# Partial Conditioning for Inference of Many-Normal-Means with Hölder Constraints 

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#### Abstract

Inferential models have been proposed for valid and efficient prior-free probabilistic inference. As it gradually gained popularity, this theory is subject to further developments for practically challenging problems. This paper considers the many-normal-means problem with the means constrained to be in the neighborhood of each other, formally represented by a Hölder space. A new method, called partial conditioning, is proposed to generate valid and efficient marginal inference about the individual means. It is shown that the method outperforms both a fiducial-counterpart in terms of validity and a conservative-counterpart in terms of efficiency. We conclude the paper by remarking that a general theory of partial conditioning for inferential models deserves future development.


Keywords - Dempster-Shafer theory, Fiducial argument, Hölder space, Lipschitz space, Nonparametric regression

## 1 Introduction

The framework of inferential models (IMs) has been developed to provide valid and efficient prior-free probabilistic inference. In its simplest form Martin and Liu, 2013), it is obtained by modifying R. A. Fisher's fiducial argument, which suffers from the mathematical difficulties discussed, for example, in Liu and Martin (2015). It has also been extended for efficient inference by combining information (Martin and Liu, 2015a) and marginalization (Martin and Liu, 2015b). For certain practically challenging problems, however, this theory is subject to further developments, similar to other existing schools of thought. One noticeable class of such challenging problems is inference on constrained parameters, which is challenging for all existing inferential methods (see, e.g., Leaf and Liu (2012) and references therein).

In this paper, we consider the many-normal-means problem with the means constrained to be in the neighborhood of each other. For both conceptual and representational simplicity, we describe the problem as a special case of the familiar classic formulation of a nonparametric regression model (Lepskii, 1991). More precisely, to motivate the problem, we consider the following model

$$
\begin{equation*}
Y_{i}=\vartheta_{0}\left(t_{i}\right)+\sigma U_{i}, \quad i=1, \cdots, n, \tag{1.1}
\end{equation*}
$$

where the $t_{i}$ denotes the design points, the $Y_{i}$ denotes the responses, and $U_{i} \stackrel{i . i . d .}{\sim} \mathcal{N}(0,1)$ denote the random error terms. Here, $\vartheta_{0}$ is the unknown regression function that is assumed to reside in the Hölder space

$$
\begin{equation*}
\Theta_{M, \gamma}=\left\{\vartheta:[0,1] \rightarrow \mathbb{R}| | \vartheta(t)-\vartheta(s)|\leq M| t-\left.s\right|^{\gamma}, \forall t, s \in[0,1]\right\}, \tag{1.2}
\end{equation*}
$$

with the Hölder exponent $0<\gamma \leq 1$. We assume $M, \gamma$, and $\sigma$ to be known constants and, without loss of generality, we set $\sigma=1$. In subsequent discussions, we suppress the subscripts and denote the Hölder space
$\Theta_{M, \gamma}$ simply by $\Theta$, unless otherwise noted. The motivation for this problem stems from the challenging setting in nonparametric regression, which we briefly discuss in the following remark.

Remark 1.1. In nonparametric regression, the objective is to estimate $f$ from a nonparametric class of functions $\mathcal{F}$, under the assumption that $f$ belongs to this class. For instance, $\mathcal{F}$ can be the set of all continuous functions on $[0,1]$. In this paper, we focus on the Hölder class $\Theta_{M, \gamma}$. A function in $\Theta_{M, \gamma}$ with $\gamma>1$ is constant, while a function satisfying $\gamma=1$ meets a Lipschitz condition with a Lipschitz constant of $M$. Thus, the space of Lipschitz continuous functions is a subset of $\Theta_{M, \gamma}$. It is very challenging to construct confidence intervals for $f$ either locally or globally. For example, consider the Nadaraya-Watson estimator $\hat{\vartheta}_{n}$ with kernel-bandwidth $h$. A Taylor expansion shows that it has bounded bias and variance

$$
\left|\mathbb{E}\left(\hat{\vartheta}_{n}(x)\right)-\vartheta_{0}(x)\right| \leq C_{1} h^{\gamma}, \quad \operatorname{Var}\left(\hat{\vartheta}_{n}(x)\right)=\frac{C_{2}}{n h}
$$

for all $x \in[0,1]$ with some positive constants $C_{1}$ and $C_{2}$. By taking $h=O\left(n^{-1 /(2 \gamma+1)}\right)$, the $L_{2}$ error of $\hat{\vartheta}_{n}$ achieves the optimal convergence rate (cf. Section 1.6.1 of Tsybakov (2009))

$$
\mathbb{E}\left\|\hat{\vartheta}_{n}-\vartheta_{0}\right\|_{L_{2}}^{2}=O\left(n^{-2 \gamma /(2 \gamma+1)}\right)
$$

or

$$
\begin{equation*}
\sqrt{\mathbb{E}\left\|\hat{\vartheta}_{n}-\vartheta_{0}\right\|_{L_{2}}^{2}}=O\left(n^{-\gamma /(2 \gamma+1)}\right) . \tag{1.3}
\end{equation*}
$$

However, under the optimal $h$, the asymptotic bias has the same order as the asymptotic standard deviation, which is of order $O\left(n^{-\gamma /(2 \gamma+1)}\right)$. To eliminate the bias, a typical strategy is to opt for a smaller order $h$, which leads to a suboptimal asymptotic confidence interval in terms of convergence rate.

Our objective here in this paper is to perform statistical inference on the unknown function $\vartheta_{0}$ within the IM framework. We propose a new method, called partial conditioning, to generate valid and efficient marginal inference about the individual means. We show that the method outperforms both a fiducial-counterpart in term of validity and a conservative-counterpart in terms of efficiency. A simple simulation-based study shows that our empirical asymptotic convergence rate of plausibility intervals for the case of $\gamma=\frac{1}{2}$ is about $O\left(n^{-0.267}\right)$, slightly better than or at least close to that in $(1.3)$, which in this case is $O\left(n^{-0.250}\right)$. While one of these two convergence rates is on point estimation and the other on interval length, such a comparison is arguably meaningful because IM plausibility intervals are valid in terms of frequency calibration.

It should be noted that IM approaches to the general nonparametric problem itself deserve in-depth investigations. This explains why our focus here is on a simple case. Our goal is to introduce an innovative idea of conditioning in the IM framework to combine information when multiple parameters entangle with each other via known constraints. As noted in the above remark on the difficulty of nonparametric regression, no existing methods can produce valid inference, even asymptotically. We believe our proposed method of partial conditioning makes a promising step in developing satisfactory solutions to nonparametric regression.

The rest of the paper is arranged as follows. Section 2 reviews basic IMs (Martin and Liu, 2013) and conditional IMs (Martin and Liu, 2015a) with the simple cases of $n=1$ and $n=2$ with $M=0$ in 1.1) and (1.2). Section 3 addresses the difficulties in the case of $n=2$ and $M \neq 0$ and introduces the proposed method of partial conditioning, with the general $n$ case considered in Section 4, where illustrative numerical examples are also provided. Finally, Section 5 concludes with a few remarks.

## 2 An Overview of Inferential Models (IMs)

Prior-free probabilistic inference is of utmost importance in scientific research and can be traced back to R. A. Fisher's inverse probability or fiducial argument (Fisher, 1930, 1973). It can even be traced back to Student (1908), especially when Fisher's fiducial argument is viewed as a way of constructing confidence intervals (Neyman, 1941). Recent developments along this direction include the generalized fiducial (Hannig,

2009, Hannig et al., 2016) and confidence distribution (Xie et al., 2011, Xie and Singh, 2013). The DempsterShafer theory of belief functions (Dempster, 1968; Shafer, 1976, 1990; Dempster et al., 2008; Dempster, 2008) can also be viewed as both an extension to consider set-valued inverse mapping and a generalization to develop methods of combining information for efficient inference; see Denœux (2016) for an excellent review.

The development of IMs for frequency-calibrated inference was inspired by the Dempster-Shafer theory Martin et al. 2010, Zhang and Liu, 2011). In particular, the IM framework makes use of the concepts of belief and plausibility functions from the theory. On a deeper mathematical level, these functions originate from the more intuitive concepts of upper and lower probabilities, which provide an intrinsic connection with well-studied imprecise probabilities (c.f. Gong and Meng, 2021; Martin, 2021, Liu and Martin, 2021, and references therein). As investigated recently by Martin (2021), the mathematical theory of IMs is also closely related to that of imprecise probabilities.

It is perhaps helpful to note that frequency-calibrated inference is desirable in scientific investigations (Denœux and Li, 2018). Such a desirable inference cannot be obtained in general without using upper and lower probabilities. Discrete data analysis and the famous benchmark Behrens-Fisher problem (Martin and Liu, 2015b, and references therein) provide excellent supporting examples. Below, we provide a brief review of IMs to set the stage for later introducing the method of partial conditioning.

### 2.1 The basic IMs

To illustrate the basic IM framework under the current problem setting, let us first consider the simple case of $n=1$, where we only make use of a single pair of observations $\left(t_{1}, y_{1}\right)$. Thus, we have

$$
\begin{equation*}
y_{1}=\vartheta_{0}\left(t_{1}\right)+u_{1}^{\star}, \tag{2.1}
\end{equation*}
$$

where $u_{1}^{\star}$ represents an unobserved realization of $U_{1} \sim \mathcal{N}(0,1)$. Using the terminology of IMs, 2.1) describes an underlying sampling model which involves the unknown parameter (function) $\vartheta \in \Theta$ and generates observed data $X_{1}=\left(t_{1}, Y_{1}\right)$ using auxiliary variable $U_{1} \sim \mathcal{N}(0,1)$. Note that the auxiliary variable is unobserved but predictable, since its distribution is fully specified.

In general, constructing an IM consists of three steps, namely, an association (A) step, a prediction (P) step, and a combination (C) step. These three steps are explained below in more detail in the context of the one-point nonparametric regression problem 2.1):
A-step. The association step can be achieved via some function or procedure $\mathcal{F}$ as

$$
X_{1}=\mathcal{F}\left(U_{1}, \vartheta\right), \quad(X \in \mathcal{X}, \vartheta \in \Theta, U \sim \mathcal{N}(0,1))
$$

This association allows for direct reasoning with the source of uncertainty $U_{1}$, which is missing. If $U_{1}=u_{1}^{\star}$ were observed, we would then be able to obtain the best possible inference for $\vartheta$, which is given by the set-valued "inverse" mapping

$$
\mathcal{G}: u_{1}^{\star} \rightarrow \Theta_{X_{1}}\left(u_{1}^{\star}\right)=\left\{\vartheta \in \Theta: \vartheta\left(t_{1}\right)=Y_{1}-u_{1}^{\star}\right\} .
$$

P-step. For inference on $\vartheta$, the discussion in the A-step suggests that we should focus our attention on accurately predicting the unobserved quantity $u_{1}^{\star}$. To predict $u_{1}^{\star}$ with a certain desired accuracy, we utilize a predictive random set $\mathcal{S}$, for example,

$$
\begin{equation*}
S\left(U_{1}\right)=\left\{\tilde{u} \in \mathbb{R}:-\left|U_{1}\right| \leq \tilde{u} \leq\left|U_{1}\right|\right\}, \quad\left(U_{1} \sim N(0,1)\right) \tag{2.2}
\end{equation*}
$$

C-step. To transfer the available information about $u^{\star}$ to the $\vartheta$-space, the last step is to combine the information in the association, the observed $x_{1}=\left(t_{1}, y_{1}\right)$, and the predictive random set $\mathcal{S}$. For this purpose, consider the expanded set

$$
\Theta_{x_{1}}(\mathcal{S})=\bigcup_{u_{1} \in \mathcal{S}} \Theta_{x_{1}}\left(u_{1}\right)=\left\{\vartheta \in \Theta:\left|\vartheta\left(t_{1}\right)-y_{1}\right| \leq\left|U_{1}\right|\right\}, \quad\left(U_{1} \sim \mathcal{N}(0,1)\right)
$$

which contains those values of $\vartheta$ that are consistent with the observed data and the sampling model for at least one candidate $u_{1}^{\star}$ value $u_{1} \in \mathcal{S}$.

The random sets obtained in the C-step are in the space of the unknown parameter, and we are ready to produce uncertainty assessment for assertions of interest. Consider an assertion $\mathcal{A}$ about the parameter of interest $\vartheta$. The assertion $\mathcal{A}$ corresponds to a set $A \subseteq \Theta$, and acts as a hypothesis in the context of classical statistics. To summarize the evidence in $x$ that supports the assertion $\mathcal{A}$, we evaluate the belief function defined by

$$
\begin{equation*}
\operatorname{bel}_{x}(A ; \mathcal{S})=\mathrm{P}_{\mathcal{S}}\left\{\Theta_{x}(\mathcal{S}) \subseteq A \mid \Theta_{x}(\mathcal{S}) \neq \varnothing\right\} \tag{2.3}
\end{equation*}
$$

and the plausibility function defined by

$$
\begin{equation*}
\mathrm{pl}_{x}(A ; \mathcal{S})=1-\operatorname{bel}_{x}\left(A^{c}, \mathcal{S}\right)=\mathrm{P}_{\mathcal{S}}\left\{\Theta_{x}(\mathcal{S}) \cap A \neq \varnothing \mid \Theta_{x}(\mathcal{S}) \neq \varnothing\right\} \tag{2.4}
\end{equation*}
$$

The belief function is subadditive in the sense that if $\varnothing \neq A \subseteq \Theta$, then $\operatorname{bel}(A ; \mathcal{S})+\operatorname{bel}\left(A^{c} ; \mathcal{S}\right) \leq 1$ with equality if and only if $\Theta_{x}(\mathcal{S})$ is a singleton with $\mathrm{P}_{\mathcal{S}}$-probability 1. Therefore, it follows that bel ${ }_{x}(A ; \mathcal{S}) \leq \mathrm{pl}_{x}(A ; \mathcal{S})$, and they are also referred to as the lower and upper probabilities, respectively. Incidentally, we note that there are continuing interests in using lower and upper probabilities for statistical inference (Gong and Meng, 2021, Liu and Martin, 2021).

Figure 1 provides a pictorial illustration of IM inference about $\theta \in \mathbb{R}$ in the sampling model $Y \sim \mathcal{N}(\theta, 1)$. The A-step is given by the data generation scheme: $y=\theta+u^{\star}$ with $u^{\star}$ a realization of the random variable $U$ following the known distribution $\mathcal{N}(\theta, 1)$. Inference about $\theta$, in terms of assertions about $\theta$, is obtained by predicting $u^{\star}$ with the predictive random set $\mathcal{S}(U)=[-|U|,|U|], U \sim \mathcal{N}(0,1)$.

To establish the frequency-calibration properties of the IM in the context of our current problem, we briefly review several results regarding validity as formally introduced by Martin and Liu (2013).

Definition 2.1 (Validity of Predictive Random Sets). A predictive random set $\mathcal{S}$ is valid for predicting the unobserved auxiliary variable $U$ if for each $\alpha \in(0,1)$,

$$
\begin{equation*}
\mathrm{P}_{U}\left\{\mathrm{P}_{\mathcal{S}}\{\mathcal{S} \not \not U\} \geq 1-\alpha\right\} \leq \alpha \tag{2.5}
\end{equation*}
$$

Intuitively, a valid predictive random set for a random variable $U$ should be sufficiently large so that its error rate in predicting realizations of $U$ is small enough in accordance with our familiar frequency calibration. The use of valid predictive randoms leads to frequency-calibrated inference on unknown parameters, which is defined by Martin and Liu (2013) as follows.

Definition 2.2 (Validity of IMs). The $I M$ is valid if, for all assertions $A$ and for any $\alpha \in(0,1)$,

$$
\begin{equation*}
\sup _{\vartheta \in A} \mathrm{P}_{X \mid \vartheta}\left\{\mathrm{pl}_{X}(A ; \mathcal{S}) \leq \alpha\right\} \leq \alpha \tag{2.6}
\end{equation*}
$$

Using (2.4), we can rewrite (2.6) as

$$
\sup _{\vartheta \in A} \mathrm{P}_{X \mid \vartheta}\left\{\operatorname{bel}_{X}\left(A^{c} ; \mathcal{S}\right) \geq 1-\alpha\right\} \leq \alpha
$$

Thus, the validity of IMs guarantees that the chance of producing large degrees of belief in the truth of a false assertion is small. The particular choice of the mathematical definition is to ensure that the belief probability is frequency-calibrated. Intuitively, it essentially states that " $5 \%$ error rate means (at most) $5 \%$ error rate" (e.g., the coverage-error of confidence/plausibility intervals or the Type-I error of significance testing). Incidentally, IMs are also directly related to significance testing (c.f. Martin and Liu, 2014, for details).

Theorem 2.1. Suppose that the predictive random set $\mathcal{S}$ is valid, and $\Theta_{x}(\mathcal{S}) \neq \varnothing$ with $\mathrm{P}_{\mathcal{S}}$-probability 1 for all $x$. Then the IM is valid.

## Data (y) Generation:



Inference: probabilistic uncertainty assessment on the assertion:


Figure 1: A pictorial illustration of $\operatorname{IM}$ inference about $\theta \in \mathbb{R}$ in the sampling model $Y \sim \mathcal{N}(\theta, 1)$. The A-step is given by the data generation process. The P-step uses a predictive random set $\mathcal{S}(U)=[-|U|,|U|]$, $U \sim \mathcal{N}(0,1)$, to predict the unobserved realization $U^{*}$ in the data generation. The C-step makes use of the random set in the $\theta$ space induced by $\mathcal{S}(U)=[-|U|,|U|]$ to compute bel $y_{y}(A)$, which is the probability for the truth of $A, P(y-S \subseteq A)$. Similarly, it computes bel ${ }_{y}\left(A^{c}\right)$, which is the probability for the falsity of $A$, $P\left(y-S \subseteq A^{c}\right)$ and, thereby, the plausibility $\mathrm{pl}_{y}(A)=1-P\left(y-S \subseteq A^{c}\right)$.

In other words, a valid inference in the sense of frequency calibration is obtained as long as predictive random sets are valid to predict unobserved auxiliary variables. For more general discussion on frequency calibration of belief functions, see Denœux and Li (2018).

For the one-point case, we have the following result, followed by a numerical example to illustrate plausibility-based confidence intervals.

Proposition 2.1. The predictive random set defined by (2.2) is valid and, therefore, the IM with the belief function given by (2.3) and the plausibility function given by (2.4) is valid.

Example 2.1 (An numerical illustration). Consider the assertion $A_{\theta_{0}}=\left\{\vartheta_{0}: \vartheta_{0}\left(t_{1}\right)=\theta_{0}\right\} \subseteq \Theta_{1,1 / 2}$ and the predictive random set $\mathcal{S}=\left[-\left|U_{1}\right|,\left|U_{1}\right|\right], U_{1} \sim \mathcal{N}(0,1)$. Since this assertion constitutes a singleton in the parameter space, the belief function bel ${y_{1}}_{1}\left(A_{\theta_{0}} ; \mathcal{S}\right)=0$, but the plausibility function

$$
\begin{equation*}
\mathrm{pl}_{y_{1}}\left(A_{\theta_{0}} ; \mathcal{S}\right)=\mathrm{P}_{\mathcal{S}}\left\{\theta_{0} \in \Theta_{x_{1}}(\mathcal{S})\right\}=\mathrm{P}_{U_{1}}\left\{\left|U_{1}\right| \geq\left|\theta_{0}-y_{1}\right|\right\}=2\left(1-\Phi\left(\left|\theta_{0}-y_{1}\right|\right)\right) . \tag{2.7}
\end{equation*}
$$



Figure 2: The plausibility function $\mathrm{pl}_{y_{1}}\left(A_{\theta_{0}} ; \mathcal{S}\right)$ defined in (2.7) as a function of $\theta_{0}$ with $y_{1}=0$.

Figure 2 shows the graphs of $\mathrm{pl}_{y_{1}}\left(A_{\theta_{0}} ; \mathcal{S}\right)$ as a function of $\theta_{0}$ for $y_{1}=0$.
Remark 2.1 (Plausibility intervals). In Example 2.1, the assertion $A=\left\{\vartheta_{0}=\theta_{0}\right\}, \vartheta_{0} \in \Theta$ constitutes a singleton in the parameter space. In this case, we can define the $100(1-\alpha) \%$ plausibility region given by

$$
\begin{equation*}
\Pi_{x}(\alpha)=\left\{\vartheta: \operatorname{pl}_{x}(\vartheta) \geq \alpha\right\}, \tag{2.8}
\end{equation*}
$$

where $\mathrm{p}_{x}(\vartheta):=\mathrm{p}_{x}(\{\vartheta\} ; \mathcal{S})$. It was shown in Martin and Liu (2013) that the plausibility region (2.8) provides an exact $100(1-\alpha) \%$ confidence interval.

### 2.2 Remarks on predictive random sets

It is easy to construct valid predictive random sets, for example, by simply defining a nested predictive random set

$$
\mathcal{S}(U)=\{u: b(u) \leq b(U)\}, U \sim \mathbb{P}_{U},
$$

with a specified boundary function $b(u), u \in \mathbb{U}$, where $\mathbb{U}$ denotes the sampling space of the auxiliary variable $U$ and $\mathbb{P}_{U}$, the distribution of $U$. Martin and Liu (2013) argue for the use of nested predictive random sets. This makes sense intuitively because we want to use small set to cover unobserved $u^{\star}$ with large probability for the sake of efficiency.

More research is still needed to investigate the efficiency issue with respect to assertions of interest. Perhaps, investigations along the line of mathematical decision theory are a possibility. The current practice of IMs is more or less intuition based. For example, we choose centered predictive random sets such as $\mathcal{S}(U)=[-|U|,|U|]$ when $U \sim \mathcal{N}(0,1))$ so that the resulting plausibility intervals are efficient in terms of interval length. Such centered predictive random sets are referred to as "default" predictive random sets.

### 2.3 The marginal IMs

As seen in the previous section for the simple one-point case, the basic IMs framework is similar to the frequentist pivotal method of constructing confidence intervals. However, the prediction of unobserved auxiliary variables can help perform efficient inference. Two such methods, called conditional IMs and marginal IMs, have been proposed in Martin and Liu (2015b|a) based on dimension-reduction. Below, we describe how marginal IMs can be used to produce efficient inference in the special cases of the two-point problem when $B=\infty$, where $B=M\left|t_{1}-t_{2}\right|^{\gamma}$.

In the $n=2$ case, we have two pairs of observations $\left(t_{1}, y_{1}\right)$ and $\left(t_{2}, y_{2}\right)$. The sampling model for this case can be written as

$$
\left\{\begin{array}{l}
y_{1}=\vartheta_{0}\left(t_{1}\right)+u_{1}^{\star} ;  \tag{2.9}\\
y_{2}=\vartheta_{0}\left(t_{2}\right)+u_{2}^{\star},
\end{array}\right.
$$

where $u_{1}^{\star}$ and $u_{2}^{\star}$ represent unobserved realizations of $U_{1}, U_{2} \stackrel{i i d}{\sim} \mathcal{N}(0,1)$. The IM framework for the current problem repeats the three steps introduced in the previous section. However, we notice that in this twopoint case, we have two auxiliary variables $U_{1}, U_{2}$, which would need to be predicted using a two-dimensional predictive random set. As Martin and Liu (2015b) pointed out, a more efficient inference procedure can be obtained by reducing the dimension of the auxiliary variable.

It is easy to understand marginal IMs for the $B=\infty$ case. In this case, we have two distinct parameters $\theta_{1}=\vartheta\left(t_{1}\right)$ and $\theta_{2}=\vartheta\left(t_{2}\right)$. This is a problem that is discussed in Martin and Liu (2015b). An intuitive approach is to construct a valid predictive random set, denoted by $\mathcal{S}_{U_{1}, U_{2}}$, for $\left(u_{1}^{\star}, u_{2}^{\star}\right)$ in such a way that the resulting inference on, for example, $\theta_{2}$ is efficient. By efficient in this case, we mean that the plausibility intervals for $\theta_{2}$ are as small as possible. It is easy to see that this is achieved if the projection of $\mathcal{S}\left(U_{1}, U_{2}\right)$ to the space of $u_{2}^{\star}$ is minimized, that is, $\mathcal{S}\left(U_{1}, U_{2}\right)$ has the form

$$
\mathcal{S}\left(U_{1}, U_{2}\right)=\left\{\left(u_{1}, u_{2}\right):\left|u_{2}\right| \leq\left|U_{2}\right|\right\}, \quad U_{1}, U_{2} \stackrel{i i d}{\sim} \mathcal{N}(0,1) .
$$

Thus, this two-dimensional predictive random set becomes effectively a one-dimensional predictive random set

$$
\mathcal{S}\left(U_{2}\right)=\left\{\left(u_{1}, u_{2}\right):\left|u_{2}\right| \leq\left|U_{2}\right|\right\}, \quad U_{2} \sim \mathcal{N}(0,1) .
$$

for $u_{1}^{\star}$. Marginal inference about $\theta_{2}=\vartheta\left(t_{2}\right)$ proceeds with the second equation in 2.9 as the association with the predictive random set $\mathcal{S}\left(U_{2}\right)$. See Martin and Liu (2015b) for more discussion on marginal IMs and its more sophisticated methods.

It is seen that the above marginal inference about $\theta_{2}$ effectively ignores the first equation in 2.9. Although his marginal inference is still valid in the case of $B \neq \infty$, it can be improved because the constraint $\left|\vartheta\left(t_{1}\right)-\vartheta\left(t_{2}\right)\right| \leq B$ provides a connection that allows for the observed information of $y_{1}$ to be used.

### 2.4 The conditional IMs

Here we explain conditional IMs for efficient inference in the special $B=0$ case of the two-point problem. A new method of taking the strengths of conditional IMs and marginal IMs is proposed for the general case of the two-point problem in Section 3 .

In the $n=2$ case, we have two pairs of observations $\left(t_{1}, y_{1}\right)$ and $\left(t_{2}, y_{2}\right)$ with the sampling model given by the association 2.9. Again, the basic IM framework for the current problem repeats the three steps introduced in the previous section. However, we notice that in this two-point case, we have two auxiliary variables $U_{1}, U_{2}$, which would need to be predicted using a two-dimensional predictive random set. As Martin and Liu 2015a) pointed out, a more efficient inference procedure can be obtained by reducing the dimension of the auxiliary variable. In particular, if some functions of the original auxiliary variable are fully observed, we can condition on the fully observed information to sharpen our prediction of the unobserved $u_{2}^{\star}$.

Notice that from 2.9) we can obtain

$$
\begin{equation*}
y_{2}-y_{1}=\vartheta_{0}\left(t_{2}\right)-\vartheta_{0}\left(t_{1}\right)+u_{2}^{\star}-u_{1}^{\star} \tag{2.10}
\end{equation*}
$$

which motivates us to introduce a new auxiliary variable $V=U_{2}-U_{1} \sim \mathcal{N}(0,2)$. By the Hölder condition (1.2), we have

$$
\begin{equation*}
\left|\vartheta_{0}\left(t_{1}\right)-\vartheta_{0}\left(t_{2}\right)\right| \leq M\left|t_{1}-t_{2}\right|^{\gamma} . \tag{2.11}
\end{equation*}
$$

Letting $B:=M\left|t_{1}-t_{2}\right|^{\gamma}$, from 2.10 and 2.11 we obtain

$$
\begin{equation*}
v^{\star}=u_{2}^{\star}-u_{1}^{\star}=y_{2}-y_{1}-\left(\vartheta_{0}\left(t_{2}\right)-\vartheta_{0}\left(t_{1}\right)\right) \in\left[y_{2}-y_{1}-B, y_{2}-y_{1}+B\right] . \tag{2.12}
\end{equation*}
$$

For each fixed $v \in\left[y_{2}-y_{1}-B, y_{2}-y_{1}+B\right]$, the conditional distributions of a linear function of $U_{1}$ and $U_{2}$, for example, $U_{2}$, given $V=v$ can be derived as follows:

$$
\left.\left(U_{2}, V\right) \sim \mathcal{N}_{2}\left(\left[\begin{array}{l}
0  \tag{2.13}\\
0
\end{array}\right],\left[\begin{array}{ll}
1 & 1 \\
1 & 2
\end{array}\right]\right) \quad \Longrightarrow \quad U_{2} \right\rvert\, V=v \sim \mathcal{N}\left(\frac{v}{2}, \frac{1}{2}\right)
$$

Thus, these conditional distributions, sharper than the corresponding marginal distributions, motivate us to construct more efficient predictive random sets for more efficient inference about $\left(\vartheta_{0}\left(t_{1}\right), \vartheta_{0}\left(t_{2}\right)\right)$.

For the special case of $B=0$, the equation 2.10 simplifies to

$$
y_{2}-y_{1}=u_{2}^{\star}-u_{1}^{\star}
$$

Most important, $u_{2}^{\star}-u_{1}^{\star}$ in this extreme case is fully observed and, therefore, can be easily used to predict $u_{1}^{\star}$ and $u_{2}^{\star}$. Formally, we can construct a conditional inferential model as proposed by Martin and Liu (2015a). Similar to basic IMs, conditional IMs also have their three steps:

A-step. Under the original IM framework, the association step is achieved via the baseline association (2.9). In the case of $B=0$, this baseline association can be decomposed as

$$
\left\{\begin{array}{l}
y_{2}-y_{1}=u_{2}^{\star}-u_{1}^{\star}  \tag{2.14}\\
y_{2}=\vartheta\left(t_{2}\right)+u_{2}^{\star}
\end{array}\right.
$$

This decomposition immediately suggests an alternative association. Let $\mathrm{P}_{U_{2} \mid y_{2}-y_{1}}$ denote the conditional distribution of $U_{2}$, given $U_{2}-U_{1}=y_{2}-y_{1}$. Since $y_{2}-y_{1}$ does not provide information on the parameter $\vartheta$, we can establish a new association

$$
\begin{equation*}
Y_{2}=\vartheta\left(t_{2}\right)+\tilde{U}_{2}, \quad\left(\tilde{U}_{2} \sim \mathrm{P}_{U_{2} \mid y_{2}-y_{1}}\right) \tag{2.15}
\end{equation*}
$$

which is referred to as the conditional association. Using 2.15), we can associate the observed information $Y_{2}$ and the parameter $\vartheta$ with the new auxiliary variable $\tilde{U}_{2} \sim \mathrm{P}_{U_{2} \mid Y_{2}-Y_{1}}$ to get the collection of sets

$$
\Theta_{y_{2}}\left(\tilde{u}_{2}^{\star}\right)=\left\{\vartheta \in \Theta: y_{2}=\vartheta\left(t_{2}\right)+\tilde{u}_{2}^{\star}\right\}
$$

P-step. Fixing the observed value $Y_{2}-Y_{1}=y_{2}-y_{1}$, we predict the unobserved value $\tilde{u}_{2}^{\star}$ of $\tilde{U}_{2}$ with a conditionally valid predictive random set $\mathcal{S}_{2} \sim \mathrm{P}_{\mathcal{S}_{2} \mid y_{2}-y_{1}}$. Notice that equation 2.13) implies that $\tilde{U}_{2} \sim \mathcal{N}\left(\frac{y_{2}-y_{1}}{2}, \frac{1}{2}\right)$, which gives rise to the default predictive random set (see Section 2.2 given by

$$
\mathcal{S}_{2}=\left\{\tilde{u}_{2}:\left|\tilde{u}_{2}-\frac{y_{2}-y_{1}}{2}\right| \leq\left|\tilde{U}_{2}-\frac{y_{2}-y_{1}}{2}\right|\right\}
$$

C-step. We combine the results of the association and prediction steps to get

$$
\Theta_{y_{2}}\left(\mathcal{S}_{2}\right)=\bigcup_{u_{2} \in \mathcal{S}_{2}} \Theta_{y_{2}}\left(u_{2}\right)=\left\{\vartheta \in \Theta:\left|\vartheta\left(t_{2}\right)-\frac{y_{1}+y_{2}}{2}\right| \leq\left|\tilde{U}_{2}-\frac{y_{2}-y_{1}}{2}\right|\right\}
$$

where $\tilde{U}_{2} \sim \mathcal{N}\left(\left(y_{2}-y_{1}\right) / 2,1 / 2\right)$. For any assertion $A \subseteq \Theta$, the corresponding conditional belief function is given by

$$
\begin{equation*}
\operatorname{cbel}_{y_{2} \mid y_{2}-y_{1}}\left(A ; \mathcal{S}_{2}\right)=\mathrm{P}_{\mathcal{S}_{2} \mid y_{2}-y_{1}}\left\{\Theta_{y_{2}}\left(\mathcal{S}_{2}\right) \subseteq A \mid \Theta_{y_{2}}\left(\mathcal{S}_{2}\right) \neq \varnothing\right\} \tag{2.16}
\end{equation*}
$$

and the conditional plausibility function is given by

$$
\begin{equation*}
\operatorname{cpl}_{y_{2} \mid y_{2}-y_{1}}\left(A ; \mathcal{S}_{2}\right)=1-\operatorname{cbel}_{y_{2} \mid y_{2}-y_{1}}\left(A^{c} ; \mathcal{S}_{2}\right)=\mathrm{P}_{\mathcal{S}_{2} \mid y_{2}-y_{1}}\left\{\Theta_{y_{2}}\left(\mathcal{S}_{2}\right) \cap A \neq \varnothing \mid \Theta_{y_{2}}\left(\mathcal{S}_{2}\right) \neq \varnothing\right\} \tag{2.17}
\end{equation*}
$$

Figure 3 provides a pictorial illustration of the above conditional IM for $\theta=\vartheta\left(t_{1}\right)=\vartheta\left(t_{2}\right)$ when $B=0$. Note that, for any singleton assertion $A=\left\{\vartheta_{0}\right\}, \vartheta_{0} \in \Theta$, the conditional belief function cbel $y_{y_{2} \mid y_{2}-y_{1}}\left(A ; \mathcal{S}_{2}\right)=$ 0 , while the conditional plausibility function takes the form of

$$
\operatorname{cpl}_{y_{2}}\left(A ; \mathcal{S}_{2}\right)=\mathrm{P}_{\mathcal{S}_{2} \mid y_{2}-y_{1}}\left\{\vartheta_{0} \in \Theta_{y_{2}}\left(\mathcal{S}_{2}\right)\right\}=2\left[1-\Phi\left(\sqrt{2}\left|\vartheta_{0}\left(t_{2}\right)-\frac{y_{1}+y_{2}}{2}\right|\right)\right] .
$$



Figure 3: A pictorial illustration of conditional $I M$ inference about $\theta \in \mathbb{R}$ in the sampling model $Y_{i} \stackrel{i . i . d .}{\sim}$ $\mathcal{N}(\theta, 1), i=1,2$. What is important is that $U_{1}-U_{2}=y_{1}-y_{2}$ is fully observed and, thereby, it is unnecessary to predict $\left(U_{1}^{*}, U_{2}^{*}\right)$ in the two-dimensional space of $\left(U_{1}, U_{2}\right)$. The observed information is used to further improve prediction accuracy.

## 3 Partial Conditioning

### 3.1 The general $B=M\left|t_{2}-t_{1}\right|$ case and its challenges

When $B$ is large enough, neither observation can provide information on the location of the other observation. In this case, we may simply choose to make inference based on associated single observation alone. The idea is supported by the method of marginal IMs Martin and Liu, 2015b). However, for the general case with $0<B<\infty$, it remains challenging to construct efficient IMs.

A seemingly attractive approach is the fiducial-type approach. In this case, inference is obtained using the conditional distribution of $U_{2}$ given $U_{2}-U_{1}$ in the constrained interval by taking the distribution of $U_{2}-U_{1}$ as its original distribution restricted to the constrained interval in 2.12 , i.e., $\left[y_{2}-y_{1}-B, y_{2}-y_{1}+B\right]$. Since this constrained interval is implicitly dependent on the unobserved realizations of $U_{2}$ and $U_{1}$, such a probability operation is questionable. In fact, it generates an uncertainty assessment without the guarantee of the desired frequency calibration. More precisely, this is because in repeated experiments for evaluating desired frequency calibration, the interval $\left[y_{2}-y_{1}-B, y_{2}-y_{1}+B\right]$ varies from experiment to experiment and depends on the experiment-specific realizations of $U_{2}-U_{1}$. Therefore, the use of the distribution of $U_{2}-U_{1}$ as its original distribution restricted to the interval $\left[y_{2}-y_{1}-B, y_{2}-y_{1}+B\right]$ is purely subjective and has no intended frequency interpretation mathematically. See Liu and Martin (2015) for more discussion of this problem for fiducial in general.

A conservative but simple approach to making valid inference is to weaken the conditional belief function (2.16) by taking its infimum and the conditional plausibility function 2.17) by taking its supremum over all possible values of the conditioning variable $U_{2}-U_{1}$. While valid inference is produced, this leads to inefficient inference, which can be even worse than that based on a single data point alone.

### 3.2 A new approach based on partial conditioning

We aim to construct nested predictive random sets for predicting $U_{2}$ based on a distribution of the form

$$
\begin{equation*}
\mathcal{N}\left(\frac{\lambda}{2} v^{\star}, 1-\lambda+\frac{\lambda^{2}}{2}\right) \tag{3.1}
\end{equation*}
$$

where $\lambda \in[0,1]$ is some function of $B$ and $v^{\star}$ is a realization of $V=U_{2}-U_{1}$. The objective is to be able to make valid inference about $\vartheta\left(t_{2}\right)$ by predicting $u_{2}^{\star}$ using 3.1 based on the fact that $U_{2}-U_{1}$ is known to lie in the constrained interval. For this reason, we call (3.1) a predictive distribution. Note that the conditional and marginal IMs are obtained as two extreme cases with $\lambda=1$ for fully conditional inference and $\lambda=0$ for completely marginal inference. The case of $0<\lambda<1$ thus resembles an approach we refer to as partial conditioning or, more exactly, partial regression of, say, $U_{2}$ on $V=U_{2}-U_{1}$. The intuition and theoretical support for the use of (3.1) for IM-based inference are explained below, with a remark on its connection with the familiar method of shrinkage estimation.

The key idea is to take into account the strengths of both conditional and marginal IMs. That is, we consider conditional IMs but conservative for validity when $B$ is small, and marginal IMs when conditioningbased conservative IMs are inefficient. Technically, this amounts to utilizing the partial regression

$$
U_{2}=\frac{\lambda}{2} V+\varepsilon
$$

to predict $U_{2}$ through prediction of $\varepsilon$ in such a way that the prediction of $\varepsilon$ is valid marginally, and thereby prediction of $U_{2}$ is valid conditionally, given $V$ in some interval. Marginally, $\varepsilon$ is normal with mean zero and variance

$$
\operatorname{Var}(\varepsilon)=\operatorname{Var}\left(U_{2}-\frac{\lambda}{2} V\right)=\left(1-\frac{\lambda}{2}\right)^{2}+\left(\frac{\lambda}{2}\right)^{2}=1-\lambda+\frac{\lambda^{2}}{2}
$$

This leads to the use of (3.1) as a valid inference.
To predict $U_{2}$, we utilize the predictive random sets

$$
\mathcal{S}_{v^{*}}=\left\{z:\left|z-\frac{\lambda}{2} v^{\star}\right| \leq\left|Z-\frac{\lambda}{2} v^{\star}\right|\right\}, \quad Z \sim \mathcal{N}\left(\frac{\lambda}{2} v^{\star}, 1-\lambda+\frac{\lambda^{2}}{2}\right)
$$

when $V$ is known to be $v^{*}$. For the two-point problem, $V$ is known to be some $v^{*}$ in the interval $\left[y_{2}-y_{1}-\right.$ $\left.B, y_{2}-y_{1}+B\right]$. Thus, the use of the conservative prediction random set

$$
\mathcal{S}=\bigcup_{v^{*} \in\left[y_{2}-y_{1}-B, y_{2}-y_{1}+B\right]} \mathcal{S}_{v^{*}}
$$

provides valid inference. For constructing confidence intervals, for example, the plausibility region

$$
\bigcup_{v^{\star} \in\left[y_{2}-y_{1}-B, y_{2}-y_{1}+B\right]}\left[\frac{\lambda}{2} v^{\star}-z_{1-\alpha / 2} \sqrt{1-\lambda+\frac{\lambda^{2}}{2}}, \frac{\lambda}{2} v^{\star}+z_{1-\alpha / 2} \sqrt{1-\lambda+\frac{\lambda^{2}}{2}}\right]
$$

covers $u_{2}^{\star}$ with probability at least $100(1-\alpha) \%$, for all $\lambda \in[0,1]$.
Note that the length of the plausibility interval is given by

$$
\begin{equation*}
\left[\frac{\lambda}{2}\left(y_{2}-y_{1}+B\right)-\frac{\lambda}{2}\left(y_{2}-y_{1}-B\right)\right]+2 z_{1-\alpha / 2} \sqrt{1-\lambda+\frac{\lambda^{2}}{2}}=\lambda B+2 z_{1-\alpha / 2} \sqrt{1-\lambda+\frac{\lambda^{2}}{2}} \tag{3.2}
\end{equation*}
$$

Using the usual measure of efficiency in terms of interval length, we minimize $\sqrt[3.2]{ }$ over $\lambda \in[0,1]$. This leads to the choice of $\lambda$ :

$$
\hat{\lambda}_{B}= \begin{cases}1-\frac{B}{\sqrt{2 z_{1-\alpha / 2}^{2}-B^{2}}}, & \text { if } 0 \leq B<z_{1-\alpha / 2} \\ 0, & \text { if } B \geq z_{1-\alpha / 2}\end{cases}
$$



Figure 4: Comparison of the widths of the plausibility intervals constructed for the assertion $A(t)=\left\{\vartheta_{0}:[0,1] \rightarrow \mathbb{R} \mid \vartheta_{0}(t)=\sqrt{t}\right\} \subseteq \Theta_{1,1 / 2}$ using the three different approaches: Marginal IMs, Partial Conditioning ("Mixture"), and Conservative Conditional IMs.

The corresponding widths are equal to $B+\sqrt{2 z_{1-\alpha / 2}^{2}-B^{2}}$ and $2 z_{1-\alpha / 2}$, respectively.
The optimal value of $\hat{\lambda}_{B}$ suggests a full conditional IM when $B$ is small relative to the width of the target confidence interval or the confidence level. Figure 4 shows the numerical results of a simulation with 100 trials. We observe that the plausibility intervals obtained using partial conditioning are the narrowest.

### 3.3 Partial conditioning versus shrinkage estimation

The use of (3.1) for inference about $\vartheta\left(t_{2}\right)$ makes a connection with the familiar method of shrinkage estimation. At a high level, both partial conditioning and the method of shrinkage estimation use information in $y_{1}$ about $\vartheta\left(t_{2}\right)$. Here we compare the two methods in terms of point estimation. The IM method of point estimation is the set of parameter values with maximum plausibility.

The basic idea of partial conditioning is to improve inference about the mean $\vartheta\left(t_{2}\right)$ of $y_{2}$ by making use of partial information of the observed $y_{1}$ and the constraint on the distance between $\vartheta\left(t_{2}\right)$ and the mean $\vartheta\left(t_{1}\right)$ of $y_{1}$. The particular construction (3.1) is proposed for this purpose. As a result, the maximum plausibility estimate of $\vartheta\left(t_{2}\right)$ is the set given by

$$
\begin{equation*}
\left[\frac{\left(2-\lambda_{B}\right) y_{2}+\lambda_{B} y_{1}}{2}-\frac{\lambda_{B}}{2} B, \frac{\left(2-\lambda_{B}\right) y_{2}+\lambda_{B} y_{1}}{2}+\frac{\lambda_{B}}{2} B\right] \tag{3.3}
\end{equation*}
$$

when the predictive random set $\mathcal{S}_{v^{\star}}$ is used.
If the inequality constraint on the distance between $\vartheta\left(t_{2}\right)$ and the mean $\vartheta\left(t_{1}\right)$ is replaced by the probabilistic condition $\vartheta\left(t_{1}\right) \mid \vartheta\left(t_{2}\right) \sim N\left(\vartheta\left(t_{2}\right), \tau^{2}\right)$ for some $\tau>0$, that is, $y_{1} \mid \vartheta\left(t_{2}\right) \sim N\left(\vartheta\left(t_{2}\right), \tau^{2}+1\right)$, the maximum likelihood estimate of $\vartheta\left(t_{2}\right)$ is then given by the shrinkage estimate

$$
\begin{equation*}
\frac{\left(1+\tau^{2}\right) y_{2}+y_{1}}{\left(1+\tau^{2}\right)+1} . \tag{3.4}
\end{equation*}
$$

It is interesting to see that the shrinkage estimate 3.4 is the center of the maximum plausibility set (3.3) when $\tau^{2}$ is taken to be $2\left(1-\lambda_{B}\right) / \lambda_{B}$. In general, the maximum plausibility estimate 3.3 is different from the shrinkage estimate (3.4), except for the special case of $B=0$, where the maximum plausibility estimate (3.3) with $\lambda_{B}=0$ is the shrinkage estimate derived with the corresponding assumption $\tau^{2}=0$.

## 4 The general $n$ case

In this section, we extend the proposed partial conditioning approach for the $n=2$ case to the general $n$ case. In order to be easily comprehensible, we provide detailed investigation on the three-point $(n=3)$ case in Section 4.1 with an illustrative numerical example. The general $n$ case follows as a straightforward generalization of the three-point problem and is summarized in Section 4.2, followed by numerical examples. Although the technical details are somewhat tedious, the main results for applications are given by the plausibility intervals 4.19. Finally, we perform a simulation study investigating the asymptotic performance of partial conditioning.

### 4.1 The $n=3$ case

For the case of $n=3$, we have three pairs of observations $\left(t_{i}, y_{i}\right), i=1,2,3$ with the data generation model:

$$
\left\{\begin{array}{l}
y_{1}=\vartheta_{0}\left(t_{1}\right)+u_{1}^{\star} ;  \tag{4.1}\\
y_{2}=\vartheta_{0}\left(t_{2}\right)+u_{2}^{\star} ; \\
y_{3}=\vartheta_{0}\left(t_{3}\right)+u_{3}^{\star}
\end{array}\right.
$$

where $u_{1}^{\star}, u_{2}^{\star}, u_{3}^{\star}$ represent unobserved realizations of $U_{1}, U_{2}, U_{3} \stackrel{i i d}{\sim} \mathcal{N}(0,1)$. The problem of interest is to make a (marginal) inference on $\vartheta_{0}\left(t_{i}\right), i=1,2,3$. Without loss of generality, assume that $t_{1} \leq t_{2} \leq t_{3}$. To proceed, we first find the relevant conditional distributions in Section 4.1.1, and then provide the optimal solution under a partial conditioning framework in Section 4.1.2.

### 4.1.1 Conditional Distributions

The system of pairwise differences from (4.1), that is,

$$
\left\{\begin{array}{l}
y_{2}-y_{1}=\vartheta_{0}\left(t_{2}\right)-\vartheta_{0}\left(t_{1}\right)+u_{2}^{\star}-u_{1}^{\star}  \tag{4.2}\\
y_{3}-y_{1}=\vartheta_{0}\left(t_{3}\right)-\vartheta_{0}\left(t_{1}\right)+u_{3}^{\star}-u_{1}^{\star} \\
y_{3}-y_{2}=\vartheta_{0}\left(t_{3}\right)-\vartheta_{0}\left(t_{2}\right)+u_{3}^{\star}-u_{2}^{\star}
\end{array}\right.
$$

motivates us to introduce the potential conditioning auxiliary variables

$$
V_{21}=U_{2}-U_{1}, \quad V_{31}=U_{3}-U_{1}, \quad V_{32}=U_{3}-U_{2}
$$

Clearly, we have that

$$
\left(V_{21}, V_{31}, V_{32}\right) \sim \mathcal{N}_{3}\left(\left[\begin{array}{l}
0 \\
0 \\
0
\end{array}\right],\left[\begin{array}{ccc}
2 & 1 & -1 \\
1 & 2 & 1 \\
-1 & 1 & 2
\end{array}\right]\right)
$$

By the Hölder condition (1.2), we have

$$
\left\{\begin{array}{l}
\left|\vartheta_{0}\left(t_{2}\right)-\vartheta_{0}\left(t_{1}\right)\right| \leq M\left|t_{2}-t_{1}\right|^{\gamma}  \tag{4.3}\\
\left|\vartheta_{0}\left(t_{3}\right)-\vartheta_{0}\left(t_{1}\right)\right| \leq M\left|t_{3}-t_{1}\right|^{\gamma} \\
\left|\vartheta_{0}\left(t_{3}\right)-\vartheta_{0}\left(t_{2}\right)\right| \leq M\left|t_{3}-t_{2}\right|^{\gamma}
\end{array}\right.
$$

For notational convenience, define

$$
B_{i j}:=M\left|t_{i}-t_{j}\right|^{\gamma}, 1 \leq i, j \leq 3
$$

From (4.2 and 4.3 we obtain the observed constraints on the pairwise differences $v_{i,}^{\star}$ :

$$
\begin{equation*}
v_{i j}^{\star}=u_{i}^{\star}-u_{j}^{\star}=y_{i}-y_{j}-\left(\vartheta_{0}\left(t_{i}\right)-\vartheta_{0}\left(t_{j}\right)\right) \in\left[y_{i}-y_{j}-B_{i j}, y_{i}-y_{j}+B_{i j}\right], 1 \leq j<i \leq 3 . \tag{4.4}
\end{equation*}
$$

A complete characterization of the conditional distributions of the original auxiliary variables $U_{1}, U_{2}, U_{3}$ given the new auxiliary variables $V_{21}, V_{31}, V_{32}$ is provided in Propositions 4.1 and 4.2, with proofs given in the appendix. Direct applications of these propositions yield the corresponding results with respect to the parameter $\vartheta \in \Theta$, which are summarized in Corollaries 4.1 and 4.2.

Proposition 4.1. For any fixed $v_{i j}=u_{i}-u_{j}$ with $i>j$, the conditional distributions of $U_{1}, U_{2}, U_{3}$ given $V_{i j}=v_{i j}$ take the following forms:

$$
\begin{align*}
& U_{1}\left|v_{21} \sim \mathcal{N}\left(\frac{u_{1}-u_{2}}{2}, \frac{1}{2}\right), U_{1}\right| v_{31} \sim \mathcal{N}\left(\frac{u_{1}-u_{3}}{2}, \frac{1}{2}\right), U_{1} \mid v_{32} \sim \mathcal{N}(0,1)  \tag{4.5}\\
& U_{2}\left|v_{21} \sim \mathcal{N}\left(\frac{u_{2}-u_{1}}{2}, \frac{1}{2}\right), U_{2}\right| v_{32} \sim \mathcal{N}\left(\frac{u_{2}-u_{3}}{2}, \frac{1}{2}\right), U_{2} \mid v_{31} \sim \mathcal{N}(0,1)  \tag{4.6}\\
& U_{3}\left|v_{31} \sim \mathcal{N}\left(\frac{u_{3}-u_{1}}{2}, \frac{1}{2}\right), U_{3}\right| v_{32} \sim \mathcal{N}\left(\frac{u_{3}-u_{2}}{2}, \frac{1}{2}\right), U_{3} \mid v_{21} \sim \mathcal{N}(0,1) \tag{4.7}
\end{align*}
$$

Corollary 4.1. Proposition 4.1 yields the following predictive random sets:

$$
\begin{align*}
& \vartheta\left(t_{1}\right)=Y_{1}-U_{1}: \mathcal{N}\left(\frac{Y_{1}+Y_{2}}{2} \pm \frac{B_{21}}{2}, \frac{1}{2}\right) \cap \mathcal{N}\left(\frac{Y_{1}+Y_{3}}{2} \pm \frac{B_{31}}{2}, \frac{1}{2}\right) ;  \tag{4.8}\\
& \vartheta\left(t_{2}\right)=Y_{2}-U_{2}: \mathcal{N}\left(\frac{Y_{1}+Y_{2}}{2} \pm \frac{B_{21}}{2}, \frac{1}{2}\right) \cap \mathcal{N}\left(\frac{Y_{2}+Y_{3}}{2} \pm \frac{B_{32}}{2}, \frac{1}{2}\right) ;  \tag{4.9}\\
& \vartheta\left(t_{3}\right)=Y_{3}-U_{3}: \mathcal{N}\left(\frac{Y_{1}+Y_{3}}{2} \pm \frac{B_{31}}{2}, \frac{1}{2}\right) \bigcap \mathcal{N}\left(\frac{Y_{2}+Y_{3}}{2} \pm \frac{B_{32}}{2}, \frac{1}{2}\right) . \tag{4.10}
\end{align*}
$$

where we abuse notation slightly and denote by $\mathcal{N}\left(\mu, \sigma^{2}\right)$ the "default" predictive random set (c.f. Section 2.2) given by $\{u:|u-\mu| \leq|U-\mu|\}$ with $U \sim \mathcal{N}\left(\mu, \sigma^{2}\right)$.

Proposition 4.2. For any fixed $v_{i_{1} j_{1}}=u_{i_{1}}-u_{j_{1}}$ and $v_{i_{2} j_{2}}=u_{i_{2}}-u_{j_{2}}$ with $i_{1}>j_{1}$ and $i_{2}>j_{2}$, the conditional distributions of $U_{1}, U_{2}, U_{3}$ given $V_{i_{1} j_{1}}=v_{i_{1} j_{1}}$ and $V_{i_{2} j_{2}}=v_{i_{2} j_{2}}$ take the following forms:

$$
\begin{align*}
& U_{1}\left|v_{21}, v_{31}, U_{1}\right| v_{21}, v_{32}, U_{1} \mid v_{31}, v_{32} \sim \mathcal{N}\left(\frac{2 u_{1}-u_{2}-u_{3}}{3}, \frac{1}{3}\right)  \tag{4.11}\\
& U_{2}\left|v_{21}, v_{31}, U_{2}\right| v_{21}, v_{32}, U_{2} \mid v_{31}, v_{32} \sim \mathcal{N}\left(\frac{2 u_{2}-u_{1}-u_{3}}{3}, \frac{1}{3}\right)  \tag{4.12}\\
& U_{3}\left|v_{21}, v_{31}, U_{3}\right| v_{21}, v_{32}, U_{3} \mid v_{31}, v_{32} \sim \mathcal{N}\left(\frac{2 u_{3}-u_{1}-u_{2}}{3}, \frac{1}{3}\right) \tag{4.13}
\end{align*}
$$

Corollary 4.2. Proposition 4.2 yields the following predictive random sets:

$$
\begin{aligned}
& \vartheta\left(t_{1}\right)=Y_{1}-U_{1}: \mathcal{N}\left(\frac{Y_{1}+Y_{2}+Y_{3}}{3} \pm \frac{B_{21}+B_{31}}{3}, \frac{1}{3}\right) \\
& \vartheta\left(t_{2}\right)=Y_{2}-U_{2}: \mathcal{N}\left(\frac{Y_{1}+Y_{2}+Y_{3}}{3} \pm \frac{B_{21}+B_{32}}{3}, \frac{1}{3}\right) \\
& \vartheta\left(t_{3}\right)=Y_{3}-U_{3}: \mathcal{N}\left(\frac{Y_{1}+Y_{2}+Y_{3}}{3} \pm \frac{B_{31}+B_{32}}{3}, \frac{1}{3}\right)
\end{aligned}
$$

where we again abuse notation and denote by $\mathcal{N}\left(\mu, \sigma^{2}\right)$ the "default" predictive random set $\{u:|u-\mu| \leq$ $|U-\mu|\}$ with $U \sim \mathcal{N}\left(\mu, \sigma^{2}\right)$. More compactly, the predictive random set for $\vartheta\left(t_{i}\right)$ can be written as

$$
\mathcal{N}\left(\bar{Y} \pm \bar{B}_{i}, \frac{1}{3}\right), \quad i=1,2,3
$$

where $\bar{Y}=\left(Y_{1}+Y_{2}+Y_{3}\right) / 3, \bar{B}_{i}=\left(B_{i 1}+B_{i 2}+B_{i 3}\right) / 3$, and we define $B_{i i}=0$ for $i=1,2,3$.
Remark 4.1. For the sake of completeness, we remark that the covariance matrices in

$$
\begin{aligned}
\left(U_{1}, V_{21}, V_{31}, V_{32}\right) \sim & \left(\left[\begin{array}{l}
0 \\
0 \\
0 \\
0
\end{array}\right],\left[\begin{array}{cccc}
1 & -1 & -1 & 0 \\
-1 & 2 & 1 & -1 \\
-1 & 1 & 2 & 1 \\
0 & -1 & 1 & 2
\end{array}\right]\right) \\
\left(U_{2}, V_{21}, V_{31}, V_{32}\right) \sim & \left.\sim\left[\begin{array}{l}
0 \\
0 \\
0 \\
0
\end{array}\right],\left[\begin{array}{cccc}
1 & 1 & 0 & -1 \\
1 & 2 & 1 & -1 \\
0 & 1 & 2 & 1 \\
-1 & -1 & 1 & 2
\end{array}\right]\right) \\
\left(U_{3}, V_{21}, V_{31}, V_{32}\right) & \sim\left(\left[\begin{array}{l}
0 \\
0 \\
0 \\
0
\end{array}\right],\left[\begin{array}{cccc}
1 & 0 & 1 & 1 \\
0 & 2 & 1 & -1 \\
1 & 1 & 2 & 1 \\
1 & -1 & 1 & 2
\end{array}\right]\right)
\end{aligned}
$$

are all degenerate, due to the fact that $V_{21}, V_{31}, V_{32}$ are linearly dependent.

### 4.1.2 Prediction via partial conditioning

Recall that our goal is to predict $u_{2}^{\star}$ to make a valid probabilistic inference. Without loss of generality, assume that $t_{2}-t_{1} \leq t_{3}-t_{2}$. Motivated by the idea of partial conditioning for the two-point problem, we construct nested predictive random sets by introducing the partial regression of $u_{2}^{\star}$ on pairwise differences:

$$
\begin{equation*}
\mathcal{N}\left(\lambda_{1}\left(\frac{u_{2}^{\star}-u_{1}^{\star}}{2}\right)+\lambda_{2}\left(\frac{2 u_{2}^{\star}-u_{1}^{\star}-u_{3}^{\star}}{3}\right),\left(\frac{\lambda_{1}}{2}+\frac{\lambda_{2}}{3}\right)^{2}+\left(1-\frac{\lambda_{1}}{2}-\frac{2}{3} \lambda_{2}\right)^{2}+\left(\frac{\lambda_{2}}{3}\right)^{2}\right) \tag{4.14}
\end{equation*}
$$

where the mixing proportions $\lambda_{1}, \lambda_{2}$ are functions of $B_{i j}$ satisfying $\lambda_{1}, \lambda_{2} \geq 0$ and $\lambda_{1}+\lambda_{2} \leq 1$. Note that the variance expression arises naturally from the fact that

$$
\operatorname{Var}\left[U_{2}-\frac{\lambda_{1}}{2}\left(U_{2}-U_{1}\right)-\frac{\lambda_{2}}{3}\left(2 U_{2}-U_{1}-U_{3}\right)\right]=\left(\frac{\lambda_{1}}{2}+\frac{\lambda_{2}}{3}\right)^{2}+\left(1-\frac{\lambda_{1}}{2}-\frac{2}{3} \lambda_{2}\right)^{2}+\left(\frac{\lambda_{2}}{3}\right)^{2}
$$

To predict $U_{2}$, we take the predictive random sets

$$
\mathcal{S}_{v_{21}^{\star}, v_{32}^{\star}}=\left\{z:\left|z-\left[\lambda_{1}\left(\frac{v_{21}^{\star}}{2}\right)+\lambda_{2}\left(\frac{v_{21}^{\star}-v_{32}^{\star}}{3}\right)\right]\right| \leq\left|Z-\left[\lambda_{1}\left(\frac{v_{21}^{\star}}{2}\right)+\lambda_{2}\left(\frac{v_{21}^{\star}-v_{32}^{\star}}{3}\right)\right]\right|\right\}
$$

where we recall that

$$
v_{i j}^{\star}=u_{i}^{\star}-u_{j}^{\star}=y_{i}-y_{j}-\left(\vartheta_{0}\left(t_{i}\right)-\vartheta_{0}\left(t_{j}\right)\right) \in\left[y_{i}-y_{j}-B_{i j}, y_{i}-y_{j}+B_{i j}\right], 1 \leq j<i \leq 3
$$

and $Z$ follows the Gaussian distribution (4.14). The predictive random sets are clearly marginally valid, and thus the corresponding IMs are valid. Therefore, the we utilize the conservative predictive random set

$$
\mathcal{S}=\bigcup_{\substack{v_{2}^{\star} \in\left[y_{2}-y_{1} \pm B_{21}\right] \\ v_{32}^{\star} \in\left[y_{3}-y_{2} \pm B_{32}\right]}} \mathcal{S}_{v_{21}^{\star}, v_{32}^{\star}}
$$

For constructing confidence intervals, this suggests the plausibility region

$$
\bigcup_{\substack{v_{21}^{\star} \in\left[y_{2}-y_{1} \pm B_{21}\right] \\ v_{32}^{\star} \in\left[y_{3}-y_{2} \pm B_{32}\right]}}\left\{\left[\left(\frac{\lambda_{1}}{2}+\frac{\lambda_{2}}{3}\right) v_{21}^{\star}-\left(\frac{\lambda_{2}}{3}\right) v_{32}^{\star}\right] \pm z_{1-\alpha / 2} \sqrt{\left(\frac{\lambda_{1}}{2}+\frac{\lambda_{2}}{3}\right)^{2}+\left(1-\frac{\lambda_{1}}{2}-\frac{2}{3} \lambda_{2}\right)^{2}+\left(\frac{\lambda_{2}}{3}\right)^{2}}\right\},
$$

which covers $u_{2}^{\star}$ with probability at least $100(1-\alpha) \%$. It can be easily verified that the width of this plausibility region is given by

$$
2\left[\left(\frac{\lambda_{1}}{2}+\frac{\lambda_{2}}{3}\right) B_{21}+\left(\frac{\lambda_{2}}{3}\right) B_{32}\right]+2 z_{1-\alpha / 2} \sqrt{\left(\frac{\lambda_{1}}{2}+\frac{\lambda_{2}}{3}\right)^{2}+\left(1-\frac{\lambda_{1}}{2}-\frac{2}{3} \lambda_{2}\right)^{2}+\left(\frac{\lambda_{2}}{3}\right)^{2}}
$$

The optimal mixing proportions $\lambda_{1}, \lambda_{2}$ are the solutions to the constrained optimization problem

$$
\begin{align*}
\min _{\lambda_{1}, \lambda_{2}} & {\left[\left(\frac{\lambda_{1}}{2}+\frac{\lambda_{2}}{3}\right) B_{21}+\left(\frac{\lambda_{2}}{3}\right) B_{32}\right]+z_{1-\alpha / 2} \sqrt{\left(\frac{\lambda_{1}}{2}+\frac{\lambda_{2}}{3}\right)^{2}+\left(1-\frac{\lambda_{1}}{2}-\frac{2}{3} \lambda_{2}\right)^{2}+\left(\frac{\lambda_{2}}{3}\right)^{2}} } \\
\text { s.t. } & \lambda_{1}, \lambda_{2} \geq 0 ; \quad \lambda_{1}+\lambda_{2} \leq 1 \tag{4.15}
\end{align*}
$$

Solving optimization problem 4.15) analytically requires finding the roots of a set of quadratic equations, a process which can become quite burdensome. Instead, we adopt numerical optimization methods such as the Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm (Nocedal and Wright, 2006; Liu and Vander Wiel, 2008) to solve problem 4.15 in order to obtain the optimal mixing proportions which minimize the length of the plausibility interval. Figure 5 shows the numerical results of a simulation with 500 trials. Again, we find that the plausibility intervals obtained using optimal mixture distributions are the narrowest.


Figure 5: Comparison of the widths of the plausibility intervals constructed for the assertion $A(t)=\left\{\vartheta_{0}:[0,1] \rightarrow \mathbb{R} \mid \vartheta_{0}(t)=\sqrt{t}\right\} \subseteq \Theta_{1,1 / 2}$ using four different approaches: Marginal IMs, Partial Conditioning ("Mixture"), Conditional IMs on the nearest point (labeled as "Cond'l (1 pt)" in the plots), and Conditional IMs on both points (labeled as "Cond'l ( 2 pts )" in the plots).

### 4.2 The general $n$ case

In this section, we extend our previous discussions to the general case of $n$ observations, and provide a generic method to construct valid and efficient pointwise plausibility intervals. Denote our sequence of observations by $\left(t_{i}, y_{i}\right), i=1, \cdots, n$, where it is not necessary to assume that the $t_{i}$ 's are sorted. With the assumption $\sigma=1$ made in Section 1. the association 1.1) can be written as

$$
\begin{equation*}
y_{i}=\vartheta_{0}\left(t_{i}\right)+u_{i}^{\star}, i=1, \cdots, n . \tag{4.16}
\end{equation*}
$$

In the rest of this section, we outline the process of conducting inference on $U_{i}$ for an arbitrary $1 \leq i \leq n$. As a preprocessing step, we sort the $n$ observations in ascending order of their distance from the $i$-th observation, such that $0=\left|t_{i}-t_{0}^{(i)}\right| \leq\left|t_{i}-t_{1}^{(i)}\right| \leq \cdots \leq\left|t_{i}-t_{n-1}^{(i)}\right|$, correspondingly. Denote by $U_{0}^{(i)}, U_{1}^{(i)}, \cdots, U_{n-1}^{(i)}$ the corresponding $U$ variables, and by $y_{0}^{(i)}, y_{1}^{(i)}, \cdots, y_{n-1}^{(i)}$ the corresponding $y$ observations. Notice that $U_{0}^{(i)}=U_{i}$ holds trivially.

To predict $U_{i}$, we can utilize the marginal distribution $\mathcal{N}(0,1)$, and the conditional distributions $U_{i} \mid u_{i}-$ $u_{1}^{(i)} ; U_{i}\left|u_{i}-u_{1}^{(i)}, u_{i}-u_{2}^{(i)} ; \cdots ; U_{i}\right| u_{i}-u_{1}^{(i)}, u_{i}-u_{2}^{(i)}, \cdots, u_{i}-u_{n-1}^{(i)}$. This motivates us to consider the partial regression of $U_{i}$ on the pairwise differences of the $U$ variables:

$$
\begin{equation*}
\mathcal{N}\left(\sum_{k=1}^{n-1} \frac{\lambda_{k}}{k+1}\left[k u_{i}-\sum_{j=1}^{k} u_{j}^{(i)}\right],\left(1-\sum_{k=1}^{n-1} \frac{\lambda_{k} k}{k+1}\right)^{2}+\sum_{j=1}^{n-1}\left(\sum_{k=j}^{n-1} \frac{\lambda_{k}}{k+1}\right)^{2}\right) \tag{4.17}
\end{equation*}
$$

where the mixing proportions $\lambda_{1}, \lambda_{2}, \cdots, \lambda_{n-1}$ are functions of $B_{i j}$ satisfying $\lambda_{1}, \lambda_{2}, \cdots, \lambda_{n-1} \geq 0$ and $\sum_{k=1}^{n-1} \lambda_{k} \leq 1$. The variance expression in Equation 4.17) is obtained by routine algebraic operations:

$$
\begin{align*}
& \operatorname{Var}\left\{U_{i}-\sum_{k=1}^{n-1} \frac{\lambda_{k}}{k+1}\left[k U_{i}-\sum_{j=1}^{k} U_{j}^{(i)}\right]\right\} \\
= & \left(1-\sum_{k=1}^{n-1} \frac{\lambda_{k} k}{k+1}\right)^{2}+\operatorname{Var}\left\{\sum_{k=1}^{n-1}\left[\frac{\lambda_{k}}{k+1} \sum_{j=1}^{k} U_{j}^{(i)}\right]\right\} \tag{4.18}
\end{align*}
$$

To deal with the second term on the right-hand-side of the last equation, we notice that the summation signs can be exchanged in the following manner:

$$
\sum_{k=1}^{n-1}\left[\frac{\lambda_{k}}{k+1} \sum_{j=1}^{k} U_{j}^{(i)}\right]=\sum_{k=1}^{n-1} \sum_{j=1}^{k} \frac{\lambda_{k}}{k+1} U_{j}^{(i)}=\sum_{j=1}^{n-1} \sum_{k=1}^{n-1} \frac{\lambda_{k}}{k+1} U_{j}^{(i)}=\sum_{j=1}^{n-1}\left(\sum_{k=j}^{n-1} \frac{\lambda_{k}}{k+1}\right) U_{j}^{(i)}
$$

Therefore, we have

$$
\operatorname{Var}\left\{\sum_{k=1}^{n-1}\left[\frac{\lambda_{k}}{k+1} \sum_{j=1}^{k} U_{j}^{(i)}\right]\right\}=\operatorname{Var}\left\{\sum_{j=1}^{n-1}\left(\sum_{k=j}^{n-1} \frac{\lambda_{k}}{k+1}\right) U_{j}^{(i)}\right\}=\sum_{j=1}^{n-1}\left(\sum_{k=j}^{n-1} \frac{\lambda_{k}}{k+1}\right)^{2}
$$

where the last step follows from the independence of $U_{j}^{(i)}, j=1, \cdots, n$. Substituting this result into equation (4.18) yields the variance expression shown in equation 4.17).

To predict $U_{i}$, we take the predictive random sets

$$
\mathcal{S}_{\left\{v_{j}^{(i)}: j=1, \ldots, k\right\}}=\left\{z:\left|z-\sum_{k=1}^{n-1} \frac{\lambda_{k}}{k+1} \sum_{j=1}^{k} v_{j}^{(i)}\right| \leq\left|Z-\sum_{k=1}^{n-1} \frac{\lambda_{k}}{k+1} \sum_{j=1}^{k} v_{j}^{(i)}\right|\right\}, Z \sim(4.17),
$$

where

$$
v_{j}^{(i)}:=u_{i}-u_{j}^{(i)}=y_{i}-y_{j}^{(i)}-\left(\vartheta_{0}\left(t_{i}\right)-\vartheta_{0}\left(t_{j}^{(i)}\right)\right) \in\left[y_{i}-y_{j}^{(i)}-B_{j}^{(i)}, y_{i}-y_{j}^{(i)}+B_{j}^{(i)}\right]
$$

with $B_{j}^{(i)}:=\left|t_{j}^{(i)}-t_{i}\right|, 1 \leq i, j \leq n$. For valid inference, we take the (conservative) predictive random sets

$$
\mathcal{S}=\bigcup_{\left\{v_{j}^{(i)} \in\left[y_{i}-y_{j}^{(i)} \pm B_{j}^{(i)}\right], j=1, \ldots, k\right\}} \mathcal{S}_{\left\{v_{j}^{(i)}: j=1, \ldots, k\right\}}
$$

This predictive random set is clearly marginally valid. Let

$$
\Delta_{\lambda}=\sqrt{\left(1-\sum_{k=1}^{n-1} \frac{\lambda_{k} k}{k+1}\right)^{2}+\sum_{j=1}^{n-1}\left(\sum_{k=j}^{n-1} \frac{\lambda_{k}}{k+1}\right)^{2}}
$$

the standard deviation of the partial regression. The plausibility region

$$
\bigcup_{v_{j}^{(i)} \in\left[y_{i}-y_{j}^{(i)} \pm B_{j}^{(i)}\right]}\left\{\sum_{k=1}^{n-1} \frac{\lambda_{k}}{k+1} \sum_{j=1}^{k} v_{j}^{(i)} \pm z_{1-\alpha / 2} \Delta_{\lambda}\right\}
$$

or equivalently,

$$
\begin{equation*}
\left[\sum_{k=1}^{n-1} \frac{\lambda_{k}}{k+1} \sum_{j=1}^{k}\left[y_{i}-y_{j}^{(i)}-B_{j}^{(i)}\right]-z_{1-\alpha / 2} \Delta_{\lambda}, \sum_{k=1}^{n-1} \frac{\lambda_{k}}{k+1} \sum_{j=1}^{k}\left[y_{i}-y_{j}^{(i)}+B_{j}^{(i)}\right]+z_{1-\alpha / 2} \Delta_{\lambda}\right] \tag{4.19}
\end{equation*}
$$

covers $u_{i}^{\star}$ with probability at least $100(1-\alpha) \%$. The width of the plausibility region is given by

$$
2 \sum_{k=1}^{n-1} \frac{\lambda_{k}}{k+1} \sum_{j=1}^{k} B_{j}^{(i)}+2 z_{1-\alpha / 2} \Delta_{\lambda}
$$

The optimal set of mixing proportions $\lambda_{1}, \lambda_{2}, \cdots, \lambda_{n-1}$ are the solutions to the constrained optimization problem

$$
\begin{align*}
\min _{\lambda_{1}, \lambda_{2}, \cdots, \lambda_{n-1}} & \sum_{k=1}^{n-1} \frac{\lambda_{k}}{k+1} \sum_{j=1}^{k} B_{j}^{(i)}+z_{1-\alpha / 2} \Delta_{\lambda}  \tag{4.20}\\
\text { subject to } & \lambda_{1}, \lambda_{2}, \cdots, \lambda_{n-1} \geq 0 ; \sum_{k=1}^{n-1} \lambda_{k} \leq 1
\end{align*}
$$

We solve problem 4.20) numerically and obtain the optimal mixing proportions. Figures 6, 7, and 9 show numerical results for various configurations of $n, M$, and $\gamma$ : the blue crosses denote observations from the underlying function $\vartheta_{0}(t)$ (dashed red line); and the plausibility intervals are displayed alongside.

### 4.3 Asymptotic studies of convergence rate of interval lengths

To investigate the asymptotic behavior of our algorithm in terms of both statistical efficiency and computational time, we conducted a sequence of experiments using the simple assertion $A(t)=\left\{\vartheta_{0}:[0,1] \rightarrow\right.$ $\left.\mathbb{R} \mid \vartheta_{0}(t)=0\right\} \subseteq \Theta_{1,1 / 2}$ with equally spaced observations.

The results of the experiments are summarized in Table 1 and plotted in Figure 8. We observe that the median widths of the plausibility intervals obtained using our algorithm and the algorithm's runtime exhibit clearly anticipated trends as $n$ increases. In order to model these two relationships, we apply the logarithm transformation and fit two linear models on the log-scale.

For median width, the fitted regression equation is

$$
\log (\text { median width })=1.378-0.267 \log (n)
$$



Figure 6: Visualization of the plausibility intervals constructed using the mixture distribution for the assertion $A(t)=\left\{\vartheta_{0}:[0,1] \rightarrow \mathbb{R} \mid \vartheta_{0}(t)=\sqrt{t}\right\} \subseteq \Theta_{1,1 / 2}$ under various settings.


Figure 7: Visualization of the plausibility intervals constructed using the mixture distribution for the assertion $A(t)=\left\{\vartheta_{0}:[0,1] \rightarrow \mathbb{R} \mid \vartheta_{0}(t)=0\right\} \subseteq \Theta_{1,1 / 2}$ under various settings.
with $R^{2}=99.97 \%$. We empirically conclude that the widths of the plausibility intervals decrease at a rate of $\mathcal{O}\left(n^{-0.267}\right)$. It is interesting to see that this simulation-based study shows that our empirical asymptotic convergence rate of plausibility intervals for the case of $\gamma=\frac{1}{2}$ is slightly better than or at least close to that
in 1.3), which in this case is $O\left(n^{-0.250}\right)$. Although one of these two convergence rates is on point estimation and the other on interval length, such a comparison is arguably meaningful to some extent because IM plausibility intervals are valid in terms of frequency calibration.

For elapsed time, the fitted regression equation is

$$
\log (\text { median width })=-10.42+4.01 \log (n)
$$

with $R^{2}=98.67 \%$. We empirically conclude that our algorithm exhibits $\mathcal{O}\left(n^{4}\right)$ time complexity.

| $n$ | Median width | Elapsed time |
| ---: | :---: | ---: |
| 5 | 2.5941 | 0.04 |
| 10 | 2.1482 | 0.37 |
| 15 | 1.9283 | 1.26 |
| 20 | 1.7772 | 3.19 |
| 25 | 1.6695 | 7.41 |
| 30 | 1.5930 | 16.38 |
| 35 | 1.5307 | 32.40 |
| 40 | 1.4756 | 58.26 |
| 45 | 1.4290 | 105.70 |
| 50 | 1.3910 | 186.41 |
| 60 | 1.3273 | 624.11 |
| 70 | 1.2756 | 1276.15 |
| 80 | 1.2331 | 2014.44 |
| 90 | 1.1976 | 2441.45 |
| 100 | 1.1671 | 3169.96 |

Table 1: Simulation results using the assertion $A(t)=\left\{\vartheta_{0}:[0,1] \rightarrow \mathbb{R} \mid \vartheta_{0}(t)=0\right\} \subseteq \Theta_{1,1 / 2}$ with equally-spaced observations.

## 5 Discussion

We have developed a partial conditioning method to extend IMs for producing valid and efficient inference about many-normal-means when the means are subject to Hölder constraints. The problem was motivated by the challenging setting in nonparametric regression, where no known methods are available to produce valid confidence intervals, even asymptotically. Thus, we make no attempt to compare the proposed method to existing methods, as our focus here is to extend conditional IMs to tackle many-normal-means with Hölder constraints (including the Lipschitz condition as a special case).

Nevertheless, the simple simulation-based study in Section 4 shows that our empirical asymptotic convergence rate of plausibility intervals for the case of $\gamma=\frac{1}{2}$ is about $O\left(n^{-0.267}\right)$. This is slightly better than or at least close to that in 1.3 , which in this case is $O\left(n^{-0.250}\right)$. As augued in Section 4 , such a comparison is arguably meaningful. Perhaps, the implication is of extreme importance to tackling the challenging problem of constructing confidence interval in nonparametric regression, which has been perceived as unsovable with existing approaches.

In addition, we expect the proposed method to inspire novel applications and theoretical developments, such as in nonparametric regression (c.f. Lepskii, 1991; Wang and Shen, 2013, and references therein). In the case when point estimation is of interest, which is often the starting-point for frequentist methods, our


Figure 8: Visualization of the simulation results in Table 1.


Figure 9: Visualization of the plausibility intervals constructed using the mixture distribution for the assertion $A(t)=\left\{\vartheta_{0}:[0,1] \rightarrow \mathbb{R} \mid \vartheta_{0}(t)=0\right\} \subseteq \Theta_{1,1 / 2}$ for various number of observations.
results 4.19) for the general $n$ case can be used to provide an alternative local shrinkage or smoothing scheme, in the same manner as described in Section 3.3.

Furthermore, while we have focused on developing the method for many-normal-means in this work, it can be extended to handle many-binomial-means and many-Poisson-means with Hölder constraints. Nevertheless,
a more general formulation of partial conditioning and its mathematical theory deserve future development.

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## Appendix: Proofs

## Proof of Proposition 2.1

Proof. For the predictive random set $\mathcal{S}=\left[-\left|U_{1}\right|,\left|U_{1}\right|\right], U_{1} \sim \mathcal{N}(0,1)$, we have

$$
\mathrm{P}_{\mathcal{S}}\left\{\mathcal{S} \not \supset u^{\star}\right\}=\mathrm{P}\left\{\left|U_{1}\right|<\left|u_{1}^{\star}\right|\right\}=2 \Phi\left(\left|u_{1}^{\star}\right|\right)-1
$$

where $\Phi(\cdot)$ denotes the standard Normal CDF. It follows that for each $\alpha \in(0,1)$,

$$
\mathrm{P}_{U_{1}}\left\{\mathrm{P}_{\mathcal{S}}\left\{\mathcal{S} \not \supset U_{1}\right\} \geq 1-\alpha\right\}=\mathrm{P}_{U_{1}}\left\{2 \Phi\left(\left|U_{1}\right|\right)-1 \geq 1-\alpha\right\}=\mathrm{P}_{U_{1}}\left\{\Phi\left(\left|U_{1}\right|\right) \geq 1-\alpha / 2\right\}=\alpha,
$$

which verifies that 2.5 holds. Thus, we have shown that the predictive random set $\mathcal{S}$ is valid, and Theorem 2.1 implies that the IM previously defined is valid as well.

## Proof of Proposition 4.1

Proof. The conditional distributions for $U_{1}$ can be derived as follows:

$$
\begin{cases}\left(U_{1}, V_{21}\right) \sim \mathcal{N}_{2}\left(\left[\begin{array}{l}
0 \\
0
\end{array}\right],\left[\begin{array}{cc}
1 & -1 \\
-1 & 2
\end{array}\right]\right) & \Rightarrow \quad U_{1} \left\lvert\, V_{21}=v_{21} \sim \mathcal{N}\left(-\frac{v_{21}}{2}, \frac{1}{2}\right)\right. \\
\left(U_{1}, V_{31}\right) \sim \mathcal{N}_{2}\left(\left[\begin{array}{l}
0 \\
0
\end{array}\right],\left[\begin{array}{cc}
1 & -1 \\
-1 & 2
\end{array}\right]\right) & \Rightarrow \quad U_{1} \left\lvert\, V_{31}=v_{31} \sim \mathcal{N}\left(-\frac{v_{31}}{2}, \frac{1}{2}\right)\right. \\
\left(U_{1}, V_{32}\right) \sim \mathcal{N}_{2}\left(\left[\begin{array}{l}
0 \\
0
\end{array}\right],\left[\begin{array}{cc}
1 & 0 \\
0 & 2
\end{array}\right]\right) & \Rightarrow \quad U_{1} \mid V_{32}=v_{32} \sim \mathcal{N}(0,1)\end{cases}
$$

The conditional distributions for $U_{2}$ can be derived as follows:

$$
\begin{cases}\left(U_{2}, V_{21}\right) \sim \mathcal{N}_{2}\left(\left[\begin{array}{l}
0 \\
0
\end{array}\right],\left[\begin{array}{ll}
1 & 1 \\
1 & 2
\end{array}\right]\right) & \Rightarrow \quad U_{2} \left\lvert\, V_{21}=v_{21} \sim \mathcal{N}\left(\frac{v_{21}}{2}, \frac{1}{2}\right)\right. \\
\left(U_{2}, V_{31}\right) \sim \mathcal{N}_{2}\left(\left[\begin{array}{l}
0 \\
0
\end{array}\right],\left[\begin{array}{ll}
1 & 0 \\
0 & 2
\end{array}\right]\right) & \Rightarrow \quad U_{2} \mid V_{31}=v_{31} \sim \mathcal{N}(0,1) \\
\left(U_{2}, V_{32}\right) \sim \mathcal{N}_{2}\left(\left[\begin{array}{l}
0 \\
0
\end{array}\right],\left[\begin{array}{cc}
1 & -1 \\
-1 & 2
\end{array}\right]\right) & \Rightarrow \quad U_{2} \left\lvert\, V_{32}=v_{32} \sim \mathcal{N}\left(-\frac{v_{32}}{2}, \frac{1}{2}\right) .\right.\end{cases}
$$

The conditional distributions for $U_{3}$ can be derived as follows:

$$
\begin{cases}\left(U_{3}, V_{21}\right) \sim \mathcal{N}_{2}\left(\left[\begin{array}{l}
0 \\
0
\end{array}\right],\left[\begin{array}{ll}
1 & 0 \\
0 & 2
\end{array}\right]\right) & \Rightarrow \quad U_{3} \mid V_{21}=v_{21} \sim \mathcal{N}(0,1) \\
\left(U_{3}, V_{31}\right) \sim \mathcal{N}_{2}\left(\left[\begin{array}{l}
0 \\
0
\end{array}\right],\left[\begin{array}{ll}
1 & 1 \\
1 & 2
\end{array}\right]\right) & \Rightarrow \quad U_{3} \left\lvert\, V_{31}=v_{31} \sim \mathcal{N}\left(\frac{v_{31}}{2}, \frac{1}{2}\right)\right. \\
\left(U_{3}, V_{32}\right) \sim \mathcal{N}_{2}\left(\left[\begin{array}{l}
0 \\
0
\end{array}\right],\left[\begin{array}{ll}
1 & 1 \\
1 & 2
\end{array}\right]\right) & \Rightarrow \quad U_{3} \left\lvert\, V_{32}=v_{32} \sim \mathcal{N}\left(\frac{v_{32}}{2}, \frac{1}{2}\right)\right.\end{cases}
$$

The proof is complete by recalling that $v_{i j}=u_{i}-u_{j}$ for $1 \leq j<i \leq 3$.

## Proof of Corollary 4.1

Proof. We notice that the results in (4.4) imply the following:

$$
\begin{array}{r}
u_{1}-u_{2}=-v_{21} \in\left[Y_{1}-Y_{2}-B_{21}, Y_{1}-Y_{2}+B_{21}\right], \\
u_{1}-u_{3}=-v_{31} \in\left[Y_{1}-Y_{3}-B_{31}, Y_{1}-Y_{3}+B_{31}\right] ; \\
u_{2}-u_{1}=v_{21} \in\left[Y_{2}-Y_{1}-B_{21}, Y_{2}-Y_{1}+B_{21}\right], \\
u_{2}-u_{3}=-v_{32} \in\left[Y_{2}-Y_{3}-B_{32}, Y_{2}-Y_{3}+B_{32}\right] ; \\
u_{3}-u_{1}=v_{31} \in\left[Y_{3}-Y_{1}-B_{31}, Y_{3}-Y_{1}+B_{31}\right], \\
u_{3}-u_{2}=v_{32} \in\left[Y_{3}-Y_{2}-B_{32}, Y_{3}-Y_{2}+B_{32}\right] .
\end{array}
$$

Combining these results with Proposition 4.1 completes the proof of the corollary.

## Proof of Proposition 4.2

Proof. The conditional distributions for $U_{1}$ can be derived as follows:

$$
\begin{aligned}
& \left(\left.\left(U_{1}, V_{21}, V_{31}\right) \sim \mathcal{N}_{3}\left(\left[\begin{array}{l}
0 \\
0 \\
0
\end{array}\right],\left[\begin{array}{ccc}
1 & -1 & -1 \\
-1 & 2 & 1 \\
-1 & 1 & 2
\end{array}\right]\right) \quad \Rightarrow \quad U_{1} \right\rvert\, v_{21}, v_{31} \sim \mathcal{N}\left(-\frac{v_{21}+v_{31}}{3}, \frac{1}{3}\right) ;\right. \\
& \left\{\left.\left(U_{1}, V_{21}, V_{32}\right) \sim \mathcal{N}_{3}\left(\left[\begin{array}{l}
0 \\
0 \\
0
\end{array}\right],\left[\begin{array}{ccc}
1 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 2
\end{array}\right]\right) \quad \Rightarrow \quad U_{1} \right\rvert\, v_{21}, v_{32} \sim \mathcal{N}\left(-\frac{2 v_{21}+v_{32}}{3}, \frac{1}{3}\right) ;\right. \\
& \left.\left(U_{1}, V_{31}, V_{32}\right) \sim \mathcal{N}_{3}\left(\left[\begin{array}{l}
0 \\
0 \\
0
\end{array}\right],\left[\begin{array}{ccc}
1 & -1 & 0 \\
-1 & 2 & 1 \\
0 & 1 & 2
\end{array}\right]\right) \quad \Rightarrow \quad U_{1} \right\rvert\, v_{31}, v_{32} \sim \mathcal{N}\left(\frac{v_{32}-2 v_{31}}{3}, \frac{1}{3}\right) .
\end{aligned}
$$

The conditional distributions for $U_{2}$ can be derived as follows:

$$
\begin{cases}\left(U_{2}, V_{21}, V_{31}\right) \sim \mathcal{N}_{3}\left(\left[\begin{array}{l}
0 \\
0 \\
0
\end{array}\right],\left[\begin{array}{ccc}
1 & 1 & 0 \\
1 & 2 & 1 \\
0 & 1 & 2
\end{array}\right]\right) & \Rightarrow \quad U_{2} \mid v_{21}, v_{31} \sim \mathcal{N}\left(\frac{2 v_{21}-v_{31}}{3}, \frac{1}{3}\right) \\
\left(U_{2}, V_{21}, V_{32}\right) \sim \mathcal{N}_{3}\left(\left[\begin{array}{l}
0 \\
0 \\
0
\end{array}\right],\left[\begin{array}{ccc}
1 & 1 & -1 \\
1 & 2 & -1 \\
-1 & -1 & 2
\end{array}\right]\right) & \Rightarrow \quad U_{2} \mid v_{21}, v_{32} \sim \mathcal{N}\left(\frac{v_{21}-v_{32}}{3}, \frac{1}{3}\right) \\
\left(U_{2}, V_{31}, V_{32}\right) \sim \mathcal{N}_{3}\left(\left[\begin{array}{l}
0 \\
0 \\
0
\end{array}\right],\left[\begin{array}{ccc}
1 & 0 & -1 \\
0 & 2 & 1 \\
-1 & 1 & 2
\end{array}\right]\right) & \Rightarrow \quad U_{2} \mid v_{31}, v_{32} \sim \mathcal{N}\left(\frac{v_{31}-2 v_{32}}{3}, \frac{1}{3}\right)\end{cases}
$$

The conditional distributions for $U_{3}$ can be derived as follows:

$$
\begin{cases}\left(U_{3}, V_{21}, V_{31}\right) \sim \mathcal{N}_{3}\left(\left[\begin{array}{l}
0 \\
0 \\
0
\end{array}\right],\left[\begin{array}{lll}
1 & 0 & 1 \\
0 & 2 & 1 \\
1 & 1 & 2
\end{array}\right]\right) & \Rightarrow \quad U_{3} \mid v_{21}, v_{31} \sim \mathcal{N}\left(\frac{2 v_{31}-v_{21}}{3}, \frac{1}{3}\right) ; \\
\left(U_{3}, V_{21}, V_{32}\right) \sim \mathcal{N}_{3}\left(\left[\begin{array}{l}
0 \\
0 \\
0
\end{array}\right],\left[\begin{array}{ccc}
1 & 0 & 1 \\
0 & 2 & -1 \\
1 & -1 & 2
\end{array}\right]\right) & \Rightarrow \quad U_{3} \mid v_{21}, v_{32} \sim \mathcal{N}\left(\frac{v_{21}+2 v_{32}}{3}, \frac{1}{3}\right) ; \\
\left(U_{3}, V_{31}, V_{32}\right) \sim \mathcal{N}_{3}\left(\left[\begin{array}{l}
0 \\
0 \\
0
\end{array}\right],\left[\begin{array}{ccc}
1 & 1 & 1 \\
1 & 2 & 1 \\
1 & 1 & 2
\end{array}\right]\right) \quad & \Rightarrow \quad U_{3} \mid v_{31}, v_{32} \sim \mathcal{N}\left(\frac{v_{31}+v_{32}}{3}, \frac{1}{3}\right)\end{cases}
$$

It can be easily verified that the three conditional distributions in each group are actually equivalent, and that the results $4.11-4.13$ hold.

## Proof of Corollary 4.2

Proof. We notice that the results in (4.4) implies the following:

$$
\begin{aligned}
2 u_{1}-u_{2}-u_{3}=-\left(v_{21}+v_{31}\right) & \in\left[2 Y_{1}-Y_{2}-Y_{3}-\left(B_{21}+B_{31}\right), 2 Y_{1}-Y_{2}-Y_{3}+\left(B_{21}+B_{31}\right)\right] \\
2 u_{2}-u_{1}-u_{3}=v_{21}-v_{32} & \in\left[2 Y_{2}-Y_{1}-Y_{3}-\left(B_{21}+B_{32}\right), 2 Y_{2}-Y_{1}-Y_{3}+\left(B_{21}+B_{32}\right)\right] \\
2 u_{3}-u_{1}-u_{2}=v_{31}+v_{32} & \in\left[2 Y_{3}-Y_{1}-Y_{2}-\left(B_{31}+B_{32}\right), 2 Y_{1}-Y_{2}-Y_{3}+\left(B_{31}+B_{32}\right)\right]
\end{aligned}
$$

Combining these results with Proposition 4.2 completes the proof of the corollary.

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