq\frac{1}{m}-\frac{1}{n}\right)+\delta$$

for an arbitrary $n>m$.

Step 2. Note that

$$\begin{aligned}&\nu\left(|X-Y|\geq\frac{1}{m}-\frac{1}{n}\right)\\=\;&\nu\left(\left(|X-Y|\geq\frac{1}{m}\right)\cup\left(\left(|X-Y|<\frac{1}{m}\right)\cap\left(|X-Y|\geq\frac{1}{m}-\frac{1}{n}\right)\right)\right).\end{aligned}$$

If 'uniform autocontinuity from above' holds, there exists some n_ε such that

$$\nu\left(\left(\left(|X-Y|<\frac{1}{m}\right)\cap\left(|X-Y|\geq\frac{1}{m}-\frac{1}{n}\right)\right)\right)<\frac{1}{n_\varepsilon}\tag{19}$$

implies

$$\nu\left(|X-Y|\geq\frac{1}{m}-\frac{1}{n}\right)<\nu\left(|X-Y|\geq\frac{1}{m}\right)+\varepsilon.\tag{20}$$

Write

$$\begin{aligned}A_n&=\left(|X-Y|<\frac{1}{m}\right)\cap\left(|X-Y|\geq\frac{1}{m}-\frac{1}{n}\right)\\&=\left(|X-Y|\subseteq\left[\frac{1}{m}-\frac{1}{n},\frac{1}{m}\right)\right)\end{aligned}$$

and observe that

$$A_n\searrow\emptyset\ .$$

If 'order continuity' holds, we thus have that

$$\lim_{n\to\infty}\nu\left(\left(|X-Y|<\frac{1}{m}\right)\cap\left(|X-Y|\geq\frac{1}{m}-\frac{1}{n}\right)\right)=0.$$

This convergence result implies, by (19), the existence of some sufficiently large n^* such that

$$\nu\left(\left(\left(|X-Y|<\frac{1}{m}\right)\cap\left(|X-Y|\geq\frac{1}{m}-\frac{1}{n^*}\right)\right)\right)<\frac{1}{n_\varepsilon}$$

holds. 'Uniform autocontinuity from above' combined with 'order continuity' therefore gives us, by (20),

$$\nu\left(|X-Y|\geq\frac{1}{m}-\frac{1}{n^*}\right)<\nu\left(|X-Y|\geq\frac{1}{m}\right)+\varepsilon \tag{21}$$

for some sufficiently large n^*.[8]

Step 3. Let $m_Y=\max\{m+1,n_\delta,n^*\}$. Observe that (i) $m_Y\geq m+1$ ensures $m_Y>m$ and (ii) $m_Y\geq n_\delta$ ensures that

$$\nu\left(|Y-Z|\geq\frac{1}{m_Y}\right)<\frac{1}{m_Y} \tag{22}$$

implies

$$\nu\left(|Y-Z|\geq\frac{1}{m_Y}\right)<\frac{1}{n_\delta}.$$

By Step 1, we thus obtain that there exists some m_Y such that (22) implies

$$\nu\left(|X-Z|\geq\frac{1}{m}\right)<\nu\left(|X-Y|\geq\frac{1}{m}-\frac{1}{m_Y}\right)+\delta. \tag{23}$$

Next observe that $m_Y\geq n^*$ ensures, by Step 2, that

$$\nu\left(|X-Y|\geq\frac{1}{m}-\frac{1}{m_Y}\right)<\nu\left(|X-Y|\geq\frac{1}{m}\right)+\varepsilon. \tag{24}$$

By combining (18), (23), and (24), we obtain that (22) implies

$$\begin{aligned}\nu\left(|X-Z|\geq\frac{1}{m}\right) &< \nu\left(|X-Y|\geq\frac{1}{m}\right)+\varepsilon+\delta \\ &= \nu\left(|X-Y|\geq\frac{1}{m}\right)+\delta^* \\ &= \frac{1}{m}.\end{aligned} \tag{25}$$

That is, under the conditions of Theorem 1 there exists some m_Y such that

$$\begin{aligned}\nu\left(|Y-Z|\geq\frac{1}{m_Y}\right) &< \frac{1}{m_Y} \\ &\Leftrightarrow \\ Z &\in B_{\frac{1}{m_Y},\frac{1}{m_Y}}(Y)\end{aligned}$$

[8] Of course, we could have obtained (21) directly by assuming 'continuity from above', which is implied by '(uniform) autocontinuity from above' combined with 'order continuity'.

implies

$$
\begin{aligned}
\nu\left(|X - Z| \geq \frac{1}{m}\right) &< \frac{1}{m} \\
&\Leftrightarrow \\
Z &\in B_{\frac{1}{m},\frac{1}{m}}(X),
\end{aligned}
$$

which is the desired Condition YX of Lemma 2.□□

B Proof of Theorem 2

Preliminaries. Fix an arbitrary $c \in \mathbb{Q}_{>0}$ and define the event

$$
\begin{aligned}
A &= (|X - Y| \geq c) \\
&= \left(\frac{|X - Y|}{1 + |X - Y|} \geq \frac{c}{1 + c}\right).
\end{aligned}
\tag{26}
$$

Note that we have for the complement of A that

$$
A^c = \left\{\frac{|X - Y|}{1 + |X - Y|} < \frac{c}{1 + c}\right\}.
$$

Our proof will employ the following properties of the Choquet integral (cf., e.g., Schmeidler 1986; König 2003):[9]

- Positive homogeneity:

$$
(C)\int a \cdot X d\nu = a \cdot (C)\int X d\nu \text{ for } a \geq 0
$$

- Translation invariance:

$$
(C)\int (X + c)\, d\nu = (C)\int X d\nu + c \text{ for } c \in \mathbb{R}
$$

- Monotonicity (in the payoff-dominance order):

$$
X \leq Y \text{ implies } (C)\int X d\nu \leq (C)\int Y d\nu
$$

[9] Schmeidler (1986, Theorem) characterizes the Choquet integral through (i) Comonotonic additivity combined with (ii) Monotonicity in the payoff-dominance order. Positive homogeneity and Translation invariance are implied by these two properties (cf. Remark 1 in Schmeidler 1986).

In what follows we show that the respective weak bases

$$\left\{ B^{d}_{\frac{1}{n}}(X) \mid n \in \mathbb{N}^{+}, X \in L \right\}$$

and

$$\left\{ B_{\frac{1}{m},\frac{1}{m}}(X) \mid m \in \mathbb{N}^{+}, X \in L \right\}$$

induce the same topology. To this purpose, Part 1 establishes that we can find for every fixed $m \in \mathbb{N}^{+}$ some $n \in \mathbb{N}^{+}$ such that

$$B^{d_0}_{\frac{1}{n}}(X) \subseteq B_{\frac{1}{m},\frac{1}{m}}(X);$$

conversely, Part 2 establishes that we can find for every fixed $n \in \mathbb{N}^{+}$ some $m \in \mathbb{N}^{+}$ such that

$$B_{\frac{1}{m},\frac{1}{m}}(X) \subseteq B^{d_0}_{\frac{1}{n}}(X).$$

Part 1. We show that there exists for every $m \in \mathbb{N}^{+}$ some $n \in \mathbb{N}^{+}$ such that

$$d_0(X,Y) < \frac{1}{n} \text{ implies } \nu\left(|X-Y| \geq \frac{1}{m}\right) < \frac{1}{m}.$$

Step 1. By the properties of the Choquet integral, we obtain

$$\begin{aligned}
d_0(X,Y) &= (C)\int \left(\frac{|X-Y|}{1+|X-Y|} 1_A + \frac{|X-Y|}{1+|X-Y|} 1_{A^c} \right) d\nu \\
&\geq (C)\int \frac{|X-Y|}{1+|X-Y|} 1_A d\nu, \text{ by monotonicity} \\
&\geq (C)\int \frac{c}{1+c} 1_A d\nu, \text{ by monotonicity} \\
&= \frac{c}{1+c}\nu(A), \text{ by homogeneity} \\
&= \frac{c}{1+c}\nu(|X-Y| \geq c).
\end{aligned}$$

Step 2. By Step 1, we have that

$$\frac{1+c}{c} d_0(X,Y) < c$$

implies

$$\nu(|X-Y| \geq c) < c.$$

Or equivalently, for $c = \frac{1}{m}$,

$$\begin{aligned} (m+1)\, d_0(X,Y) &< \frac{1}{m} \\ &\Leftrightarrow \\ d_0(X,Y) &< \frac{1}{(m+1)\,m} \end{aligned}$$

implies

$$\nu\left(|X-Y| \geq \frac{1}{m}\right) < \frac{1}{m}.$$

Choosing n such that

$$\begin{aligned} \frac{1}{n} &\leq \frac{1}{(m+1)\,m} \\ &\Leftrightarrow \\ m^2+m &\leq n \end{aligned}$$

gives us the desired relationship that

$$Y \in B^{d_0}_{\frac{1}{n}}(X) \text{ implies } Y \in B_{\frac{1}{m},\frac{1}{m}}(X).$$

Part 2. We show that there exists for every $n \in \mathbb{N}^+$ some $m \in \mathbb{N}^+$ such that

$$\nu\left(|X-Y| \geq \frac{1}{m}\right) < \frac{1}{m} \text{ implies } d_0(X,Y) < \frac{1}{n}. \tag{27}$$

Step 1. Recall the definition of event A by (26) and of it's complement event. Because of

$$\begin{aligned} \frac{|X-Y|}{1+|X-Y|} &\leq 1 \text{ for all } \omega \in A, \text{ and} \\ \frac{|X-Y|}{1+|X-Y|} &< \frac{c}{1+c} \text{ for all } \omega \in A^c, \end{aligned}$$

the properties of the Choquet integral imply

$$\begin{aligned} d_0(X,Y) &= (C)\int\left(\frac{|X-Y|}{1+|X-Y|}1_A + \frac{|X-Y|}{1+|X-Y|}1_{A^c}\right)d\nu \\ &\leq (C)\int\left(1\cdot 1_A + \frac{c}{1+c}\cdot 1_{A^c}\right)d\nu, \text{ by monotonicity} \\ &\leq (C)\int\left(1\cdot 1_A + \frac{c}{1+c}\right)d\nu, \text{ by monotonicity} \\ &= \nu(A) + \frac{c}{1+c}, \text{ by translation invariance} \\ &= \nu(|X-Y| \geq c) + \frac{c}{1+c}. \end{aligned}$$

Step 2. By Step 1, we have that

$$\nu\left(|X-Y| \geq c\right) + \frac{c}{1+c} < \frac{1}{n}$$

implies

$$d_0\left(X, Y\right) < \frac{1}{n}. \tag{28}$$

Since $c > 0$ was arbitrary, pick some sufficiently small c^* such that

$$\frac{1}{n} - \frac{c^*}{1+c^*} > 0$$

so that (28) holds if

$$\nu\left(|X-Y| \geq c^*\right) < \frac{1}{n} - \frac{c^*}{1+c^*}. \tag{29}$$

Let $\frac{1}{m} \leq \min\left\{c^*, \frac{1}{n} - \frac{c^*}{1+c^*}\right\}$ to obtain, by (3) and (29), that

$$\begin{aligned} \nu\left(|X-Y| \geq \frac{1}{m}\right) &< \frac{1}{m} \\ &\Rightarrow \\ \nu\left(|X-Y| \geq c^*\right) + \frac{c^*}{1+c^*} &< \frac{1}{n}. \end{aligned} \tag{30}$$

Consequently, (30) implies $d_0\left(X, Y\right) < \frac{1}{n}$ for sufficiently large m, e.g.,

$$c^* = \frac{1}{n} \Rightarrow m = n\left(n+1\right)$$

will do.□□

Remark 4. Our Theorem 1 establishes that (i) ‘uniform autocontinuity from above’ and (ii) ‘order continuity’ together ensure that all $\left(\frac{1}{m}, \frac{1}{m}\right)$-balls are open sets in the finest topology $\tau^* \in T_{\mathcal{C}}$. Theorem 4.1 in Ouyang and Zhang (2011) implies, by our Theorem 2, that ‘uniform autocontinuity from above’ alone ensures the weaker condition that all $\left(\frac{1}{m}, \frac{1}{m}\right)$-balls are neighborhoods in $\tau^* \in T_{\mathcal{C}}$, which is sufficient for τ^* being a ‘topology of convergence in ν’. Provided that the analysis in Ouyang and Zhang (2011) as well as our Theorem 2 are correct, there are two possibilities. Either there are situations such that all d_0-balls but not all $\left(\frac{1}{m}, \frac{1}{m}\right)$-balls are open sets in τ^* whereby ν is ‘uniformly autocontinuous from above’ but not ‘continuous from above’. Or one should be able to prove the conclusion of Theorem 1–according to which all $\left(\frac{1}{m}, \frac{1}{m}\right)$-balls are open sets in τ^*–for all ν that are ‘uniformly autocontinuous from above’ without the additional assumption of ‘order continuity’. We would like to leave this open issue to specialists or/and to future research.

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