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Pierre Andrieu[a], Sarah Cohen-Boulakia[a], Miguel Couceiro[c], Alain Denise[a,b], Adeline Pierrot[a]

[a] *Université Paris-Saclay, CNRS, Laboratoire Interdisciplinaire des Sciences du Numérique, Orsay, 91405, France*
[b] *Université Paris-Saclay, CEA, CNRS, Institute for Integrative Biology of the Cell (I2BC), Orsay, 91198, France*
[c] *Université Lorraine, CNRS, LORIA, Nancy, 91198, France*

Abstract

The aggregation of multiple rankings into a consensus ranking is a crucial task in various domains such as search engine results or user-based ratings. This task poses significant challenges due to its inherent complexity. The complexity of the problem stems not only from the need for exactness and efficiency, but also from the diversity of real-world scenarios, which often involve incomplete rankings and ties.

Most existing methods propose a specific way to aggregate rankings. However, these methods often do not take into account different real use-case scenarios which can impact the relevance of their final result, as the congruence between the aggregated output and the expected outcome inherently depends on the context. To address the issue of context-dependency in ranking aggregation, we introduce a unifying framework that subsumes a variety of generalizations of the Kemeny score for incomplete rankings with ties and enables the design of new ones if a specific context requires it. Our framework is parameterized, allowing for different behaviors depending on the specific use case.

We provide a broader scope of application to the methods encompassed by our approach, augmenting them with a larger theoretical and algorithmic structure. We establish an axiomatic study to better understand each method within our framework and present an algorithmic approach that includes exact methods, partitioning algorithms, and heuristics.

Finally, we demonstrate the practical relevance of our approach through an empirical study on both real and synthetic datasets. Notably, the synthetic datasets are generated based on devised real-world scenarios, highlighting the context-dependent applicability of different Kemeny-based rank aggregation methods within our framework.

Keywords: Consensus ranking; Kemeny rank aggregation; incomplete rankings; rankings with ties; partitioning methods.

Highlights

- Kemeny–Young method, Kemeny rule, the maximum likelihood method, the median relation
- Rank aggregation, Aggregation of incomplete rankings and/or rankings with ties
- Parameterized framework allowing different interpretations of missing data
- Generalized axioms: Majority criterion, Condorcet criterion / winner, Smith criterion
- Efficient algorithm for rank aggregation, ILP algorithm to compute consensus rankings

1. Introduction

The problem of rank aggregation, where multiple rankings are aggregated into one *consensus ranking*, is studied and applied in various communities: in social choice theory [1, 2], within computer science as in algorithmics [3], databases [4, 5, 6], artificial intelligence [7, 8], but also in application domains such as physics [9] and biology [10, 11, 12]. A substantial part of literature has focused on the aggregation of complete rankings without ties (*i.e.*, linear orderings, *a.k.a* permutations) [13, 14, 3, 15, 7, 16, 17, 18, 19, 20]. In this context, the Kemeny rule consists in computing a Kemeny consensus, *i.e.*, a ranking that is as "close" as possible to the input rankings using the *Kendall-τ distance*. This problem is NP-hard in most cases [21, 22, 18]. Several exact algorithms have been designed [23, 24] none of them can handle more than a few dozen elements. The current most efficient solution introduced by Kuhlman *et al.* [20] can handle until one hundred elements on permutations. To aggregate massive rankings (hundreds of elements), heuristics and approximation algorithms are classically used (*e.g.* [14, 3, 25, 26]) and several space reduction methods have been introduced [27, 28, 22, 13, 29, 30, 31] including *partitioning methods* to divide the initial problem into independent sub-problems [32, 27, 31, 22, 13].

In real life applications, rankings may have *ties* (elements ranked *ex aequo* at the same position) and may be *incomplete* (the rankings to aggregate are not all on the same set of elements). On the one hand, Kemeny presented in [33] a generalisation of the Kendall-τ distance to provide a distance function between two complete rankings with ties [33], extended by Fagin *et al.* who also designed an approximation algorithm [34]. On the other hand, several methods have been designed to face the problem of incomplete rankings [35, 27, 21, 36, 13, 11, 37, 38] (some of them dealing both with incomplete rankings and ties). However, these various methods applied to the same input may give different results, and there is no mention of which one should be used according to the context. In this article, we claim that the choice of the method has to be done carrefully, according to the use-case context, in particular the interpretation of non-ranked elements.

Let us consider Example 1 as a demonstration of how the interpretation of non-ranked elements can significantly vary according to the context, making the handling of incomplete rankings particularly complex. Here, we present two distinct use cases.

Example 1. *Consider the following 5 rankings (r_1 to r_5) over eight elements (A to H) to aggregate. The rankings can be incomplete (e.g., F, G and H are non-ranked in r_1) and with ties (e.g., G and C share the same ex aequo position in r_3). For instance, r_3 can be seen as $A \prec (G \equiv C) \prec D \prec B$.*

$$
\begin{aligned}
r_1 &:= [\{A\}, \{B, C, D\}, \{E\}] \\
r_2 &:= [\{A\}, \{B\}, \{G\}, \{C\}, \{F\}, \{E\}] \\
r_3 &:= [\{A\}, \{G, C\}, \{D\}, \{B\}] \\
r_4 &:= [\{H\}, \{A\}, \{C\}, \{B\}, \{G\}, \{D\}, \{F\}, \{E\}] \\
r_5 &:= [\{H\}, \{B\}, \{A\}, \{D\}, \{F\}, \{C\}, \{G\}, \{E\}]
\end{aligned}
$$

The goal is to compute a consensus ranking, which represents the best the rankings of the input. Note that the consensus ranking is required to be complete, that is to say, it should constitute a ranking of all elements that appear in at least one ranking of the input (here, letters from A to H). This requirement stems from the perspective of decision-making support: we do not want a relevant element to be excluded from the consensus ranking, and it is highly challenging to define a systematic, absolute criterion for excluding certain elements from the consensus ranking.

Let us now consider two use cases and focus on the position of elements A and H to show that Example 1 is a concrete example where, depending on the context, one may prefer obtaining a consensus either with A before H or the contrary.

Use case 1 - Non-ranked elements are less relevant than ranked elements

Consider a first use case which mimics the behavior of biologist users when they query the NCBI EntrezGene database (one of the major databases for genes). Given a keyword denoting a disease name, EntrezGene provides genes known to be associated with such a disease. To get a maximal amount of information, biologists usually query EntrezGene several times, with alternative synonyms of the original keyword (e.g., *breast cancer*, *mammalian carcinoma*, etc.). Each query provides a ranking of genes, based on the number of occurrences of the keyword in the annotation file of each gene. The relevance and effectiveness considering multiple synonyms for the disease of interest and then aggregating the resulting rankings has been demonstrated in [39]. This is a real use case in the context of aggregating incomplete rankings, more detailled in [11] and [39].

Here, non-ranked genes in a ranking have no occurrence of the corresponding query in their annotation files. In this context, non-ranked elements for a given input ranking are less relevant with respect to the corresponding query than ranked elements. For instance, in r_1, gene A is more related to the synonym associated with r_1 than gene H.

Now, consider the elements A and H: who should be ranked ahead? Notice that A is ranked three times in the first position (in r_1, r_2 and r_3), once at the second position (in r_4) and once at the third position (in r_5). H is ranked twice in the first position but it does not appear in the other rankings, H is thus considered as not relevant in a majority of rankings. As a consequence, <u>A should be ranked before H</u> in the consensus.

Use case 2 - Non-ranked elements were not evaluated.

Consider the same dataset (Example 1) but in a different context. Users were invited to rank the movies they have watched, from a provided selection, according to their personal preference.

Here, the rankings are incomplete as not all movies have been seen by a given user. A non-ranked element corresponds to an unknown information: a user cannot compare an unseen movie to the other movies.

Let us focus again on elements A and H. Note that each time A and H are comparable (both ranked, both seen by the same user) H is ranked before A hence preferred over A. As a consequence, <u>H should be ranked before A</u> in the consensus.

These two use cases underscore the vital role of contextual properties in defining the consensus ranking. In use case 1, where non-ranked elements are perceived as less interesting than ranked ones, we might expect a principle like the majority criterion: if an element is ranked first in a majority of rankings, it should also top the consensus ranking. However, in use case 2, where the distinction between watched and unwatched movies takes on a different significance, this principle may not apply. This brings into spotlight the concept of axiomatic criteria, rooted in social choice theory, not only as guiding principles but also as tools that help to understand why in certain contexts these principles might be consciously disregarded. Moreover, Use Case 2 highlights the advantage of formulating a complete consensus ranking, one that contains every element ranked in at least one input ranking. This property allows elements that are ranked infrequently, yet highly when they are, to emerge as significant candidates in decision-making contexts.

In this paper, we argue that incomplete rankings warrant substantial and careful consideration. More specifically, we emphasize the necessity of accounting for the context-dependent significance of non-ranked elements. Rather than introducing yet another rank aggregation function, our aim is to establish a context-flexible framework capable of aggregating incomplete rankings with ties within a Kemeny paradigm.

More precisely, our contributions are the following ones:

- a **unifying framework for aggregation of incomplete rankings with ties** subsuming a variery of generalizations of the Kemeny score and enabling the conception of new methods on demand; Our framework is parameterized to allow different kinds of behaviours depending on the use cases (signification of non-ranked elements, etc.)
- a **formal analysis** of the framework, establishing links with the social choice theory, and between the parameters of the framework and some qualitative features of the datasets;
- an **algorithmic setting** composed of an efficient Integer Linear Programming (ILP) exact algorithm, a set of partitioning properties and several heuristics;
- a **series of experiments** to assess the practical applicability and effectiveness of the framework on both real and synthetic datasets with the latter being generated based on real-world use case scenarios.

The remainder of this paper is organized as follows. Section 2 focuses on related work. Section 3 provides the definitions of the major concepts underlying rank aggregation. Section 4 introduces a new unifying framework to handle incomplete rankings with ties. Section 5 performs a formal study of this framework. Section 6 introduces the algorithmic setting. Section 7 evaluates our framework on a very large number of datasets. Section 8 further discusses these results, establishes a user guide for rank aggregation of incomplete rankings with ties and outlines perspectives for future work. The complete version of the exact algorithm and the technical proofs of our results are given in the appendices.

2. Related work

2.1. Kemeny rule for rankings

As stated in the introduction, the Kemeny rule (also called Kemeny–Young method) for the rank aggregation problem [33, 40] has been largely studied in the context where rankings are permutations *i.e. complete rankings without ties* [15, 27, 41, 23, 17, 29, 25, 26, 19, 20] for which important theoretical contributions are given. The importance of the Kemeny-Young-method is well-detailed in [42]. The result of the aggregation is set to be the ranking that minimizes the *Kemeny score*, which is the sum of the Kendall-τ distances to the rankings to be aggregated. The *Kendall-τ distance* counts the number of pairwise order disagreements between two permutations (precise definitions are given in Section 3).

Fagin *et al.* [34] have presented a parameterized model to aggregate complete *rankings with ties* which has been used in several works (*e.g.*, [43, 44]). This model extends the generalization for ties already designed by Kemeny in [33] by introducing a real parameter $p \in]0, 1]$ which is intuitively the cost of creating or breaking ties in a ranking. In several other works, ties are treated differently. For example, the cost of creating or breaking ties is 0 in [36, 35] whereas [33] and [43] suggest setting a penalty of 2 for a strict order disagreement and 1 for breaking ties.

As computing an optimal consensus is NP-hard, several *space reduction methods* have been proposed [32, 27, 28, 29, 30, 31, 39] including *partitioning methods* able to divide the initial problem into independent sub-problems [27, 28, 31]. However, all of them need the rankings to be complete. Furthermore, many heuristics and approximation algorithms have been designed for complete rankings (*e.g.*, [14, 3, 44, 35, 44, 19, 5, 29]). A benchmark is provided in [45].

Considering incomplete rankings is highly important in social choice theory [46, 47, 48]. Several methods have been designed to aggregate *incomplete rankings* in the context of the Kemeny rule. They can be divided into two groups: (i) the methods which make the incomplete rankings artificially complete and (ii) the methods which tune the Kendall-τ distance to handle incomplete rankings.

The first group contains the *unification method* [36] which consists in appending at the end of each incomplete ranking the non-ranked elements in a *unifying bucket*, and the *projection method* [13] which consists in removing all the elements which are not in all the input rankings. The underlying assumption related to the unification method is that non-ranked elements are less relevant towards a given ranking than ranked elements. The projection method will not be considered in this paper for two reasons. First, removing an element in the input rankings can modify the relative position of other pairs of elements in a consensus ranking, except for the aggregation methods that respect the so called *independence of irrelevant alternatives* [49]. Second, removing all the elements that are not ranked in all the input rankings can cause a drastic loss of information and prevent a relevant choice in a decision making context.

The second group contains the *induced Kendall-τ distance* [21], called "Kendall-τ distance with incomplete votes" in [27], the *pseudometric with penalty parameter* p [11], the *extended Kendall-τ distance* [35] and the *generalized induced Kendall-τ distance with penalty parameter* p [37] which combines the induced Kendall-τ distance [21] for incomplete rankings without ties and the model of Fagin *et al.* for complete rankings with ties [34]. The generalized induced Kendall-τ distance with penalty parameter p and the extended Kendall-τ distance do not penalize non-ranked elements whereas the pseudometric with penalty parameter p penalizes non-ranked elements towards ranked elements in a given ranking.

Difference between methods to aggregate incomplete rankings. Table 1 highlights the differences between the methods of this group. More precisely, the penalty for placing A before H, H before A, B before C, F before H in the consensus ranking regarding the input ranking r_1 of Example 1 is given for each method.

Table 1: Cost induced by relative orderings in a consensus ranking towards r_1 according to the chosen method. $p \in]0, 1]$ is Fagin's parameter [34]. ND stands for "not defined".

method / in consensus	[36]	[21, 27]	[11]	[37]	[35]	[34]	[43]
B before A	1	1	1	1	1	1	2
H before A	1	0	1	0	0	ND	ND
B before C	0	ND	p	p	0	p	1
F before H	0	0	0	0	0	ND	ND
A tied with B	ND	ND	p	p	p	p	1

Table 1 highlights the huge variety of penalties from a method to another.

More importantly, there is a lack of solutions provided when dealing with the large majority of real datasets which are incomplete.

In conclusion, given a tuple of incomplete input rankings to aggregate, four major difficulties are encountered: (i) choosing which method to use, (ii) finding an exact algorithm (none of the methods in Table 1 provides an exact algorithm), (iii) improve efficiency by using partitioning techniques (none of the partitioning methods used in the context of complete rankings are usable for incomplete rankings), (iv) determining which axiomatic criteria are respected in this setting.

2.2. Other approaches to handling incomplete rankings and/or rankings with ties

While this review primarily focuses on extensions of the Kemeny Young method, it is worth noticing that there are alternative paradigms to handling incomplete rankings, each with their own unique perspectives and techniques.

- Rank correlation coefficient: One alternative is the rank correlation coefficient introduced by Emond and Mason [50]. Unlike the Kemeny approach, which seeks to minimize the total number of disagreement between rankings, the rank correlation coefficient measures the degree of agreement between rankings, by substracting the number of disagreements to the number of agreements, and normalize by the total number of pairs of elements. This measure has been extended to handle incomplete rankings and ties in [38].

- Binary relations: An incomplete ranking can be interpreted as a preorder, a binary relation that is reflexive and transitive. The field of binary relations aggregation (e.g. [51]) provides a diverse range of methods to handle incomplete rankings, offering a different perspective on the problem, where the output is not necessarily a complete ranking.

- Social choice theory: For each voting system that allows computing a complete ranking of candidates (e.g. Borda count [25], Copeland's method [26]...) there are articles discussing their application to incomplete and/or tied rankings. These voting systems can also be used as heuristics within a Kemeny-like context (see section 6.3 for example), providing a bridge between these different perspectives.

Despite the diversity and complementarity of these approaches, what sets our framework apart is its capacity to adapt to various real-world data contexts. We highly believe that the aggregation methods which are relevant to apply truly depend on the context, in particular (but not only) to the interpretation of non-ranked elements. The flexibility in contextual adaptation, the theoretical study giving hints how to choose the parameters depending of the context, and the quite efficient algorithms provided form the core of our contribution and distinguish our approach from others in the field.

3. Preliminaries

We introduce the basic notions and terminology related to rankings and the rank aggregation problem.

3.1. Basic notions and terminology

Given a *universe* U (a set of elements to rank), a *ranking* r on U is a list $r = [b_1, ..., b_k]$ of disjoint subsets of U, *i.e.*, for each $i \in \{1, \ldots, k\}$, $b_i \subseteq U$ and $b_i \cap b_j = \emptyset$, whenever $i \neq j$. Each b_i represents a so-called *bucket*. Intuitively, a bucket represents a subset of U whose elements are ranked *ex aequo*. The *domain* of r is defined by $dom(r) = b_1 \cup ... \cup b_k$. In Example 1, we have $dom(r_1) = \{A, \ldots, E\}$ whereas $dom(r_4) = \{A, \ldots, H\}$. We say that $x \in U$ is *ranked* in r if $x \in dom(r)$. Otherwise, x is *non-ranked* in r.

We consider different types of rankings. By a *ranking without ties* we mean a ranking in which each bucket contains exactly one element, that is, $|b_i| = 1$ for every $i \in \{1, \ldots, k\}$. If the latter condition is not satisfied, then we say that r is a *ranking with ties*. In Example 1, r_1 and r_3 are rankings with ties ($\{B, C, D\}$ is a bucket of size 3 in r_1) whereas r_2, r_4 and r_5 are rankings without ties. Moreover, r is said to be a *complete ranking* if $dom(r) = U$. Otherwise, r is said to be an *incomplete ranking*. In Example 1, r_4 and r_5 are complete whereas r_1, r_2 and r_3 are incomplete. Complete rankings without ties are also known as *linear orderings* or *permutations*.

Notation. Let $r = [b_1, ..., b_k]$ be a ranking of U, and let x,y be two elements of U. We say that x is *ranked before* y (or that y is *ranked after* x) *in* r, denoted by $x \prec_r y$, if there exists $i < j$ such that $x \in b_i$ and $y \in b_j$. We also say that x is *tied* with y in r, denoted by $x \equiv_r y$, if there exists i such that $x, y \in b_i$. Moreover, $x \diamond_r y$ will denote $x \in dom(r) \wedge y \notin dom(r)$, *i.e.*, x is ranked in r while y is not.

Let $r = [b_1, ..., b_k]$ be an incomplete ranking on U. The *unified ranking of* r, denoted $unif(r)$, is the complete ranking such that $unif(r) = [b_1, ..., b_k, U \backslash dom(r)]$. In Example 1, $unif(r_1) = [\{A\}, \{B, C, D\}, \{E\}, \{F, G, H\}]$.

The *position* $r(x)$ of $x \in dom(r)$ in r is defined by $r(x) = 1 + |\{y \in dom(r) : y \prec_r x\}|$. Note that $r(x)$ is not defined whenever $x \notin dom(r)$.

Throughout the paper, we will also adopt the following notation.

- R denotes a tuple $(r_1, ..., r_m)$ of m rankings,
- U denotes a finite set and we will assume, without loss of generality, that $U = \{1, ..., n\}$.
- $\mathscr{L}(U)$ denotes the set of all complete rankings without ties on U (*i.e.* linear orderings),
- $\mathscr{C}(U)$ denotes the set of all complete rankings on U,
- $\mathscr{A}(U)$ denotes the set of all rankings on U (complete or incomplete, and with or without ties).
- $\mathscr{W}(U)$ denotes the set of all rankings without ties on U (complete or incomplete).
- Given a set X, $X^{<\infty}$ denotes the set of all finite tuples over X, *i.e.*, $X^{<\infty} = \bigcup\limits_{i \geq 1} X^i$.

3.2. Kemeny rank aggregation

Given two complete rankings without ties $r_1, r_2 \in \mathscr{L}(U)$, their *Kendall-$\tau$ distance* is denoted by $K(r_1, r_2)$ and is defined by

$$K(r_1, r_2) = \sum_{1 \leq x < y \leq n} \overline{K}_{x,y}(r_1, r_2), \tag{1}$$

where $\overline{K}_{x,y}(r_1, r_2) = 1$ if $x \prec_{r_1} y$ and $y \prec_{r_2} x$ or $y \prec_{r_1} x$ and $x \prec_{r_2} y$, and $\overline{K}_{x,y}(r_1, r_2) = 0$, otherwise [34]. Back to Example 1, we have $\overline{K}_{A,B}(r_4, r_5) = 1$ as A is before B in r_4 whereas B is before A in r_5 and $\overline{K}_{A,C}(r_4, r_5) = 0$ as A is before C in both r_4 and r_5. Finally, $K(r_4, r_5) = 1_{\{A,B\}} + 1_{\{B,C\}} + 1_{\{C,D\}} + 1_{\{C,F\}} + 1_{\{D,G\}} + 1_{\{F,G\}} = 6$.

Based on this distance, the *Kemeny score* $S(c, R)$ of a complete ranking without ties $c \in \mathscr{L}(U)$ with respect to a tuple of complete rankings without ties $R \in \mathscr{L}(U)^m$ is then defined as the sum of the Kendall-τ distances between c and each ranking in R. Formally,

$$S(c, R) = \sum_{r \in R} K(c, r).$$

In this context, the rank aggregation problem consists of finding a complete ranking without ties $c \in \mathscr{L}(U)$ that minimizes the Kemeny score with respect to R, that is, such that $S(c, R) \leq S(r, R)$, for each $r \in \mathscr{L}(U)$. Such a ranking is called a *median* (also called *Kemeny consensus* or *optimal consensus*). The median may not be unique. The problem of finding a median is NP-hard as soon as $|U| > 2$ for an even number of rankings $m \geq 4$ [21, 22] and for an odd number of rankings $m \geq 7$ [18].

Following the setting introduced by Kemeny, Fagin *et al.* [34] expanded the approach by bringing in a *penalty parameter* $0 < p \leq 1$ within the framework of the Kendall-τ distance. This real number can be interpreted as the cost to pay when creating or breaking ties within the rankings. Formally, for $r_1, r_2 \in \mathscr{C}(U)$,

$$K^p(r_1, r_2) = \sum_{1 \leq x < y \leq n} \overline{K}^p_{x,y}(r_1, r_2),$$

where $\overline{K}^p_{x,y}(r_1, r_2)$ is

- 0 if x and y are in the same order or both tied in r_1 and r_2,
- 1 if x and y are in the reversed order in r_1 and r_2, and
- p if x and y are tied in one ranking but not in the other one.

For example, in Example 1, we have $\overline{K}^p_{B,C}(r_1, r_3) = p$ as B is tied with C in r_1 and C is before B in r_3. Note that $K^p(r_1, r_3)$ cannot be fully computed as K^p is not defined on incomplete rankings. The *Kemeny score with penalty parameter* p of a complete ranking c and a tuple R of complete rankings is naturally defined by

$$S^p(c, R) = \sum_{r \in R} K^p(c, r). \tag{2}$$

In this paper, we call *consensus ranking* any complete ranking (without assumption of optimality).

4. Unifying rank aggregation framework

We propose a unifying Kemeny-based framework for rank aggregation that can handle complete or incomplete rankings, with or without ties. Our framework subsumes all the methods of the literature that are based of a Kemeny score we are aware of, in particular the one presented in [36, 21, 34, 27, 11, 35, 37]. Our framework enables a contextual interpretation and thus an assessment of the significance of non-ranked elements through two hyper-parameters (two 6-tuples of real values) whose values indicate, among other things, how a non-ranked element should be considered and compared to a ranked element (less important or incomparable).

4.1. Penalty tuples of the framework

When the input rankings are complete, the Kemeny rule consists of finding all complete rankings that minimize the Kemeny score (with penalty parameter p if there are ties in the input rankings). When extending the framework to account for possibly incomplete rankings, the goal is to find all complete rankings that minimize a Kemeny-based score designed to handle ties and/or incomplete rankings. As previously mentioned, even if the input rankings are incomplete, we want the output rankings to be complete since we want all the elements of U to be ranked.

Let c be a consensus ranking, and let (x, y) be a pair of elements of U. There are three possible cases for the relative position of x and y with respect to c, namely, $x \equiv_c y$, $x \prec_c y$ or $y \prec_c x$. Let now r be a, possibly incomplete, input ranking. As r may be incomplete, there are now six possible cases to consider:

1: $x \prec_r y$,

2: $y \prec_r x$,

3: $x \equiv_r y$,

4: $x \diamond_r y$,

5: $y \diamond_r x$, and

6: $x \notin dom(r) \wedge y \notin dom(r)$.

Following the tracks of Fagin *et al.* [34], we associate a penalty for each of the cases above. For each pair of elements (x, y), we define two 6-tuples:

- B for the case when $x \prec_c y$ ("before"), and whose component $B[\ell]$ (denoted B_ℓ), $1 \leq \ell \leq 6$, is the penalty associated with case ℓ above, and

- T for the case when $x \equiv_c y$ ("tied"), and whose component $T[\ell]$ (denoted T_ℓ), $1 \leq \ell \leq 6$, is the penalty associated with case ℓ above.

For example, B_2 is the penalty associated to the case when $x \prec_c y$ and $y \prec_r x$, while T_5 is the penalty associated to the case when $x \equiv_c y$ and y is in the domain of r while x is not. All these penalties are summarized in Table 2.

Table 2: Table of penalties

$c \in \mathscr{C}(U)$ / $r \in R$	$x \prec_c y$	$x \equiv_c y$
$x \prec_r y$	B_1	T_1
$y \prec_r x$	B_2	T_2
$x \equiv_r y$	B_3	T_3
$x \diamond_r y$	B_4	T_4
$y \diamond_r x$	B_5	T_5
$x \notin dom(r) \wedge y \notin dom(r)$	B_6	T_6

4.2. Rank aggregation within the framework

We now define a family of functions that generalizes the Kemeny score S^p with penalty parameter p. Essentially, instead of considering a single real parameter p, we now have two hyperparameters $B, T \in \mathbb{R}^6$ whose components indicate the penalties listed in Table 2. We now define the *Kemeny score* $S^{(B,T)}$ *with penalty* B *and* T as follows: for all $(c, R) \in \mathscr{C}(U) \times \mathscr{A}(U)^{<\infty}$,

$$S^{(B,T)}(c, R) = \sum_{\substack{x,y \in U \\ x \prec_c y}} \langle B, \Omega^R_{x,y} \rangle \ + \ \frac{1}{2} * \sum_{\substack{x \neq y \in U \\ x \equiv_c y}} \langle T, \Omega^R_{x,y} \rangle \tag{3}$$

where $\Omega^R_{x,y}$ is the vector $(\omega^R_{x \prec y}, \omega^R_{y \prec x}, \omega^R_{x \equiv y}, \omega^R_{x.\overline{y}}, \omega^R_{y.\overline{x}}, \omega^R_{\emptyset})$ defined by

- $\omega^R_{x \prec y} = |\{r \in R : x \prec_r y\}|$,
- $\omega^R_{x \equiv y} = |\{r \in R : x \equiv_r y\}|$,

- $\omega^R_{x.\overline{y}} = |\{r \in R : x \diamond_r y\}|$,
- $\omega^R_{\emptyset} = |\{r \in R : x \notin dom(r) \wedge y \notin dom(r)\}|$,

and where $\langle B, \Omega^R_{x,y}\rangle$ (resp. $\langle T, \Omega^R_{x,y}\rangle$) denotes the dot product of B (resp. T) with $\Omega^R_{x,y}$. Intuitively, the value of $\langle B, \Omega^R_{x,y}\rangle$ represents the cost of placing x before y in c with respect to R, and the value of $\langle T, \Omega^R_{x,y}\rangle$ represents the cost of tying x and y in c with respect to R.

The choice of the coefficient $1/2$ is motivated by the fact that for any two elements x and y such that $x \equiv_c y$, we also have $y \equiv_c x$. In this way, we prevent the situation $x \equiv_c y$ from being counted twice.

Moreover, using the properties of the dot product, it is easy to see that the set of Kemeny scores with penalty tuples B and T constitutes a vector space with real scalars.

Definition 1. *Given a pair $(x, y) \in U^2$, we define:*

- $before^{(B,T)}_R(x, y) = \langle B, \Omega^R_{x,y}\rangle$ *the cost of placing x before y in a consensus ranking with respect to R.*
- $tied^{(B,T)}_R(x, y) = \langle T, \Omega^R_{x,y}\rangle$ *the cost of tying x and y in a consensus ranking with respect to R.*
- $minc^{(B,T)}_R(x, y) = min(before^{(B,T)}_R(x, y), before^{(B,T)}_R(y, x), tied^{(B,T)}_R(x, y))$ *the cost of the cheapest relative ordering of x and y in a consensus ranking with respect to R.*

We use the notation $before(x, y)$, $tied(x, y)$ and $minc(x, y)$ when there is no ambiguity concerning R and the penalty tuples (B, T).

4.3. Kemeny-compatible function.

We now set restrictions on the parameters to conserve a Kemeny prism and define the notion of Kemeny-compatible function.

Definition 2. *We say that a mapping $f : dom(f) \to \mathbb{R}$ is a Kemeny-compatible function (KCF) if*

$$\mathscr{L}(U) \times \mathscr{L}(U)^{<\infty} \subseteq dom(f) \subseteq \mathscr{C}(U) \times \mathscr{A}(U)^{<\infty},$$

and if there exist $B, T \in \mathbb{R}^6_{\geq 0}$ such that *(i)* $B_1 = T_3 = 0$,

(ii) $B_2 > 0$,

(iii) $T_1 = T_2$,

(iv) $T_4 = T_5$,

(v) $B_4 \leq B_5$.

and such that $f(c, R) = S^{(B,T)}(c, R)$ *for all* $(c, R) \in dom(f)$.

Restriction (i) ensures that no penalty is applied to a given pair (x, y) if their relative order is the same between the consensus c and an input ranking r. On the contrary, restriction (ii) ensures that a penalty is applied if $x \prec_c y$ while $y \prec_r x$. Restrictions (iii) and (iv) ensure that $tied(x, y) = tied(y, x)$: the cost of tying x with y must be the same as the cost of tying y with x. Finally, restriction (v) means that non-ranked elements are considered either incomparable or less pertinent than ranked elements ($B_4 > B_5$ would mean that non-ranked elements are considered more pertinent than ranked ones).

Proposition 1. *Let* $f(c, R) = S^{(B,T)}(c, R)$ *be a KCF. Then, there exists a function* g *such that for all* $(c, R) \in dom(f)$,

$$f(c, R) = \sum_{r \in R} g(c, r). \tag{4}$$

Informally, $g(c, r)$ is the sum of the penalties for disagreement between c and r for every pair of elements (a formal proof is given in Appendix A.1).

Suppose that f is a KCF and let c_1, c_2 be two consensus rankings such that $f(c_1, R) < f(c_2, R)$. Then c_1 represents R better than c_2 (with respect to f).

Definition 3. *Let* R *be a tuple of rankings (complete or incomplete, with or without ties) and let* f *be a KCF. A median of* R *with respect to* f *is a complete ranking* c *such that* $(c, R) \in dom(f)$ *and* $f(c, R) \leq f(r, R)$ *for every complete ranking* r *such that* $(r, R) \in dom(f)$. *The set of all the medians of* R *with respect to* f *is denoted by* $M_f(R)$.

In other words, a median of R with respect to a KCF f is a complete ranking such that no other complete ranking can represent R better with respect to f. Note however that such a median is not necessarily unique.

Table 3 shows, by exhibiting the corresponding penalty tuples, that the model of KCF subsumes the Kemeny-based methods of the literature we are aware of.

method	domain	tuples B and T
KS [33]	$\mathscr{L}(U) \times \mathscr{L}(U)^{<\infty}$	$B = (0, 1, \infty, \infty, \infty, \infty)$ $T = (\infty, \infty, \infty, \infty, \infty, \infty)$
KSP [34]	$\mathscr{C}(U) \times \mathscr{C}(U)^{<\infty}$	$B = (0, 1, p, \infty, \infty, \infty)$ $T = (p, p, 0, \infty, \infty, \infty)$
KSE [43]	$\mathscr{C}(U) \times \mathscr{C}(U)^{<\infty}$	$B = (0, 2, 1, \infty, \infty, \infty)$ $T = (1, 1, 0, \infty, \infty, \infty)$
UKSP [36] followed by [34]	$\mathscr{C}(U) \times \mathscr{A}(U)^{<\infty}$	$B = (0, 1, p, 0, 1, 1)$ $T = (p, p, 0, p, p, 0)$
IKS [21], [27]	$\mathscr{L}(U) \times \mathscr{W}(U)^{<\infty}$	$B = (0, 1, \infty, 0, 0, \infty)$ $T = (\infty, \infty, \infty, \infty, \infty, \infty)$
GPDP [11]	$\mathscr{C}(U) \times \mathscr{A}(U)^{<\infty}$	$B = (0, 1, p, 0, 1, 0)$ $T = (p, p, 0, p, p, 0)$
EKS [35]	$\mathscr{L}(U) \times \mathscr{A}(U)^{<\infty}$	$B = (0, 1, 0, 0, 0, 0)$ $T = (\infty, \infty, \infty, \infty, \infty, \infty)$
IGKS [37]	$\mathscr{C}(U) \times \mathscr{A}(U)^{<\infty}$	$B = (0, 1, p, 0, 0, 0)$ $T = (p, p, 0, 0, 0, 0)$

Table 3: Penalty tuples for KCFs in related work. The ∞ symbol denotes coefficients not useful in view of the domain of the KCF (they may be replaced by any value above 1 to use the algorithms and properties presented in the following). KS stands for Kemeny score, KSP stands for Kemeny score with penalty p, KSE stands for Kemeny score for elections, $UKSP$ stands for Unification process followed by Kemeny score with penalty p, IKS stands for Induced Kemeny score, GPDP stands for Generalized pseudo-distance with penalty p, EKS stands for Extended Kemeny score, IGKS stands for Induced generalized Kemeny score.

5. Formal study of the model of KCF

Our general framework is parametrized to allow different kinds of behaviours depending on the use cases. We now address the question of how the choice of the parameters influence the output. In particular, we have seen in the introduction that depending on the signification of non-ranked elements, we want to respect or not the majority criterion (cf. Section 1). Subsection 5.3 defines axiomatic criteria and states when these are satisfied or not, depending on the choice of the parameters. This shows that different parameters may lead to different behaviours: respecting or not some criteria. On the contrary, different KCFs may provide the same output. We define an equivalence relation on KCFs to take this into account.

The choice of the parameters may also have an influence of some properties like having a linear ordering among the set of medians, or the algorithmic complexity of the problem of finding a median.

5.1. Equivalent classes and complexity

In the remainder of the paper, $\mathscr{X}$ will stand for $\mathscr{L}(U)$ or $\mathscr{C}(U)$, and $\mathscr{Y}$ will stand for $\mathscr{L}(U)$, $\mathscr{C}(U)$ or $\mathscr{A}(U)$.

Definition 4. *Two KCFs f and g are equivalent on $\mathscr{D} = \mathscr{X} \times \mathscr{Y}^{<\infty}$ if and only if for all $R \in \mathscr{Y}^{<\infty}$, $M_{f|_{\mathscr{D}}}(R) = M_{g|_{\mathscr{D}}}(R)$.*

In other words, two KCFs are equivalent if they produce the same set of medians for each tuple of input rankings.

Theorem 1. *Let f and $g = S^{(B,T)}$ be two KCFs such that $B_2 \leq 2 * T_2$ and suppose that f is equivalent to g on $dom(f)$. Then, finding a median of $R \in \mathscr{Y}^{<\infty}$ with respect to f is NP-Hard as soon as $|U| > 2$.*

In particular, when $p \geq \frac{1}{2}$, finding a median for the Kemeny score with penalty parameter p is NP-Hard (see Table 3).

Note that when $T_i = 0$ for all $1 \leq i \leq 6$, finding a median is trivial since the ranking where all the elements are tied is a median (the associated score is zero). The complexity of finding a median is an open problem when $B_2 > 2{*}T_2$. The proof of Theorem 1 and of every statement of this subsection is given in Appendix A.2.

We now present two sufficient conditions, the first one ensures at least one median is a complete ranking without ties and the second one that all the medians are complete rankings without ties. These results allow to save time by space reduction in algorithms providing a median: we know that in this case we can concentrate on complete rankings without ties instead of exploring all rankings.

Proposition 2. *Let $f = S^{(B,T)}$ and let R be a tuple of rankings. If $before(x, y) + before(y, x) \leq 2 * tied(x, y)$ for all $x \neq y \in U$, then there exists $\mu \in M_f(R)$ such that $\mu \in \mathscr{L}(U)$. Moreover, if the above inequality is strict for all $x \neq y \in U$, then $M_f(R) \subseteq \mathscr{L}(U)$.*

5.2. Equivalent classes and choice of parameters

Here we present results that can help a user to choose the penalty tuples. The goal is not only to provide comparisons among existing KCFs, but also to guide users in the creation of new ones. For instance, Theorem 2 provides insights into questions such as, 'Will the consensus rankings change if we modify B4=0 and B5 = 1 to B4 = 1 and B5 = 2?'

First, we give a sufficient condition to determine when two KCFs are equivalent. A consequence is that w.l.o.g. we can take $B_2 = 1$.

Proposition 3. *If there exists $\phi\colon \mathbb{R} \to \mathbb{R}$ strictly increasing such that $g(c, R) = \phi(f(c, R))$, for all $(c, R) \in \mathscr{X} \times \mathscr{Y}^{<\infty}$, then f and g are equivalent on $\mathscr{X} \times \mathscr{Y}^{<\infty}$.*

For example, if $g(c, R) = a \cdot f(c, R)) + b$ with $a > 0$, then f and g are equivalent. In particular, a KCF $f = S^{(B,T)}$ is equivalent to the KCF $g = S^{(B',T')}$ where $B'_2 = 1$ and for all $1 \leq i \leq 6$, $B'_i = B_i/B_2$ and $T'_i = T_i/B_2$. Hence, w.l.o.g., we can set $B_2 = 1$, which is done in the experiments presented in Section 7.

By taking $B_2 = 1$, we are basically considering an "homothety". Now, let us explore the "translation" part. For that, we define another equivalence relation that is directly related to the penalty vectors, and that reveals the groups of penalties that are in correspondence.

Definition 5. *Two KCFs f and g are $\ominus-$equivalent on $\mathscr{X} \times \mathscr{Y}^{<\infty}$ if and only if $f(c_1, R) - f(c_2, R) = g(c_1, R) - g(c_2, R)$ $\forall c_1, c_2 \in \mathscr{X}$, $R \in \mathscr{Y}^{<\infty}$.*

Obviously, two KCFs which are $\ominus$-equivalent have the same set of medians, since in this case $g(c, R) = f(c, R) + b$.

Proposition 4. *Two KCFs f and g that are $\ominus-$equivalent have the same set of medians.*

Theorem 2. *Let $f = S^{(B,T)}$ and $g = S^{(B',T')}$ be two KCFs defined on $\mathscr{C}(U) \times \mathscr{A}(U)^{<\infty}$. They are $\ominus$-equivalent on $\mathscr{C}(U) \times \mathscr{A}(U)^{<\infty}$ if and only if the following conditions hold:*

- $B_2 = B'_2$,
- $T_2 = T'_2$,
- $B_3 = B'_3$,
- $B_4 - B'_4 = B_5 - B'_5 = T_5 - T'_5$,
- $B_6 - B'_6 = T_6 - T'_6$.

Interestingly, only the penalties concerning the treatment of non-ranked elements (indices 4 to 6 in the penalty vectors) can be modified while keeping a $\ominus-$equivalent. Moreover, notice that the penalties at index 4 (T_4, B_4) and index 5 form a first group, and the penalties at index 6 form a second group to consider independently.

How to prevent ties in the consensus? For the situation where there is a need to avoid ties in the consensus, we present a family of KCFs that ensures there is at least one median which is a complete ranking without ties.

Proposition 5. *Let $f = S^{(B,T)}$ be a KCF and R be a tuple of rankings. If all the following conditions hold, then there exists $m \in M_f(R)$ such that $m \in \mathscr{L}(U)$:*

1. $B_2 \leq 2 * T_2$,
2. $B_3 = 0$,
3. $B_4 + B_5 \leq T_4 + T_5$,
4. $B_6 \leq T_6$.

Moreover:

(i) if the rankings in R are complete, (3) and (4) are no longer necessary, and

(ii) if the rankings in R do not have ties, then (2) is no longer necessary and, by making strict all remaining inequalities, we get $M_f(R) \subseteq \mathscr{L}(U)$.

In particular, if the rankings in R are complete and without ties, then $M_f(R) \subseteq \mathscr{L}(U)$ as soon as $B_2 < 2 * T_2$.

Importance of $B_5 - B_4$. When the rankings to aggregate are incomplete, it is especially important to be careful with the value of $B_5 - B_4$. Using Lemma 1 (see Appendix), we can notice that $before(x, y) - before(y, x)$ only depends on (i) the vector $\Omega^R_{x,y}$, (ii) the penalties B_2 (that can be fixed to 1) and (iii) $B_4 - B_5$. Moreover, the following proposition gives a sufficient condition to ensure that two KCF are not equivalent.

Proposition 6. *Let $f = S^{(B,T)}$ and $g = S^{(B',T)}$ be two KCFs such that*

1. $\frac{B'_5 - B'_4}{B'_2} \neq \frac{B_5 - B_4}{B_2}$,
2. $max(B_2, B'_2) \leq 2 * T_2$, *and*
3. $max(B_4 + B_5, B'_4 + B'_5) \leq 2 * T_4$.

Then f and g are not equivalent on $\mathscr{C}(U) \times \mathscr{A}(U)^{<\infty}$.

For instance, IGKS and GPDP of Table 3 are not equivalent when $p \geq 0.5$ (from Proposition 6). Moreover, KSP and IGKS are equivalent on the domain of KSP by replacing the ∞ symbol (unused due to the domain of KSP) with the values of IGKS (from Theorem 2).

5.3. Axiomatic settings

In this subsection, we adapt some classical criteria of the social choice theory (originally defined for complete rankings without ties) for incomplete rankings, with or without ties. We will then determine when such criteria are satisfied, and discuss their meaning. Let us first introduce some notions.

- An element $x \in U$ is *first-ranked* (resp. *strictly first-ranked*) by ranking r if $x \in r[1]$ (resp. if $r[1] = \{x\}$);
- An element $x \in U$ is the *unique winner* if x is strictly first-ranked by any median.

We first introduce the following variants of well known axioms, namely, the *Majority*, *Condorcet* and *Smith* axioms. We will then state that the framework of KCF we propose induces an equivalence relation between the three generalized axioms, whereas there is only an implication relation in general in social choice theory for the original axioms.

Axiom 1 (Majority-like axiom). *If there exists an element $x \in U$ such that x is strictly first-ranked by a strict majority of rankings, then x is the unique winner.*

Axiom 2 (**Condorcet-like axiom**). *If there exists an element $x \in U$ such that for each element $y \in U \backslash \{x\}$, x is before y in a strict majority of rankings, then x is the unique winner.*

Such an element will be called *Condorcet-like winner* in the remaining of the paper. Determining if there is a Condorcet-like winner and computing it can be done in $O(|U|^2.|R|)$.

Axiom 3 (**Smith-like axiom**). *For any tuple R of rankings, the unique winner (if such element exists) must come from the smallest non-empty subset $S \subseteq U$ such that for every element $x \in S$ and every element $y \in U \backslash S$, x is before y in a strict majority of rankings.*

Note that in a classical social choice context where the input rankings and the consensus ranking are complete and without ties, the Smith axiom induces the Condorcet axiom which induces the Majority axiom. Moreover, it is known that the classical Kemeny rule for complete rankings is a method that fulfills these three classical axioms.

As stated in Theorem 3 below, in our framework of KCF, the three generalized axioms coincide, but some KCF do not fulfill them.

Theorem 3. *The following assertions are equivalent:*

- $S^{(B,T)}$ *satisfies the Majority-like axiom on* $\mathscr{C}(U) \times \mathscr{A}(U)^{<\infty}$
- $S^{(B,T)}$ *satisfies the Condorcet-like axiom on* $\mathscr{C}(U) \times \mathscr{A}(U)^{<\infty}$
- $S^{(B,T)}$ *satisfies the Smith-like axiom on* $\mathscr{C}(U) \times \mathscr{A}(U)^{<\infty}$
- $B_2 = B_5 - B_4$ *and* $\underset{i \in \{1,4\}}{min}(T_i - B_i) \geq \underset{j \in \{2,3,5,6\}}{max}(B_j - T_j)$.

For example, the parametrized Kemeny score introduced by Fagin *et al.* for rankings with ties fulfills any of the three above axioms if and only if $p \geq 0.5$. Moreover, UKSP et GPDP of Table 3 fulfills any of the three above axioms if and only if $p \geq 0.5$.

In which context is it relevant to satisfy these three axioms when rankings may be incomplete? In Example 1, we presented a same tuple of rankings within two different use cases. The Majority-like criterion was fair in one context and unwanted in the second one. With Theorem 3, we can see that a KCF does not satisfy the Majority-like criterion on $\mathscr{C}(U) \times \mathscr{A}(U)^{<\infty}$

if $B_2 \neq B_5 - B_4$. For example, if a non-ranked element is not penalized "enough" with respect to a ranked element in an input ranking, then the element x who is ranked first for the majority is not protected with respect to another element y if y was non-ranked when x was ranked first. That is precisely what we need when the non-ranked elements are not comparable with the ranked elements in an input ranking. More concretely, if the "voters" do not have an exhaustive view of the universe, it makes no sense to wish this axiom. In the same way, if several people are asked to rank their favorite movies, it makes no sense to consider that each voter has seen exactly the same set of movies. Finally it may be interesting to violate the Majority-like axiom by setting $B_5 = 0$ if the objective is to highlight rare pearls (elements that may be non-ranked many times, but very well-ranked when they are ranked).

We now introduce a fourth axiom that is inspired by the *local independence of irrelevant alternatives* criterion in social choice theory, which is pertaining to partitioning methods.

Axiom 4 (**local independence-of-irrelevant-alternatives-like (LIIAL) axiom**). *For any function f, any tuple R of rankings in the domain of f,*

and any median $[P_1, \ldots, P_k]$ of R with respect to f, the two following conditions hold:

- *if the elements in P_k are removed (i.e., projection into $U' = U \setminus P_k$), then $[P_1, \ldots, P_{k-1}]$ is a median for the new tuple of votes,*
- *if the elements in P_1 are removed (i.e., projection into $U' = U \setminus P_1$), then $[P_2, ..., P_k]$ is a median for the new tuple of votes.*

We first show that every KCF respects the LLIAL axiom. We then explain how it can be used in an algorithmic setting.

Proposition 7. $\forall B, T,\ S^{(B,T)}$ *respects the local independence of irrelevant alternatives criterion on $\mathscr{C}(U) \times \mathscr{A}(U)^{<\infty}$.*

On the usefulness of LIIAL. The LIIAL axiom states that after computing a median, the first elements will keep their relative order if the least interesting elements are removed from the input rankings and, conversely, the last elements will keep their relative order if the first elements are removed. This criterion is interesting in some concrete situations, for instance, in scheduling experiments to further explore the importance of some genes related to a given disease. If the researchers conducting the experiments do not want to consider the two first elements given the median ranking (because they are famous genes, extensively studied already for example), they can directly consider the third one in the median ranking without computing a a new consensus after removing the top-2 elements from the input rankings.

Also, it is noteworthy to mention the advantageous interaction between the LIIAL and the Condorcet-like criterion. Indeed, suppose that x is Condorcet-like winner. Then all medians will place x in first position. Moreover, under LIIAL, we can consider the input rankings without x. If we find a new Condorcet-like winner, we can place it in second position. This process can be iterated as long as a Condorcet-like winner is found . If this process can been applied $|U|$ times, there is a unique median and this latter can be computed using this process in $O(|U|^3.|R|)$.

6. Algorithmic approach

This section presents algorithms that can be integrated into our framework (Section 4). First, we present an exact algorithm for any KCF that uses Integer Linear Programming (ILP). Second, we state that the partitioning methods able to divide the initial problem into smaller sub-problems can be generalized for any KCF. Third, we discuss the generalization to any KCF of heuristics classically used in a Kemeny framework.

6.1. ILP generic exact algorithm

We have designed an exact generic ILP algorithm able to handle any KCF $f = S^{(B,T)}$. The algorithm takes as input a tuple of rankings and the penalty tuples (B, T) and returns a median with respect to f. The ILP part is given in Table 4 and the complete algorithm is given in Appendix B (Algorithm 1).

The first constraint states that the variables $b_{x,y}$, $b_{y,x}$ and $t_{x,y}$ are binary values. They are respectively set to 1 if and only if x is before y, y is before x and x is tied with y in the consensus.

The second constraint states that in the consensus ranking c, for a given pair (x, y), either x is before y, or y is before x, or x and y are tied. Moreover, in c, the following transitivity constraints hold for any x, y, $z \in U$:

- The third constraint states that if x is before y and y is before z or tied with z, then x is before z.

$$
\begin{array}{ll}
\text{minimize} & \sum_{x<y} before(x,y) * b_{(x,y)} \\
 & +before(y,x) * b_{(y,x)} + tied(x,y) * t_{\{x,y\}} \\
\text{subject to} & b_{(x,y)},\, b_{(y,x)} \text{ and } t_{\{x,y\}} \in \{0,1\},\ \forall x < y \\
 & b_{(x,y)} + b_{(y,x)} + t_{\{x,y\}} = 1,\ \forall x < y \\
 & b_{(x,y)} + b_{(y,z)} + t_{\{y,z\}} - b_{(x,z)} \leq 1,\ \forall x \neq y \neq z \neq x \\
 & b_{(x,y)} + t_{\{x,y\}} + b_{(y,z)} - b_{(x,z)} \leq 1,\ \forall x \neq y \neq z \neq x \\
 & t_{\{x,y\}} + t_{\{y,z\}} - t_{\{x,z\}} \leq 1,\ \forall x < y < z
\end{array}
$$

Table 4: ILP part of exact algorithm

- The fourth constraint states that if x is before y or tied with y and y is before z then x is before z.
- The fifth constraint states that if x is tied with y and y is tied with z, then x is tied with z.

In practice, exact algorithms cannot be used if there are more than a few dozens of elements, even when the input rankings are complete [45]. As we will see in the experiments, our exact algorithm can handle until a bit more than one hundred elements (and even more with some optimization). The efficiency of exacts algorithms can be improved by using partitioning methods.

6.2. Partitioning methods

Since partitioning methods for KCFs are based on graphs, we first briefly recall some concepts rooted in graph theory that will be of use in the section.

6.2.1. Basic background in graph theory

A *directed graph* G is denoted as $G = (V, A)$, where V is the set of *vertices* and $A \subseteq V^2$ is the set of *arcs* (or directed edges): each arc in A is an ordered pair of vertices.

A *strongly connected component* (SCC) of a directed graph is a maximal subgraph in which there is a directed path from any vertices to every other vertex. In other words, an SCC is a part of the graph where every node can reach every other node within the same component and this subgraph cannot be further expanded without losing this property.

Every directed graph can be uniquely decomposed into its SCCs: the SCCs form a partition of the set of vertices. Given a directed graph G, we define the *strongly connected component graph* of G, denoted as G_c. In G_c, each vertex represents a SCC of G. There exists a directed edge from vertex v_1 to v_2 in G_c if and only if there exists a directed path in G from some vertex in SCC v_1 to some vertex in SCC v_2 (by definition of SCCs, this is equivalent to have a directed path in G from any vertex in SCC v_1 to any vertex in SCC v_2). Interestingly, G_c inherently forms a directed acyclic graph (DAG). Given a directed graph G, Tarjan's algorithm [52] returns the SCCs of G.

Finally, a *topological sort* of a DAG is a linear ordering of its vertices such that for every directed edge (u, v) from vertex u to vertex v, u comes before v in the ordering. Given a DAG G, Kahn's algorithm [53] returns a topological sort of G.

6.2.2. Partitioning methods for KCFs

Previous works, especially [13, 39] present graph-based methods to divide the initial problem into independent sub-problems for two *ad-hoc* KCFs. In this subsection, we generalize these works for any KCF.

The *graph of elements* defined in [39] can be generalized for any KCF in the following way.

Definition 6. *Let $f = S^{(B,T)}$ be a KCF. The graph of elements of R with respect to f is the directed graph $G_{(R,f)} = (V, E)$ defined by:*

1. $V = U$
2. $E = \{(x, y) \in V^2 : before^{(B,T)}(y, x) > minc^{(B,T)}(x, y)\}$

Notation. For a set $U' \subseteq U$, we denote $R(U')$ the tuples of rankings built from the input tuple of rankings R by removing all the elements which are not in U'.

Theorem 4 below enables to divide the initial problem into several independent sub-problems. It subsumes [39] as it is usable for any KCF. This theorem uses the notions of strongly connected components which can be computed with Tarjan's algorithm [52], and of topological sort which can be computed with Kahn's algorithm [53].

Theorem 4. *Let f be a KCF, R be a tuple of rankings, $G^c_{(R,f)}$ be the graph of the strongly connected components of $G_{(R,f)}$, $\mathscr{T} = [\mathscr{T}_1, \mathscr{T}_2, \ldots, \mathscr{T}_k]$ be a topological sort of $G^c_{(R,f)}$ ($\mathscr{T}_i$ are the strongly connected components of $G_{(R,f)}$) and μ_i be a median for $R(\mathscr{T}_i)$ for $1 \leq i \leq k$. Then the concatenation $\mu_1.\mu_2 \ldots \mu_k$ is a median for R.*

Note that the graph $G_{(R,f)}$ often admits several topological sorts, each of which leading to a distinct median.

Corollary 1. *Let f be a KCF, R be a tuple of rankings and $G^c_{(R,f)}$ be the graph of the strongly connected components of $G_{(R,f)}$ the graph of elements. The number of topological sorts of $G^c_{(R,f)}$ is a lower bound of the number of medians of R with respect to f.*

The following theorem allows to compute a partition respected by all the medians. This is particularly interesting for two tasks: (i) reducing the computation time when computing all the optimal medians and (ii) representing common points between all the medians without needing to compute them. It generalizes the approach [39] for any KCF.

Theorem 5. *Let $G_{(R,f)} = (V, A)$ be the graph of elements and let $\mathscr{R}$ be the set of all the pairs $(x, y) \in E$ such that $before(y, x) > minc(x, y)$ and $tied(x, y) > minc(x, y)$. Let $P = [P_1, P_2, ..., P_k]$ be an ordered partition of V such that:*

1. $\forall i < j$, $\forall x \in P_i$, $\forall y \in P_j$, $(y, x) \notin E$, *and*
2. $\forall i$, $\forall x \in P_i$, $\forall y \in P_{i+1}$, $(x, y) \in \mathscr{R}$.

Then each optimal consensus respects P.

Both partitions defined in Theorems 4 and 5 can be computed in $O(|R| * |U|^2)$. The partitions are obtained using the strongly connected components of the graph of elements (Definition 6). In this graph, there is both an arc from x to y and an arc from y to x if and only if $tied(x, y) < before(x, y)$ and $tied(x, y) < before(y, x)$. Intuitively, having low values for T_i compared to B_i reduces the chance to get many sub-problems as low values of T_i increase the chance that $tied(x, y) < before(x, y)$ and $tied(x, y) < before(y, x)$ that is two arcs for the pair $\{x, y\}$.

Proposition 8. *Let $x \in U$, $y \in U$. Let $f = S^{(B,T)}$ and $g = S^{(B',T')}$ be two KCFs such that the following conditions hold:*

1. $B = B'$,
2. $T_i \geq T'_i$ *for all* $1 \leq i \leq 6$, *and*
3. *there exists* $1 \leq i \leq 6$ *such that* $T_i > T'_i$ *and* $\Omega^R_{x,y}[i] > 0$.

Then,

(*i*) $before^{(B,T)}(x,y) = before^{(B',T')}(x,y)$, *and*

(*ii*) $tied^{(B,T)}(x,y) > tied^{(B',T')}(x,y)$

A consequence of the latter proposition is that the set of arcs of the graph of elements with respect to f is included in the set of arcs of the graph of elements with respect to g. The consequences of having low values of T (compared to B) on the ability to partition the initial problem into many several sub-problems are investigated in the experiments (Subsection 7.3).

The sub-problems obtained by partitioning methods may not be small enough to use an exact algorithm. In this situation, it is necessary to use a heuristic. We now present heuristics that can be used for KCFs.

6.3. Adapted Kemeny Heuristics for KCFs

Several heuristics are used in the context of rank aggregation of complete rankings without ties. Several of them like KwikSort [14], CopelandMethod [26], BioConsert [44], Borda [25], MedRank [5], Pick-A-Perm [14] have been generalized in [45] to handle rankings with ties. Most of them are also easily adaptable to fit with any KCF whereas some others are not. Basically, heuristics that are based on pairwise comparisons can easily be generalized to KCFs although other methods cannot. We provide here an intuition of how to adapt a given heuristic.

Adaptation of KwikSort to any KCF. KwikSort [14] is a 2- approximation in the context of rank aggregation of complete rankings without ties. It is a divide-and-conquer algorithm. It is defined in [14] and adapted by [45] to handle ties. The principle is the following one: first select a pivot, then every remaining element y must be placed (i) in the left group if y is before the pivot in a majority of rankings, (ii) in the right group if the pivot is before y in a majority of rankings, otherwise (iii) in the same group of the pivot if the pivot and y are tied in a majority of rankings. Then, repeat recursively this procedure on the left group and the right group. The following modification makes KwikSort available for any KCF: y must be placed (i) in the left group if $before_R(y, pivot) = minc(pivot, y)$, (ii) otherwise in the right group if $before_R(pivot, y) = min_R(pivot, y)$, and (iii) otherwise in the same group of the pivot if $tied_R(pivot, y) = min_R(pivot, y)$. The complexity of KwikSort is $O(|R| * |U| * \log |U|)$.

Adaptation of Copeland Method to any KCF. The Copeland method [26] consists in associating a score to each element $x \in U$:

$$Cop(x) = w(x) - l(x),$$

where $w(x)$ (resp. $l(x)$) is the number of elements y such that x is before y (resp. y is before x) in a strict majority of rankings. In our framework, $w(x)$ (resp. $l(x)$) denotes the number of elements y such that $before(x, y) < before(y, x)$ and $before(x, y) < tied(x, y)$ (resp. $before(y, x) < before(x, y)$ and $before(y, x) < tied(x, y)$). The complexity of the Copeland method is $O(|R| * |U|^2)$.

In a similar way, the local search heuristics BioConsert [44] can be generalized to fit with any KCF. Some other methods cannot be naturally generalized for every KCF, but they may be adapted for some KCFs that are used in the state of the art. For example, it makes sense to extend Borda to fit with the generalization of Kemeny presented in [21] by associating to each element its mean position on the rankings where it is ranked (in [21], non-ranked elements are not penalized). The elements are then sorted by increasing score. MedRank can also be extended to handle [21].

The framework presented in Section 4 and the algorithms presented in this section, including the exact algorithm, have been implemented within a Python packaged named corankco available at `https://pypi.org/project/corankco/`.

7. Experiments

We present three series of experiments in this section. First, we provide an evaluation of the scalability of our exact algorithm and the benefits of Proposition 2 and Theorem 4 to reduce the computation time of the exact algorithm (Subsection 7.2). Second, we highlight links between the values of the penalty vectors of our model and the ability to partition the initial problem into smaller independent sub-problems using Theorem 4 (Subsection 7.3). Finally, in Subsection 7.4, we present experiments which underline the importance of correctly setting parameters to interpret non-ranked elements. We consider two datasets where non-ranked elements should be interpreted as either irrelevant or not fairly comparable to ranked elements. Without loss of generality (cf. subsection 5.2), we set $B_2 = 1$ in all the experiments.

7.1. Setting and Datasets

Setting. Experiments were conducted on a four dual-core processor Intel Core 2.9GHz with 32GB memory desktop using python 3.8. The source code for the experiments is hosted in a separate project, distinct from the main Corankco Python package. This project can be accessed at `https://github.com/pierreandrieu/experiments-corankco`. It includes the datasets, a Dockerfile, and a readme file for reproducing the experiments.

Datasets. Two datasets have been used for these experiments. The **biological dataset** consists of disease names and for each disease name, a tuple of rankings of genes obtained by querying the database Gene [54] using different formulations (synonyms) of the disease name. Based of the work of [55], we considered a total of 1,968 diseases (that is, 1,968 tuples of rankings). The rankings are (naturally) incomplete and without ties. The number of rankings in each tuple of rankings varies from 3 to 63 with a mean of 10.9 rankings, and the number of elements (genes) in each tuple of rankings varies from 30 to 1,121 with a mean of 215.3 genes.

min	25%	50% (median)	75%	max
30	84	201	308	1121

Table 5: Distribution of the number of elements (genes) in the tuples of rankings of the biological dataset.

The **student dataset** is a generated dataset inspired by a real use case involving Master students. We consider here 100 groups of 300 students. A list of 17 teaching units are proposed to each group of students. Depending on the track they are registered on, they should validate a number of units: 280 students registered to track A have to validate 14 units (out of 17) while 20 students registered to track B have to validate 9 units (out of the same list of 17 units). Students registration to units have been uniformly generated and grades have been obtained following two normal laws $\mathcal{N}(\mu, \sigma^2)$ (one for each track) described here-after: (i) $\mathcal{N}(10, 25)$ for the 280 students of track A ; (ii) $\mathcal{N}(16, 16)$ for the 20 students of track B. Grades higher than 20 (resp. smaller than 0) are set to 20 (resp. 0).

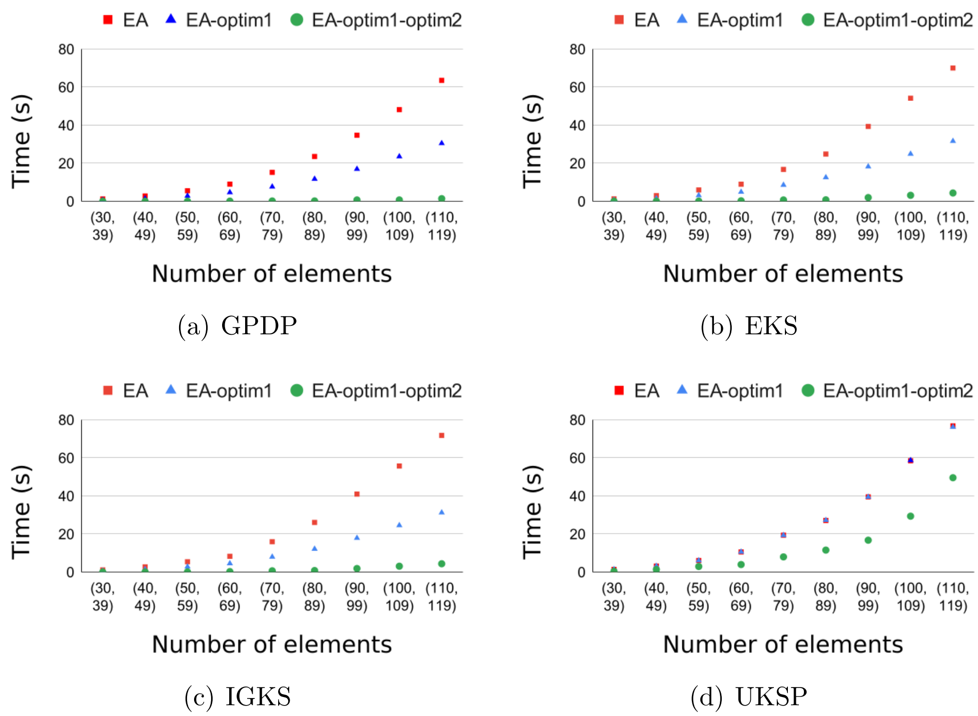
By ranking the students according to their grade in each unit, this dataset can be seen as 100 tuples of 17 rankings over the 300 students. The rankings are incomplete as all the students are not involved in all the units and the rankings contain ties as students who have the same grade for a unit (rounded to one digit after the point) are considered as tied.

7.2. Scalability of the exact algorithm

In this subsection, we focus on the number of elements to rank since it is the parameter which induces the exponential complexity in the Kemeny framework. Classically, exact algorithms can barely handle datasets containing more than a few dozens of elements to rank [23, 45]. Interestingly, an ILP and a Branch and Bound algorithm have been recently introduced [20] allowing to rank until one hundred of elements, but only for complete rankings without ties. Our implementation employs the well-regarded optimization solver, IBM ILOG CPLEX Optimization Studio (v.20.10) [56], recognized for its performance in optimization problems. This can potentially provide us with a competitive edge in achieving superior results compared to other studies, such as the matrix-based approach outlined in [57]. While this latter method is constrained, capable of handling only up to 17 elements within an hour, it nonetheless has the benefit of being self-contained.

We evaluate the scalability of our exact algorithm by considering the exact algorithm without optimization (EA) presented in Subsection 6.1, the optimisation offered by Proposition 2 (we call the corresponding algorithm EA-optim1) and by Theorem 4 (EA-optim2). We use a subset of the biological dataset: we consider the diseases for which the total number of genes in rankings range from 30 to 119. We compute a consensus ranking for each disease name and its synonyms, using the four generalizations of the Kemeny score in the literature able to handle incomplete rankings, namely UKSP, GPDP, EKS, IGKS (see Table 3 p.18). We ignored IKS as IGKS is a generalization of IKS for rankings with ties. We set $p = 1$ (see Table 3) to avoid creating artificial ties in the consensus (as there are no ties in the biological dataset).

Results. Figure 1 shows that the penalty vectors have no effect on the computation time of EA (without optimization). Moreover, Proposition 2 (EA-optim1) allows to reduce by half the computation time of EA in a very regular way for the first three KCFs and has almost no effect on average on the fourth KCF. We can see that optim1-optim2 (the combination of Proposition 2 and Theorem 4) is strongly efficient on the first three KCFs since it divides the computation time of the EA by approximately 100 on the smallest datasets (30-49 elements) and by approximately 20 on the larger



(a) GPDP (b) EKS

(c) IGKS (d) UKSP

Figure 1: Experiment 1 - Computation time of EA, EA-optim1 and EA-optim1-optim2 with the four generalizations of the Kemeny score able to handle incomplete rankings: GPDP, EKS, IGKS, UKSP (see Table 3 p.18).

datasets (50-119). Finally, optim1-optim2 is also quite efficient on the fourth KCF as it reduces the computation time in a regular way by 1.5. Note that even if Proposition 2 appears less efficient than Theorem 4, both are useful as several datasets can only benefit from Proposition 2 (they have only one strongly connected components in the sense of Theorem 4).

We now consider a second experiment where we measure the computation time of EA in its fully optimized version on the biological dataset (that is, we consider EA-optim1-optim2) with the four same KCFs, considering the diseases for which the total number of genes in the rankings is more than 130. For each dozen of number of elements (130-139, etc.) if all the corresponding disease have a computation time < 600s, we try the following range, otherwise we stop the experiment for this KCF.

Results. EA-optim1-optim2 can handle until 159 elements with UKSP

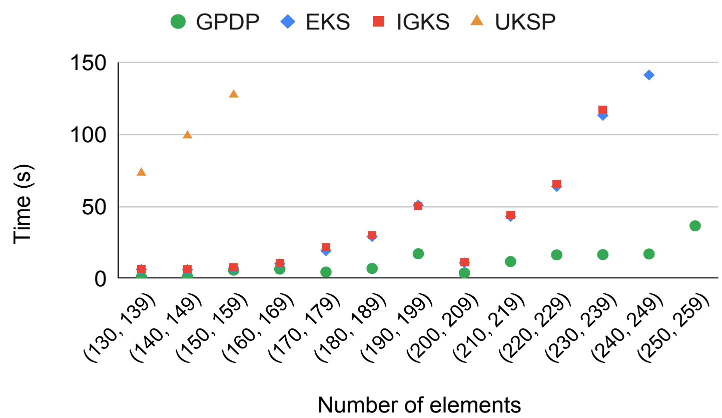


Figure 2: Experiment 2 - Average computation time of EA with optim1 + optim2 for the four generalizations of the Kemeny score able to handle incomplete rankings: GPDP, EKS, IGKS, UKSP (see Table 3 p.18).

which is few compatible with the optimizations. Interestingly, our optimized EA could manage very big datasets with GPDP, EKS and IKGS (until 259 elements for GPDP, with a mean time computation of 36 seconds). We are not aware of any exact algorithm in the related work able to deal with such real big datasets.

7.3. On the ability to partition

The previous experiment highlighted that the efficiency of optim1 and optim2 depends on the values of the penalty vectors. Here we investigate the correlations between the values of the penalty vectors and the ability to partition the initial problem into sub-problems using Theorem 4 (optim2).

Impact of penalty vector T. As discussed around Proposition 8, having low values for T compared to B reduces the chance to get many sub-problem. Here, for a given tuple of rankings and a given KCF, we focus when T decreases on the size of the largest SCC (strongly connected component) *i.e.* the size of the biggest sub-problem to resolve. The lower this number, the easiest the computation of a median.

We fix $B = (0, 1, 1, 0, 1, 0)$ as in GPDP and make T_i vary from 0 to 1 by steps of 0.25 for all i (except T_3 whose value is set to 0, as required by our model). We split the 1,968 biological datasets into four groups based on their size: less than 60 elements (gr_1), between 60 and 99 elements (gr_2) between 100 and 299 elements (gr_3) and 300 elements or more (gr_4). For each value of T and each group, we compute the average size of the biggest SCC.

T_i ($i \neq 3$) / Nb of elements	0	0.25	0.5	0.75	1
gr_1 : (30, 60)	40.4	10.2	3.0	3.0	3.0
gr_2 : (60, 99)	72.0	21.3	8.1	8.1	8.1
gr_3 : (100, 299)	188.8	57.6	34.3	34.3	34.3
gr_4 : (300, 1121)	418.7	187.0	165.1	165.1	165.1

Table 6: Experiment 3 - Average size of the biggest sub-problem to resolve using Theorem 4 in biological dataset according to the value of T_i for $i \neq 3$ ($T_3 = 0$).

Results. Table 6 shows that high values in T make the problems highly partitionable into sub-problems as they are related to low maximal size of SCCs (compared to the number of elements). On the contrary, low values in T reduce the ability to divide the initial problem into small sub-problems.

Impact of $B_6 - T_6$. We now investigate the influence of $B_6 - T_6$ on the ability to partition. This value is especially important as it is related to the unification process very commonly used in the literature [44, 15, 36] to transform real (incomplete) datasets into complete(d) datasets by appending at the end of each incomplete ranking the non-ranked elements in a unifying bucket.

Recall that B_6 (resp. T_6) is the penalty for placing x before y (resp. tying x and y) in a consensus for each input ranking such that x and y are both non-ranked. The value of $B_6 - T_6$ indicates how much the tied elements in the unifying bucket should really be considered as "strongly" tied. Note that B_6 is the only penalty that differs between GPDP, the KCF with the most efficient EA-optim1-optim2, and UKSP, the KCF with the least efficient EA-optim1-optim2 (see Table 3 and Figures 1 and 2).

For this experiment, we consider the penalty vectors of GPDP (with $p = 1$) except B_6 that varies from 0 to 1 by steps of 0.25. From Theorem 2, two KCFs with the same values of B_i and T_i for $i \leq 5$ are equivalent if they have the same value of $B_6 - T_6$ so that there is no need to make also vary T_6. We use the same tuples of rankings as in the previous experiment.

$B_6 - T_6$ / Nb of elements	0	0.25	0.5	0.75	1
gr_1 : (30, 60)	3.0	21.4	32.4	39.0	39.4
gr_2 : (60, 99)	8.1	40.3	57.7	68.7	69.1
gr_3 : (100, 299)	34.3	125.8	168.9	186.0	186.6
gr_4 : (300, 1121)	165.1	357.5	392.7	408.0	408.4

Table 7: Experiment 4 - Average size of the biggest sub-problem to resolve using Theorem 4 in biological dataset according to the value of $B_6 - T_6$.

Results. Table 7 shows that if B_6 is equal to T_6, the problems are highly partitionable. On the contrary, if B_6 is too high compared to T_6, then there is no benefit to use the partitioning. The ability to partition decreases particularly quickly. This explains the efficiency of Theorem 4 to reduce the computation time of the exact algorithm (see Subsection 7.2) for the three KCFs having $B_6 = T_6$ and the weakened efficiency of EA-optim1-optim2 with UKSP having $B_6 - T_6 = 1$. As a conclusion, B_6 should not be higher than T_6, unless the use case requires the non-ranked elements to be strongly considered as tied.

7.4. Signification of non-ranked elements

This subsection focuses on the role of B_4 and B_5 and shows that these coefficients should be set according to the signification of non-ranked elements. Recall that B_4 is the cost for placing x before y in the consensus for each input ranking where x is ranked, whereas y is non-ranked in an input ranking. B_5 is the cost for placing x before y in the consensus for each input ranking where y is ranked whereas x is non-ranked in an input ranking. In this context, $B_5 - B_4$ expresses how strongly non-ranked elements should be penalized. $B_5 - B_4 = 0$ means that non-ranked elements are not penalized at all (non-ranked elements cannot be compared to ranked elements) and $B_5 - B_4 > 0$ means that non-ranked elements should be penalized (non-ranked elements are interpreted as less relevant as ranked ones).

Theorem 2 shows that B_4, B_5 and T_5 are coefficients that should be considered together. Moreover, the KCF model requires $B_4 \leq B_5$ and $T_4 = T_5$. The coefficients of (B, T) for the experiments of this subsection are stated in Table 8. We set $B_4 = 0$ and make $B_5 = T_5$ vary from 0 to 2. Proposition 6 shows that this variation indeed gives KCFs that are not equivalent.

B	0	1	1	0	B_5	0
T	1	1	0	B_5	B_5	0

Table 8: Penalty vectors for experiments 5 and 6. B_5 varies from 0 to 2.

To compute the consensus for the experiments of this subsection, we first use Theorem 4 to divide the initial problems into sub-problems, then EA-optim1-optim2 is used for the sub-problems with 150 or less elements to rank, and the heuristics BioConsert [44] (adapted for the KCFs), which provided good results in a previous benchmark on complete rankings [45], is used for bigger sub-problems.

7.4.1. Experiment 5: When non-ranked elements are less relevant than ranked ones in a given ranking

For this experiment, we use the biological dataset. Recall that for each disease name, a consensus ranking of genes is computed based on several rankings of genes obtained by different formulations of queries on the database Gene [54]. Here, we use a goldstandard (GS) where for each disease, a set of genes known to be involved with this disease are provided. Such a GS is given by the database Orphanet [58] and available for 125 diseases. We make B_5 vary from 0 to 2. For each obtained KCF and for k varying between 10 and 100, we evaluate the consensus ranking obtained on the 125 datasets by counting the number of genes from the GS retrieved in the top-k elements of the consensus. The sum of the size of the GS over the 125 diseases is 515 (occurrences of genes).

Signification of non-ranked elements. In this context, elements (genes) which are absent in a ranking are strictly less interesting that genes that appear in such a ranking as the database considers these elements as irrelevant toward the formulation of the disease. In other words, non-ranked elements should be penalized.

Results. Result are presented in Figure 3. Finding 50% of the genes of the goldstandard in the consensus ranking is performed (i) already in the top-20 for KCFs with $B_5 - B_4$ positive and (ii) only in the top-90 for KCFs with $B_5 - B_4 = 0$. Not penalizing the non-ranked elements whereas it should be done can thus lead to poor quality aggregation.

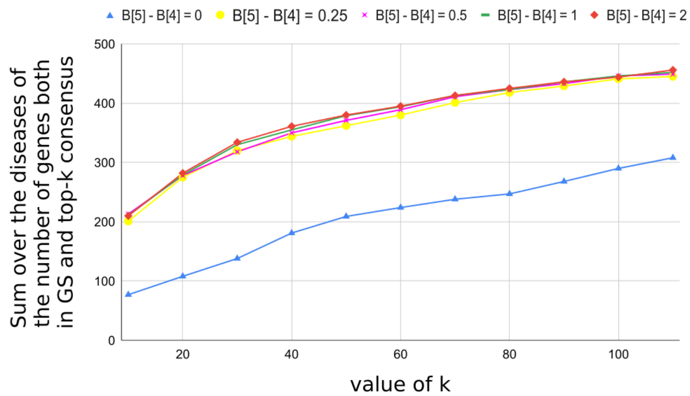


Figure 3: Experiment 5 - Ability to rank well the genes of the GS according to $B_5 - B_4$.

7.4.2. Experiment 6: When non-ranked elements are not fairly comparable towards ranked ones in a given ranking

In this experiment, we consider the student datasets. To evaluate the quality of the consensus obtained we count the number of students in the top-20 of the consensus who are also in the top-20 of the overall average. Here, the overall average is thus our gold standard. $B_5 - B_4$ still varies from 0 to 2.

Signification of non-ranked elements. Here, students which are absent in a ranking have not followed a class, they cannot be compared with other students of that class. non-ranked elements can be interpreted as unknown (incomparable) and should thus not be penalized.

Results. Results are provided in Figure 4. When $B_5 - B_4 = 0$, on average 19 students (out of 20) are both in the top-20 consensus ranking and in the top-20 of the gold standard (overall average). Interestingly, this number declines significantly with the increase of $B_5 - B_4$. Here, fixing $B_5 - B_4 = 0$ provides much better quality results than $B_5 - B_4 > 0$. As a conclusion, the quality of the results is degraded when the penalties for non-ranked elements values are not properly set, that is here when $B_5 - B_4 > 0$.

8. Concluding discussion

Aggregating multiple rankings into one consensus ranking is needed in a very large number of domains. This task is very challenging, because

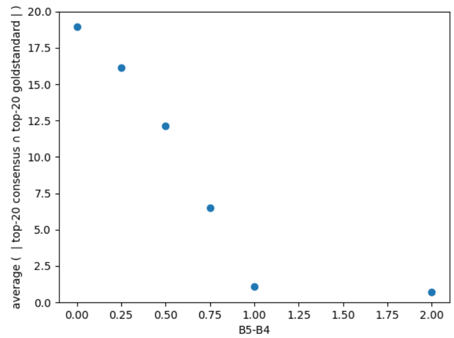


Figure 4: Experiment 6 - Mean number of students both in top-20 consensus and overall average ranking (goldstandard) according to $B_5 - B_4$

of its algorithmic complexity, and because rankings encountered may have ties, and more importantly non-ranked elements. Moreover, the method to aggregate rankings has to be chosen carefully, since different methods may given different results, not all relevant depending on the context.

In this paper, we presented an unifying framework for Kemeny rank aggregation considering any rankings with/out ties and with/out non-ranked elements. All the existing generalizations of the Kemeny score for rankings in the related work can be represented in this framework (see Table 3 p.18). We have extensively studied this framework both theoretically and experimentally, providing contributions on axiomatic and algorithmic settings. In particular, we have provided an exact algorithm able to manage big real dataset.

Our framework has two vectors B and T of six real positive coefficients as parameters (see Table 2 p.15) so that it can be well fitted to different use cases. We summarize below some restrictions and guidelines for the choice of parameter values.

First of all, three restrictions have to be considered to stay within a Kemeny framework: $B_1 = T_3 = 0$, $T_1 = T_2$ and $T_4 = T_5$. We now discuss the suitable values for the parameters regarding four major questions:

- Do we need to **prevent ties** (ex aequo elements) in the consensus provided?

- How should we **manage the ties**? Are elements tied because they are indifferently before/after one another or rather because they are intrinsically linked to one another, and it would make no sense to dissociate them?

- How should we **manage non-ranked elements**? Are non-ranked elements interpreted as data less relevant than ranked elements or rather data that are unknown/incomparable with ranked data?

- Do we want to generate a consensus that **respects 3 classical axioms** related to the social choice? (namely, Majority-like axiom, Condorcet-like axiom, Smith-like axiom, see Theorem 3).

Preventing ties in the consensus. We imposed the consensus rankings to be complete. If in a use case a consensus without ties is expected, then the values should verify the following conditions:

- $B_2 \leq 2 * T_2$,

- $B_3 = 0$,

- $B_4 + B_5 \leq T_4 + T_5$, and

- $B_6 \leq T_6$.

This choice ensures at least one median is a complete ranking without ties, *i.e.*, a linear ordering (see Proposition 5)

Penalizing or not penalizing ties. If tied elements are interpreted as elements that could be indifferently before/after one another then set $B_3 = 0$. If tied elements means that it makes no sense to dissociate them, then set $B_3 > 0$ (see Section 4.1).

signification of non-ranked elements and parameters. The value $B_5 - B_4$ represents how much the non-ranked elements must be penalized with respect to the ranked elements. If non-ranked elements are interpreted as less relevant than the ranked ones, then set $B_5 > B_4$. If the non-ranked elements are incomparable with the ranked elements, then set $B_5 = B_4$ (cf Subsection 5.2 and Subsection 7.4).

Respecting or not some axioms. To respect the Majority-like, the Condorcet-like and the Smith like axioms defined in Subsection 5.3, the following conditions should be respected: $B_2 = B_5 - B_4$ and

$$\min_{i\in\{1,4\}}(T_i - B_i) \geq \max_{j\in\{2,3,5,6\}}(B_j - T_j).$$

These axioms guarantee that under some conditions on the rankings, all the medians ranks alone in position 1 the same element x (cf Subsection 5.3).

Efficiency. Parameter values have an impact on the ability to partition the initial problem into independent sub-problems (cf. Proposition 8 and Subsection 7.3). If a value of T_i is too low compared to B_i, then the ability to partition is low, that is, the number of sub-problems obtained using Theorem 4 is small.

Our framework allows to make an informed choice of the method to aggregate real rankings according to the context, and to have available an exact algorithm (with optimization) and heuristics. Empirical results with real and synthetic datasets demonstrate that the framework is effective.

Future work. Future works will be both theoretical and experimental. We aim to investigate further the complexity of the rank aggregation problem when consensus rankings may have ties, which remains an open problem since 2004 with the work of Fagin *et al.*. We also plan to extend works about generation of complete rankings without ties to generate synthetic datasets of incomplete rankings with ties while controlling several parameters such as similarity and proportion of non-ranked elements. Such datasets will be helpful to highlight correlations between these parameters and the importance of the parameters of our framework with respect to the consensus rankings.

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Appendix A. Proofs

This appendix is organized as follows: Appendix A.1 gives the proofs of Subsection 4.3, Appendix A.2 gives the proofs of Subsection 5.1, Appendix A.3 gives the proofs of Subsection 5.2, Appendix A.4 gives the proofs of Subsection 5.3 and Appendix A.5 gives the proofs of Subsection 6.2.

Appendix A.1. Proof of Proposition 1 (Subsection 4.3)

Proposition 1. *Let $f(c,R) = S^{(B,T)}(c,R)$ be a KCF. Then, there exists a function g such that for all $(c,R) \in dom(f)$,*

$$f(c,R) = \sum_{r \in R} g(c,r). \tag{4}$$

Proof. Following Definition 2, $\forall (c,R) \in dom(f)$,

$$\begin{aligned} f(c,R) &= \sum_{\substack{x \neq y \in U \\ x \prec_c y}} \langle B, \Omega^R_{x,y} \rangle \ + \ \frac{1}{2} * \sum_{\substack{x \neq y \in U \\ x \equiv_c y}} \langle T, \Omega^R_{x,y} \rangle \\ &= \sum_{x \neq y \in U} \mathbf{1}_{x \prec_c y} * \langle B, \Omega^R_{x,y} \rangle \ + \frac{1}{2} * \mathbf{1}_{x \equiv_c y} * \langle T, \Omega^R_{x,y} \rangle \end{aligned}$$

By definition of B, T and $\Omega^R_{x,y}$ (see Table 2 and Definition 2), we obtain:

$$\begin{aligned} f(c,R) = \sum_{r \in R} \quad & \sum_{x \neq y \in U} \mathbf{1}_{x \prec_c y} * (\mathbf{1}_{y \prec_r x} * B_2 + \mathbf{1}_{x \equiv_r y} * B_3 \\ & + \mathbf{1}_{x \diamond_r y} * B_4 + \mathbf{1}_{y \diamond_r x} * B_5 + \mathbf{1}_{x \notin dom(r) \wedge y \notin dom(r)} * B_6) \\ & + \frac{1}{2} * \mathbf{1}_{x \equiv_c y} * (\mathbf{1}_{x \prec_r y \ \vee \ y \prec_r x} * T_1 + \mathbf{1}_{x \diamond_r y \ \vee \ y \diamond_r x} * T_4 \\ & + \mathbf{1}_{x \notin dom(r) \ \wedge \ y \notin dom(r)} * T_6) \\ = & \sum_{x \neq y \in U} \quad \sum_{r \in R} \mathbf{1}_{x \prec_c y} * (\mathbf{1}_{y \prec_r x} * B_2 + \mathbf{1}_{x \equiv_r y} * B_3 \\ & + \mathbf{1}_{x \diamond_r y} * B_4 + \mathbf{1}_{y \diamond_r x} * B_5 + \mathbf{1}_{x \notin dom(r) \wedge y \notin dom(r)} * B_6) \\ & + \frac{1}{2} * \mathbf{1}_{x \equiv_c y} * (\mathbf{1}_{x \prec_r y \ \vee \ y \prec_r x} * T_1 + \mathbf{1}_{x \diamond_r y \ \vee \ y \diamond_r x} * T_4 \\ & + \mathbf{1}_{x \notin dom(r) \ \wedge \ y \notin dom(r)} * T_6) \\ = & \sum_{r \in R} g(c,r) \end{aligned}$$

with

$$
\begin{aligned}
g(c,R) = \sum_{x \neq y \in U} \mathbf{1}_{x \prec_c y} * (& \mathbf{1}_{y \prec_r x} * B_2 + \mathbf{1}_{x \equiv_r y} * B_3 + \mathbf{1}_{x \diamond_r y} * B_4 \\
& + \mathbf{1}_{y \diamond_r x} * B_5 + \mathbf{1}_{x \notin dom(r) \wedge y \notin dom(r)} * B_6) \\
& + \frac{1}{2} * \mathbf{1}_{x \equiv_c y} * (\mathbf{1}_{x \prec_r y \,\vee\, y \prec_r x} * T_1 \\
& + \mathbf{1}_{x \diamond_r y \,\vee\, y \diamond_r x} * T_4 \\
& + \mathbf{1}_{x \notin dom(r) \,\wedge\, y \notin dom(r)} * T_6)
\end{aligned}
$$

□

Appendix A.2. Proofs of Subsection 5.1

Appendix A.2.1. Additional content

Definition 7. *Let R be a tuple of possibly incomplete rankings and let f be a KCF. An ℓ-**median** of R with respect to f is a ranking $c \in \mathscr{L}(U)$ s.t. $(c, R) \in dom(f)$ and $f(c, R) \leq f(c', R)$ for every $c' \in \mathscr{L}(U)$ such that $(c', R) \in dom(f)$.*

In other words, an ℓ-median is a complete ranking without ties on U such that no other complete ranking without ties on U can represent R better.

Notation. Let f be a KCF and R be a tuple of rankings. We denote by $\boldsymbol{\Lambda_f(R)}$ the set of all the ℓ-medians of R with respect to f.

Proposition 9. *All KCFs are equivalent on $\mathscr{L}(U) \times \mathscr{L}(U)^{<\infty}$.*

Proof. Let $f = S^{(B,T)}$, $g = S^{(B',T')}$ be two KCFs and $k = \frac{B'_2}{B_2}$. Let $R \in \mathscr{L}(U)^{<\infty}$ *i.e.* R is a tuple of complete rankings without ties. In other words, $\Omega^R_{x,y}[i] = 0$ for $i \geq 3$. Also, using definition 2, we have $B_1 = 0$. Thus, from (3), we have for any $c \in \mathscr{L}(U)$:

$$
f(c,R) = \sum_{\substack{x,y \in U \\ x \prec_c y}} B_2 * \Omega^R_{x,y}[2] + \frac{1}{2} * \sum_{\substack{x \neq y \in U \\ x \equiv_c y}} T_1 * (\Omega^R_{x,y}[1] + \Omega^R_{x,y}[2]).
$$

Since $c \in \mathscr{L}(U)$, there are no distinct $x, y \in U$ such that $x \equiv_c y$, and hence

$$
f(c,R) = \sum_{\substack{x,y \in U \\ x \prec_c y}} B_2 * \Omega^R_{x,y}[2].
$$

Similarly, we obtain

$$g(c,R) = \sum_{\substack{x,y\in U\\ x\prec_c y}} B_2' * \Omega^R_{x,y}[2] = \sum_{\substack{x,y\in U\\ x\prec_c y}} k * B_2 * \Omega^R_{x,y}[2]$$
$$= k * f(c,R),$$

which completes the proof. $\square$

Corollary 2. *Let f and g be two KCF. If $R \in \mathscr{L}(U)^{<\infty}$, then $\Lambda_f(R) = \Lambda_g(R)$.*

Appendix A.2.2. Proof of Proposition 2

Proposition 2. *Let $f = S^{(B,T)}$ and let R be a tuple of rankings. If $before(x,y) + before(y,x) \leq 2*tied(x,y)$ for all $x \neq y \in U$, then there exists $\mu \in M_f(R)$ such that $\mu \in \mathscr{L}(U)$. Moreover, if the above inequality is strict for all $x \neq y \in U$, then $M_f(R) \subseteq \mathscr{L}(U)$.*

Proof. We show that for every consensus c such that $c \in \mathscr{C}(U)\setminus \mathscr{L}(U)$ (complete ranking with ties), there exists $c' \in \mathscr{L}(U)$ such that $f(c,R) \geq f(c',R)$. Let $l_1, l_2 \in \mathscr{L}(U)$ be two complete rankings without ties such that, for every distinct $x, y \in U$, the following two implications hold:

- $x \prec_c y \Rightarrow x \prec_{l_1} y$ and $x \prec_{l_2} y$.
- $x \equiv_c y \Rightarrow (l_1(x) - l_1(y)) * (l_2(x) - l_2(y)) < 0$ (x and y are in reversed order in l_1 and l_2 whenever they are tied in c).

Note that such complete rankings without ties always exist. Intuitively, l_1 and l_2 are complete rankings without ties compatible with c where the ties have been randomly broken and in reversed order in l_1 and l_2.

We will show that that l_1 (or l_2) represents R as least as well as c with respect to f. So let $\Delta = 2 * f(c,R) - f(l_1,R) - f(l_2,R)$. Thus,

$$\Delta = \sum_{\substack{x\neq y\in U\\ x\equiv_c y\\ x\prec_{l_1} y}} 2 * tied(x,y) - before(x,y) - before(y,x)$$

Let us first consider the inequality

$$before(x,y) + before(y,x) \leq 2 * tied(x,y).$$

We can conclude that $f(c, R) \geq f(l_1, R)$ or $f(c, R) \geq f(l_2, R)$. If c is an optimal consensus, then both l_1 and l_2 are optimal consensus.

Let us now consider the strict inequality in the proposition, *i.e.*,

$$before(x, y) + before(y, x) < 2 * tied(x, y).$$

We can conclude that $f(c, R) > f(l_1, R)$ or $f(c, R) > f(l_2, R)$. Here c can not be an optimal consensus. $\square$

Appendix A.2.3. Additional content (2)

Corollary 3. *Let f and $g = S^{(B,T)}$ be two KCFs such that $B_2 \leq 2 * T_2$ and f is equivalent to g on $dom(f)$. If $R \in \mathscr{L}(U)^{<\infty}$, then there exists $\mu \in M_f(R)$ such that $\mu \in \mathscr{L}(U)$.*

Moreover, if $B_2 < 2 * T_2$, we have $M_f(R) \subseteq \mathscr{L}(U)$.

Proof. Let g is a KCF and $x \neq y \in U$, and set

$$\Delta = 2 * tied(x, y) - before(x, y) - before(y, x).$$

Since g is a KCF, we know that $B_1 = 0$ and $T_1 = T_2$. Moreover, as $R \in \mathscr{L}(U)^{<\infty}$, if $x, y \in U$ are distinct, then $\Omega^R_{x,y}[1] + \Omega^R_{x,y}[2] = |R|$, $\Omega^R_{y,x}[2] = \Omega^R_{x,y}[1]$ and $\Omega^R_{x,y}[i] = \Omega^R_{y,x}[i] = 0$ for $i \geq 3$. Hence, we have:

$$\begin{aligned}\Delta =& 2\,T_2 * (\Omega^R_{x,y}[1] + \Omega^R_{x,y}[2]) - B_2(\Omega^R_{x,y}[2] + \Omega^R_{y,x}[2]) \\ &= (2\,T_2 * |R| - B_2 * |R|) \geq 0.\end{aligned}$$

Suppose that the inequality

$$B_2 \leq 2 * T_2$$

holds. Then $before(x, y) + before(y, x) \leq 2 * tied(x, y)$, and it follows from Proposition 2 that there exists $\mu \in M_g(R)$ such that $\mu \in \mathscr{L}(U)$. As f and g are equivalent on the domain of f, we obtain that there exists $\mu \in M_f(R)$ such that $\mu \in \mathscr{L}(U)$.

Let us now that the strict inequality $B_2 < 2{*}T_2$ holds. Then $before(x, y)+ before(y, x) < 2 * tied(x, y)$, and it follows from Proposition 2 that $M_g(R) \subseteq \mathscr{L}(U)$. As f and g are equivalent on the domain of f, we obtain that $M_f(R) \subseteq \mathscr{L}(U)$. $\square$

A direct consequence is that any two KCFs $f = S^{(B,T)}$ and $g = S^{(B',T')}$ give the same set of medians for any tuple of complete rankings without ties, if the two following conditions holds:

$$B_2 < 2 * T_2 \quad \text{and} \quad B'_2 < 2 * T'_2 \tag{A.1}$$

Corollary 4. *Let $f = S^{(B^f,T_f)}$ and $g = S^{(B^g,T_g)}$ be two KCFs such that* (A.1) *holds. If $R \in \mathscr{L}(U)^{<\infty}$, then $M_f(R) = M_g(R)$.*

Proof. As f and g are two KCFs, it follows from Corollary 2 that $\Lambda_f(R) = \Lambda_g(R)$. Since $B^g[2] \leq 2{*}T_g[2]$ and $B^f[2] \leq 2{*}T_f[2]$, it follows from Corollary 3 that $\Lambda_g(R) = M_g(R)$ and $\Lambda_f(R) = M_f(R)$, and thus $M_f(R) = M_g(R)$. □

Appendix A.2.4. Proof of Theorem 1

Theorem 1. *Let f and $g = S^{(B,T)}$ be two KCFs such that $B_2 \leq 2 * T_2$ and suppose that f is equivalent to g on dom(f). Then, finding a median of $R \in \mathscr{Y}^{<\infty}$ with respect to f is NP-Hard as soon as $|U| > 2$.*

Proof. Let set f be a KCF. We denote C the set of all the possible consensus according to the domain of f.

Instance: $< f, R, k >$ where $k \geq 0$. Question : Is there a consensus $c \in C$ such that $f(c, R) \leq k$?

Suppose that $B_2 \leq 2 * T_2$. Then, it follows from Corollary 3 that this problem is equivalent to determining whether there is a complete ranking without ties c such that $f(c, R) \leq k$. This is the rank aggregation problem for complete rankings without ties, which is NP-hard for an even number of rankings $m \geq 4$ [21, 22] and for an odd number of rankings $m \geq 7$ [18]. □

Appendix A.3. Proofs of Subsection 5.2

Appendix A.3.1. Proof of Proposition 3

Proposition 3. *If there exists $\phi\colon \mathbb{R} \to \mathbb{R}$ strictly increasing such that $g(c, R) = \phi(f(c, R))$, for all $(c, R) \in \mathscr{X} \times \mathscr{Y}^{<\infty}$, then f and g are equivalent on $\mathscr{X} \times \mathscr{Y}^{<\infty}$.*

Proof. Let $c \in M_f(R)$. Then, for all $c' \in \mathscr{X}$, we have $f(c, R) \leq f(c', R)$. As ϕ is increasing, then $g(c, R) \leq g(c', R)$, which means that $c \in M_g(R)$. Thus $M_f(R) \subset M_g(R)$. Moreover, since ϕ is strictly increasing, if $g(c, R) = g(c', R)$ then $f(c, R) = f(c', R)$, which implies that $M_g(R) \subset M_f(R)$. Finally $M_f(R) = M_g(R)$, so f and g are equivalent. □

Appendix A.3.2. Proof of Proposition 4

Lemma 1. *Let $S^{(B,T)}$ be a KCF. Then, for every $x, y \in U$ and $R \in \mathscr{A}(U)^{<\infty}$, we have that:*

$$\begin{aligned} before(x,y) - before(y,x) = B_2 * (\Omega^R_{x,y}[2] - \Omega^R_{x,y}[1]) \\ + (B_5 - B_4) * (\Omega^R_{x,y}[5] - \Omega^R_{x,y}[4]). \end{aligned} \tag{A.2}$$

Moreover, we have that

$$before(x,y) - tied(x,y) = \sum_{1 \le i \le 6} \Omega^R_{x,y}[i] * (B_i - T_i). \tag{A.3}$$

Proposition 4. *Two KCFs f and g that are $\ominus$−equivalent have the same set of medians.*

Proof. Let c be a median of R with respect to f. Then, using Definition 3, $f(c,R) \le f(c',R)$ that is $f(c,R) - f(c',R) \le 0$ for all $c \in \mathscr{C}(U)$. We know by hypothesis that $g(c,R) - g(c',R) = f(c,R) - f(c',R)$ that is $g(c,R) - g(c',R) = 0$. Hence, c is also a median of R with respect to g. □

Appendix A.3.3. Proof of Theorem 2

Theorem 2. *Let $f = S^{(B,T)}$ and $g = S^{(B',T')}$ be two KCFs defined on $\mathscr{C}(U) \times \mathscr{A}(U)^{<\infty}$. They are $\ominus$-equivalent on $\mathscr{C}(U) \times \mathscr{A}(U)^{<\infty}$ if and only if the following conditions hold:*

- $B_2 = B'_2$,
- $T_2 = T'_2$,
- $B_3 = B'_3$,
- $B_4 - B'_4 = B_5 - B'_5 = T_5 - T'_5$,
- $B_6 - B'_6 = T_6 - T'_6$.

Proof. We first prove the condition is sufficient that is f and g are $\ominus$-equivalent if the above condition is respected.

Let $f = S^{(B,T)}$ and $g = S^{(B',T')}$ be two KCFs defined on $\mathscr{C}(U) \times \mathscr{A}(U)^{<\infty}$ such that the following conditions hold.

- $B_2 = B'_2$

- $T_2 = T_2'$
- $B_3 = B_3'$
- $B_4 - B_4' = B_5 - B_5' = T_5 - T_5'$
- $B_6 - B_6' = T_6 - T_6'$

Note that for any two consensus rankings (complete rankings) c and c', the value of $f(c, R) - f(c', R)$ does not depend on the pairs of elements such that their relative order is the same between c and c' (they induce the same cost). To improve readability we will omit the superscript R, and we will write $before_f(x, y)$ for $before_R^{(B,T)}$, $tied_f(x, y)$ for $tied_R^{(B,T)}$, $before_g(x, y)$ for $before_R^{(B',T')}$, and $tied_g(x, y)$ for $tied_R^{(B',T')}$.

Set $A = f(c, R) - f(c', R)$. We have

$$
\begin{aligned}
A = & \sum_{\substack{x,y \in U \\ x \prec_c y \\ y \prec_{c'} x}} before_f(x, y) - before_f(y, x) \\
& + \sum_{\substack{x,y \in U \\ x \prec_c y \\ x \equiv_{c'} y}} before_f(x, y) - tied_f(x, y) \\
& + \sum_{\substack{x,y \in U \\ x \equiv_c y \\ x \prec_{c'} y}} tied_f(x, y) - before_f(x, y).
\end{aligned}
$$

Using Lemma 1, we obtain:

$$
\begin{aligned}
A = & \sum_{\substack{x,y \in U \\ x \prec_c y \\ y \prec_{c'} x}} B_2 * (\Omega_{x,y}^R[2] - \Omega_{x,y}^R[1]) \\
& + (B_5 - B_4) * (\Omega_{x,y}^R[5] - \Omega_{x,y}^R[4]) \\
& + \sum_{\substack{x,y \in U \\ x \prec_c y \\ x \equiv_{c'} y}} \sum_{1 \le i \le 6} \Omega_{x,y}^R[i] * (B_i - T_i) \\
& + \sum_{\substack{x,y \in U \\ x \equiv_c y \\ x \prec_{c'} y}} \sum_{1 \le i \le 6} \Omega_{x,y}^R[i] * (T_i - B_i).
\end{aligned}
$$

Similarly, for $B = g(c, R) - g(c', R)$, we have:

$$B = \sum_{\substack{x,y \in U \\ x \prec_c y \\ y \prec_{c'} x}} B'_2 * (\Omega^R_{x,y}[2] - \Omega^R_{x,y}[1])$$
$$+ (B'_5 - B'_4) * (\Omega^R_{x,y}[5] - \Omega^R_{x,y}[4])$$
$$+ \sum_{\substack{x,y \in U \\ x \prec_c y \\ x \equiv_{c'} y}} \sum_{1 \leq i \leq 6} \Omega^R_{x,y}[i] * (B'_i - T'_i)$$
$$+ \sum_{\substack{x,y \in U \\ x \equiv_c y \\ x \prec_{c'} y}} + \sum_{1 \leq i \leq 6} \Omega^R_{x,y}[i] * (T'_i - B'_i).$$

Set $D = A - B = f(c, R) - f(c', R) - (g(c, R) - g(c', R))$. We have:

$$D = \sum_{\substack{x,y \in U \\ x \prec_c y \\ y \prec_{c'} x}} (B_2 - B'_2) * (\Omega^R_{x,y}[2] - \Omega^R_{x,y}[1])$$
$$+ (B_5 - B_4 - B'_5 + B'_4) * (\Omega^R_{x,y}[5] - \Omega^R_{x,y}[4])$$
$$+ \sum_{\substack{x,y \in U \\ x \prec_c y \\ x \equiv_{c'} y}} \sum_{1 \leq i \leq 6} \Omega^R_{x,y}[i] * (B_i - T_i - B'_i + T'_i).$$

By assumption, $B_2 = B'_2$ and $B_5 - B'_5 = B_4 - B'_4$, and thus:

$$\sum_{\substack{x,y \in U \\ x \prec_c y \\ y \prec_{c'} x}} (B_2 - B'_2) * (\Omega^R_{x,y}[2] - \Omega^R_{x,y}[1])$$
$$+ (B_5 - B_4 - B'_5 + B'_4) * (\Omega^R_{x,y}[5] - \Omega^R_{x,y}[4]) = 0$$

Moreover, as f and g are KCFs, we know that $B_1 = B'_1 = T_3 = T'_3 = 0$, $T_1 = T_2$, $T_4 = T_5$. Using the assumption $B_2 = B'_2$, $T_2 = T'_2$, $B_3 = B'_3$, $B_5 - B'_5 = B_4 - B'_4 = T_5 - T'_5$, $B_6 - B'_6 = T_6 - T'_6$, We thus obtain $\Omega^R_{x,y}[i] * (B_i - T_i - B'_i + T'_i) = 0$ for all $1 \leq i \leq 6$.

Hence, $f(c, R) - f(c', R) - (g(c, R) - g(c', R)) = 0$, *i.e.*, f and g are $\ominus$-equivalent.

To prove necessity, we use for the following a tuple of rankings R that contains only one ranking. The result can be generalized by repeating this ranking m times in R. Without loss of generality, suppose $U = \{1, \ldots, n\}$. Let set $f = S^{(B,T)}$ and $g = S^{(B',T')}$ are two KCFs.

To prove the necessity of $B_2 = B'_2$, suppose that $B_2 \neq B'_2$. Let r be a ranking such that $r \in \mathscr{L}(U)$, and let r' be the complete ranking without ties of U such that $x \prec_{r'} y$ if and only if $y \prec_r x$.

Consider $R = (r)$. Then, we have $f(r, R) = 0$ and $f(r', R) = \frac{n*(n-1)}{2} * B_2$. Moreover, we have $g(r, R) = 0$ and $g(r', R) = \frac{n*(n-1)}{2} * B'_2$. As $B_2 \neq B'_2$, we obtain $f(r, R) - f(r', R) \neq g(r, R) - g(r', R)$, that is, f and g are not $\ominus$-equivalent.

To prove the necessity of $T_2 = T'_2$, suppose that $T_2 \neq T'_2$, and let $R = (r)$.

Let r be a ranking such that $r \in \mathscr{L}(U)$ and r' be the complete ranking such that all elements are tied, that is, $r' = [\{1, \ldots, n\}]$.

Let $R = (r)$. Then, we have $f(r, R) = 0$ and $f(r', R) = \frac{n*(n-1)}{2} * T_2$. Moreover, we have $g(r, R) = 0$ and $g(r', R) = \frac{n*(n-1)}{2} * T'_2$. As $T_2 \neq T'_2$, we obtain $f(r, R) - f(r', R) \neq g(r, R) - g(r', R)$, that is, f and g are not $\ominus$-equivalent.

To prove the necessity of $B_3 = B'_3$, suppose that $B_3 \neq B'_3$, and consider $R = (r)$. Let r be the complete ranking such that all the elements are tied, that is, $r = [\{1, \ldots, n\}]$, and let r' be a ranking such that $r' \in \mathscr{L}(U)$.

Let $R = (r)$. Then, we have that $f(r, R) = 0$ and $f(r', R) = \frac{n*(n-1)}{2} * B_3$. Moreover, we have $g(r, R) = 0$ and $g(r', R) = \frac{n*(n-1)}{2} * B'_3$. As $B_2 \neq B'_2$, we obtain $f(r, R) - f(r', R) \neq g(r, R) - g(r', R)$, that is, f and g are not $\ominus$-equivalent.

To prove the necessity of $B_5 - B'_4 = B_4 - B'_4$ or, equivalently, $B_5 - B_4 = B'_5 - B'_4$, suppose that $B_5 - B_4 \neq B'_5 - B'_4$.

Let $r = [\{1\}, \{2\}, \ldots, \{n-1\}]$, $r_1 = [\{1\}, \{2\}, \ldots, \{n\}]$ and $r_2 = [\{n\}, \{1\}, \{2\}, \ldots, \{n-1\}]$, and let $R = (r)$. Then, we have $f(r_1, R) = (n-1) * B_4$ and $f(r_2, R) = (n-1) * B_5$. Moreover, we have $g(r1, R) = (n-1) * B'_4$ and $g(r_2, R) = (n-1) * B'_5$. As $B_5 - B_4 \neq B'_5 - B'_4$, we obtain $f(r_1, R) - f(r_2, R) \neq g(r_1, R) - g(r_2, R)$, that is, f and g are not $\ominus$-equivalent.

To prove the necessity of $B_5 - B'_5 = T_5 - T'_5$ or, equivalently, $B_5 - T_5 = B'_5 - T'_5$, suppose that $B_5 - T_5 \neq B'_5 - T'_5$.

Let $r = [\{1\}, \ldots, \{n-1\}]$. Set $R = (r)$, and let $r_1 = [\{1\}, \ldots, \{n-2\}, \{n\}, \{n-1\}]$ and $r_2 = [\{1\}, \ldots, \{n-2\}, \{n-1, n\}]$. Then, we have $f(r_1, R) = (n-2) * B_4 + B_5$ and $f(r_2, R) = (n-2) * B_4 + T_5$. Moreover, we have $g(r_1, R) = (n-2) * B'_4 + B'_5$ and $g(r_2, R) = (n-2) * B'_4 + T'_5$. As $B_5 - T_5 = B'_5 - T'_5$, we thus obtain $f(r_1, R) - f(r_2, R) \neq g(r_1, R) - g(r_2, R)$, that is, f and g are not $\ominus$-equivalent.

To prove the necessity of $B_4 - B'_4 = T_5 - T'_5$ or, equivalently, $B_4 - T_5 = B'_4 - T'_5$, suppose that $B_4 - T_5 \neq B'_4 - T'_5$.

Let $r = [\{1\}, \ldots, \{n-1\}]$, and set $R = (r)$ and let $r_1 = [\{n-1\}, \{n\}, \{n-2\}, \ldots, \{1\}]$ and $r_2 = [\{n-1, n\}, \{n-2\}, \ldots, \{1\}]$. Then, we have $f(r_1, R) = (n-2) * B_5 + B_4$ and $f(r_2, R) = (n-2) * B_5 + T_5$. Moreover, we have $g(r_1, R) = (n-2) * B'_5 + B'_4$ and $g(r_2, R) = (n-2) * B'_5 + T'_5$. As $B_4 - T_5 \neq B'_4 - T'_5$, we thus obtain $f(r_1, R) - f(r_2, R) \neq g(r_1, R) - g(r_2, R)$, that is, f and g are not $\ominus$-equivalent.

To prove the necessity of $B_6 - B'_6 = T_6 - T'_6$, that is, $B_6 - T_6 = B'_6 - T'_6$. Let set $B_6 - T_6 \neq B'_6 - T'_6$.

Let $r = []$ be the ranking such that $dom(r) = \emptyset$, and set $R = (r)$. Let $r_1 \in \mathscr{L}(U)$ a complete ranking without ties on U, and let r_2 be the complete ranking such that $r_2 = [\{1, \ldots, n\}]$. Then, we have $f(r_1, R) = \frac{n*(n-1)}{2} * B_6$ and $f(r_2, R) = \frac{n*(n-1)}{2} * T_6$. Moreover, we have $g(r_1, R) = \frac{n*(n-1)}{2} * B'_6$ and $g(r_2, R) = \frac{n*(n-1)}{2} * T'_6$. As $B_6 - T_6 \neq B'_6 - T'_6$, we thus obtain $f(r_1, R) - f(r_2, R) \neq g(r_1, R) - g(r_2, R)$, that is, f and g are not $\ominus$-equivalent.

This completes the proof of Theorem 2. $\square$

Appendix A.3.4. Proof of Proposition 5

Proposition 5. *Let $f = S^{(B,T)}$ be a KCF and R be a tuple of rankings. If all the following conditions hold, then there exists $m \in M_f(R)$ such that $m \in \mathscr{L}(U)$:*

1. $B_2 \leq 2 * T_2$,
2. $B_3 = 0$,
3. $B_4 + B_5 \leq T_4 + T_5$,
4. $B_6 \leq T_6$.

Moreover:

(i) if the rankings in R are complete, (3) and (4) are no longer necessary, and

(ii) if the rankings in R do not have ties, then (2) is no longer necessary and, by making strict all remaining inequalities, we get $M_f(R) \subseteq \mathscr{L}(U)$.

In particular, if the rankings in R are complete and without ties, then $M_f(R) \subseteq \mathscr{L}(U)$ as soon as $B_2 < 2 * T_2$.

Proof. Let $f = S^{(B,T)}$ be a KCF, and let set $x \in U$ and $y \in U$. By the definition of $\Omega^R_{x,y}$ in Equation (3), we can notice that

$$\Omega^R_{y,x} = (\Omega^R_{x,y}[2], \Omega^R_{x,y}[1], \Omega^R_{x,y}[3], \Omega^R_{x,y}[5], \Omega^R_{x,y}[4], \Omega^R_{x,y}[6]).$$

Using Definition 1, we obtain

$$before(y, x) = \langle (B_2, B_1, B_3, B_4, B_5, B_6), \Omega^R_{x,y} \rangle,$$

and hence

$$\begin{aligned} &before(x, y) + before(y, x) \\ &= \langle (B_1 + B_2, B_1 + B_2, 2 * B_3, B_4 + B_5, B_4 + B_5, 2 * B_6), \Omega^R_{x,y} \rangle. \end{aligned}$$

Using Definition 1, we obtain $tied(x, y) = \langle T, \Omega^R_{x,y} \rangle$. Finally,

$$\begin{aligned} &before(x, y) + before(y, x) - 2 * tied(x, y) \\ &= \langle (B_1 + B_2, B_1 + B_2, 2 * B_3, B_4 + B_5, B_4 + B_5, 2 * B_6) - 2 * T, \Omega^R_{x,y} \rangle. \end{aligned}$$

We know that $\Omega[i] \geq 0$ for all $1 \leq i \leq 6$. Hence, if the following conditions hold:

1. $B_2 \leq 2 * T_2$
2. $B_3 = 0$
3. $B_4 + B_5 \leq T_4 + T_5$
4. $B_6 \leq T_6$

then $before(x, y) + before(y, x) - 2 * tied(x, y) \leq 0$ *i.e* $before(x, y) + before(y, x) \leq 2 * tied(x, y)$. We know thanks to Proposition 2 that there exists a median $m \in M_f$ such that $m \in \mathscr{L}(U)$. Moreover, the three following points are to notice:

- if $R \in \mathscr{C}(U)^{<\infty}$, then $\Omega^R_{x,y}[i] = 0$ for all $4 \leq i \leq 6$: (3) and (4) are not necessary anymore.
- if $R \in \mathscr{W}(U)^{<\infty}$, then $\Omega^R_{x,y}[3] = 0$ and (2) is not necessary anymore. Moreover, if the remaining inequalities are strict, by Proposition 2, we obtain $before(x, y) + before(y, x) - 2 * tied(x, y) < 0$ and $M_f \subseteq \mathscr{L}(U)$
- if $R \in \mathscr{L}(U)^{<\infty}$, then $\Omega^R_{x,y}[i] = 0$ for all $3 \leq i \leq 6$ and (2), (3), (4) are not necessary anymore. Moreover, if the remaining inequality that is (1) is strict, we obtain $before(x, y) + before(y, x) - 2 * tied(x, y) < 0$ and $M_f \subseteq \mathscr{L}(U)$

□

Appendix A.3.5. Proof of Proposition 6

Lemma 2. *Let $f = S^{(B,T)}$ be a KCF. Let set $x \neq y \in U$. Then,*

$$\begin{aligned}&before_R^{(B,T)}(x,y) + before_R^{(B,T)}(y,x) - 2 * tied_R^{(B,T)}(x,y)\\ &= \langle (B_1 + B_2, B_1 + B_2, 2 * B_3, B_4 + B_5, B_4 + B_5, 2 * B_6) - 2 * T, \Omega_{x,y}^R \rangle.\end{aligned}$$

Proof. By the definition of $\Omega_{x,y}^R$ in Equation (3), we can notice that

$$\Omega_{y,x}^R = (\Omega_{x,y}^R[2], \Omega_{x,y}^R[1], \Omega_{x,y}^R[3], \Omega_{x,y}^R[5], \Omega_{x,y}^R[4], \Omega_{x,y}^R[6]).$$

Using Definition 1, we obtain

$$\begin{aligned}before(y,x) &= \langle (B_2, B_1, B_3, B_4, B_5, B_6), \Omega_{x,y}^R \rangle, \text{ and hence}\\ before(x,y) + before(y,x) &= \langle (B_1 + B_2, B_1 + B_2, 2 * B_3,\\ &\quad B_4 + B_5, B_4 + B_5, 2 * B_6), \Omega_{x,y}^R \rangle.\end{aligned}$$

Using Definition 1, we obtain $tied(x,y) = \langle T, \Omega_{x,y}^R \rangle$. Finally,

$$\begin{aligned}&before(x,y) + before(y,x) - 2 * tied(x,y)\\ = \;&\langle (B_1 + B_2, B_1 + B_2, 2 * B_3, B_4 + B_5, B_4 + B_5, 2 * B_6) - 2 * T, \Omega_{x,y}^R \rangle. \quad \square\end{aligned}$$

Proposition 6. *Let $f = S^{(B,T)}$ and $g = S^{(B',T)}$ be two KCFs such that*

1. $\frac{B'_5 - B'_4}{B'_2} \neq \frac{B_5 - B_4}{B_2}$,
2. $max(B_2, B'_2) \leq 2 * T_2$, *and*
3. $max(B_4 + B_5, B'_4 + B'_5) \leq 2 * T_4$.

Then f and g are not equivalent on $\mathscr{C}(U) \times \mathscr{A}(U)^{<\infty}$.

Proof. To improve readability, if $h = S^{(B_1,T_1)}$ is a KCF and x, y are two elements in U, we will omit the subscript R, and we will write $before_h(x,y)$ for $before_R^{(B_1,T_1)}(X,Y)$, $tied_h(x,y)$ for $tied_R^{(B_1,T_1)}(x,y)$, and $before_h(x,y)$ for $before_R^{(B_1,T_1)}(x,y)$.

Let $f = S^{(B,T)}$ and $g = S^{(B',T)}$ with $B'_5 - B'_4 \neq B_5 - B_4$. Suppose $U = \{X,Y\}$. Then, there are only three possible consensus: $c_1 = [\{X\},\{Y\}]$, $c_2 = [\{Y\},\{X\}]$ and $c_3 = [\{X,Y\}]$. Using Definition 1, $f(c_1,R) = before_f(X,Y)$, $f(c_2,R) = before_f(Y,X)$ and $f(c_3,R) = tied_f(X,Y)$.

We will prove that there exist $k_1 \in \mathbb{N}^*, k_2 \in \mathbb{N}^*$ such that if R contains exactly k_1 times the ranking $[\{Y\},\{X\}]$ and exactly k_2 times the ranking

$[\{X\}]$ (and no other ranking), then f and g do not have the same set of medians for R.

We first prove that c_3 can neither be the unique median of R with respect to f, nor be the unique median of R and with respect to g. Using Lemma 2, we obtain

$$\begin{aligned}&before_f(X,Y) + before_f(Y,X) - 2 * tied_f(X,Y)\\&= \langle (B_1+B_2, B_1+B_2, 2*B_3, B_4+B_5, B_4+B_5, 2*B_6) - 2*T, \Omega^R_{X,Y}\rangle.\end{aligned}$$

By construction of R, we have $\Omega^R_{X,Y}[i] = 0$ for $i \in \{1,3,5,6\}$. Moreover, as f is a KCF, $B_1 = 0$. Finally,

$$\begin{aligned}&before_f(X,Y) + before_f(Y,X) - 2 * tied_f(X,Y)\\&= \Omega^R_{X,Y}[2] * (B_2 - 2*T_2) + \Omega^R_{X,Y}[4] * (B_4 + B_5 - 2*T_4).\end{aligned}$$

As $B_2 \leq 2 * T_2$ and $B_4 + B_5 \leq 2 * T_4$,

$$before_f(X,Y) + before_f(Y,X) - 2 * tied_f(X,Y) \leq 0,$$

that is, either $before_f(X,Y) \leq tied_f(X,Y)$, or $before_f(Y,X) \leq tied_f(X,Y)$. Finally, c_3 cannot be the unique median of R with respect to f. By a similar reasoning, we obtain that c_3 cannot be the unique median of R with respect to g. As a consequence,

$$before_f(X,Y) - before_f(Y,X) < 0 \quad \text{(resp. } before_f(X,Y) - before_f(Y,X) > 0\text{)}$$

implies that c_1 is a median of R with respect to f and c_2 is not a median of R with respect to f (resp. c_2 is a median of R with respect to f and c_1 is not a median of R with respect to f). Using Lemma 1, we have

$$\begin{aligned}&before_f(X,Y) - before_f(Y,X)\\&= B_2 * (\Omega^R_{X,Y}[2] - \Omega^R_{X,Y}[1]) + (B_5 - B_4) * (\Omega^R_{X,Y}[5] - \Omega^R_{X,Y}[4]).\end{aligned}$$

By construction of R, $\Omega^R_{X,Y}[1] = \Omega^R_{X,Y}[5] = 0$, $\Omega^R_{X,Y}[2] = k_1$ and $\Omega^R_{X,Y}[4] = k_2$. We deduce that

$$before_f(X,Y) - before_f(Y,X) = k_1 * B_2 - k_2 * (B_5 - B_4).$$

We have $before_f(X,Y) - before_f(Y,X) < 0 \Leftrightarrow \frac{k1}{k2} < \frac{B_5 - B_4}{B_2}$. With a similar reasoning, we obtain that $before_g(X,Y) - before_g(Y,X) > 0 \Leftrightarrow \frac{k1}{k2} > \frac{B'_5 - B'_4}{B'_2}$.

As a consequence, if $\frac{B'_5-B'_4}{B'_2} < \frac{k1}{k2} < \frac{B_5-B_4}{B_2}$, then f and g do not have the same set of medians for R. With a similar reasoning, if $\frac{B'_5-B'_4}{B'_2} > \frac{k1}{k2} > \frac{B_5-B_4}{B_2}$, then f and g do not have the same set of medians for R. As $\mathbb{Q}$ is dense in $\mathbb{R}$, there exist k_1 and k_2 such that f and g do not have the same set of medians for R. We can conclude that f and g are not equivalent.

Note that it is possible to extend this proof to any size of U, by choosing R such that R contains exactly k_1 times the ranking $r \bigcup\{Y\} \bigcup\{X\}$ and k_2 times the ranking $r \bigcup\{X\}$, where r is a complete ranking without ties of $U\backslash\{X, Y\}$ and $r \bigcup\{Y\} \bigcup\{X\}$ is the ranking obtained by adding two buckets at the end of r, the first one containing Y and the last one containing X. Intuitively, using Theorem 5, X and Y define an independent sub-problem (the elements in $dom(r)$ are before X and Y in all the medians). $\square$

Appendix A.4. Proofs of Subsection 5.3

Appendix A.4.1. Preliminary results

We first present some preliminary lemmas that are useful for the proof of Theorem 3.

Lemma 3. *Let $f = S^{(B,T)}$ be a KCF such that $B_2 = B_5 - B_4$ and $x \neq y \in U$. Then $\sum_{i\in\{1,4\}} \Omega^R_{x,y}[i] > \sum_{j\in\{2,3,5,6\}} \Omega^R_{x,y}[j]$ implies $before(x, y) < before(y, x)$.*

Proof. Let $A = before(x, y) - before(y, x)$. By Lemma 1, we get

$$\begin{aligned} A &= B_2 * (\Omega^R_{x,y}[2] - \Omega^R_{x,y}[1]) \\ &\quad + (B_5 - B_4) * (\Omega^R_{x,y}[5] - \Omega^R_{x,y}[4]) \\ &\quad = B_2 * (\Omega^R_{x,y}[2] - \Omega^R_{x,y}[1] + \Omega^R_{x,y}[5] - \Omega^R_{x,y}[4]) < 0. \end{aligned}$$ $\square$

The intuition behind the following lemma is the following: if for example T_1 and T_4 are high enough, the cost of tying two elements x and y in a consensus will increase significantly each time that $x \prec_r y$ (component 1) and $x \diamond_r y$ (component 4) in an input ranking r. We obtain the guarantee that if in a strict majority of rankings $x \prec_r y$ or $x \diamond_r y$, then the cost of placing x before y in the consensus will be lower than the cost of tying them (which does not mean that x cannot be tied with y as there may be other elements to rank). This intuition can be easily generalized, and thus entailing the following lemma.

Lemma 4. *Let $f = S^{(B,T)}$ be a KCF and let x and y be two distinct elements of U. Set $P = \{1, 2, 3, 4, 5, 6\}$, and consider two disjoint subsets P_1 and P_2 of P. If the following conditions hold*

- $\underset{i \in P_1}{min}(T_i - B_i) \geq \underset{j \in P_2}{max}(B_j - T_j)$,
- $\sum_{i \in P_1} \Omega^R_{x,y}[i] > \sum_{j \in P_2} \Omega^R_{x,y}[j]$,
- *for every* $k \notin P_1 \cup P_2$, $\Omega^R_{x,y}[k] = 0$,

then $before(x, y) < tied(x, y)$.

Proof. Suppose that $\sum_{i \in P_1} \Omega^R_{x,y}[i] > \sum_{j \in P_2} \Omega^R_{x,y}[j]$. Let $A = before(x, y) - tied(x, y)$. By (A.3), we have:

$$\begin{aligned} A &= \sum_{i \in P_1} \Omega^R_{x,y}[i] * (B_i - T_i) + \sum_{j \in P_2} \Omega^R_{x,y}[j] * (B_j - T_j) \\ &= \sum_{j \in P_2} \Omega^R_{x,y}[j] * (B_j - T_j) - \sum_{i \in P_1} \Omega^R_{x,y}[i] * (T_i - B_i). \end{aligned}$$

Let $A_1 = \underset{i \in P_1}{min}(T_i - B_i)$ and $A_2 = \underset{j \in P_2}{max}(B_j - T_j)$. Then

$$\sum_{i \in P_1} \Omega^R_{x,y}[i] * (T_i - B_i) \geq A_1 * \sum_{i \in P_1} \Omega^R_{x,y}[i]$$

and

$$\sum_{j \in P_2} \Omega^R_{x,y}[j] * (B_j - T_j) \leq A_2 * \sum_{j \in P_2} \Omega^R_{x,y}[j].$$

Since $A_1 \geq A_2$ and $\sum_{i \in P_1} \Omega^R_{x,y}[i] > \sum_{j \in P_2} \Omega^R_{x,y}[j]$, $A < 0$. □

Lemma 5. *Suppose U contains only two elements i.e. $U = \{X, Y\}$. Let $f = S^{(B,T)}$ be a KCF such that $B_2 \neq B_5 - B_4$. Then, there exists $R = (r_1, \ldots, r_m) \in \mathscr{A}(U)^{<\infty}$ such that*

- $\Omega^R_{X,Y}[3] = \Omega^R_{X,Y}[6] = 0$, *in particular X and Y are never tied.*
- $\sum_{i \in \{1,4\}} \Omega^R_{X,Y}[i] > \dfrac{m}{2}$.
- $[\{X\}, \{Y\}]$ *is not a median of R with respect to f.*

Proof. As $U = \{X, Y\}$, there are only possible consensus: $c_1 = [\{X\}, \{Y\}]$, $c_2 = [\{Y\}, \{X\}]$ and $c_3 = [\{X, Y\}]$. Using Definition 2, $f(c_1, R) = \langle B, \Omega^R_{X,Y} \rangle$ and $f(c_2, R) = \langle B, \Omega^R_{Y,X} \rangle$. Suppose that $B_2 \neq B_5 - B_4$. We isolate the two possible cases.

Case 1: $B_2 < B_5 - B_4$. We construct an R such that it contains $[\{X\}, \{Y\}]$ $t + 1$ times and $[\{Y\}]$ t times, *i.e.*,

$$\Omega^R_{X,Y} = (t + 1, 0, 0, 0, t, 0).$$

By construction,

$$\sum_{i \in \{1,4\}} \Omega^R_{X,Y}[i] > \sum_{j \in \{2,5\}} \Omega^R_{X,Y}[j].$$

Let $A_1 = \langle B, \Omega^R_{X,Y} \rangle - \langle B, \Omega^R_{Y,X} \rangle = f(c_1, R) - f(c_2, R)$. From Lemma 1 Equation (A.2), it then follows that

$$\begin{aligned} A_1 &= -B_2 * (t + 1) + t(B_5 - B_4) \\ &= t(B_5 - B_4 - B_2) - B_2 > 0, \end{aligned}$$

which is equivalent to $t > \dfrac{B_2}{B_5 - B_4 - B_2}$.

Case 2: $B_2 > B_5 - B_4$. We now construct R such that it contains $[\{X\}]$ $t + 1$ times and $[\{Y\}, \{X\}]$ t times, *i.e.*,

$$\Omega^R_{X,Y} = (0, t, 0, t + 1, 0, 0).$$

By construction,

$$\sum_{i \in \{1,4\}} \Omega^R_{X,Y}[i] > \sum_{j \in \{2,5\}} \Omega^R_{X,Y}[j].$$

Let $A_2 = \langle B, \Omega^R_{X,Y} \rangle - \langle B, \Omega^R_{Y,X} \rangle$.

Again, by Lemma 1 Equation (A.2), we have

$$\begin{aligned} A_2 &= t * B_2 + (B_5 - B_4) * (-t - 1) \\ &= t(B_2 + B_4 - B_5) + B_4 - B_5 > 0, \end{aligned}$$

which is equivalent to $t > max(\dfrac{B_5 - B_4}{B_2 + B_4 - B_5}, 0)$.

In both cases, we have that $c_1 = [\{X\}, \{Y\}]$ cannot be a a median of R with respect to f and, since the first to conditions are clearly satisfied, the proof is now complete. □

Lemma 6. *Let $P = \{1, 2, 3, 4, 5, 6\}$, and consider two disjoint subsets P_1 and P_2 of P. Suppose that $U = \{X, Y\}$, and let $f = S^{(B,T)}$ be a KCF such that $\underset{i \in P_1}{min}(T_i - B_i) < \underset{j \in P_2}{max}(B_j - T_j)$. Then there exists $R \in \mathscr{A}(U)^{<\infty}$ such that*

- $\sum_{i \in P_1} \Omega^R_{X,Y}[i] > \sum_{j \in P_2} \Omega^R_{X,Y}[j]$,
- *for every $k \notin P_1 \cup P_2$, $\Omega^R_{X,Y}[k] = 0$, and*
- $[\{X\}, \{Y\}]$ *is not a median of R with respect to f.*

Proof. Again there are only the following three possible consensus: $c_1 = [\{X\}, \{Y\}]$, $c_2 = [\{Y\}, \{X\}]$ and $c_3 = [\{X, Y\}]$.

From Definition 2, we have that

$$f(c_1, R) = \langle B, \Omega^R_{X,Y} \rangle \quad \text{and} \quad f(c_3, R) = \langle T, \Omega^R_{X,Y} \rangle.$$

We show that if $\underset{i \in P_1}{min}(T_i - B_i) < \underset{j \in P_2}{max}(B_j - T_j)$, then there is $R \in \mathscr{A}(U)^{<\infty}$ such that

$$\sum_{i \in P_1} \Omega^R_{X,Y}[i] > \sum_{j \in P_2} \Omega^R_{X,Y}[j],$$

$\Omega^R_{X,Y}[k] = 0$ whenever $k \notin P_1 \cup P_2$, and $f(c_1, R) > f(c_3, R)$, and thus c_1 is not a median of R with respect to f. So let

$$\begin{aligned} r_1 &= [\{X\}, \{Y\}], \quad r_2 = [\{Y\}, \{X\}], \quad r_3 = [\{X, Y\}], \\ r_4 &= [\{X\}], \quad r_5 = [\{Y\}] \quad \text{and} \quad r_6 = []. \end{aligned}$$

Take $\alpha \in P_1$ such that $T_\alpha - B_\alpha = \underset{i \in P_1}{min}(T_i - B_i)$, and take $\beta \in P_2$ such that $T_\beta - B_\beta = \underset{j \in P_2}{max}(B_j - T_j)$.

We construct an R such that it contains r_β t times and r_α $t + 1$ times. By construction, we have that

$$\sum_{i \in P_1} \Omega^R_{X,Y}[i] > \sum_{j \in P_2} \Omega^R_{X,Y}[j],$$

and that $\Omega^R_{X,Y}[k] = 0$, whenever $k \notin P_1 \cup P_2$.

Set $A = \langle B, \Omega^R_{X,Y}\rangle - \langle T, \Omega^R_{X,Y}\rangle = f(c_1, R) - f(c_3, R)$. By Lemma A.3, it follows that

$$\begin{aligned} A &= \Omega^R_{X,Y}[\alpha] * \min_{i \in P_1}(T_i - B_i) + \Omega^R_{X,Y}[\beta] * \max_{j \in P_2}(B_j - T_j) \\ &= (t+1) * \min_{i \in P_1}(T_i - B_i) + t * \max_{j \in P_2}(B_j - T_j) > 0, \end{aligned}$$

which implies that

$$t > max\left(\frac{-\min_{i \in P_1}(T_i - B_i)}{\min_{i \in P_1}(T_i - B_i) + \max_{j \in P_2}(B_j - T_j)}, 0\right).$$ □

Appendix A.4.2. Proof of Theorem 3

Theorem 3. *The following assertions are equivalent:*

- $S^{(B,T)}$ *satisfies the Majority-like axiom on* $\mathscr{C}(U) \times \mathscr{A}(U)^{<\infty}$
- $S^{(B,T)}$ *satisfies the Condorcet-like axiom on* $\mathscr{C}(U) \times \mathscr{A}(U)^{<\infty}$
- $S^{(B,T)}$ *satisfies the Smith-like axiom on* $\mathscr{C}(U) \times \mathscr{A}(U)^{<\infty}$
- $B_2 = B_5 - B_4$ *and* $\min_{i \in \{1,4\}}(T_i - B_i) \geq \max_{j \in \{2,3,5,6\}}(B_j - T_j)$.

Proof. Note that the majority like axiom is included in the Condorcet like axiom which is included in the Smith like axiom.

We first prove that if $B_2 \neq B_5 - B_4$ or $\min_{i \in \{1,4\}}(T_i - B_i) < \max_{j \in \{2,3,5,6\}}(B_j - T_j)$, f does not respects the majority like axiom (and thus the Condorcet like axiom and the Smith like axiom). We prove in a second time that if $B_2 = B_5 - B_4$ and $\min_{i \in \{1,4\}}(T_i - B_i) \geq \max_{j \in \{2,3,5,6\}}(B_j - T_j)$, f respect the Smith like axiom (and thus the Condorcet like axiom and the majority like axiom).

We show the necessity of both conditions by contraposition in the particular case $U = \{X, Y\}$; the generalization to any U is then not difficult by taking R in which each ranking has only 2 equivalence classes represented by X and Y.

First, suppose that $B_2 \neq B_5 - B_4$. Then, according to Lemma 5, there exists $R = (r_1, \ldots, r_m) \in \mathscr{A}(U)^{<\infty}$ such that

$$\sum_{i \in \{1,4\}} \Omega^R_{X,Y}[i] > \frac{m}{2}$$

but $[\{X\},\{Y\}] \notin M_f$, *i.e.*, X is not a unique winner. As a consequence, f does not respect the Smith like axiom.

Second, suppose that $\underset{i\in\{1,4\}}{min}(T_i - B_i) < \underset{j\in\{2,3,5,6\}}{max}(B_j - T_j)$. According to Lemma 6, there exists $R \in \mathscr{L}(U)^{<\infty}$ such that

$$\sum_{i\in\{1,4\}} \Omega^R_{X,Y}[i] > \sum_{j\in\{2,3,5,6\}} \Omega^R_{X,Y}[j]$$

but $[\{X\},\{Y\}] \notin M_f$, *i.e.*, again X is not a unique winner. As a consequence, f does not respect the Smith like axiom.

Now to see that both conditions are sufficient, suppose that $B_2 = B_5 - B_4$ and $\underset{i\in\{1,4\}}{min}(T_i - B_i) \geq \underset{j\in\{2,3,5,6\}}{max}(B_j - T_j)$. We show that f respects the Smith like axiom.

Let $\mathscr{S} \subseteq U$ be the smallest non-empty subset such that for every element $x \in \mathscr{S}$ and every element $y \in U \backslash \mathscr{S}$, x is before y in a strict majority of rankings.

Let $c = [P_1, ..., P_k]$ be a median for R with respect to f such that there exists $(x) \in \mathscr{S}$, $y \notin \mathscr{S}$ such that $y \preceq_c x$. Consider a complete ranking c' such that :

- for all $x \neq y \in S$, $x \prec_{c'} y$ iif $x \preceq_c y$
- for all $x \neq y \in U \backslash S$, $x \prec_{c'} y$ iif $x \preceq_c y$
- for every $x \in S$ $y \in U \backslash S$, $x \leq_{c'} y$.

c' is the concatenation between c'_1 and c'_2 where c'_1 is the ranking c in which all the elements not in S have been removed and c'_2 is the ranking c in which all the in S have been removed.

Set $A = f(c', R) - f(c, R)$. Using Definition 2,

$$\begin{aligned} A = & \sum_{x\in S, y\in U\backslash S, x\equiv_c y} \langle B, \Omega^R_{x,y}\rangle \; - \langle T, \Omega^R_{x,y}\rangle \\ + & \sum_{x\in S, y\in U\backslash S, y\prec_c x} \langle B, \Omega^R_{x,y}\rangle \; - \langle B, \Omega^R_{y,x}\rangle. \end{aligned}$$

From Lemma 4, it follows that for every $x \in \mathscr{S}, y \in U \backslash \mathscr{S}$,

$$before(x, y) - tied(x, y) < 0,$$

and from Lemma 3, it follows that

$$before(x, y) - before(y, x) < 0.$$

Hence, $f(c', R) < f(c, R)$, and thus c cannot be a median of R with respect to f.

Finally, we proved that in all the medians of R with respect to f, all the elements of $\mathscr{S}$ are before all the remaining elements: f respects the Smith like axiom (and thus the Condorcet like axiom and the Smith like axiom). $\square$

Appendix A.4.3. Proof of Proposition 7

Proposition 7. $\forall B, T$, $S^{(B,T)}$ *respects the local independence of irrelevant alternatives criterion on* $\mathscr{C}(U) \times \mathscr{A}(U)^{<\infty}$.

Proof. Let $f = S^{(B,T)}$ be a KCF, $c = [P_1, ..., P_k]$ be a median of R with respect to f, $U' = U \backslash P_1$, R' be the projection of R on U' and $c_2 = [P'_1, ..., P'_{k'}] \in C(U')$.

Let $c' = [P_1, P'_1, ..., P'_{k'}]$ be a consensus ranking of R as it is a complete ranking on U. Let set $A = f(c, R) - f(c', R) \leq 0$ as c is a median for R (definition 3). We know that

$$A = \sum_{\substack{x,y \in U \\ x \prec_c y}} before(x, y) + \frac{1}{2} \sum_{\substack{x \neq y \in U \\ x \equiv_c y}} tied(x, y)$$

$$- \sum_{\substack{x,y \in U \\ x \prec_{c'} y}} before(x, y) - \frac{1}{2} \sum_{\substack{x \neq y \in U \\ x \equiv_{c'} y}} tied(x, y)$$

As for every $x, y \in P_1$, $x \equiv_c y$ and $x \equiv_{c'} y$, we can ignore the pairs of elements in P_1 when calculating A:

$$A = \sum_{\substack{x,y \in U' \\ x \prec_c y}} before(x, y) + \frac{1}{2} \sum_{\substack{x \neq y \in U' \\ x \equiv_c y}} tied(x, y) - \sum_{\substack{x,y \in U' \\ x \prec_{c'} y}} before(x, y)$$

As $A \leq 0$, we can conclude that $f([P_2, ..., P_k], R') \leq f(c_2, R')$. Finally, $[P_2, \ldots, P_k]$ is necessarily a median for R'. By a similar reasoning, $[P_1, \ldots, P_{k-1}]$ is a median for the projection R' of R into $U' = U \setminus P_k$. Finally, f respects the LIIAL axiom. $\square$

Appendix A.5. Proofs of Subsection 6.2

Notation. For a KCF f, a tuple or rankings R, two elements $x, y \in U$ and a consensus $c \in C(U)$, $\overline{f}_c(x,y)$ represents the cost induced by the relative order of (x,y) in the consensus c with respect to f. More precisely, $\overline{f}_c(x,y) =$

- $before(x,y)$ if x is before y in c.
- $before(y,x)$ if y is before x in c.
- $tied(x,y)$ if x and y are tied c.

Recalls on graph theory. $G = (V, A)$ is a directed graph whose V represent the set of vertices and A represent the arcs.

A strongly connected component of a directed graph $G = (V, E)$ is a subset V' of V (possibly V itself) such that (i) for any two vertices (x, y) of V', there exists a directed path from x to y, and (ii) V' is maximal for (i) *i.e.* there is no subset V'' of V such that $V' \subset V''$ and V'' respects (i).

Let $G = (V, A)$ be a directed graph and $(c_1, \ldots, c_k)$ be the strongly components of G. The graph of the strongly connected components of G, denoted $G^c = (V^c, A^c)$ is the directed graph such that

- V^c is the set of the strongly connected components of G;
- $(c_i, c_j) \in A^c$ if and only if there is at least one element x of c_i and one element y of c_j such that (x, y) is an arc of G.

Computing the strongly connected components of G can be done with Tarjan's strongly connected components algorithm [52]. Note that by definition, G^c is a directed acyclic graph (DAG).

Finally, a DAG $G = (V, E)$ admits at least one topological sort that is a list $\mathscr{T} = [\mathscr{T}_1, \mathscr{T}_2, \ldots, \mathscr{T}_k]$ of all the vertices of G such that $\mathscr{T}_i$ is not reachable from $\mathscr{T}_j$ for each $i < j$. A topological sort can be computed with Kahn's algorithm [53].

Appendix A.5.1. Proof of Theorem 4

Theorem 4. *Let f be a KCF, R be a tuple of rankings, $G^c_{(R,f)}$ be the graph of the strongly connected components of $G_{(R,f)}$, $\mathscr{T} = [\mathscr{T}_1, \mathscr{T}_2, \ldots, \mathscr{T}_k]$ be a topological sort of $G^c_{(R,f)}$ ($\mathscr{T}_i$ are the strongly connected components of $G_{(R,f)}$) and μ_i be a median for $R(\mathscr{T}_i)$ for $1 \leq i \leq k$. Then the concatenation $\mu_1.\mu_2 \ldots \mu_k$ is a median for R.*

Proof. Following Definition 2, $\forall (c, R) \in dom(f)$,

$$f(c,R) = \sum_{\substack{x \neq y \in U \\ x \prec_c y}} \langle B, \Omega^R_{x,y} \rangle \ + \ \frac{1}{2} * \sum_{\substack{x \neq y \in U \\ x \equiv_c y}} \langle T, \Omega^R_{x,y} \rangle$$
$$= \sum_{x \neq y \in U} \mathbf{1}_{x \prec_c y} * \langle B, \Omega^R_{x,y} \rangle \ + \frac{1}{2} * \mathbf{1}_{x \equiv_c y} * \langle T, \Omega^R_{x,y} \rangle$$

Let μ be the concatenation $\mu_1.\mu_2 \dots \mu_k$ and c be any consensus ranking. We prove that μ is an optimal consensus by showing that $S(\mu, R) \leq S(c, R)$. The score $S^{(B,T)}$ is defined in Equation (3) (in Section 4) as a sum over $x, y \in U$. We cut this sum in $k+1$ parts and show that each part is smaller or equal for μ than for c: for each i from 1 to k, we consider the part of the sum over (x, y) such that x and y both belong to $\mathscr{T}_i$. Then this part is smaller or equal for μ than for c since μ_i is an optimal consensus for $R(\mathscr{T}_i)$. Now consider the remaining part of the sum. It is over (x, y) such that x and y do not belong to the same $\mathscr{T}_i$. Assume that $x \in \mathscr{T}_i$, $y \in \mathscr{T}_j$, $i < j$ (the proof is similar if $i > j$). As $\mathscr{T}_i$ is before $\mathscr{T}_j$ in $\mathscr{T}$, there is no arc from y to x in G_e. By construction of G_e, we can conclude that $before(x, y) = min(x, y)$. In other words, the cost induced by (x, y) in μ cannot be higher than the cost induced by (x, y) in c. Finally, $S(\mu, R) \leq S(c, R)$ as claimed. □

Appendix A.5.2. Proof of Corollary 1

Corollary 1. *Let f be a KCF, R be a tuple of rankings and $G^c_{(R,f)}$ be the graph of the strongly connected components of $G_{(R,f)}$ the graph of elements. The number of topological sorts of $G^c_{(R,f)}$ is a lower bound of the number of medians of R with respect to f.*

Proof. Let f be a KCF, R be a tuple of rankings, $G^c_{(R,f)}$ be the graph of the strongly connected components of $G_{(R,f)}$. According to Theorem 4, for any topological sort $\mathscr{T} = [\mathscr{T}_1, \mathscr{T}_2, \dots, \mathscr{T}_k]$ of $G^c_{(R,f)}$, the concatenation $\mu_1.\mu_2 \dots \mu_k$ forms a median for R (recall that μ_i is a median for $R(\mathscr{T}_i)$). We now prove that two different topological sorts lead to two different medians. Let $\mathscr{T} = [\mathscr{T}_1, \mathscr{T}_2, \dots, \mathscr{T}_k]$ and $\mathscr{T}' = [\mathscr{T}'_1, \mathscr{T}'_2, \dots, \mathscr{T}'_{k'}]$ be two different topological sorts of $G^c_{(R,f)}$. Let μ_i be a median for $R(\mathscr{T}_i)$ for all $1 \leq i \leq k$, μ'_i be a median for $R(\mathscr{T}'_i)$ for all $1 \leq i \leq k'$, μ be the concatenation $\mu_1.\mu_2 \dots \mu_k$ and μ' be the concatenation $\mu'_1.\mu'_2 \dots \mu'_{k'}$. As $\mathscr{T} \neq \mathscr{T}'$, there necessarily exists $x \in U$, $y \in U$, $i < i'$, $j \geq j'$ such that either $x \in \mathscr{T}_i$, $y \in \mathscr{T}[i']$, $x \in \mathscr{T}'_j$, $y \in \mathscr{T}'[j']$

(case 1), or $x \in \mathscr{T}_j$, $y \in \mathscr{T}[j']$, $x \in \mathscr{T}'_i$, $y \in \mathscr{T}'[i']$ (case 2). In the first case, x is before y in μ whereas x is tied with y or y is before x in μ'. In the second case, x is before y in μ' whereas x is tied with y or y is before x in μ. We can conclude that $\mu \neq \mu'$.

□

Appendix A.5.3. Proof of Theorem 5

Theorem 5. *Let $G_{(R,f)} = (V, A)$ be the graph of elements and let $\mathscr{R}$ be the set of all the pairs $(x, y) \in E$ such that $before(y, x) > minc(x, y)$ and $tied(x, y) > minc(x, y)$. Let $P = [P_1, P_2, ..., P_k]$ be an ordered partition of V such that:*

1. $\forall i < j$, $\forall x \in P_i$, $\forall y \in P_j$, $(y, x) \notin E$, *and*
2. $\forall i$, $\forall x \in P_i$, $\forall y \in P_{i+1}$, $(x, y) \in \mathscr{R}$.

Then each optimal consensus respects P.

Proof. Consider a consensus ranking c which does not respect P. We will show that c is not optimal. From c, we build a consensus c' as follows: take first the elements of P_1 in the same order as they are in c, then append the elements of P_2 in the same order as they are in c, and repeat the operation for the elements of P_3, ..., until P_k. By construction, c' respects P, thus c' is different from c. Now, let us compare $f(c, R)$ and $f(c', R)$. Each pair of elements (x, y) such that x and y are in the same relative order in c and c' induces the same cost for c and c' (*i.e.* $\overline{f}_c(x, y) = \overline{f}_{c'}(x, y)$).

Now consider pairs (x, y) such that x and y are not in the same relative order in c and c' (there is at least one such pair). By construction of c', x and y are in different groups of P. Suppose $x \in P_i$ and $y \in P_j$ with $i < j$. Then by hypothesis on P, there is no arc from y to x in $G_{(R,f)}$, *i.e.* $before(x, y) = minc(x, y)$. Hence, $\overline{f}_{c'}(x, y) \leq \overline{f}_c(x, y)$.

Moreover, since $y \in P_j$ is before $x \in P_i$ in c, there exists $i \leq k < j$ and $z \in P_k$, $z' \in P_{k+1}$ such that z' is before z in c. But by construction of c', z is before z' in c'. By the theorem hypotheses, $(z, z') \in \mathscr{R}$, *i.e.* $before(z, z')$ is the unique minimum, hence $\overline{f}_{c'}(z, z') < \overline{f}_c(z, z')$.

Finally, $\overline{f}_{c'}(x, y) \leq \overline{f}_c(x, y)$ for every pair (x, y) and there is a strict inequality for at least one pair, so overall $f(c, R) > f(c', R)$, hence c is not optimal, concluding the proof. □

Appendix A.5.4. Proof of Proposition 8

Proposition 8. *Let $x \in U$, $y \in U$. Let $f = S^{(B,T)}$ and $g = S^{(B',T')}$ be two KCFs such that the following conditions hold:*

1. $B = B'$,
2. $T_i \geq T'_i$ *for all* $1 \leq i \leq 6$, *and*
3. *there exists* $1 \leq i \leq 6$ *such that* $T_i > T'_i$ *and* $\Omega^R_{x,y}[i] > 0$.

Then,

(i) $before^{(B,T)}(x, y) = before^{(B',T')}(x, y)$, *and*

(ii) $tied^{(B,T)}(x, y) > tied^{(B',T')}(x, y)$

We first prove that

$$before_R^{(B,T)}(x, y) = before_R^{(B',T')}(x, y).$$

We know from Definition 1 that $before_R^{(B,T)}(x, y) = \langle B, \Omega^R_{x,y}\rangle$. We obtain

$$before_R^{(B,T)}(x, y) - before_R^{(B',T')}(x, y) = \langle B - B', \Omega^R_{x,y}\rangle.$$

As $B = B'$, we can conclude that $before_R^{(B,T)}(x, y) - before_R^{(B',T')}(x, y) = 0$, that is,

$$before_R^{(B,T)}(x, y) = before_R^{(B',T')}(x, y).$$

We now prove that $tied^{(B,T)}(x, y) > tied^{(B',T')}(x, y)$. We know from Definition 1 that $tied_R^{(B,T)}(x, y) = \langle T, \Omega^R_{x,y}\rangle$. We obtain

$$tied^{(B,T)}(x, y) - tied^{(B',T')}(x, y) = \langle T - T', \Omega^R_{x,y}\rangle.$$

Recall $\Omega^R_{x,y}[i] \geq 0$, for all $1 \leq i \leq 6$ (Equation (3) in Section 4). As $T_i \geq T'_i$ for all $1 \leq i \leq 6$, we know that $(T - T')_i \geq 0$ for all $1 \leq i \leq 6$. We obtain $\langle T - T', \Omega^R_{x,y}\rangle \geq 0$. Moreover, as there exists i such that $T_i - T'_i > 0$ and $\Omega^R_{x,y}[i] > 0$, then $\langle T - T', \Omega^R_{x,y}\rangle \neq 0$. Finally, $\langle T - T', \Omega^R_{x,y}\rangle > 0$, that is,

$$tied^{(B,T)}(x, y) > tied^{(B',T')}(x, y).$$

Appendix B. Exact algorithm for KCFs

We now present the complete exact algorithm for KCFs in Algorithm 1. Without loss of generality, we consider $U = \{1, ..., n\}$.

```
Input: R: a tuple of rankings, f: a KCF
begin
    U ← AllDistinctElements(R)
    for x ≠ y ∈ U do
        compute and save before(x, y), before(y, x) and tied(x, y)
          with respect to f
     minimize    Σ_{x<y} before(x, y) * b_(x,y)
                 +before(y, x) * b_(y,x) + tied(x, y) * t_{x,y}
    subject to   b_(x,y), b_(y,x) and t_{x,y} ∈ {0, 1}, ∀x < y
                 b_(x,y) + b_(y,x) + t_{x,y} = 1, ∀x < y
                 b_(x,y) + b_(y,z) + t_{y,z} − b_(x,z) ≤ 1,
                 ∀x ≠ y ≠ z ≠ x
                 b_(x,y) + t_{x,y} + b_(y,z) − b_(x,z) ≤ 1,
                 ∀x ≠ y ≠ z ≠ x
                 t_{x,y} + t_{y,z} − t_{x,z} ≤ 1, ∀x < y < z
    consensus ← EmptyListOfSets()
    /* pos[x] = position of x in the consensus                */
    pos ← Array(size(U)) filled with 1
    for x ∈ U do
        /* there is at most one bucket for each elem ∈ U       */
        append an empty set in consensus
        for y ∈ U\{x} do
            if b_(x,y) = 1 then
                pos[y] += 1 // pos[y] = 1 + nb of elems before y
    /* if pos[x] = k, x goes in k^th bucket                    */
    for x ∈ U do
        add x in consensus[pos[x]]
    /* for each bucket of size k > 1, the k − 1 following
       buckets are empty                                      */
    remove the empty sets in consensus
    return consensus
```

.
Algorithm 1: Exact ILP algorithm to compute a median