



HAL
open science

Optimization Problems with Evidential Linear Objective

Tuan-Anh Vu, Sohaib Afifi, Eric Lefèvre, Frédéric Pichon

► **To cite this version:**

Tuan-Anh Vu, Sohaib Afifi, Eric Lefèvre, Frédéric Pichon. Optimization Problems with Evidential Linear Objective. International Journal of Approximate Reasoning, 2023, 161, 10.1016/j.ijar.2023.108987 . hal-04178376

HAL Id: hal-04178376

<https://hal.science/hal-04178376>

Submitted on 7 Aug 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial - NoDerivatives 4.0 International License

Optimization Problems with Evidential Linear Objective

Tuan-Anh Vu^a, Sohaib Afifi^a, Eric Lefèvre^a, Frédéric Pichon^a

^a*Univ. Artois, UR 3926, Laboratoire de Genie Informatique et d'Automatique de l'Artois (LGI2A), F-62400 Bethune, France*

Abstract

We investigate a general optimization problem with a linear objective in which the coefficients are uncertain and the uncertainty is represented by a belief function. We consider five common criteria to compare solutions in this setting: generalized Hurwicz, strong dominance, weak dominance, maximality and E-admissibility. We provide characterizations for the non-dominated solutions with respect to these criteria when the focal sets of the belief function are Cartesian products of compact sets. These characterizations correspond to established concepts in optimization. They make it possible to find non-dominated solutions by solving known variants of the deterministic version of the optimization problem or even, in some cases, simply by solving the deterministic version.

Keywords: Belief function, Robust optimization, Combinatorial optimization, Linear programming.

1. Introduction

Our paper focuses on a very general class of optimization problems where the objective function is linear (LOP). LOP covers a broad range of practical problems in diverse areas such as transportation, scheduling, network design, and profit planning, to name only a few important domains. In many realistic situations, one often encounters uncertainty on the coefficients of the objective function. Various approaches have been developed to model the uncertainty on coefficients, including robust optimization frameworks that represent uncertainty using discrete scenario sets [19, 10, 12] and intervals [16, 17, 10, 19, 5]. In the former representation, all possible realizations or scenarios of coefficients are explicitly listed to obtain the so-called scenario set. In the interval representation, each coefficient is constrained to lie within a given closed interval, and the scenario set is the Cartesian product of these intervals.

Email addresses: tanh.vu@univ-artois.fr (Tuan-Anh Vu),
sohaib.afifi@univ-artois.fr (Sohaib Afifi), eric.lefevre@univ-artois.fr (Eric Lefèvre), frederic.pichon@univ-artois.fr (Frédéric Pichon)

14 In this paper, we investigate the case where the uncertainty on the coeffi-
15 cients is *evidential*, *i.e.*, modelled by a belief function [21]. More specifically, we
16 assume that each so-called focal set of the considered belief function is a Carte-
17 sian product of compact sets, with each compact set describing possible values
18 of each coefficient. Such a belief function is a direct and natural generalization
19 of the interval representation, which arises when intervals are extended to com-
20 pact sets and probabilities are assigned to scenario sets. It can be illustrated as
21 follows: in a network with three cities A, B, and C, under good weather condi-
22 tions, it may take 20 to 30 minutes to travel from A to B, and 10 to 20 minutes
23 to travel from B to C; however under bad weather conditions, the travel times
24 from A to B (resp. B to C) takes 30 to 40 minutes (resp. 15 to 25 minutes) and
25 the forecast tells us that the probability of good weather (resp. bad weather) is
26 0.8 (resp. 0.2).

27 In the presence of evidential uncertainty on coefficients, the notion of best,
28 *i.e.*, optimal, solutions becomes ill-defined. In our preliminary work¹ [25], which
29 considered the shortest path problem (SPP) where each path has an evidential
30 weight, we drew inspiration from [10] and utilized decision theory under eviden-
31 tial uncertainty [7], to define the best paths as those that are non-dominated
32 with respect to some preference relation over paths built on the notions of their
33 lower and upper expected weights. Specifically, we studied the cases of the
34 preference relations obtained from three common criteria for decision-making,
35 namely generalized Hurwicz, strong dominance, and weak dominance.

36 Besides [25], optimization problems under evidential uncertainty were ex-
37 plored recently in [15, 22, 12]. The authors of [15, 22] considered various vari-
38 ants of the vehicle routing problem with different uncertainty factors. In the
39 resulting optimization problems, solutions had evidential costs and were com-
40 pared according to their upper expected costs, *i.e.*, using a particular case of the
41 generalized Hurwicz criterion. Guillaume *et al.* [12] considered the LOP prob-
42 lem with evidential coefficients, where each focal set of the belief function on
43 the coefficients can be any discrete scenario set. They defined best solutions as
44 the non-dominated ones according to the generalized Hurwicz criterion and they
45 provided complexity results regarding the problem of finding such solutions.

46 In this paper, we expand upon the work [25] by investigating a much broader
47 class of problems, *i.e.*, LOP, and by incorporating two additional well-known cri-
48 teria from the literature [2]: maximality and E-admissibility. More specifically,
49 this paper's primary contributions are summarized as follows:

- 50 1. We propose models for LOPs in which the coefficients in the objective are
51 subject to evidential uncertainty. Here, each feasible solution is regarded
52 as an act, which is a fundamental concept in decision theory. These models
53 are based on five common criteria from the literature for comparing acts,
54 namely generalized Hurwicz, strong dominance, weak dominance, maxi-
55 mality, and E-admissibility. A key feature of these models is that they

¹This paper is an extended and revised version of [25].

56 make use of the expressive nature of the belief function framework as they
 57 allow for incomparability of some solutions due to a lack of information.

58 2. We provide a characterization for the non-dominated solutions of each
 59 criterion, given our assumption about the focal sets. These characteriza-
 60 tions correspond to established concepts of optimization. This makes it
 61 possible to find non-dominated solutions by solving known variants of the
 62 deterministic version of the LOP or even, in some cases (e.g., the case of
 63 the generalized Hurwicz criterion), simply by solving its deterministic ver-
 64 sion. For instance, we can use SPP-related algorithms to efficiently find
 65 non-dominated solutions for the five criteria in the case of the SPP. In our
 66 opinion, this is the main advantage of our works compared to [15, 22, 12],
 67 where finding non-dominated solutions with respect to the Hurwicz crite-
 68 rion was much harder in general than solving the deterministic version.

69 We note that the idea of using decision theory under uncertainty, and specif-
 70 ically maximality and a special case of the generalized Hurwicz criterion, to
 71 formalize optimization problems under (severe) uncertainty was first proposed
 72 in [20], where the very general theory of coherent lower previsions is used as
 73 the uncertainty representation framework. However, the resulting models were
 74 studied in detail and connected to their deterministic counterparts only in a few
 75 special uncertainty cases, such as the case of intervals (vacuous previsions); the
 76 case of the evidential representation of uncertainty was not investigated.

77 The rest of this paper is organized as follows. Sections 2 and 3 present nec-
 78 essary background material on the LOP and belief function theory, respectively.
 79 Sections 4 and 5 are devoted to the formalization and resolution of the LOP
 80 with evidential coefficients, respectively. The paper ends with a conclusion in
 81 Section 6.

82 2. Optimization problems with a linear objective (LOP)

83 Many real-world problems have variables that are either integers or a mixture
 84 of integers and real numbers. In this paper, we mainly focus on the following
 85 optimization problem:

$$\begin{aligned} & \max / \min \quad c^T x \\ & \text{s.t. } x \in \mathcal{X} \subseteq \mathbb{Z}_{\geq 0}^{n_1} \times \mathbb{R}_{\geq 0}^{n_2} \text{ with } n_1 + n_2 = n. \end{aligned} \quad (\text{LOP})$$

86 where $\mathcal{X} \neq \emptyset$ is a set of feasible solutions and c is a vector of objective function
 87 coefficients $c_i \in \mathbb{R}$.

A very important class of Problem LOP is *linear mixed-integer programming*
 (MIP) problems:

$$\begin{aligned} & \max / \min \quad c^T x \\ & \text{s.t. } Mx \leq b, x \in \mathbb{Z}_{\geq 0}^{n_1} \times \mathbb{R}_{\geq 0}^{n_2}. \end{aligned} \quad (\text{MIP})$$

88 where M is a $m \times n$ matrix and b is a m -vector. We require that M and b have
 89 rational entries [28]. A practical instance of Problem MIP is the uncapacitated
 90 lot sizing problem (Example 1).

91 **Example 1** (Uncapacitated lot sizing). *The problem is to decide on a produc-*
 92 *tion plan for an n -period horizon for a single product. The parameters of the*
 93 *problem are:*

- 94 • f_t , which is the fixed cost of producing in period t ;
- 95 • p_t , which is the production cost in period t ;
- 96 • h_t , which is the unit storage cost in period t ;
- 97 • d_t , which is the demand in period t .

The problem can be modelled by the following optimization problem:

$$\begin{aligned} \min \quad & \sum_{t=1}^n p_t x_t + \sum_{t=1}^n h_t s_t + \sum_{t=1}^n f_t y_t \\ & s_{t-1} + x_t = d_t + s_t \quad (t = 1, 2, \dots, n) \\ & x_t \leq M y_t \quad (t = 1, 2, \dots, n) \\ & s_0 = 0, s_t, x_t \geq 0, y_t \in \{0, 1\} \quad (t = 1, 2, \dots, n) \end{aligned} \quad (\text{ULS})$$

98 where the decision variables are:

- 99 • x_t , which is the amount produced in period t ;
- 100 • s_t , which is the stock at the end of period t ;
- 101 • $y_t = 1$ if production occurs in t and $y_t = 0$ otherwise;

102 and where M is a big constant value.

Problem LOP is referred to as a 0-1 combinatorial optimization problem (01COP) when $\mathcal{X} \subseteq \{0, 1\}^n$:

$$\begin{aligned} \max / \min \quad & c^T x \\ \text{s.t.} \quad & x \in \mathcal{X} \subseteq \{0, 1\}^n. \end{aligned} \quad (\text{01COP})$$

103 This class includes many important problems. Below, we provide two of the
 104 most notable examples.

105 **Example 2** (The shortest path problem (SPP)). *Let $G = (V, A)$ be a directed*
 106 *graph with set of vertices V , set of arcs A and weight $c_{ij} \geq 0$ for each arc (i, j)*
 107 *in A . Let s and t be two vertices in V called the source and the destination,*
 108 *respectively.*

Finding a s - t shortest path, i.e., a s - t path of lowest weight, can be modelled as the following optimization problem:

$$\begin{aligned}
& \min \sum_{(i,j) \in A} c_{ij} x_{ij} \\
& \sum_{(s,i) \in A} x_{si} - \sum_{(j,s) \in A} x_{js} = 1 \\
& \sum_{(t,i) \in A} x_{ti} - \sum_{(j,t) \in A} x_{jt} = -1 \quad (\text{SPP}) \\
& \sum_{(k,i) \in A} x_{ki} - \sum_{(j,k) \in A} x_{jk} = 0, \quad \forall k \in V \setminus \{s, t\} \\
& x_{ij} \in \{0, 1\}, \quad \forall (i, j) \in A
\end{aligned}$$

109 where each s - t path is identified with a set $x = \{x_{ij} | (i, j) \in A\}$ of which element
110 $x_{ij} = 1$ if arc (i, j) is in the path and $x_{ij} = 0$ otherwise.

Example 3 (The 0-1 knapsack problem (01KP)). Suppose a company has a budget of W and needs to choose which items to manufacture from a set of n possible items, each with a production cost of w_i and fixed profit of p_i (all values are numbers in unit €). The 01KP involves selecting a subset of items to manufacture that maximizes the total profit while keeping the total production costs below W . The 01KP can be formulated as

$$\begin{aligned}
& \max \sum_{i=1}^n p_i x_i \\
& \text{s.t.} \quad \sum_{i=1}^n w_i x_i \leq W \\
& x_i \in \{0, 1\} \quad (i = 1, 2, \dots, n).
\end{aligned} \quad (\text{01KP})$$

111 The sets of feasible solution in Examples 2 and 3 are described by linear
112 constraints. However, it should be noted that Problem 01COP is not limited to
113 problems with linear constraints as \mathcal{X} can be any set.

When \mathcal{X} is a convex subset of $\mathbb{R}_{\geq 0}^n$, Problem LOP becomes a convex optimization problem (CV):

$$\begin{aligned}
& \max / \min \quad c^T x \\
& \text{s.t.} \quad x \in \mathcal{X} \subseteq \mathbb{R}_{\geq 0}^n \text{ is convex.}
\end{aligned} \quad (\text{CV})$$

114 This class includes linear programming as a particular case.

115 3. Belief function theory

116 Let Ω be the set, called frame of discernment, of all possible values of a
117 variable of interest ω . In belief function theory [21], adapting the presentation

118 of [27], partial knowledge about the true (unknown) value of ω , when Ω is
 119 a closed subset of \mathbb{R}^n as will be the case in this paper, is represented by a
 120 mapping $m : \mathcal{C} \mapsto [0, 1]$ called mass function, where \mathcal{C} is assumed here to be a
 121 finite collection of closed subsets of Ω , such that $\sum_{A \in \mathcal{C}} m(A) = 1$ and $m(\emptyset) = 0$.
 122 Mass $m(A)$ quantifies the amount of belief allocated to the fact of knowing only
 123 that $\omega \in A$. A subset $A \subseteq \Omega$ is called a focal set of m if $m(A) > 0$. The set of
 124 all focal sets of m is denoted by \mathcal{F} .

The mass function m induces a belief function Bel and a plausibility function Pl defined on $\mathcal{B}(\Omega)$ the Borel subsets of Ω :

$$Bel(A) = \sum_{B \in \mathcal{F}: B \subseteq A} m(B), \quad Pl(A) = \sum_{B \in \mathcal{F}: B \cap A \neq \emptyset} m(B). \quad (1)$$

A probability measure P on $\mathcal{B}(\Omega)$ is compatible with m if $Bel(A) \leq P(A) \forall A \in \mathcal{B}(\Omega)$. We denote by $\mathcal{P}(m)$ the set of all probability measures that are compatible with m . The *upper expected value* $\overline{E}_m(h)$ and *lower expected value* $\underline{E}_m(h)$ of a bounded, measurable function $h : \Omega \rightarrow \mathbb{R}$, relative to m , are defined as

$$\overline{E}_m(h) := \sup_{P \in \mathcal{P}(m)} E_P(h), \quad \underline{E}_m(h) := \inf_{P \in \mathcal{P}(m)} E_P(h). \quad (2)$$

A well-known result [27, Section 2.4] states that the upper and lower expected values of h can be computed as:

$$\overline{E}_m(h) = \sum_{A \in \mathcal{F}} m(A) \sup_{\omega_i \in A} h(\omega_i), \quad (3)$$

$$\underline{E}_m(h) = \sum_{A \in \mathcal{F}} m(A) \inf_{\omega_i \in A} h(\omega_i). \quad (4)$$

125 When mass function m is clear from the context, $\overline{E}_m(h)$ and $\underline{E}_m(h)$ may be
 126 simply written $\overline{E}(h)$ and $\underline{E}(h)$, respectively.

127 Assume Ω represents the state of nature and its true value is known in the
 128 form of some mass function m . Assume further that a decision maker (DM)
 129 needs to choose an act (decision) f from a set \mathcal{Q} . The outcome of each act can
 130 vary based on the prevailing state of nature. Denoting by \mathcal{O} the set of possible
 131 outcomes, each act can thus be formalized as a mapping $f : \Omega \rightarrow \mathcal{O}$.

132 Depending on the context, outcomes induce either utilities or costs. Utilities
 133 (resp. costs) of outcomes can be quantified by an utility function $u : \mathcal{O} \rightarrow \mathbb{R}$
 134 (resp. cost function $l : \mathcal{O} \rightarrow \mathbb{R}$). We assume that for any f , $u \circ f$ (resp. $l \circ f$)
 135 is a bounded real-valued map. In the following, to keep the discussion concise,
 136 we concentrate on presenting the treatment when the outcomes are associated
 137 with an utility function since a cost minimization can be turned in a utility
 138 maximization by taking the negative. Moreover, to enhance comprehension, we
 139 will use a specific problem, the SPP, to illustrate the results of the cost function
 140 case in Section 5.

141 In this framework, the DM's preference over acts is denoted by \succeq , where
 142 $f \succeq g$ means that act f is preferred to act g . The preference relation is typically

143 assumed to satisfy the reflexivity property ($f \succeq f$ for any f) and the transitivity
 144 property (if $f \succeq g$ and $g \succeq k$, then $f \succeq k$ for any $f, g,$ and k), making it a
 145 preorder. Furthermore, if the relation is antisymmetric ($f = g$ for any f and g
 146 such that $f \succeq g$ and $g \succeq f$), then it becomes an order. Relation \succeq is complete if
 147 for any two acts f and g , $f \succeq g$ or $g \succeq f$, otherwise, it is partial. Additionally,
 148 f is strictly (resp. equally) preferred to g , which is denoted by $f \succ g$ (resp.
 149 $f \sim g$), if $f \succeq g$ but not $g \succeq f$ (resp. if $f \succeq g$ and $g \succeq f$).

Typically, the DM seeks solutions in the set Opt of non-dominated acts:

$$Opt = \{f \in \mathcal{Q} : \nexists g \text{ such that } g \succ f\}. \quad (5)$$

150 If relation \succeq is complete, finding one solution in Opt is enough since solutions in
 151 Opt are preferred equally between each other and strictly preferred to the rest
 152 $\mathcal{Q} \setminus Opt$. In this case, solutions in Opt are also called optimal acts. On the other
 153 hand, if relation \succeq is partial, the DM may need to identify all solutions in Opt .

154 Usually, the DM constructs his preference over acts based on some criterion.
 155 We denote by \succeq_{cr} his preference according to some criterion cr and by Opt_{cr}
 156 its associated set of non-dominated (or best) acts. In this paper, we consider
 157 five common criteria defined as follows for any two acts f and g [7]:

1. Generalized Hurwicz criterion: $f \succeq_{hu}^\alpha g$ if

$$\alpha \bar{E}_m(u \circ f) + (1 - \alpha) \underline{E}_m(u \circ f) \geq \alpha \bar{E}_m(u \circ g) + (1 - \alpha) \underline{E}_m(u \circ g) \quad (6)$$

158 for some fixed parameter $\alpha \in [0, 1]$, representing an optimism/pessimism
 159 degree, and where $\bar{E}_m(u \circ f)$ and $\underline{E}_m(u \circ f)$ denote, respectively, the
 160 upper and lower expected utilities of act f with respect to mass function
 161 m . Relation \succeq_{hu}^α is complete and we have $f \succ_{hu}^\alpha g$ if (6) is strict. The set
 162 of non-dominated acts with respect to \succeq_{hu}^α is denoted by Opt_{hu}^α .

2. Strong dominance criterion: $f \succeq_{str} g$ if

$$\underline{E}_m(u \circ f) \geq \bar{E}_m(u \circ g). \quad (7)$$

163 Relation \succeq_{str} is partial and we have $f \succ_{str} g$ if (7) is strict. The set of
 164 non-dominated acts with respect to \succeq_{str} is denoted by Opt_{str} .

3. Weak dominance criterion: $f \succeq_{weak} g$ if

$$\bar{E}_m(u \circ f) \geq \bar{E}_m(u \circ g) \text{ and } \underline{E}_m(u \circ f) \geq \underline{E}_m(u \circ g). \quad (8)$$

165 Relation \succeq_{weak} is partial and we have $f \succ_{weak} g$ if at least one inequality
 166 in (8) is strict. The set of non-dominated acts with respect to \succeq_{weak} is
 167 denoted by Opt_{weak} .

4. Maximality criterion: $f \succeq_{max} g$ if

$$\underline{E}_m(u \circ f - u \circ g) \geq 0 \iff \forall P \in \mathcal{P}(m), E_P(u \circ f) \geq E_P(u \circ g), \quad (9)$$

168 Relation \succeq_{max} is partial and we have $f \succ_{max} g$ if $\underline{E}_m(u \circ f - u \circ g) > 0$.
 169 The set of non-dominated acts with respect to \succeq_{max} is denoted by Opt_{max} .

170 5. E-admissibility criterion: Let Opt_{adm} be the set of non-dominated solu-
 171 tions with respect to E-admissibility criterion, then $f \in Opt_{adm}$ iff there
 172 exists $P \in \mathcal{P}(m)$ such that $E_P(u \circ f) \geq E_P(u \circ g)$ for any act g .

173 Note that $Opt_{adm} \subseteq Opt_{max}$ and $Opt_{weak} \subseteq Opt_{max} \subseteq Opt_{str}$ with usually
 174 strict inclusions (see [8]).

175 We can observe that E-admissibility differs from other decision criteria, as
 176 it directly defines a set of non-dominated acts (*choice set*), without the need
 177 for explicitly defining a preference relation. However, we can still construct a
 178 preference relation from the choice set (see [7]).

179 Given these criteria, a relevant question for the DM is which criterion should
 180 be chosen. The choice of the criterion depends on factors such as its proper-
 181 ties or its associated computational cost of determining non-dominated acts.
 182 For instance, when comparing strong dominance and maximality, the computa-
 183 tional cost associated with maximality is generally higher than that of strong
 184 dominance, but strong dominance is more conservative than maximality since
 185 $Opt_{max} \subseteq Opt_{str}$. However, dealing with this question is beyond the scope of our
 186 paper. We refer to the excellent review papers of Troffaes [23] and Denoeux [7]
 187 for comprehensive discussions of these criteria.

188 4. LOP with evidential coefficients: modelling

189 In this section, we formalize what we mean by best solutions of Problem LOP
 190 when coefficients in the objective function are evidential, *i.e.*, are known in the
 191 form of a mass function, and we also describe a particular assumption about
 192 the focal sets of this mass function.

193 Let us assume that the coefficients c_i , for all $i \in 1, \dots, n$, in the objective
 194 of Problem LOP are only partially known. More specifically, we consider the
 195 case where information about the coefficients is modelled by a mass function.
 196 Formally, let Ω_i be the frame of discernment for the variable c_i , *i.e.*, the set of
 197 possible values for the coefficients c_i and let $\Omega := \times_{i=1}^n \Omega_i$. Any $c \in \Omega$ will be
 198 called a scenario: it represents a possible assignment of values for all coefficients
 199 in the objective function. A mass function m on Ω , with set of focal sets denoted
 200 by $\mathcal{F} = \{F_1, \dots, F_K\}$, represents uncertainty about the coefficients.

201 **Example 4.** Consider the Problem SPP, let c^1 and c^2 be the two scenarios
 202 represented by Figures 1a and 1b, respectively. The mass function m such that
 203 $m(F_1) = 0.4$ and $m(F_2) = 0.6$, with $F_1 = \{c^1, c^2\}$ and $F_2 = \{c^1\}$, represents
 204 partial knowledge about arc weights.

205 As will be seen, making a particular assumption about the nature of the
 206 focal sets of m is useful. This assumption relies on the following definition.

207 **Definition 1.** Given a subset $A \subseteq \Omega$, we denote by $A^{\setminus i}$ its projection on Ω_i .
 208 We say that A is a rectangle iff it can be expressed as the Cartesian product of
 209 its projections, that is: $A = \times_{i=1}^n A^{\setminus i}$.

210 The assumption about the focal sets of m is the following:

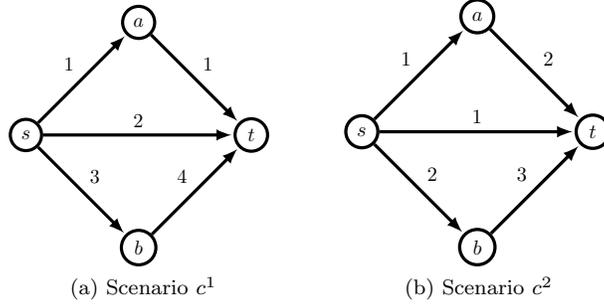


Figure 1: Two possible assignments of values, *i.e.*, two scenarios, for the arc weights.

211 **Assumption 1** (Rectangular with Compact projections (RC)). *Each focal set*
 212 *of m is a rectangle where each of its projection is a compact subset of \mathbb{R} .*

213 Let m be a mass function satisfying the RC assumption and let F_r be a
 214 focal set of m . The minimum and maximum values of its projection $F_r^{\downarrow i}$ will be
 215 denoted hereafter l_i^r and u_i^r , respectively.

216 While assuming focal sets to be rectangular may seem restrictive, it has
 217 been argued in [1] that such focal sets arise in many practical situations, such
 218 as in the example given in the Introduction and, for instance, it results from the
 219 combination of marginal mass functions m^i defined on Ω_i under the assumption
 220 of independence [6]. The compactness assumption is also rather mild as it allows
 221 $F^{\downarrow i}$ to be, *e.g.*, any closed (real) interval or any finite set of real numbers (and
 222 thus the practical situation of independent marginal mass functions m^i having
 223 closed intervals or finite sets as focal sets, fits the RC assumption). RC focal
 224 sets are further illustrated by Example 5 in a particular case where they are
 225 Cartesian products of intervals.

Example 5. *Consider the Problem SPP. Let m be the mass function such that*
 $m(F_1) = 0.5$ and $m(F_2) = 0.5$ with focal sets F_1 and F_2 , depicted in Figure 2,
such that

$$\begin{aligned} F_1 &= [l_{sa}, u_{sa}] \times [l_{sb}, u_{sb}] \times [l_{st}, u_{st}] \times [l_{at}, u_{at}] \times [l_{bt}, u_{bt}] \\ &= [2, 3] \times [1, 3] \times [4, 5] \times [1, 2] \times [2, 4]. \end{aligned}$$

and, similarly,

$$F_2 = [3, 4] \times [2, 4] \times [5, 6] \times [2, 3] \times [3, 5].$$

226 *Each focal set is a subset of Ω . For instance, the scenario $c = \{c_{sa}, c_{sb}, c_{st}, c_{at}, c_{bt}\}$*
 227 *with $c_{sa} = 2, c_{sb} = 3, c_{st} = 4, c_{at} = 1$ and $c_{bt} = 2$ is included in F_1 .*

228 When coefficients are evidential, *i.e.*, there is some uncertainty about them
 229 in the form of a mass function m on Ω , the preference over feasible solutions

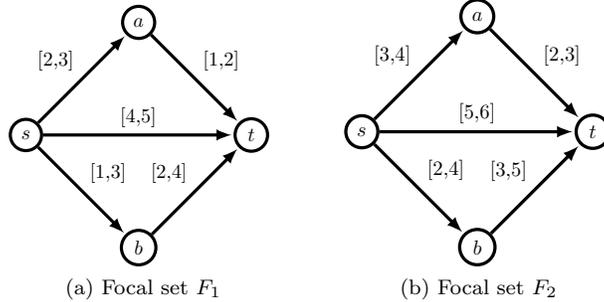


Figure 2: Two focal sets which are Cartesian products of intervals.

230 with respect to the (uncertain) coefficients can be established using the decision-
 231 making framework recalled in Section 3. Specifically, the set Ω of scenarios
 232 represents the possible states of nature. The set of feasible solutions \mathcal{X} represents
 233 the possible acts. By a slight abuse of notation, each solution x can be
 234 interpreted as a function $x : \Omega \rightarrow \mathcal{O}$ such that $x(c) = c^T x$, and the intended
 235 interpretation should be clear from the context.

236 If Problem LOP is a maximization problem (resp. minimization), the value
 237 $\sum_{i=1}^n c_i x_i$ of $x \in \mathcal{X}$ under scenario $c = \{c_i | i = 1, \dots, n\} \in \Omega$ represents the
 238 utility $u \circ x(c)$ (resp. cost $l \circ x(c)$) of solution (act) x for the scenario (state of
 239 nature) c , with u (resp. l) being the identity function. From here on, we will
 240 use the notation x to represent $u \circ x$ and $l \circ x$ for convenience.

241 The preference over feasible solutions, and the associated best solutions, can
 242 then be defined using any of the five criteria recalled in Section 3. In the next
 243 section, we provide the main results of this paper, which concern best solutions
 244 with respect to these five criteria and under assumption RC.

245 **Remark 1.** In [12], Problem LOP with evidential coefficients is also considered.
 246 The essential difference² between [12] and the present paper is the nature of the
 247 focal sets of the mass function m on the coefficients: in [12], they are assumed
 248 to be discrete scenario sets, whereas here we assume them to be RC. Hence, for
 249 instance, the mass function in Example 4 fits the setting of [12] but does not fit
 250 ours, whereas the mass function in Example 5 fits our setting but does not fit
 251 the one of [12].

252 5. LOP with evidential coefficients: solving

253 In this section, we provide methods for finding best (non-dominated) solu-
 254 tions, with respect to the five criteria presented in Section 3, of Problem LOP

²Another important difference with [12] is that only the generalized Hurwicz criterion is considered in this latter paper, whereas we consider four additional criteria.

255 when coefficients in the objective function are evidential, *i.e.*, are known in the
 256 form of some mass function m on Ω with set of focal sets $\mathcal{F} = \{F_1, \dots, F_K\}$.

257 For $i \in \{1, \dots, n\}$, let π_i be the map from Ω to \mathbb{R} such that $\pi_i(c) = c_i$, *i.e.*,
 258 $\pi_i(c)$ is nothing but coefficient c_i of scenario $c \in \Omega$. As will be seen, the upper
 259 $\overline{E}(\pi_i)$ and lower $\underline{E}(\pi_i)$ expected values of π_i with respect to m are central in
 260 our characterizations of the non-dominated solutions for the five criteria. These
 261 values can be computed easily under assumption RC:

Proposition 1. *Under assumption RC, we have*

$$\overline{E}(\pi_i) = \sum_{r=1}^K m(F_r) u_i^r, \quad (10)$$

$$\underline{E}(\pi_i) = \sum_{r=1}^K m(F_r) l_i^r. \quad (11)$$

Proof. We have

$$\overline{E}(\pi_i) = \sum_{r=1}^K m(F_r) \max_{c \in F_r} \pi_i(c) \quad (12)$$

$$= \sum_{r=1}^K m(F_r) \max_{c_i \in F_r^{\downarrow i}} c_i. \quad (13)$$

262 Similarly, we obtain $\underline{E}(\pi_i) = \sum_{r=1}^K m(F_r) \min_{c_i \in F_r^{\downarrow i}} c_i$. The proposition follows
 263 from the fact that under assumption RC, the projection $F_r^{\downarrow i}$ of focal set $F_r \in \mathcal{F}$,
 264 has maximum value u_i^r and minimum value l_i^r . \square

To simplify the exposition of our results, $\overline{E}(\pi_i)$ and $\underline{E}(\pi_i)$ under assumption
 RC will be denoted hereafter by \bar{u}_i and \bar{l}_i , respectively, *i.e.*, we have

$$\bar{u}_i := \sum_{r=1}^K m(F_r) u_i^r, \quad (14)$$

$$\bar{l}_i := \sum_{r=1}^K m(F_r) l_i^r. \quad (15)$$

Example 6 (Example 5 continued). *Consider the Problem SPP and the mass
 function in Example 5, with evidential weighted graph in Figure 2. We have for
 instance for arc s - a :*

$$\bar{u}_{sa} = m(F_1) \cdot u_{sa}^1 + m(F_2) \cdot u_{sa}^2 \quad (16)$$

$$= 0.5 \cdot 3 + 0.5 \cdot 4 = 3.5, \quad (17)$$

$$\bar{l}_{sa} = 0.5 \cdot 2 + 0.5 \cdot 3 = 2.5. \quad (18)$$

265 We treat in this section the five criteria in the order that they were introduced
 266 in Section 3. Note that, as is the case for Proposition 1 above, all the following
 267 Propositions require assumption RC to hold, and thus, for conciseness, we will
 268 no longer explicitly state this assumption in the Propositions.

269 *5.1. Generalized Hurwicz criterion*

270 We give a characterization for non-dominated solutions with respect to the
271 generalized Hurwicz criterion.

272 First, we can remark that this criterion relies on the notions of upper and
273 lower expected utilities of acts, acts being here feasible solutions. The upper
274 $\overline{E}(x)$ and lower $\underline{E}(x)$ expected utilities of a solution x can be computed easily
275 under assumption RC:

Proposition 2. *(Under assumption RC) We have*

$$\overline{E}(x) = \sum_{i=1}^n \bar{u}_i x_i, \quad (19)$$

$$\underline{E}(x) = \sum_{i=1}^n \bar{l}_i x_i. \quad (20)$$

Proof. By definition and since each focal set is compact, the upper and lower expected utilities of x are

$$\overline{E}(x) = \sum_{r=1}^K m(F_r) \max_{c^r \in F_r} \left(\sum_{i=1}^n c_i^r x_i \right), \quad (21)$$

$$\underline{E}(x) = \sum_{r=1}^K m(F_r) \min_{c^r \in F_r} \left(\sum_{i=1}^n c_i^r x_i \right). \quad (22)$$

276 The inner maximum and minimum in (21) and (22) are obtained when each
277 component c_i^r in c^r equals u_i^r and l_i^r , respectively. By regrouping terms we get
278 the desired result. \square

Since \succeq_{hu}^α is complete, it is sufficient to find one solution of the set Opt_{hu}^α , as explained in Section 3. To find one such solution, we need to solve the optimization problem,

$$\max / \min \quad \alpha \overline{E}_m(x) + (1 - \alpha) \underline{E}_m(x) \\ x \in \mathcal{X}, \quad (23)$$

279 for some specified value of $\alpha \in [0, 1]$.

280 In the case of general focal sets, solving Problem (23) is usually much more
281 challenging than solving its deterministic counterpart Problem LOP. For in-
282 stance, the deterministic Problem SPP can be solved efficiently in polynomial
283 time, but if $\alpha = 1$ the Problem (23) is weakly NP-hard already in the case
284 when mass function m has a single focal set containing two elements [30]. The
285 situation worsens if $\alpha = 0$, as the problem becomes strongly NP-hard and not
286 approximable [12, Theorem 1]. However, under assumption RC, the complexity
287 of Problem (23) remains unchanged compared to Problem LOP, since it is a
288 direct consequence of the following characterization.

289 **Proposition 3.** A solution x is in Opt_{hu}^α iff x is an optimal solution of Prob-
 290 lem LOP with coefficients $c_i = \alpha \bar{u}_i + (1 - \alpha) \bar{l}_i$.

Proof. Using Proposition 2, the Problem (23) becomes

$$\max / \min \sum_{i=1}^n (\alpha \bar{u}_i + (1 - \alpha) \bar{l}_i) x_i \quad (24)$$

$$x \in \mathcal{X} \quad (25)$$

291

□

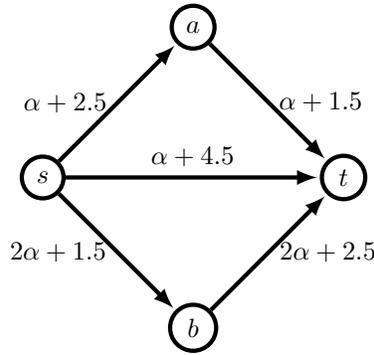


Figure 3: The parametric weighted graph associated with Opt_{hu}^α .

292 **Example 7** (Example 5 continued). To find a best path in Opt_{hu}^α for the ev-
 293 idential weighted graph in Figure 2, we need to solve the deterministic SPP in
 294 the graph showed in Figure 3, for some specified value of α (we have for instance
 295 for arc s - a , using Example 6: $\alpha \bar{u}_{sa} + (1 - \alpha) \bar{l}_{sa} = \alpha \cdot 3.5 + (1 - \alpha) \cdot 2.5 = \alpha + 2.5$).
 296 For example, if $\alpha = 0$ then the corresponding shortest paths are s - a - t and s - b - t ,
 297 while the shortest one is s - t , if $\alpha = 1$.

298 **Remark 2.** Thanks to Proposition 3, we can establish that best acts with respect
 299 to the generalized Hurwicz criterion for various α are solutions of a parametric
 300 LOP. Hence, methods from parametric optimization can help to solve a whole
 301 family of problems parameterized by α . For instance, the standard approach for
 302 solving parametric linear programming is the parametric simplex method [24,
 303 Chapter 7]. In the parametric SPP from Figure 3, as the DM varies his opti-
 304 mism/pessimism degree from 0 to 1, the break-point (point where a change in
 305 the parameter α causes a sudden change in the solutions) is 0.5. More precisely,
 306 for all $\alpha \in [0, 0.5]$ the best path is s - a - t , while for all $\alpha \in [0.5, 1]$ the optimal one
 307 is s - t . We refer to the work of Gusfield [13] for a comprehensive discussion of
 308 parametric combinatorial optimization problems.

309 5.2. Strong dominance criterion

310 In the same spirit as Proposition 3, we give now a characterization for non-
 311 dominated solutions with respect to the strong dominance criterion when Prob-
 312 lem LOP is a maximization problem.

Proposition 4. *A solution x is in Opt_{str} iff x is feasible with respect to the following constraints:*

$$x \in \mathcal{X} \quad (26)$$

$$\sum_{i=1}^n \bar{u}_i x_i \geq z \quad (27)$$

313 where z is the optimal value of Problem LOP in which $c_i = \bar{l}_i, i = 1, 2, \dots, n$.

Proof. By definition,

$$x \in Opt_{str} \Leftrightarrow \nexists y \in \mathcal{X} \text{ such that } \underline{E}(y) > \bar{E}(x) \quad (28)$$

$$\Leftrightarrow \forall y \in \mathcal{X} \text{ then } \underline{E}(y) \leq \bar{E}(x) \quad (29)$$

$$\Leftrightarrow \max_{y \in \mathcal{X}} \underline{E}(y) \leq \bar{E}(x) \quad (30)$$

314 As a special case of Proposition 3, when $\alpha = 0, z = \max_{y \in \mathcal{X}} \underline{E}(y)$ is obtained
 315 by solving Problem LOP with $c_i = \bar{l}_i$. From Proposition 2, we have $\bar{E}(x) =$
 316 $\sum_{i=1}^n \bar{u}_i x_i$, and thus the result follows. \square

317 We also have a similar result when Problem LOP is a minimization problem.

318

Proposition 5. *A solution x is in Opt_{str} iff x is a feasible with respect to the following constraints:*

$$x \in \mathcal{X} \quad (31)$$

$$\sum_{i=1}^n \bar{l}_i x_i \leq z \quad (32)$$

319 where z is the optimal value of Problem LOP in which $c_i = \bar{u}_i, (i = 1, 2, \dots, n)$.

320 Problem (26 -27) is called a *lower bound feasibility problem* since it is the
 321 feasibility problem with the additional constraint $\sum_{i=1}^n \bar{u}_i x_i \geq z$ (see [28, Section
 322 I.5.5]).

323 Since the relation \succeq_{str} is partial, it may be necessary to identify all solutions
 324 in the set Opt_{str} , meaning all feasible solutions of (26 -27). The complexity of
 325 this task depends on the structure of Problem LOP itself. In a specific case
 326 mentioned in our previous works [25], enumerating Opt_{str} for the SPP amounts
 327 to finding all paths in G with arc weights $c_{ij} = \bar{l}_{ij}$, whose weights are lower
 328 than or equal to the lowest weight of a s - t path in G with arc weights $c_{ij} = \bar{u}_{ij}$.
 329 Hence, we can use efficient algorithms such as the ones in [3, 4], where the

330 authors studied a problem of determining near optimal paths; for example, they
 331 wished to find all s - t paths in a directed graph whose weights do not exceed
 332 more than 5% the lowest weight, which is equivalent to finding all paths whose
 weights are less than or equal to a given threshold.

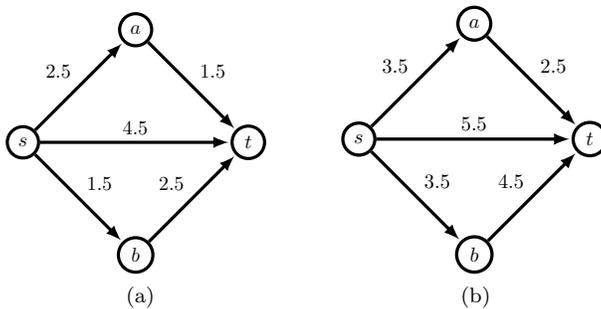


Figure 4: Two graphs associated with Opt_{str} .

333

334 **Example 8** (Example 5 continued). *To find the paths in Opt_{str} for the evi-*
 335 *dential weighted graph in Figure 2, according to Proposition 5 we first compute*
 336 *the lowest weight of a s - t path in the graph in Figure 4b, which is 5.5. The set*
 337 *Opt_{str} comprises then the s - t paths in the graph in Figure 4a that have weights*
 338 *no more than 5.5, which are the paths s - t , s - a - t , and s - b - t .*

339 **5.3. Weak dominance criterion**

There is a strong connection between the weak dominance criterion and bi-objective optimization. A bi-objective optimization problem can be expressed as

$$\max / \min f_1(x) \tag{33}$$

$$\max / \min f_2(x) \tag{34}$$

$$x \in \mathcal{X} \tag{35}$$

340 As the objectives (33-34) are typically conflicting, there is usually no solution
 341 x that maximizes (resp. minimizes) simultaneously $f_1(x)$ and $f_2(x)$. Instead,
 342 we seek to find all so-called efficient solutions of (33-35): a solution x is efficient
 343 if there is no feasible solution $y \in \mathcal{X}$ such that $f_1(y) \geq f_1(x)$ and $f_2(y) \geq f_2(x)$
 344 (resp. $f_1(y) \leq f_1(x)$ and $f_2(y) \leq f_2(x)$) where at least one of the inequalities is
 345 strict.

Example 9. *The bi-objective SPP is a particular bi-objective optimization problem. Assume that each arc (i, j) in G has two deterministic attributes c_{ij} and t_{ij} that describes, e.g., the cost and the travel time from i to j , respectively. The*

goal is to find all efficient solutions, i.e., s - t paths, of the following problem:

$$\min \sum_{(i,j) \in A} c_{ij} x_{ij} \quad (36)$$

$$\min \sum_{(i,j) \in A} t_{ij} x_{ij} \quad (37)$$

$$x \text{ is a } s\text{-}t \text{ path} \quad (38)$$

346 We now give a characterization for solutions in Opt_{weak} in terms of efficient
347 solutions of a bi-objective optimization problem.

Proposition 6. *A solution x is in Opt_{weak} iff x is a efficient solution of the problem:*

$$\begin{aligned} \max / \min \sum_{i=1}^n \bar{l}_i x_i \\ \max / \min \sum_{i=1}^n \bar{u}_i x_i \\ x \in \mathcal{X} \end{aligned} \quad (39)$$

348 *Proof.* It is easy to see that $x \in Opt_{weak}$ iff x is an efficient solutions with
349 objectives $f_1(x) := \bar{E}(x)$ and $f_2(x) := \underline{E}(x)$, which, using Proposition 2, leads
350 to Problem (39). \square

351 From Proposition 6, identifying solutions in Opt_{weak} is equivalent to finding
352 solutions for Problem (39). Considering again Problem SPP as an example, we
353 can remark that the bi-objective SPP has been extensively studied in the liter-
354 ature. Hence, we can apply off-the-shelf fast methods developed specifically for
355 the bi-objective SPP, such as [11], to find solutions in Opt_{weak} for Problem SPP.

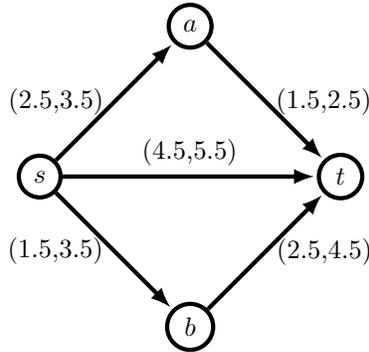


Figure 5: The graph associated with Opt_{weak} of which each arc (i, j) has two attributes $(\bar{l}_{ij}, \bar{u}_{ij})$.

356 **Example 10** (Example 5 continued). *Each path in Opt_{weak} is an efficient s - t*
 357 *path in the graph in Figure 5. Opt_{weak} consists of paths s - t and s - a - t (s - b - t is*
 358 *dominated by s - a - t).*

359 **Remark 3.** *It should be noted that any generalized Hurwicz optimal solution*
 360 *with $0 < \alpha < 1$ is also a solution of Opt_{weak} . As a result, determining such solu-*
 361 *tions for various α values can provide an inner approximation of Opt_{weak} . This*
 362 *stems from bi-objective optimization theory, where these solutions are known as*
 363 *supported efficient solutions: they are the solutions of $\min_{x \in \mathcal{X}} \lambda_1 f_1(x) + \lambda_2 f_2(x)$*
 364 *for some $\lambda_1, \lambda_2 > 0$.*

365 5.4. Maximality and E-admissibility criteria

366 Contrary to the other criteria, identifying characterizations for maximality
 367 and E-admissibility relies on the nature of Problem LOP. As will be seen, solu-
 368 tions in Opt_{max} and Opt_{adm} are closely related to the notion of *possibly optimal*
 369 *solution* in robust optimization, where a solution x is referred to as possibly
 370 optimal if it is an optimal solution to a problem \mathcal{P} for at least one scenario
 371 in the set of all possible scenarios Γ . This notion appears in various works in
 372 the realm of minimax regret optimization with interval data, such as in [16]
 373 for linear programming problems, in [29] for the minimum spanning tree prob-
 374 lem (where the authors called a possibly optimal spanning tree a weak tree),
 375 and in [17] for other combinatorial optimization problems. To emphasize the
 376 importance of the notion, we frame it in the following definition.

377 **Definition 2.** *A solution x is a possibly optimal solution of Problem LOP with*
 378 *respect to the set $\mathcal{C} := \times_{i=1}^n [\bar{l}_i, \bar{u}_i]$ if x is an optimal solution for at least one*
 379 *vector c in \mathcal{C} . The set of these possibly optimal solutions is denoted by $Opt_{pos}^{\mathcal{C}}$.*

380 5.4.1. The general case

381 In the general case, *i.e.*, the Problem LOP with evidential coefficients, we
 382 are not able to provide similar characterizations for solutions in Opt_{max} and
 383 Opt_{adm} as for previous criteria. Instead, we offer partial answers by providing
 384 a sufficient condition for solutions of Opt_{max} (Proposition 7) and a necessary
 385 condition for solutions of Opt_{adm} (Proposition 8).

386 **Proposition 7.** *If $x \in Opt_{pos}^{\mathcal{C}}$ then $x \in Opt_{max}$.*

Proof. If x is optimal under c^o where $c_i^o \in [\bar{l}_i, \bar{u}_i]$, for all $i \in \{1, \dots, n\}$ then,

$$\forall y \in \mathcal{X}, 0 \geq \sum_{i=1}^n c_i^o(y_i - x_i) = \sum_{i:y_i \geq x_i} c_i^o(y_i - x_i) + \sum_{i:y_i < x_i} c_i^o(y_i - x_i) \quad (40)$$

$$\Rightarrow 0 \geq \sum_{i:y_i \geq x_i} \bar{l}_i(y_i - x_i) + \sum_{i:y_i < x_i} \bar{u}_i(y_i - x_i) \quad (41)$$

On the other hand,

$$\underline{E}(y - x) = \sum_{r=1}^K m(F_r) \min_{c \in F_r} \sum_{i=1}^n c_i (y_i - x_i) \quad (42)$$

$$= \sum_{r=1}^K m(F_r) \left(\sum_{i: y_i \geq x_i} l_i^r (y_i - x_i) + \sum_{i: y_i < x_i} u_i^r (y_i - x_i) \right) \quad (43)$$

$$= \sum_{i: y_i \geq x_i} \bar{l}_i (y_i - x_i) + \sum_{i: y_i < x_i} \bar{u}_i (y_i - x_i) \quad (44)$$

387 From (41) and (44), we have that $\forall y \in \mathcal{X}$, $\underline{E}(y - x) \leq 0$, and thus $x \in Opt_{max}$.
 388 \square

389 **Proposition 8.** *If $x \in Opt_{adm}$ then $x \in Opt_{pos}^C$.*

Proof. Recall that an act x is a map from Ω to \mathbb{R} such that $x(c) = \sum_{i=1}^n x_i c_i$. Note that $x(c) = \sum_{i=1}^n x_i \pi_i(c)$. Let $P \in \mathcal{P}(m)$. By linearity of integration, we have

$$E_P(x) = \int_{\Omega} x(c) dP(c) = \sum_{i=1}^n x_i \int_{\Omega} \pi_i(c) dP(c) = \sum_{i=1}^n x_i E_P(\pi_i). \quad (45)$$

Since $P \in \mathcal{P}(m)$, we have $\underline{E}(\pi_i) \leq E_P(\pi_i) \leq \bar{E}(\pi_i)$, *i.e.*,

$$\bar{l}_i \leq E_P(\pi_i) \leq \bar{u}_i. \quad (46)$$

390 If $x \in Opt_{adm}$ then $\exists P \in \mathcal{P}(m)$ such that $E_P(x) \geq E_P(y) \forall y$. From Equa-
 391 tion (45), $\sum_{i=1}^n E_P(\pi_i) x_i \geq \sum_{i=1}^n E_P(\pi_i) y_i$, and thus x is optimal under c^o
 392 where $c_i^o := E_P(\pi_i)$. By Equation (46), we have $c_i^o \in [\bar{l}_i, \bar{u}_i]$. \square

393 A direct consequence of Propositions 7 and 8 is the following result.

394 **Corollary 1.** $Opt_{adm} \subseteq Opt_{pos}^C \subseteq Opt_{max}$.

395 In the important case of Problem CV the sets Opt_{pos}^C , Opt_{adm} , and Opt_{max}
 396 coincide:

397 **Proposition 9.** *For Problem CV, $Opt_{adm} = Opt_{pos}^C = Opt_{max}$.*

398 *Proof.* As the set of acts \mathcal{X} is convex, by the result in [26, Section 3.9.5],
 399 $Opt_{adm} = Opt_{max}$. The result follows from Corollary 1. \square

400 In the following two sections, we study these inclusions in Corollary 1 with
 401 respect to two other wide class of optimization problems besides Problem CV,
 402 namely Problems MIP and 01COP. As will be shown, the three sets also coincide
 403 for 01COP, whereas only the sets Opt_{pos}^C and Opt_{adm} coincide for MIP. There-
 404 fore, overall, our findings are that the inclusion between Opt_{pos}^C and Opt_{max} in
 405 Corollary 1 can be strict, whereas the inclusion between Opt_{adm} and Opt_{pos}^C is
 406 actually an equality for three important particular LOPs, *i.e.*, Problems CV,
 407 MIP and 01COP; it remains an open, non-trivial, question whether there exists
 408 an instance of Problem LOP for which the inclusion between Opt_{adm} and Opt_{pos}^C
 409 in Corollary 1 is strict.

410 5.4.2. Problem MIP

411 Let S be the feasible set of Problem MIP, consider the following optimization
412 problem:

$$\begin{aligned} & \max / \min c^T x \\ & \text{s.t. } x \in \text{conv}(S) \end{aligned} \tag{CMIP}$$

413 where $\text{conv}(S)$ is the convex hull of S .

414 A fundamental result in integer programming states that Problem CMIP
415 is a linear programming problem and we can solve Problem MIP by solving
416 Problem CMIP. To make the paper self-contained, we will state the result here
417 without providing a proof. Further information and a detailed proof can be
418 found in standard textbooks such as [28, Theorems 6.2 and 6.3].

419 **Proposition 10.** *Assume that Problem MIP is a maximization problem. For
420 any $c \in \mathbb{R}^n$, if x^* is an optimal solution of Problem MIP, then x^* is an optimal
421 solution of Problem CMIP.*

422 We can now provide a characterization of E-admissibility for Problem MIP
423 by proving that the converse of Proposition 8 also holds.

424 **Proposition 11.** *For Problem MIP, $x \in \text{Opt}_{adm}$ iff $x \in \text{Opt}_{pos}^{\mathcal{C}}$.*

425 *Proof.* If x is an optimal solution of Problem MIP under some $c^o \in \mathcal{C}$ then
426 by Proposition 10, x is also an optimal solution of Problem CMIP under c^o .
427 As Problem CMIP is convex, by Proposition 9, x is an E-admissible act of
428 Problem CMIP. Moreover, since $S \subseteq \text{conv}(S)$, then x is also an E-admissible
429 act of Problem MIP. \square

430 Corollary 1 states that if $x \in \text{Opt}_{pos}^{\mathcal{C}}$ then $x \in \text{Opt}_{max}$ for Problem LOP and
431 thus also for Problem MIP. The next example shows that for Problem MIP, we
432 can have $x \in \text{Opt}_{max}$ but $x \notin \text{Opt}_{pos}^{\mathcal{C}}$ (even when the mass function has a single
433 focal set), *i.e.*, the inclusion between $\text{Opt}_{pos}^{\mathcal{C}}$ and Opt_{max} in Corollary 1 can be
434 strict.

Example 11. *Consider the following optimization problem where each coefficient c_1, c_2, c_3 and c_4 in the objective is known to lie in an interval: $c_1 \in [1, 3]$,*

$c_2 \in [1, 3]$, $c_3 = 0$ and $c_4 = 0$.

$$\begin{aligned}
& \max c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 \\
& -2x_1 - x_2 \leq -6 \\
& x_1 + x_2 \leq 5 \\
& -x_1 - 2x_2 \leq -6 \\
& x_1 - 10x_3 \leq 2 \\
& -x_1 + 10x_3 \leq 6 \\
& x_2 - 10x_4 \leq 2 \\
& -x_2 + 10x_4 \leq 6 \\
& x_1, x_2 \in \{1, 2, 3, 4\} \\
& x_3, x_4 \in \{0, 1\}
\end{aligned}$$

It can easily be checked that the set of feasible solutions \mathcal{X} is $\mathcal{X} = \{x := (2, 2, 0, 0), y := (1, 4, 0, 1), z := (4, 1, 1, 0)\}$. An easy computation gives $\underline{E}(y - x) = -1$ and $\underline{E}(z - x) = -1$, thus $x \in \text{Opt}_{max}$. Assume $x \in \text{Opt}_{pos}$, which means that there exists $c \in [1, 3] \times [1, 3] \times \{0\} \times \{0\}$ such that $c^T x \geq c^T y$ and $c^T x \geq c^T z$. It implies that

$$2c_1 + 2c_2 \geq c_1 + 4c_2 \text{ and } 2c_1 + 2c_2 \geq 4c_1 + c_2 \quad (47)$$

$$\Leftrightarrow c_1 \geq 2c_2 \text{ and } c_2 \geq 2c_1. \quad (48)$$

435 Since (48) cannot be true, we get a contradiction and thus $x \notin \text{Opt}_{pos}$.

436 5.4.3. Problem 01COP

437 We give the characterizations for non-dominated solutions with respect to
438 the maximality and E-admissibility criteria for Problem 01COP. In this case
439 the set of feasible acts \mathcal{X} is not convex. Somewhat surprisingly, as we are going
440 to show, the two sets of non-dominated solutions still coincide.

For any $x \in \mathcal{X}$, let \bar{c}^{xr} be the scenario associated to x in focal set F_r , such that

$$\bar{c}_i^{xr} = u_i^r \text{ if } x_i = 1, \bar{c}_i^{xr} = l_i^r \text{ if } x_i = 0. \quad (49)$$

441 Lemma 1 is simple but it is the key element to uncover the characterization
442 of the maximality criterion.

Lemma 1. For any $x, y \in \mathcal{X}$,

$$\min_{c \in F_r} c^T y - c^T x = (\bar{c}^{xr})^T y - (\bar{c}^{xr})^T x.$$

Proof. For any $c \in F_r$,

$$c^T y - c^T x = \sum_{i=1}^n c_i(y_i - x_i) = \sum_{i:x_i=0}^n c_i(y_i - x_i) + \sum_{i:x_i=1}^n c_i(y_i - x_i) \quad (50)$$

$$\geq \sum_{i:x_i=0}^n l_i^r(y_i - x_i) + \sum_{i:x_i=1}^n u_i^r(y_i - x_i) \quad (51)$$

$$= (\bar{c}^{xr})^T y - (\bar{c}^{xr})^T x \quad (52)$$

443 where the inequality (51) holds because if $x_i = 0$ then $y_i - x_i \geq 0$ and if $x_i = 1$
444 then $y_i - x_i \leq 0$. \square

Denote by \bar{c}^x the set of coefficients in which $\bar{c}_i^x = \sum_{r=1}^K m(F_r) \bar{c}_i^{xr}$. Hence, we have:

$$\bar{c}_i^x = \bar{u}_i \text{ if } x_i = 1, \bar{c}_i^x = \bar{l}_i \text{ if } x_i = 0. \quad (53)$$

445 A characterization of solutions in Opt_{max} is given as follows.

446 **Proposition 12.** *For Problem 01COP, a solution $x \in Opt_{max}$ iff x is an*
447 *optimal solution under \bar{c}^x .*

Proof. By definition,

$$x \in Opt_{max} \Leftrightarrow \nexists y \text{ such that } y \succ_{max} x \Leftrightarrow \nexists y \text{ such that } \underline{E}(y - x) > 0 \quad (54)$$

$$\Leftrightarrow \forall y \in \mathcal{X}, \sum_{r=1}^K m(F_r) \min_{c \in F_r} (c^T y - c^T x) \leq 0 \quad (55)$$

$$\Leftrightarrow \forall y \in \mathcal{X}, \sum_{r=1}^K m(F_r) ((\bar{c}^{xr})^T y - (\bar{c}^{xr})^T x) \leq 0 \quad (\text{Lemma 1}) \quad (56)$$

$$\Leftrightarrow \forall y \in \mathcal{X}, \sum_{r=1}^K m(F_r) \sum_{i=1}^n \bar{c}_i^{xr} y_i \leq \sum_{r=1}^K m(F_r) \sum_{i=1}^n \bar{c}_i^{xr} x_i \quad (57)$$

$$\Leftrightarrow \forall y \in \mathcal{X}, \sum_{i=1}^n \bar{c}_i^x y_i \leq \sum_{i=1}^n \bar{c}_i^x x_i \quad (58)$$

448 Hence, $x \in Opt_{max}$ iff x is an optimal solution under \bar{c}^x . \square

449 Proposition 12 offers a method to check if a given feasible solution x be-
450 longs to Opt_{max} . To do so, one first calculates the optimal value, z_x , of Prob-
451 lem 01COP with $c_i = \bar{c}_i^x$ and then compares $\sum_{i=1}^n \bar{c}_i^x x_i$ with z_x . Moreover, the
452 following characterization provides a way to identify a solution in Opt_{max} by
453 solving Problem 01COP under some $c^o \in \mathcal{C}$.

454 **Proposition 13.** *For Problem 01COP, a solution $x \in Opt_{pos}^{\mathcal{C}}$ iff x is optimal*
455 *under \bar{c}^x .*

Proof. One direction is obvious. We only need to show the other direction. Assume that x is optimal under $c^o \in \mathcal{C}$. Then for any y , we have

$$\sum_{i=1}^n c_i^o x_i \geq \sum_{i=1}^n c_i^o y_i \quad (59)$$

$$\Leftrightarrow \sum_{i:x_i=1,y_i=0} c_i^o \geq \sum_{i:y_i=1,x_i=0} c_i^o \quad (60)$$

$$\Rightarrow \sum_{i:x_i=1,y_i=0} \bar{u}_i \geq \sum_{i:y_i=1,x_i=0} \bar{l}_i \quad (61)$$

$$\Leftrightarrow \sum_{i:x_i=1,y_i=0} \bar{u}_i + \sum_{i:x_i=y_i=1} \bar{u}_i \geq \sum_{i:y_i=1,x_i=0} \bar{l}_i + \sum_{i:x_i=y_i=1} \bar{u}_i \quad (62)$$

$$\Leftrightarrow \sum_{i=1}^n \bar{c}_i^x x_i \geq \sum_{i=1}^n \bar{c}_i^x y_i. \quad (63)$$

456 Hence, x is optimal under \bar{c}^x . □

457 **Remark 4.** *The proof of Proposition 13 is essentially the same as the proof*
 458 *of [29, Theorem 2.1] where the authors characterize weak trees.*

459 We are now in the position to provide a characterization for E-admissibility.
 460 We remark here that although the feasible acts \mathcal{X} of Problem 01COP may not
 461 be in the form $Mx \leq b$, the convex hull $\text{conv}(\mathcal{X})$ is still a bounded polyhedron
 462 as \mathcal{X} is a finite set. Hence, it still follows from Propositions 10 and 11 that
 463 x is E-admissible iff $x \in \text{Opt}_{\text{pos}}^{\mathcal{C}}$. However, the nature of Problem 01COP
 464 makes it possible to derive a proof for this fact, without relying on the powerful
 465 Proposition 10. We feel that it is useful to present a simpler proof here.

466 **Proposition 14.** *For Problem 01COP, a solution x is in Opt_{adm} iff x is an*
 467 *optimal solution under \bar{c}^x .*

468 *Proof.* If $x \in \text{Opt}_{\text{adm}}$ then $x \in \text{Opt}_{\text{max}}$, by Proposition 12 x is a optimal solution
 469 under \bar{c}^x . Assume that x is an optimal solution with $c_i = \bar{c}_i^x$. We construct an
 470 allocation map a of m as:

$$a(\bar{c}^{x^r}, F_r) = m(F_r), \forall r \in \{1, \dots, K\}. \quad (64)$$

We define a discrete probability measure P such that

$$P(\{c\}) = \sum_{\bar{c}^{x^r}=c} a(\bar{c}^{x^r}, F_r). \quad (65)$$

471 Thanks to [27, Theorem 1], we have $P \in \mathcal{P}(m)$. It is easy to see that $E_P(\pi_i) = \bar{u}_i$
 472 if $x_i = 1$ and $E_P(\pi_i) = \bar{l}_i$ if $x_i = 0$. Since x is optimal and by Equation (45),
 473 $E_P(x) \geq E_P(y)$ for any y . Therefore, x is E-admissible.

474 □

475 Consequently, we arrive to the main result.

476 **Proposition 15.** *If Problem 01COP is a maximization problem then the fol-*
 477 *lowing are equivalent:*

- 478 (i) $x \in Opt_{max}$.
- 479 (ii) $x \in Opt_{adm}$.
- 480 (iii) x is an optimal solution under \bar{c}^x .
- 481 (iv) $x \in Opt_{pos}^C$.

Let \underline{c}^x be the set of coefficients, defined as follows:

$$\underline{c}_i^x = \bar{l}_i \text{ if } x_i = 1, \underline{c}_i^x = \bar{u}_i \text{ if } x_i = 0. \quad (66)$$

482 Likewise, we have the next result.

483 **Proposition 16.** *If Problem 01COP is a minimization problem then the fol-*
 484 *lowing are equivalent:*

- 485 (i) $x \in Opt_{max}$.
- 486 (ii) $x \in Opt_{adm}$.
- 487 (iii) x is an optimal solution under \underline{c}^x .
- 488 (iv) $x \in Opt_{pos}^C$.

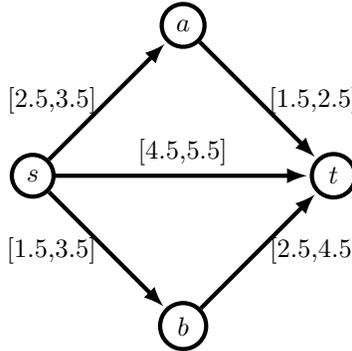


Figure 6: The graph associated with Opt_{max} and Opt_{adm} in which weights of arc (i, j) are in the interval $[\bar{l}_{ij}, \bar{u}_{ij}]$.

489 **Example 12** (Example 5 continued). *The graph in Figure 6 contains informa-*
 490 *tion about Opt_{max} (or, equivalently, Opt_{adm}). For instance, $s-a-t \in Opt_{max}$*
 491 *since it is optimal under the set of arc weights $c_{sa}=2.5$, $c_{at}=1.5$, $c_{st}=5.5$,*
 492 *$c_{sb}=3.5$, and $c_{bt}=4.5$. By setting the arc weights to $c_{sa}=3$, $c_{at}=2.5$, $c_{st}=5$,*
 493 *$c_{sb}=3$, and $c_{bt}=4$, the optimal path is $s-t$, which also belongs to Opt_{max} . The*
 494 *set Opt_{max} consists of $s-a-t$, $s-b-t$, and $s-t$.*

495 The characterization we provided is particularly valuable for E-admissibility.
 496 As noted in [2], verifying whether an act is E-admissible typically involves solv-
 497 ing a large linear programming problem. However, Propositions 15 and 16 imply
 498 that if Problem 01COP can be solved efficiently (e.g., Problem SPP), checking
 499 E-admissibility is also efficient.

500 **Remark 5.** Since \succeq_{max} is a partial relation, Opt_{max} may need to be enumer-
501 ated. For some problems, such as the SPP, the size of Opt_{weak} (and therefore,
502 the size of Opt_{max}) grows exponentially with $|V|$ [14], making the enumeration a
503 very time-consuming process. Preprocessing can be applied to speed up the pro-
504 cess by eliminating the elements x_i which are never in any solution of Opt_{max} .
505 We note that determining whether $x_i = 1$ is part of a possibly optimal solution
506 (i.e., solution in Opt_{max}) is NP-hard for many polynomially solvable problems
507 such as the SPP or the assignment problem [17]. Nonetheless, for an important
508 class of combinatorial optimization problems, i.e., the matroidal problem (which
509 includes the minimum spanning tree problem), Kasperski et al. [18] showed that
510 this determination can be done efficiently.

511 6. Conclusion

512 In this paper, we have considered a very general optimization problem with
513 a linear objective function (LOP). When coefficients of the objective are ev-
514 idential, the notion of optimal solution is ill-defined. Therefore, we propose
515 extensions of the notion of optimal solutions to this context, as the sets of non-
516 dominated solutions according to the generalized Hurwicz, strong dominance,
517 weak dominance, maximality and E-admissibility criteria. By considering the
518 particular case where focal sets are Cartesian products of compact sets, we are
519 able to characterize the non-dominated solutions in terms of various concepts in
520 optimization. This makes it possible to find non-dominated solutions by solv-
521 ing known variants of the deterministic version of the LOP or even, in some
522 cases, simply by solving the deterministic version. Specifically, non-dominated
523 acts with respect to generalized Hurwicz are solutions of the *deterministic* LOP.
524 Non-dominated acts with respect to generalized Hurwicz under unknown opti-
525 mism/pessimism degree are solutions of the *parametric* LOP. Non-dominated
526 acts with respect to strong dominance are solutions of a *lower-bound feasibil-*
527 *ity problem*. Non-dominated acts with respect to weak dominance correspond
528 exactly to the *efficient* solutions of the bi-objective LOP problem. Lastly, non-
529 dominated acts with respect to maximality and E-admissibility are linked to the
530 robust optimization framework via the concept of *possibly optimal solutions* of
531 the LOP.

532 Topics of future research include i) finding a characterization of the maxi-
533 mality criterion for linear mixed integer programming problems; ii) providing a
534 polynomial representation of all non-dominated solutions with respect to max-
535 imality and E-admissibility for combinatorial optimization problems or at least
536 for matroidal problems. Since these latter solutions are also possibly optimal,
537 one possible direction is to expand the works of [9], in which a compact repre-
538 sentation of possibly optimal solutions is given for the item selection problem
539 (a special case of matroidal problems).

540 **References**

- 541 [1] Aguirre, F., Destercke, S., Dubois, D., Sallak, M., Jacob, C.: Inclusion–
542 exclusion principle for belief functions. *Int. J. Approx. Reason.* **55**(8), 1708–
543 1727 (2014)
- 544 [2] Augustin, T., Coolen, F.P., De Cooman, G., Troffaes, M.C.: *Introduction*
545 *to imprecise probabilities*. John Wiley & Sons (2014)
- 546 [3] Byers, T.H., Waterman, M.S.: Determining all optimal and near-optimal
547 solutions when solving shortest path problems by dynamic programming.
548 *Oper. Res.* **32**(6), 1381–1384 (1984)
- 549 [4] Carlyle, W.M., Wood, R.K.: Near-shortest and k-shortest simple paths.
550 *Networks* **46**(2), 98–109 (2005)
- 551 [5] Chanas, S., Zieliński, P.: The computational complexity of the critical-
552 ity problems in a network with interval activity times. *Eur. J. Oper. Res.*
553 **136**(3), 541–550 (2002)
- 554 [6] Couso, I., Moral, S.: Independence concepts in evidence theory. *Int. J.*
555 *Approx. Reason.* **51**(7), 748–758 (2010)
- 556 [7] Denoeux, T.: Decision-making with belief functions: a review. *Int. J. Ap-*
557 *prox. Reason.* **109**, 87–110 (2019)
- 558 [8] Destercke, S.: A decision rule for imprecise probabilities based on pair-
559 wise comparison of expectation bounds. In: Borgelt, C., González-
560 Rodríguez, G., Trutschnig, W., Lubiano, M.A., Gil, M.Á., Grzegorzewski,
561 P., Hryniewicz, O. (eds.) *Combining Soft Computing and Statistical Meth-*
562 *ods in Data Analysis. Advances in Intelligent and Soft Computing*, vol. 77,
563 pp. 189–197. Springer (2010)
- 564 [9] Destercke, S., Guillaume, R.: Necessary and possibly optimal items in se-
565 lecting problems. In: Ciucci, D., Couso, I., Medina, J., Slezak, D., Petturiti,
566 D., Bouchon-Meunier, B., Yager, R.R. (eds.) *Information Processing and*
567 *Management of Uncertainty in Knowledge-Based Systems - 19th Interna-*
568 *tional Conference, IPMU 2022, Part I. Communications in Computer and*
569 *Information Science*, vol. 1601, pp. 494–503. Springer (2022)
- 570 [10] Dias, L.C., Clímaco, J.N.: Shortest path problems with partial information:
571 models and algorithms for detecting dominance. *Eur. J. Oper. Res.* **121**(1),
572 16–31 (2000)
- 573 [11] Duque, D., Lozano, L., Medaglia, A.L.: An exact method for the biobjec-
574 tive shortest path problem for large-scale road networks. *Eur. J. Oper. Res.*
575 **242**(3), 788–797 (2015)
- 576 [12] Guillaume, R., Kasperski, A., Zieliński, P.: Robust optimization with sce-
577 narios using random fuzzy sets. In: *Proc. of FUZZ-IEEE 2021*. pp. 1–6.
578 IEEE (2021)

- 579 [13] Gusfield, D.M.: Sensitivity analysis for combinatorial optimization. Tech.
580 rep., EECS Department, University of California, Berkeley (May 1980)
- 581 [14] Hansen, P.: Bicriterion path problems. In: Fandel, G., Gal, T. (eds.) Mul-
582 tiple criteria decision making theory and application. pp. 109–127. Springer
583 (1980)
- 584 [15] Helal, N., Pichon, F., Porumbel, D., Mercier, D., Lefèvre, E.: The capac-
585 itated vehicle routing problem with evidential demands. *Int. J. Approx.*
586 *Reason.* **95**, 124–151 (2018)
- 587 [16] Inuiguchi, M., Sakawa, M.: Minimax regret solution to linear programming
588 problems with an interval objective function. *Eur. J. Oper. Res.* **86**(3), 526–
589 536 (1995)
- 590 [17] Kasperski, A.: Discrete optimization with interval data. Springer (2008)
- 591 [18] Kasperski, A., Zieliński, P.: On combinatorial optimization problems on
592 matroids with uncertain weights. *Eur. J. Oper. Res.* **177**(2), 851–864 (2007)
- 593 [19] Kouvelis, P., Yu, G.: Robust discrete optimization and its applications,
594 Nonconvex Optimization and Its Applications, vol. 14. Springer US (1996)
- 595 [20] Quaeghebeur, E., Shariatmadar, K., De Cooman, G.: Constrained opti-
596 mization problems under uncertainty with coherent lower previsions. *Fuzzy*
597 *Sets Syst.* **206**, 74–88 (2012)
- 598 [21] Shafer, G.: A mathematical theory of evidence. Princeton university press
599 (1976)
- 600 [22] Tedjini, T., Afifi, S., Pichon, F., Lefèvre, E.: The vehicle routing problem
601 with time windows and evidential service and travel times: A recourse
602 model. In: Vejnárová, J., Wilson, N. (eds.) Proc. of ECSQARU 2021. pp.
603 381–395. Springer (2021)
- 604 [23] Troffaes, M.C.: Decision making under uncertainty using imprecise proba-
605 bilities. *Int. J. Approx. Reason.* **45**(1), 17–29 (2007)
- 606 [24] Vanderbei, R.J.: Linear programming. Springer (2020)
- 607 [25] Vu, T.A., Afifi, S., Lefèvre, É., Pichon, F.: On modelling and solving
608 the shortest path problem with evidential weights. In: Le Hégarat-Masclé,
609 S., Bloch, I., Aldea, E. (eds.) Belief Functions: Theory and Applications.
610 pp. 139–149. Springer International Publishing, Cham (2022)
- 611 [26] Walley, P.: Statistical reasoning with imprecise probabilities, vol. 42.
612 Springer (1991)
- 613 [27] Wasserman, L.A.: Belief functions and statistical inference. *Can. J. Stat.*
614 **18**(3), 183–196 (1990)

- 615 [28] Wolsey, L.A., Nemhauser, G.L.: Integer and combinatorial optimization,
616 vol. 55. John Wiley & Sons (1999)
- 617 [29] Yaman, H., Karaşan, O.E., Pınar, M.Ç.: The robust spanning tree problem
618 with interval data. *Oper. Res. Lett.* **29**(1), 31–40 (2001)
- 619 [30] Yu, G., Yang, J.: On the robust shortest path problem. *Comput. Oper.*
620 *Res.* **25**(6), 457–468 (1998)