

Three-way Decisions with Evaluative Linguistic Expressions

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Abstract

We propose a linguistic interpretation of three-way decisions, where the regions of acceptance, rejection, and non-commitment are constructed by using the so-called evaluative linguistic expressions, which are expressions of natural language such as small, medium, very short, quite roughly strong, extremely good, etc. Our results highlight new connections between two different research areas: three-way decisions and the theory of evaluative linguistic expressions.

Keywords— Three-way decisions, Rough sets, Probabilistic rough sets, Evaluative linguistic expressions, Explainable Artificial Intelligence

1 Introduction

The theory of three-way decisions (TWD) divides a finite and non-empty universe into three disjoint sets, which are called positive, negative, and boundary regions. These regions respectively induce positive, negative, and boundary rules: a positive rule makes a decision of acceptance, a negative rule makes a decision of rejection, and a boundary rule makes an abstained or non-committed decision [1, 2]. The concept of three-way decisions was originally introduced in Rough Set Theory [1, 3] and until today, it has been widely studied and applied to many decision-making problems (see [4, 5, 6, 7] for some examples). Thus, several approaches have been proposed to generate the three regions; one of them is based on probabilistic rough sets, which generalizes probabilistic rough sets [8, 9] where the three regions are constructed using a pair of thresholds and the notion of conditional probability (in this case, the regions are called probabilistic positive, negative, and boundary regions).

The contribution of this article is to provide a linguistic interpretation of the positive, negative, and boundary regions. So, we propose a three-way decision method based on the concept of *evaluative linguistic expressions*, which are expressions of natural language such as *small, medium, very short, quite roughly strong, extremely good, etc.* These are already considered in the majority of applications of fuzzy modelling. Since we use evaluative linguistic expressions to evaluate the size of sets, we focus on the expressions involving the adjectives *small, medium, and big* that can be preceded by an adverb; examples are *very small, roughly medium, and extremely big*. Mathematically, an evaluative linguistic expression is modelled by a function $Ev : [0, 1] \rightarrow [0, 1]$. The formal theory of evaluative linguistic expressions is introduced and explained in [10, 11, 12, 13].

The positive, negative, and boundary regions of a non-empty and finite universe U are defined here starting from a subset X of U , an equivalence relation \mathcal{R} on U (i.e. \mathcal{R} is reflexive, symmetric and transitive), an evaluative linguistic expression Ev , and

a pair of thresholds (α, β) with $0 \leq \beta < \alpha \leq 1$. Then, an object x belongs to the positive region when *the size of $[x]_{\mathcal{R}} \cap X$ evaluated w.r.t. Ev is at least α* , where $[x]_{\mathcal{R}}$ is the equivalence class of x w.r.t. \mathcal{R} . Analogously, x belongs to the negative region when *the size of $[x]_{\mathcal{R}} \cap X$ evaluated w.r.t. Ev is at most β* . Finally, the remaining elements form the boundary region. In order to obtain the three regions, the size of $X \cap [x]_{\mathcal{R}}$ is quantified using a fuzzy measure [14, 15].

The role of evaluative linguistic expressions in the context of three-way decisions can be better understood with the following example.

Example 1. *Suppose that the number of buses between the University of Buenos Aires and the rest of the city has to be increased from 7 am to 8 am. Thus, we intend to understand which city areas need buses the most, as resources are limited. Let us denote the areas of the city with A_1, \dots, A_n and map each area A_i with the set S_{A_i} made of all students of the university who live in A_i . Thus, S_{A_1}, \dots, S_{A_n} can be seen as the equivalence classes w.r.t. the relation \mathcal{R} on the set of all students of the University of Buenos Aires living in the city: $x\mathcal{R}y$ if and only if x and y live in the same area. Based on a survey, we also consider a set X made of all students that usually take a bus to the university in the slot time [7 am, 8 am]. We also choose $(\alpha, \beta) = (0.3, 0.6)$ and $Ev = \text{extremely big}$. We construct three regions in the following way. The positive region is the union of $S_{A'_1}, \dots, S_{A'_k}$ (with $\{A'_1, \dots, A'_k\} \subseteq \{A_1, \dots, A_n\}$) so that the amount of students of S_{A_i} that take a bus from 7 am to 8 am is “extremely big” with a value of at least 0.6. Similarly, the negative region is the union of $S_{A^*_1}, \dots, S_{A^*_h}$ (with $\{A^*_1, \dots, A^*_h\} \subseteq \{A_1, \dots, A_n\}$) so that the amount of students of S_{A_i} that take a bus from 7 am to 8 am is extremely big with a value of at most 0.3. All other students form the boundary region. The final decision is immediate: the buses are certainly increased for the areas A'_1, \dots, A'_k , but not for A^*_1, \dots, A^*_h . Furthermore, the decision is postponed for the remaining areas (that is, for each $A_i \notin \{A'_1, \dots, A'_k\} \cup \{A^*_1, \dots, A^*_h\}$). In order to make a decision in those areas, for example, we could take into account the workers (besides the students) that need a bus in the slot of time [7 am, 8 am].*

The choice of Ev depends on the context where the three regions are used. Indeed, in the previous example, we have chosen *extremely big* in order to select the areas where a large number of students catch the bus from 7 am to 8 am. However, if we focus on the inverse problem (namely we need to eliminate some existing bus rides), then we should identify the areas where there are fewer students taking the bus in the time slot [7 am, 8 am]. Therefore, in this case, the evaluative linguistic expression *extremely small* is more appropriate to construct the three regions.

A significant contribution of this article is providing a linguistic and novel interpretation of the positive, negative, and boundary regions already determined with probabilistic rough sets. Consequently, the reasons for decisions of acceptance, rejection, and non-commitment can be explained in terms of expressions of natural language. Of course, the advantage is that non-technical users dealing with TWD models can better understand the reliability of the procedures related to the final decisions. This is in line with the scope of *Explainable Artificial Intelligence (XAI)*, which is a new approach to AI emphasizing the ability of machines to give sound motivations about their decisions and behaviour [16].

The article is organized as follows. The next section reviews some basic notions regarding probabilistic three-way decisions and the concept of evaluative linguistic expressions. Also, the notion of fuzzy measure is recalled. Section 3 presents a new model of three-way decisions based on the theory of evaluative linguistic expressions. As a consequence, a linguistic generalization of Pawlak rough sets is introduced. Finally, Section 4 connects the TWD models based on evaluative linguistic expressions and probabilistic rough sets. In particular, confining to the evaluative linguistic expressions modelled by increasing functions, we find the class of thresholds so that the corresponding probabilistic positive, negative, and boundary regions are equal to those

generated by a given evaluative linguistic expression.

2 Preliminaries

In the following, we consider a finite universe U , a subset X of U , and an equivalence relation \mathcal{R} on U (i.e. \mathcal{R} is reflexive, symmetric, and transitive). Moreover, we indicate the equivalence class of $x \in U$ w.r.t. \mathcal{R} with $[x]_{\mathcal{R}}$.

2.1 Three-way decisions with probabilistic rough sets

This subsection recalls the fundamental notions of three-way decisions based on probabilistic rough sets.

Viewing X and $[x]_{\mathcal{R}}$ as events of U , the symbol $Pr(X|[x]_{\mathcal{R}})$ denotes the *conditional probability* of X given $[x]_{\mathcal{R}}$, i.e.

$$Pr(X|[x]_{\mathcal{R}}) = \frac{|[x]_{\mathcal{R}} \cap X|}{|[x]_{\mathcal{R}}|}. \quad (1)$$

Then, three special subsets of U are determined by using (1) and a pair of thresholds as shown by the next definition.

Definition 1. Let $\alpha, \beta \in [0, 1]$ such that $\beta < \alpha$, the (α, β) -probabilistic positive, negative and boundary regions are respectively the following:

- (i) $POS_{(\alpha, \beta)}(X) = \{x \in U \mid Pr(X|[x]_{\mathcal{R}}) \geq \alpha\}$,
- (ii) $NEG_{(\alpha, \beta)}(X) = \{x \in U \mid Pr(X|[x]_{\mathcal{R}}) \leq \beta\}$,
- (iii) $BND_{(\alpha, \beta)}(X) = \{x \in U \mid \beta < Pr(X|[x]_{\mathcal{R}}) < \alpha\}$.

We put

$$\mathcal{T}_{(\alpha, \beta)}(X) = \{POS_{(\alpha, \beta)}(X), NEG_{(\alpha, \beta)}(X), BND_{(\alpha, \beta)}(X)\} \quad (2)$$

and we say that $\mathcal{T}_{(\alpha, \beta)}(X)$ is a *tri-partition* of U due to the following remark ¹.

Remark 1. The three regions of $\mathcal{T}_{(\alpha, \beta)}(X)$ are mutually disjoint, i.e. $A \cap B = \emptyset$ for each $A, B \in \{POS_{(\alpha, \beta)}(X), NEG_{(\alpha, \beta)}(X), BND_{(\alpha, \beta)}(X)\}$ with $A \neq B$, and they cover the universe U , i.e.

$$POS_{(\alpha, \beta)}(X) \cup NEG_{(\alpha, \beta)}(X) \cup BND_{(\alpha, \beta)}(X) = U. \quad (3)$$

In the context of three-way decisions, the following rules are considered: let $x \in U$,

- if $x \in POS_{(\alpha, \beta)}(X)$, then x is accepted;
- if $x \in NEG_{(\alpha, \beta)}(X)$, then x is rejected;
- if $x \in BND_{(\alpha, \beta)}(X)$, then we abstain on x .

The values $Pr(X|[x]_{\mathcal{R}})$ represents the *accuracy* or *confidence* of the rules:

- the higher $Pr(X|[x]_{\mathcal{R}})$ is, the more confident we are that $x \in POS_{(\alpha, \beta)}(X)$ is correctly accepted,
- the lower $Pr(X|[x]_{\mathcal{R}})$ is, the more confident we are that $x \in NEG_{(\alpha, \beta)}(X)$ is correctly rejected.

Definition 1 is strictly related to the notion of probabilistic rough sets.

¹By a tri-partition, we mean a partition of U made of three equivalence classes. On the other hand, $\{POS_{(\alpha, \beta)}(X), NEG_{(\alpha, \beta)}(X), BND_{(\alpha, \beta)}(X)\}$ could collapse into a bi-partition or the whole universe when one or two of its sets are empty.

Definition 2. The (α, β) -probabilistic rough set of X is the pair

$$(\mathcal{L}_{(\alpha, \beta)}(X), \mathcal{U}_{(\alpha, \beta)}(X)),$$

where

$$\mathcal{L}_{(\alpha, \beta)}(X) = POS_{(\alpha, \beta)}(X) \quad \text{and} \quad \mathcal{U}_{(\alpha, \beta)}(X) = POS_{(\alpha, \beta)}(X) \cup BND_{(\alpha, \beta)}(X),$$

which are respectively called (α, β) - lower and upper approximations of X .

Remark 2. When $\alpha = 1$ and $\beta = 0$, $(\mathcal{L}_{(\alpha, \beta)}(X), \mathcal{U}_{(\alpha, \beta)}(X))$ is the rough set $(\mathcal{L}(X), \mathcal{U}(X))$ of X defined by Pawlak in [3], namely

$$(\mathcal{L}(X), \mathcal{U}(X)) = (\{x \in U \mid [x]_R \subseteq X\}, \{x \in U \mid [x]_R \cap X \neq \emptyset\}). \quad (4)$$

The sets $\mathcal{L}(X)$ and $\mathcal{U}(X)$ are respectively called lower and upper approximations of X w.r.t. \mathcal{R} .

2.2 Evaluative Linguistic Expressions

This subsection reviews concepts and results that are found in [10, 17] and it recalls the notion of fuzzy measure.

Evaluative linguistic expressions are special expressions of natural language, which people commonly employ to evaluate, judge, estimate, and in many other situations. Examples of evaluative linguistic expressions are *small, medium, big, about twenty-five, roughly one hundred, very short, more or less deep, not very tall, roughly warm or medium-hot*, etc. For convenience, we will often omit the adjective “linguistic” and use only the term “evaluative expressions”. The simplest evaluative expressions are called *pure evaluative expressions* and have the following structure:

$$\langle \text{linguistic hedge} \rangle \langle \text{TE-adjective} \rangle,$$

where

- a linguistic hedge is an adverbial modification such as *very, roughly, approximately, significantly* and
- a TE-adjective is an adjective such as *good, medium, big, short*, etc. TE stands for *trichotomous evaluative*, indeed TE-adjectives typically form pairs of antonyms like *small* and *big* completed by a middle member, which is *medium* in the case of *small* and *big*. Other examples are “*weak, medium-strong, and strong*” and “*soft, medium-hard, and hard*”.

The *empty linguistic hedge* is employed to deal with evaluative expressions made of only a TE-adjective; hence, *small, medium, and big* are considered evaluative expressions. Other pure evaluative expressions are the fuzzy numbers like *about twenty-five*. Two or more pure evaluative expressions can be connected to form *negative evaluative expressions* like “*NOT very small*” and *compound evaluative expressions* like “*very expensive AND extremely small*” and “*very expensive OR extremely small*”.

The semantics of evaluative expressions is based on the essential concepts of *context, intension, and extension*.

- The *context* is a state of the world at a given time and place in which an evaluative expression appears. Each context is represented by a linearly ordered scale, which is bounded by s and b . Moreover, a context is given by a triple $\omega = \langle s, m, b \rangle$, where s is the “most typical” small value, m is the “most typical” medium value, and b is the “most typical” big value. For example, suppose that evaluative expressions are used to evaluate the size of apartments. If we are thinking of apartments for one person, then we could choose $\omega_1 = \langle 40, 70, 100 \rangle$ as context, which means that flats measuring 40 m^2 , 70 m^2 , and 100 m^2 are

typically small, medium and big, respectively. On the other hand, when changing context and thinking of apartments for a family of 5 people, the context $\omega_5 = \langle 70, 120, 160 \rangle$ is more appropriate.

- The *intension* of an evaluative expression is a function mapping each context into a fuzzy set of a given universe. Taking up the previous example, we consider a universe made of four apartments $\{a_1, a_2, a_3, a_4\}$, then the intension of *small* is the map Int_{small} that assigns to the context ω_5 the fuzzy set A_{ω_5} so that $A_{\omega_5}(a_i)$ is the degree to which a_i is small in the context ω_5 (namely, a_i is small for 5 people).
- The *extension* of an evaluative expression Ev is a fuzzy set determined by the intension of Ev , given a context ω . Concerning the previous example, $Int_{small}(\omega_5) = A_{\omega_5}$ is an example of an extension of *small*.

In this article, we confine to the TE adjectives *small*, *medium*, and *big* because we use evaluative expressions to evaluate the size of sets. So, let X be a subset of a universe U , we will say that the size of X w.r.t. to the size of U is *very small*, *extremely big*, etc. Furthermore, we confine to the *standard context*, which is $\langle 0, 0.5, 1 \rangle$. Finally, since sizes are expressed by means of a fuzzy measure (by Example 2, the measure of the size of a set X is a value of $[0, 1]$), the extensions of our evaluative expressions are functions from $[0, 1]$ to $[0, 1]$, which have a specific formula. The extension of an evaluative expression like $\langle \text{linguist hedge} \rangle \langle TE - \text{adjective} \rangle$ with $TE - \text{adjective} \in \{small, medium, big\}$ is obtained by composing two functions, one models the linguistic hedge and the other models the TE-adjective. The function describing a linguistic hedge depends on three parameters, which are experimentally estimated (see [17] for more details).

In what follows, we provide the formula of $\neg Sm : [0, 1] \rightarrow [0, 1]$, $BiVe : [0, 1] \rightarrow [0, 1]$, and $BiEx : [0, 1] \rightarrow [0, 1]$, which are the extensions of the evaluative expressions *not small*, *very big*, and *extremely big*, where the context $\langle 0, 0.5, 1 \rangle$ is fixed ².

$$\neg Sm(x) = \begin{cases} 1 & \text{if } x \in [0.275, 1], \\ 1 - \frac{(0.275 - x)^2}{0.02305} & \text{if } x \in (0.16, 0.275) \\ \frac{(x - 0.0745)^2}{0.01714} & \text{if } x \in (0.0745, 0.16] \\ 0 & \text{if } x \in [0, 0.0745] \end{cases} \quad (5)$$

$$BiVe(x) = \begin{cases} 1 & \text{if } x \in [0.9575, 1], \\ 1 - \frac{(0.9575 - x)^2}{0.00796} & \text{if } x \in [0.895, 0.9575), \\ \frac{(x - 0.83)^2}{0.00828} & \text{if } x \in (0.83, 0.895), \\ 0 & \text{if } x \in [0, 0.83]. \end{cases} \quad (6)$$

$$BiEx(x) = \begin{cases} 1 & \text{if } x \in [0.995, 1], \\ 1 - \frac{(0.995 - x)^2}{0.00495} & \text{if } x \in [0.95, 0.995), \\ \frac{(x - 0.885)^2}{0.00715} & \text{if } x \in (0.885, 0.95), \\ 0 & \text{if } x \in [0, 0.885]. \end{cases} \quad (7)$$

²We have got the formulas of $BiVe$ and $BiEx$ using the function $\nu_{a,b,c}(LH(\omega^{-1}))$ and Table 5.1 given in [17]. Concerning the formula of $\neg Sm$, we have considered that $\neg Sm(x) = 1 - Sm(x)$. After that, we have found the formula of Sm using the function $\nu_{a,b,c}(RH(\omega^{-1}))$ and Table 5.1 of [17].

Remark 3. The evaluative expressions $\neg Sm$, $BiVe$, and $BiEx$ have a special role: they are respectively used to construct the formula of fuzzy quantifiers many, most, and almost all [18].

A further class of linguistic expressions is $\{\Delta_t : [0, 1] \rightarrow \{0, 1\} \mid t \in [0, 1]\}$, where given $t \in [0, 1]$ the formula of Δ_t is the following: let $a \in [0, 1]$,

$$\Delta_t(a) = \begin{cases} 1 & \text{if } a \geq t \\ 0 & \text{otherwise.} \end{cases} \quad (8)$$

In the sequel, we need the notion of fuzzy measure [14, 15].

Definition 3. Let U be a finite universe, a mapping $\varphi : 2^U \rightarrow \mathbb{R}$ is called fuzzy measure if and only if

- (a) $\varphi(\emptyset) = 0$;
- (b) if $X \subseteq Y$ then $\varphi(X) \leq \varphi(Y)$, for each $X, Y \subseteq U$ (monotonicity).

A fuzzy measure φ is called *normalized* or *regular* if $\varphi(U) = 1$.

In this paper, we focus on the normalized fuzzy measure given by the next example.

Example 2. Let U be a finite universe, the function $f : 2^U \rightarrow \mathbb{R}$ that assigns $\frac{|Y|}{|U|}$ to each $Y \subseteq U$ is a fuzzy measure.

The value $\frac{|Y|}{|U|}$ belongs to $[0, 1]$ and measures “how much Y is large with respect to U in the scale $[0, 1]$ ”.

Let us observe that in Probability theory $\frac{|Y|}{|U|}$ represents “how likely the event Y is to occur”.

3 Three-way decisions with linguistic expressions

This subsection proposes a novel model for three-way decisions, which is based on the concept of evaluative linguistic expressions previously described.

In the sequel, we use the symbol \mathcal{E} to denote the collection of the extensions of all evaluative expressions in the context $\langle 0, 0.5, 1 \rangle$.

Therefore, let $Ev \in \mathcal{E}$, let $X \subseteq U$, and let $\alpha, \beta \in [0, 1]$ with $\beta < \alpha$, three regions of U are determined. In particular, the region of a given element $x \in U$ is found by taking into account the following steps:

1. computing $Ev \left(\frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \right)$, which is the evaluation of *the size of $X \cap [x]_{\mathcal{R}}$ w.r.t. the size of $[x]_{\mathcal{R}}$* by using Ev ;
2. comparing $Ev \left(\frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \right)$ with the thresholds α and β .

For example, regarding point 1, if Ev models the evaluative expression “*significantly big*”, then $Ev \left(\frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \right)$ measures

“*how much the size of $X \cap [x]_{\mathcal{R}}$ is significantly big w.r.t. the size of $[x]_{\mathcal{R}}$* ”.

Equivalently, we are saying that

“*the size of the set of the elements of $[x]_{\mathcal{R}}$ that also belong to X is significantly big with the truth degree $Ev \left(\frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \right)$* ”.

Observe that $\frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|}$ syntactically coincides with the conditional probability (see (1)), but here it has a different interpretation: it is the fuzzy measure specified by Example 2.

Formally, the three regions of U determined by an evaluative expression are given by the following definition.

Definition 4. Let $Ev \in \mathcal{E}$, the (α, β) -linguistic positive, negative, and boundary regions induced by Ev are respectively the following:

- (i) $POS_{(\alpha, \beta)}^{Ev}(X) = \left\{ x \in U \mid Ev \left(\frac{|[x]_{\mathcal{R}} \cap X|}{|[x]_{\mathcal{R}}|} \right) \geq \alpha \right\};$
- (ii) $NEG_{(\alpha, \beta)}^{Ev}(X) = \left\{ x \in U \mid Ev \left(\frac{|[x]_{\mathcal{R}} \cap X|}{|[x]_{\mathcal{R}}|} \right) \leq \beta \right\};$
- (iii) $BND_{(\alpha, \beta)}^{Ev}(X) = \left\{ x \in U \mid \beta < Ev \left(\frac{|[x]_{\mathcal{R}} \cap X|}{|[x]_{\mathcal{R}}|} \right) < \alpha \right\}.$

We put

$$\mathcal{T}_{(\alpha, \beta)}^{Ev}(X) = \{POS_{(\alpha, \beta)}^{Ev}(X), NEG_{(\alpha, \beta)}^{Ev}(X), BND_{(\alpha, \beta)}^{Ev}(X)\} \quad (9)$$

and we say that $\mathcal{T}_{(\alpha, \beta)}^{Ev}(X)$ is a tri-partition of U .

Remark 4. The three regions of $\mathcal{T}_{(\alpha, \beta)}^{Ev}(X)$ are mutually disjoint, i.e. $A \cap B = \emptyset$ for each $A, B \in \{POS_{(\alpha, \beta)}^{Ev}(X), NEG_{(\alpha, \beta)}^{Ev}(X), BND_{(\alpha, \beta)}^{Ev}(X)\}$ with $A \neq B$, and they cover the universe U , i.e.

$$POS_{(\alpha, \beta)}^{Ev}(X) \cup NEG_{(\alpha, \beta)}^{Ev}(X) \cup BND_{(\alpha, \beta)}^{Ev}(X) = U. \quad (10)$$

Remark 5. Let us focus on the evaluative expressions not small, very big, extremely big, and utmost. The first three expressions are respectively modelled by (5), (6), and (7). According to Remark 3, $\neg Sm$, $BiVe$, and $BiEx$ respectively appear into the formula of quantifiers many, most, and almost all. Moreover, as explained in [19] (see Lemma 4.5), considering that X and $[x]_{\mathcal{R}}$ are crisp set, $\neg Sm \left(\frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \right)$,

$BiVe \left(\frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \right)$, and $BiEx \left(\frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \right)$ exactly coincide with the formula of quantifiers many, most, and almost all. Hence, they have the following meaning:

- $\neg Sm \left(\frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \right)$ is degree to which “many objects of $[x]_{\mathcal{R}}$ are in X ”,
- $BiVe \left(\frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \right)$ is degree to which “most objects of $[x]_{\mathcal{R}}$ are in X ”,
- $BiEx \left(\frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \right)$ is degree to which “almost all objects of $[x]_{\mathcal{R}}$ are in X ”.

Moreover, the function Δ_1 that is obtained by (8) and putting $t = 1$, models the evaluative expression utmost and corresponds to the quantifier all. Therefore, $\Delta^1 \left(\frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \right)$ is understood as the degree to which “all objects of $[x]_{\mathcal{R}}$ are in X ”.

Observe that here the universe of quantification coincides with $[x]_{\mathcal{R}}$, which is always non-empty, considering that \mathcal{R} is reflexive, hence $\{x\} \subseteq [x]_{\mathcal{R}}$. In mathematical logic, the assumption expressing that the universe of quantification must be non-empty is called existential import (or presupposition) [20].

Remark 6. Consider the evaluative expressions represented by (8). Let $t \in [0, 1]$, then $\Delta_t \left(\frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \right)$ is the degree to which “the size of the set of elements of $[x]_{\mathcal{R}}$ that also belong to X is at least as large as t (in the scale $[0, 1]$)”.

3.1 An illustrative example

In this subsection, we provide an example of how to use linguistic three-way decision to provide recommendations based on users's profile.

We consider a universe $U = \{u_1, \dots, u_{32}\}$ made of users of online communities and the following equivalence relation \mathcal{R} on U : let $x, y \in U$, $x\mathcal{R}y$ if and only if x and y belong to the same community. Therefore, \mathcal{R} corresponds to the partition $\{C_1, \dots, C_6\}$ of U , where C_i is the set of users of U belonging to the community i . In particular, we suppose that

$$C_1 = \{u_1, \dots, u_5\}, C_2 = \{u_6, \dots, u_{10}\}, C_3 = \{u_{11}, \dots, u_{15}\}, \\ C_4 = \{u_{16}, \dots, u_{20}\}, C_5 = \{u_{21}, \dots, u_{25}\}, \text{ and } C_6 = \{u_{26}, \dots, u_{32}\}.$$

We use the symbol X_T to denote the set of users of U interested in a specific topic T .

For example,

$$X_{Sport} = \{u_{10}, u_{11}, u_{12}, u_{18}, u_{19}, u_{20}, u_{21}, u_{22}, u_{23}, u_{24}, u_{26}\}$$

is the set of all users of U interested in the topic $Sport$.

Using three-way decisions based on evaluative expressions, we intend to select the most appropriate communities among C_1, \dots, C_6 to which propose news related to the topic T .

If we choose $(\alpha, \beta) = (0.8, 0.2)$ and the evaluative expression $\neg Sm$ corresponding to the fuzzy quantifier *many*, then we decide to assign the news about the topic T to the communities of $POS_{(0.8, 0.2)}^{\neg Sm}(X_T)$. Indeed, enough users of $POS_{(0.8, 0.2)}^{\neg Sm}(X_T)$ are interested in T : $x \in POS_{(0.8, 0.2)}^{\neg Sm}$ if and only if the degree to which

"many users of the community of x are interested in T "

is greater than or equal to 0.8.

In the sequel, we determine the communities to which provide the news about $Sport$. To do this, we firstly compute the value $\frac{|X_{Sport} \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|}$ for each $x \in U$:

$$\frac{|X_{Sport} \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} = \begin{cases} 0 & \text{if } x \in C_1, \\ 0.14 & \text{if } x \in C_6, \\ 0.2 & \text{if } x \in C_2, \\ 0.4 & \text{if } x \in C_3, \\ 0.6 & \text{if } x \in C_4, \\ 0.8 & \text{if } x \in C_5. \end{cases} \quad (11)$$

According to the definition of $\neg Sm$ that is given by (5), we get $\neg Sm(0) = 0$, $\neg Sm(0.14) = 0.25$, $\neg Sm(0.2) = 0.75$ and $\neg Sm(0.4) = \neg Sm(0.6) = \neg Sm(0.8) = 1$.

Consequently,

$$\neg Sm\left(\frac{|X_{Sport} \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|}\right) = \begin{cases} 0 & \text{if } x \in C_1, \\ 0.25 & \text{if } x \in C_6, \\ 0.75 & \text{if } x \in C_2 \\ 1 & \text{if } x \in C_3 \cup C_4 \cup C_5. \end{cases} \quad (12)$$

Then, the positive, negative and boundary regions induced by $(0.8, 0.2)$ and $\neg Sm$ are the following:

$$POS_{(0.8, 0.2)}^{\neg Sm}(X_{Sport}) = \left\{ x \in U \mid \neg Sm\left(\frac{|X_{Sport} \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|}\right) \geq 0.8 \right\} = C_3 \cup C_4 \cup C_5,$$

$$NEG_{(0.8,0.2)}^{-Sm}(X_{Sport}) = \left\{ x \in U \mid \neg Sm \left(\frac{|X_{Sport} \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \right) \leq 0.2 \right\} = C_1,$$

$$BND_{(0.8,0.2)}^{-Sm}(X_{Sport}) = \left\{ x \in U \mid 0.2 < \neg Sm \left(\frac{|X_{Sport} \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \right) < 0.8 \right\} = C_2 \cup C_6.$$

The three regions lead the following decisions. Firstly, we choose to provide the news about the sport to the communities C_3 , C_4 , and C_5 that form the positive region, considering that *these contains many users interested in sport* with a degree that we think is enough high (≥ 0.8). Moreover, we require further analysis on the communities C_2 and C_6 forming the boundary region, before providing news about sports. For example, we could evaluate the interests of their users in the future or once new users join them. Finally, we surely do not provide sports news to C_1 because we think that not enough of its users are interested in sports topics, indeed we consider the degree to which *many users of C_1 are interested in sports* low (≤ 0.2).

3.2 Linguistic Rough Sets

Definition 4 also leads to a novel generalization of Pawlak rough sets.

Definition 5. Let $Ev \in \mathcal{E}$, the (α, β) -linguistic rough set of X determined by \mathcal{R} and Ev is the pair $(\mathcal{L}_{(\alpha,\beta)}^{Ev}(X), \mathcal{U}_{(\alpha,\beta)}^{Ev}(X))$, where

$$\mathcal{L}_{(\alpha,\beta)}^{Ev}(X) = \left\{ x \in X \mid Ev \left(\frac{|[x]_{\mathcal{R}} \cap X|}{|[x]_{\mathcal{R}}|} \right) \geq \alpha \right\} \quad \text{and}$$

$$\mathcal{U}_{(\alpha,\beta)}^{Ev}(X) = \left\{ x \in X \mid Ev \left(\frac{|[x]_{\mathcal{R}} \cap X|}{|[x]_{\mathcal{R}}|} \right) > \beta \right\}.$$

$\mathcal{L}_{(\alpha,\beta)}^{Ev}(X)$ and $\mathcal{U}_{(\alpha,\beta)}^{Ev}(X)$ are respectively called (α, β) -linguistic lower and upper approximation of X determined by \mathcal{R} and Ev .

Equivalently, by Definition 4, we get

$$\mathcal{L}_{(\alpha,\beta)}^{Ev}(X) = POS_{(\alpha,\beta)}^{Ev}(X) \quad \text{and} \quad \mathcal{U}_{(\alpha,\beta)}^{Ev}(X) = POS_{(\alpha,\beta)}^{Ev}(X) \cup BND_{(\alpha,\beta)}^{Ev}(X).$$

Let $x \in U$, the value $Ev \left(\frac{|[x]_{\mathcal{R}} \cap X|}{|[x]_{\mathcal{R}}|} \right)$ is viewed as the degree of confidence expressing *how much we can trust that x belongs to X* .

The following is an illustrative example.

Example 3. Consider Example 3.1. In terms of generalized rough sets, X_{Sport} can be approximated by the $(0.8, 0.2)$ -linguistic rough set

$$(\mathcal{L}_{(0.8,0.2)}^{-Sm}(X_{Sport}), \mathcal{U}_{(0.8,0.2)}^{-Sm}(X_{Sport})) = (C_3 \cup C_4 \cup C_5, C_2 \cup C_3 \cup C_4 \cup C_5 \cup C_6) = (\{u_{11}, \dots, u_{25}\}, \{u_6, \dots, u_{32}\}).$$

Each element $x \in U$ is associated with the value $\neg Sm \left(\frac{|X_{Sport} \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \right)$, which is understood as the degree of confidence expressing how much we can trust that x is interested in sports.

4 Connection with TWD methods

In this section, we find a link between the TWD methods based on probabilistic rough sets and evaluative expressions. In particular, we fix a finite universe U , a subset X of U , an equivalence relation \mathcal{R} on U , and a pair of thresholds (α, β) such that

$0 \leq \alpha < \beta \leq 1$, and we aim to determine for each evaluative expression Ev , the class of all pairs of thresholds like (α', β') so that $\mathcal{T}_{(\alpha', \beta')}(X)$ coincides with $\mathcal{T}_{(\alpha, \beta)}^{Ev}(X)$.

In this paper, we confine to the class $\mathcal{E}^+ \subset \mathcal{E}$, which is made of all extensions that are increasing functions, i.e. let $Ev \in \mathcal{E}$, $Ev \in \mathcal{E}^+$ if and only if “ $Ev(x) \leq Ev(y)$ ” for each $x, y \in [0, 1]$ such that $x \leq y$ ”. Examples of evaluative expressions so that their extension is an increasing function, are *not small*, *very big*, and *extremely big* (see (5), (6), and (7)). However, there exist evaluative expressions like *small* that are represented by a decreasing function and others like *medium* that are represented by a non-monotonic function.

In order to obtain the results of this section, we need to define the values α_1^{Ev} , α_2^{Ev} , β_1^{Ev} , and β_2^{Ev} , which are associated with $\mathcal{T}_{(\alpha, \beta)}^{Ev}(X)$, where $Ev \in \mathcal{E}$.

Definition 6. Let $Ev \in \mathcal{E}$. If $POS_{(\alpha, \beta)}^{Ev}(X)$, $NEG_{(\alpha, \beta)}^{Ev}(X)$, $BND_{(\alpha, \beta)}^{Ev}(X) \neq \emptyset$, then we put

$$\begin{aligned} (i) \quad \alpha_1^{Ev} &= \max \left\{ \frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \mid x \in BND_{(\alpha, \beta)}^{Ev}(X) \right\}, \\ (ii) \quad \alpha_2^{Ev} &= \min \left\{ \frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \mid x \in POS_{(\alpha, \beta)}^{Ev}(X) \right\}, \\ (iii) \quad \beta_1^{Ev} &= \max \left\{ \frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \mid x \in NEG_{(\alpha, \beta)}^{Ev}(X) \right\}, \\ (iv) \quad \beta_2^{Ev} &= \min \left\{ \frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \mid x \in BND_{(\alpha, \beta)}^{Ev}(X) \right\}. \end{aligned}$$

Example 4. Consider the universe U , its subset X_{Sport} , and the pair of thresholds (α, β) that are defined by Example 3.1. Then, the corresponding positive, negative, and boundary regions are the following:

$$\begin{aligned} POS_{(0.8, 0.2)}^{\neg Sm}(X_{Sport}) &= C_3 \cup C_4 \cup C_5, \quad NEG_{(0.8, 0.2)}^{\neg Sm}(X_{Sport}) = C_1, \quad \text{and} \\ BND_{(0.8, 0.2)}^{\neg Sm}(X_{Sport}) &= C_2 \cup C_6. \end{aligned}$$

Hence, by (11), we get ³

$$\begin{aligned} \left\{ \frac{|X_{Sport} \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \mid x \in POS_{(\alpha, \beta)}^{Ev}(X_{Sport}) \right\} &= \left\{ \frac{|X_{Sport} \cap C_3|}{|C_3|}, \frac{|X_{Sport} \cap C_4|}{|C_4|}, \frac{|X_{Sport} \cap C_5|}{|C_5|} \right\} = \{0.4, 0.6, 0.8\}; \\ \left\{ \frac{|X_{Sport} \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \mid x \in NEG_{(\alpha, \beta)}^{Ev}(X_{Sport}) \right\} &= \left\{ \frac{|X_{Sport} \cap C_1|}{|C_1|} \right\} = \{0\}; \\ \left\{ \frac{|X_{Sport} \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \mid x \in BND_{(\alpha, \beta)}^{Ev}(X_{Sport}) \right\} &= \left\{ \frac{|X_{Sport} \cap C_2|}{|C_2|}, \frac{|X_{Sport} \cap C_6|}{|C_6|} \right\} \\ &= \{0.2, 0.14\}. \end{aligned}$$

Finally, by Definition 6, $\alpha_1^{\neg Sm} = \max\{0.14, 0.2\} = 0.2$, $\alpha_2^{\neg Sm} = \min\{0.4, 0.6, 0.8\} = 0.4$, $\beta_1^{\neg Sm} = \max\{0\} = 0$, and $\beta_2^{\neg Sm} = \min\{0.14, 0.2\} = 0.14$.

³Recall that the equivalence classes of $\{[x]_{\mathcal{R}} \mid x \in U\}$ are the sets C_1, C_2, C_3, C_4, C_5 , and C_6 .

If Ev is an increasing function, namely $Ev \in \mathcal{E}^+$, then we can order β_1^{Ev} , β_2^{Ev} , α_1^{Ev} , and α_2^{Ev} as shown in the following proposition.

Proposition 1. *Let $Ev \in \mathcal{E}^+$. If $POS_{(\alpha,\beta)}^{Ev}(X), NEG_{(\alpha,\beta)}^{Ev}(X), BND_{(\alpha,\beta)}^{Ev}(X) \neq \emptyset$, then $0 \leq \beta_1^{Ev} < \beta_2^{Ev} \leq \alpha_1^{Ev} < \alpha_2^{Ev} \leq 1$.*

Proof. By Definition 6, it is trivial that $0 \leq \beta_1^{Ev}, \beta_2^{Ev}, \alpha_1^{Ev}, \alpha_2^{Ev} \leq 1$.

$(\beta_1^{Ev} < \beta_2^{Ev})$. By Definition 6 ((iii) and (iv)), $\beta_1^{Ev} = \frac{|X \cap [x_1]_{\mathcal{R}}|}{|[x_1]_{\mathcal{R}}|}$ with $x_1 \in NEG_{(\alpha,\beta)}^{Ev}(X)$

and $\beta_2^{Ev} = \frac{|X \cap [x_2]_{\mathcal{R}}|}{|[x_2]_{\mathcal{R}}|}$ with $x_2 \in BND_{(\alpha,\beta)}^{Ev}(X)$. Then, by Definition 4 ((ii) and (iii)), $Ev(\beta_1^{Ev}) \leq \beta$ and $\beta < Ev(\beta_2^{Ev}) < \alpha$. Hence, $Ev(\beta_1^{Ev}) < Ev(\beta_2^{Ev})$. Thus, considering that Ev is an increasing function, we can conclude that $\beta_1^{Ev} < \beta_2^{Ev}$.

$(\alpha_1^{Ev} < \alpha_2^{Ev})$. By Definition 6 ((i) and (ii)), $\alpha_1^{Ev} = \frac{|X \cap [x_1]_{\mathcal{R}}|}{|[x_1]_{\mathcal{R}}|}$ with $x_1 \in BND_{(\alpha,\beta)}^{Ev}(X)$

and $\alpha_2^{Ev} = \frac{|X \cap [x_2]_{\mathcal{R}}|}{|[x_2]_{\mathcal{R}}|}$ with $x_2 \in POS_{(\alpha,\beta)}^{Ev}(X)$. Thus, by Definition 4 ((iii) and (i)), $\beta < Ev(\alpha_1^{Ev}) < \alpha$ and $Ev(\alpha_2^{Ev}) \geq \alpha$. Thus, $Ev(\alpha_1^{Ev}) < Ev(\alpha_2^{Ev})$. Hence, considering that Ev is an increasing function, $\alpha_1^{Ev} < \alpha_2^{Ev}$.

$(\beta_2^{Ev} \leq \alpha_1^{Ev})$. By Definition 6 ((i) and (iii)), $\alpha_1^{Ev} = \frac{|X \cap [x_1]_{\mathcal{R}}|}{|[x_1]_{\mathcal{R}}|}$ with $x_1 \in BND_{(\alpha,\beta)}^{Ev}(X)$

and $\beta_2^{Ev} = \frac{|X \cap [x_2]_{\mathcal{R}}|}{|[x_2]_{\mathcal{R}}|}$ with $x_2 \in BND_{(\alpha,\beta)}^{Ev}(X)$. Therefore, since $x_1, x_2 \in BND_{(\alpha,\beta)}^{Ev}(X)$, $\beta_2^{Ev} \leq \alpha_1^{Ev}$ clearly holds.

□

Example 5. *In Example 4, we have shown that $\alpha_1^{-Sm} = 0.2$, $\alpha_2^{-Sm} = 0.4$, $\beta_1^{-Sm} = 0$, and $\beta_2^{-Sm} = 0.14$. Then, according to Proposition 1, $0 \leq \beta_1^{Ev} < \beta_2^{Ev} \leq \alpha_1^{Ev} < \alpha_2^{Ev} \leq 1$.*

The next theorems show that the three regions generated by $Ev \in \mathcal{E}^+$ can be also obtained by using the probabilistic approach and changing the initial thresholds. We separately analyze the following cases: all three regions are non-empty (Theorem 1) and only one of the three regions is empty (Theorems 2-4). The remaining case where only one region is non-empty (namely, one of the three regions coincides with the universe) is omitted because not significant.

Theorem 1. *Let $Ev \in \mathcal{E}^+$ such that $POS_{(\alpha,\beta)}^{Ev}(X), NEG_{(\alpha,\beta)}^{Ev}(X), BND_{(\alpha,\beta)}^{Ev}(X) \neq \emptyset$ and let $\alpha', \beta' \in [0, 1]$ with $\beta' < \alpha'$. Then,*

$$\mathcal{T}_{(\alpha,\beta)}^{Ev}(X) = \mathcal{T}_{(\alpha',\beta')}^{Ev}(X) \text{ if and only if } \alpha' \in (\alpha_1^{Ev}, \alpha_2^{Ev}] \text{ and } \beta' \in [\beta_1^{Ev}, \beta_2^{Ev}).$$

Remark 7. *Before to prove Theorem 1, let us represent the intervals that contain α' and β' (i.e. the values for generating $\mathcal{T}_{(\alpha,\beta)}^{Ev}(X)$ with probabilistic rough sets) by Figure 1. By Definition 6, these intervals separates $POS_{(\alpha,\beta)}^{Ev}(X)$ from $BND_{(\alpha,\beta)}^{Ev}(X)$ and $NEG_{(\alpha,\beta)}^{Ev}(X)$ from $BND_{(\alpha,\beta)}^{Ev}(X)$. More precisely, the value $Ev\left(\frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|}\right)$ belongs to $[0, \beta_1^{Ev}]$ when $x \in NEG_{(\alpha,\beta)}^{Ev}(X)$, to $[\beta_2^{Ev}, \alpha_1^{Ev}]$ when $x \in BND_{(\alpha,\beta)}^{Ev}(X)$, and to $[\alpha_2^{Ev}, 1]$ when $x \in POS_{(\alpha,\beta)}^{Ev}(X)$.*

Proof. (\Leftarrow). Let $\alpha' \in (\alpha_1^{Ev}, \alpha_2^{Ev}]$ and let $\beta' \in [\beta_1^{Ev}, \beta_2^{Ev})$, we need to prove that $POS_{(\alpha,\beta)}^{Ev}(X) = POS_{(\alpha',\beta')}^{Ev}(X)$, $NEG_{(\alpha,\beta)}^{Ev}(X) = NEG_{(\alpha',\beta')}^{Ev}(X)$, and $BND_{(\alpha,\beta)}^{Ev}(X) = BND_{(\alpha',\beta')}^{Ev}(X)$.

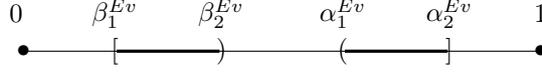


Figure 1: Intervals $[\beta_1^{Ev}, \beta_2^{Ev})$ and $(\alpha_1^{Ev}, \alpha_2^{Ev}]$.

$(POS_{(\alpha, \beta)}^{Ev}(X) = POS_{(\alpha', \beta')}^{Ev}(X))$. Let $\bar{x} \in POS_{(\alpha, \beta)}^{Ev}(X)$, then $\frac{|X \cap [\bar{x}]_{\mathcal{R}}|}{|[\bar{x}]_{\mathcal{R}}|} \geq \alpha_2^{Ev}$ from Definition 6 (ii). Moreover, $\alpha' \leq \alpha_2^{Ev}$ because $\alpha' \in (\alpha_1^{Ev}, \alpha_2^{Ev}]$. Consequently, $\frac{|X \cap [\bar{x}]_{\mathcal{R}}|}{|[\bar{x}]_{\mathcal{R}}|} \geq \alpha'$. Finally, $\bar{x} \in POS_{(\alpha', \beta')}^{Ev}(X)$ from Definition 1 (i).

Let $\bar{x} \in POS_{(\alpha', \beta')}^{Ev}(X)$, then $\frac{|X \cap [\bar{x}]_{\mathcal{R}}|}{|[\bar{x}]_{\mathcal{R}}|} \geq \alpha'$ from Definition 1 (i). By the previous inequality and $\alpha' > \alpha_1^{Ev}$, we get $\frac{|X \cap [\bar{x}]_{\mathcal{R}}|}{|[\bar{x}]_{\mathcal{R}}|} > \alpha_1^{Ev}$. Hence, considering that α_1^{Ev} is the maximum of $\left\{ \frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \mid x \in BND_{(\alpha, \beta)}^{Ev}(X) \right\}$ (see Definition 6(i)), we are sure that $\bar{x} \notin BND_{(\alpha, \beta)}^{Ev}(X)$. Furthermore, $\frac{|X \cap [\bar{x}]_{\mathcal{R}}|}{|[\bar{x}]_{\mathcal{R}}|} > \alpha_1^{Ev}$ and $\beta_1^{Ev} < \alpha_1^{Ev}$ (see Proposition 1) imply that $\frac{|X \cap [\bar{x}]_{\mathcal{R}}|}{|[\bar{x}]_{\mathcal{R}}|} > \beta_1^{Ev}$. Thus, considering that β_1^{Ev} is the maximum of $\left\{ \frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \mid x \in NEG_{(\alpha, \beta)}^{Ev}(X) \right\}$ (see Definition 6(iii)), we have $\bar{x} \notin NEG_{(\alpha, \beta)}^{Ev}(X)$. Ultimately, by (10), $\bar{x} \in POS_{(\alpha, \beta)}^{Ev}(X)$.

$(BND_{(\alpha, \beta)}^{Ev}(X) = BND_{(\alpha', \beta')}^{Ev}(X))$. Let $\bar{x} \in BND_{(\alpha, \beta)}^{Ev}(X)$. By Definition 6 ((i) and (iv)), $\beta_2^{Ev} \leq \frac{|X \cap [\bar{x}]_{\mathcal{R}}|}{|[\bar{x}]_{\mathcal{R}}|} \leq \alpha_1^{Ev}$. Moreover, by hypothesis, $\beta' < \beta_2^{Ev}$ and $\alpha' > \alpha_1^{Ev}$. Thus, we can conclude that $\beta' < \frac{|X \cap [\bar{x}]_{\mathcal{R}}|}{|[\bar{x}]_{\mathcal{R}}|} < \alpha'$, namely $\bar{x} \in BND_{(\alpha', \beta')}^{Ev}(X)$ from Definition 1 (iii).

Let $\bar{x} \in BND_{(\alpha', \beta')}^{Ev}(X)$, then $\beta' < \frac{|X \cap [\bar{x}]_{\mathcal{R}}|}{|[\bar{x}]_{\mathcal{R}}|} < \alpha'$ from Definition 1 (iii). By hypothesis, $\beta_1^{Ev} \leq \beta'$ and $\alpha' \leq \alpha_2^{Ev}$. Hence, we know that $\beta_1^{Ev} < \frac{|X \cap [\bar{x}]_{\mathcal{R}}|}{|[\bar{x}]_{\mathcal{R}}|} < \alpha_2^{Ev}$. By Definition 6 (iii), $\beta_1^{Ev} < \frac{|X \cap [\bar{x}]_{\mathcal{R}}|}{|[\bar{x}]_{\mathcal{R}}|}$ implies that $\bar{x} \notin NEG_{(\alpha, \beta)}^{Ev}(X)$.

Furthermore, by Definition 6 (ii), $\frac{|X \cap [\bar{x}]_{\mathcal{R}}|}{|[\bar{x}]_{\mathcal{R}}|} < \alpha_2^{Ev}$ implies that $\bar{x} \notin POS_{(\alpha, \beta)}^{Ev}(X)$. Ultimately, by (10), $\bar{x} \in BND_{(\alpha, \beta)}^{Ev}(X)$.

$(NEG_{(\alpha, \beta)}^{Ev}(X) = NEG_{(\alpha', \beta')}^{Ev}(X))$. We have previously shown that $POS_{(\alpha, \beta)}^{Ev}(X) = POS_{(\alpha', \beta')}^{Ev}(X)$ and $BND_{(\alpha, \beta)}^{Ev}(X) = BND_{(\alpha', \beta')}^{Ev}(X)$. So, by (3) and (10), it is clear that $NEG_{(\alpha, \beta)}^{Ev}(X) = NEG_{(\alpha', \beta')}^{Ev}(X)$.

(\Rightarrow). Let $\mathcal{T}_{(\alpha, \beta)}^{Ev}(X) = \mathcal{T}_{(\alpha', \beta')}^{Ev}(X)$, we intend to prove that $\beta_1^{Ev} \leq \beta' < \beta_2^{Ev}$ and $\alpha_1^{Ev} < \alpha' \leq \alpha_2^{Ev}$.

($\alpha' \leq \alpha_2^{Ev}$). Let $x_2 \in U$ such that $\alpha_2^{Ev} = \frac{|X \cap [x_2]_{\mathcal{R}}|}{|[x_2]_{\mathcal{R}}|}$. By Definition 6 (ii), $x_2 \in POS_{(\alpha, \beta)}^{Ev}(X)$. Hence, $\alpha' > \alpha_2^{Ev}$ means that $\frac{|X \cap [x_2]_{\mathcal{R}}|}{|[x_2]_{\mathcal{R}}|} < \alpha'$. Thus, $x_2 \notin$

$POS_{(\alpha',\beta')}(X)$ from Definition 1 (i). This contradicts that $POS_{(\alpha,\beta)}^{Ev}(X) = POS_{(\alpha',\beta')}(X)$. Thus, it must be true that $\alpha' \leq \alpha_2^{Ev}$.

$(\alpha_1^{Ev} < \alpha')$. Let $x_1 \in U$ such that $\alpha_1^{Ev} = \frac{|X \cap [x_1]_{\mathcal{R}}|}{|[x_1]_{\mathcal{R}}|}$. By Definition 6 (i), $x_1 \in BND_{(\alpha,\beta)}^{Ev}(X)$. If $\alpha_1^{Ev} \geq \alpha'$, then $\frac{|X \cap [x_1]_{\mathcal{R}}|}{|[x_1]_{\mathcal{R}}|} \geq \alpha'$. So, $x_1 \in POS_{(\alpha',\beta')}(X)$ from Definition 1 (i). This contradicts that $POS_{(\alpha,\beta)}^{Ev}(X) = POS_{(\alpha',\beta')}(X)$. Thus, it must be true that $\alpha_1^{Ev} < \alpha'$.

$(\beta_1^{Ev} \leq \beta')$. Let $x_1 \in U$ such that $\beta_1^{Ev} = \frac{|X \cap [x_1]_{\mathcal{R}}|}{|[x_1]_{\mathcal{R}}|}$. By Definition 6(iii), $x_1 \in NEG_{(\alpha,\beta)}^{Ev}(X)$. If $\beta_1^{Ev} > \beta'$, then $\frac{|X \cap [x_1]_{\mathcal{R}}|}{|[x_1]_{\mathcal{R}}|} > \beta'$, which implies that $x_1 \notin NEG_{(\alpha',\beta')}(X)$ from Definition 1(ii). This contradicts that $NEG_{(\alpha,\beta)}^{Ev}(X) = NEG_{(\alpha',\beta')}(X)$. Thus, it must be true that $\beta_1^{Ev} \leq \beta'$.

$(\beta' < \beta_2^{Ev})$. Let $x_2 \in U$ such that $\beta_2^{Ev} = \frac{|X \cap [x_2]_{\mathcal{R}}|}{|[x_2]_{\mathcal{R}}|}$. By Definition 6(iv), $x_2 \in BND_{(\alpha,\beta)}^{Ev}(X)$. Also, if $\beta' \geq \beta_2^{Ev}$, then $\frac{|X \cap [x_2]_{\mathcal{R}}|}{|[x_2]_{\mathcal{R}}|} \leq \beta'$, which implies that $x_2 \in NEG_{(\alpha',\beta')}(X)$ from Definition 1 (ii). This contradicts that $BND_{(\alpha,\beta)}^{Ev}(X) = BND_{(\alpha',\beta')}(X)$. Thus, it must be true that $\beta' < \beta_2^{Ev}$.

□

Example 6. Consider Example 3.1, $\neg Sm$ is an increasing function and all the three regions of $\mathcal{T}_{(0.8,0.2)}^{\neg Sm}(X_{Sport})$ are non-empty. In Example 4, we have found that $\alpha_1^{\neg Sm} = 0.2$, $\alpha_2^{\neg Sm} = 0.4$, $\beta_1^{\neg Sm} = 0$, and $\beta_2^{\neg Sm} = 0.14$. Therefore, according to Theorem 1, we get $\mathcal{T}_{(0.8,0.2)}^{\neg Sm}(X_{Sport}) = \mathcal{T}_{(\alpha',\beta')}(X_{Sport})$ for each (α',β') such that $\alpha' \in (0.2, 0.4]$ and $\beta' \in [0, 0.14]$

For example, we can easily verify that $\mathcal{T}_{(0.8,0.2)}^{\neg Sm}(X_{Sport}) = \mathcal{T}_{(0.3,0.1)}(X_{Sport})$. Indeed, by (11) and by Definition 1,

- $POS_{(0.3,0.1)}(X_{Sport}) = \left\{ x \in U \mid \frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \geq 0.3 \right\} = C_3 \cup C_4 \cup C_5$,
- $NEG_{(0.3,0.1)}(X_{Sport}) = \left\{ x \in U \mid \frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \leq 0.1 \right\} = C_1$, and
- $BND_{(0.3,0.1)}(X_{Sport}) = \left\{ x \in U \mid 0.1 < \frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} < 0.3 \right\} = C_2 \cup C_6$.

By Theorem 1, we can connect linguistic rough sets with classical rough sets. More precisely, the following corollary holds.

Corollary 1. Let $Ev \in \mathcal{E}^+$ with $POS_{(\alpha,\beta)}^{Ev}(X), NEG_{(\alpha,\beta)}^{Ev}(X), BND_{(\alpha,\beta)}^{Ev}(X) \neq \emptyset$. Then,

$$(\mathcal{L}_{(\alpha,\beta)}^{Ev}(X), \mathcal{U}_{(\alpha,\beta)}^{Ev}(X)) = (\mathcal{L}(X), \mathcal{U}(X)) \text{ }^4 \text{ if and only if } \beta_1^{Ev} = 0 \text{ and } \alpha_2^{Ev} = 1.$$

Proof. (\Rightarrow). Suppose that $(\mathcal{L}_{(\alpha,\beta)}^{Ev}(X), \mathcal{U}_{(\alpha,\beta)}^{Ev}(X))$ is the rough set of X . Then, by (4), let $x \in U$, $x \in POS_{(\alpha,\beta)}^{Ev}(X)$ if and only if $[x]_{\mathcal{R}} \subseteq X$. The latter means that $\frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} = 1$ for each $x \in POS_{(\alpha,\beta)}^{Ev}(X)$. Hence, By Definition 6 (ii), $\alpha_2^{Ev} = 1$.

⁴Recall that $(\mathcal{L}_{(\alpha,\beta)}^{Ev}(X), \mathcal{U}_{(\alpha,\beta)}^{Ev}(X))$ is the linguistic rough set of X given by Definition 5 and $(\mathcal{L}(X), \mathcal{U}(X))$ is the classical rough set of X given by Eq. (4).

By (4), let $x \in U$, $x \in POS_{(\alpha,\beta)}^{Ev}(X) \cup BND_{(\alpha,\beta)}^{Ev}(X)$ if and only if $[x]_{\mathcal{R}} \cap X \neq \emptyset$. Since $POS_{(\alpha,\beta)}^{Ev}(X) \cup BND_{(\alpha,\beta)}^{Ev}(X) = U \setminus NEG_{(\alpha,\beta)}^{Ev}(X)$, we know that $\frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} = 0$ for each $x \in NEG_{(\alpha,\beta)}^{Ev}(X)$. Finally, by Definition 6 (iii), $\beta_1^{Ev} = 0$.

(\Leftarrow). Suppose that $\alpha_2^{Ev} = 1$ and $\beta_1^{Ev} = 0$. Trivially, $\alpha_2^{Ev} \in (\alpha_1^{Ev}, \alpha_2^{Ev}]$ and $\beta_1^{Ev} \in [\beta_1^{Ev}, \beta_2^{Ev})$. Then, by Theorem 1, $(POS_{(\alpha,\beta)}^{Ev}(X), POS_{(\alpha,\beta)}^{Ev}(X) \cup BND_{(\alpha,\beta)}^{Ev}(X)) = (POS_{(1,0)}(X), POS_{(1,0)}(X) \cup BND_{(1,0)}(X))$. Moreover, by Remark 2, $(\mathcal{L}_{(1,0)}(X), \mathcal{U}_{(1,0)}(X)) = (POS_{(1,0)}(X), POS_{(1,0)}(X) \cup BND_{(1,0)}(X))$ coincides with the rough set $(\mathcal{L}(X), \mathcal{U}(X))$ given by (4). \square

Example 7. Consider the universe $U = \{u_1, \dots, u_{20}\}$ and the evaluative expression Very big, which is modelled by (6). We suppose that U is partitioned into three equivalence classes: $C_1 = \{u_1, \dots, u_5\}$, $C_2 = \{u_6, \dots, u_{10}\}$, and $C_3 = \{u_{11}, \dots, u_{20}\}$. If $X = \{u_6, \dots, u_{19}\}$, we can simply prove that $(\mathcal{L}_{(0.7,0.3)}^{BiVe}(X), \mathcal{E}_{(0.7,0.3)}^{BiVe}(X)) = (\mathcal{L}(X), \mathcal{U}(X))$. Indeed, we get

$$\frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} = \begin{cases} 0 & \text{if } x \in C_1, \\ 0.9 & \text{if } x \in C_3, \\ 1 & \text{if } x \in C_2. \end{cases} \text{ and } BiVe\left(\frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|}\right) = \begin{cases} 0 & \text{if } x \in C_1, \\ 0.59 & \text{if } x \in C_3, \\ 1 & \text{if } x \in C_2. \end{cases}$$

Thus,

$$POS_{(0.7,0.3)}^{BiVe}(X) = \{C_i \mid i \in \{1, 2, 3\} \text{ and } BiVe\left(\frac{|X \cap C_i|}{|C_i|}\right) \geq 0.7\} = C_2;$$

$$NEG_{(0.7,0.3)}^{BiVe}(X) = \{C_i \mid i \in \{1, 2, 3\} \text{ and } BiVe\left(\frac{|X \cap C_i|}{|C_i|}\right) \leq 0.3\} = C_1;$$

$$BND_{(0.7,0.3)}^{BiVe}(X) = \{C_i \mid i \in \{1, 2, 3\} \text{ and } 0.3 < BiVe\left(\frac{|X \cap C_i|}{|C_i|}\right) < 0.7\} = C_3.$$

Also, by Definition 6, $\beta_1^{BiVe} = 0$, $\alpha_2^{BiVe} = 1$, $\alpha_1^{BiVe} = \beta_2^{BiVe} = 0.9$. Since the hypothesis of the previous corollary is satisfied, we expect that $(\mathcal{L}(X), \mathcal{U}(X)) = (\mathcal{L}_{(0.7,0.3)}^{BiVe}(X), \mathcal{U}_{(0.7,0.3)}^{BiVe}(X)) = (C_2, C_2 \cup C_3)$. We can immediately verify that this is true: $\mathcal{L}(X) = C_2$ because C_2 is the unique class among C_1, C_2 , and C_3 that is included in X ; moreover, $\mathcal{U}(X) = C_2 \cup C_3$ because $X \cap C_2 \neq \emptyset$ and $X \cap C_3 \neq \emptyset$, while $X \cap C_1 = \emptyset$.

We are now going to deal with the cases where one of $BND_{(\alpha,\beta)}^{Ev}$, $POS_{(\alpha,\beta)}^{Ev}$, $NEG_{(\alpha,\beta)}^{Ev}$ is empty.

Theorem 2. Let $Ev \in \mathcal{E}^+$ such that $BND_{(\alpha,\beta)}^{Ev} = \emptyset$ and $POS_{(\alpha,\beta)}^{Ev}, NEG_{(\alpha,\beta)}^{Ev} \neq \emptyset$. Let $\alpha', \beta' \in [0, 1]$ such that $\beta' < \alpha'$. Then,

$$\mathcal{T}_{(\alpha,\beta)}^{Ev}(X) = \mathcal{T}_{(\alpha',\beta')}^{Ev}(X) \text{ if and only if } \beta_1^{Ev} \leq \beta' < \alpha' \leq \alpha_2^{Ev}.$$

Proof. (\Leftarrow). Let $\alpha', \beta' \in [0, 1]$ such that $\beta_1^{Ev} \leq \beta' < \alpha' \leq \alpha_2^{Ev}$, we need to prove that $POS_{(\alpha,\beta)}^{Ev}(X) = POS_{(\alpha',\beta')}^{Ev}(X)$, $NEG_{(\alpha,\beta)}^{Ev}(X) = NEG_{(\alpha',\beta')}^{Ev}(X)$, and $BND_{(\alpha,\beta)}^{Ev}(X) = BND_{(\alpha',\beta')}^{Ev}(X)$.

$(POS_{(\alpha,\beta)}^{Ev}(X) = POS_{(\alpha',\beta')}^{Ev}(X))$. Let $\bar{x} \in POS_{(\alpha,\beta)}^{Ev}(X)$. Then, $\frac{|X \cap [\bar{x}]_{\mathcal{R}}|}{|[\bar{x}]_{\mathcal{R}}|} \geq \alpha_1^{Ev}$

from Definition 6 (i). By hypothesis, $\alpha' \leq \alpha_2^{Ev}$. Finally, by the previous two inequalities, we obtain that $\frac{|X \cap [\bar{x}]_{\mathcal{R}}|}{|[\bar{x}]_{\mathcal{R}}|} \geq \alpha'$. Namely, $\bar{x} \in POS_{(\alpha',\beta')}^{Ev}(X)$ from Definition 1 (i).

Let $\bar{x} \in POS_{(\alpha',\beta')}^{Ev}(X)$. Then, $\frac{|X \cap [\bar{x}]_{\mathcal{R}}|}{|[\bar{x}]_{\mathcal{R}}|} \geq \alpha'$ by Definition 1 (i). Let $x_1 \in U$ such that $\beta_1^{Ev} = \frac{|X \cap [x_1]_{\mathcal{R}}|}{|[x_1]_{\mathcal{R}}|}$. Moreover, by hypothesis $\beta_1^{Ev} < \alpha'$. So, by the

previous two inequalities, we get $\frac{|X \cap [\bar{x}]_{\mathcal{R}}|}{|[\bar{x}]_{\mathcal{R}}|} > \beta_1^{Ev}$. Then, by Definition 6 (iii), $\bar{x} \notin NEG_{(\alpha,\beta)}^{Ev}(X)$. Lastly, by (10) and by the hypothesis $BND_{(\alpha,\beta)}^{Ev}(X) = \emptyset$, we can conclude that $\bar{x} \in POS_{(\alpha,\beta)}^{Ev}(X)$.

$(NEG_{(\alpha,\beta)}^{Ev}(X) = NEG_{(\alpha',\beta')}(X))$. Let $\bar{x} \in NEG_{(\alpha,\beta)}^{Ev}(X)$. Then, by Definition 6 (iii), $\frac{|X \cap [\bar{x}]_{\mathcal{R}}|}{|[\bar{x}]_{\mathcal{R}}|} \leq \beta_1^{Ev}$. Additionally, we know that $\beta_1^{Ev} \leq \beta'$ from hypothesis. Then, $\frac{|X \cap [\bar{x}]_{\mathcal{R}}|}{|[\bar{x}]_{\mathcal{R}}|} \leq \beta'$, namely $\bar{x} \in NEG_{(\alpha',\beta')}(X)$ from Definition 1 (ii).

Let $\bar{x} \in NEG_{(\alpha',\beta')}(X)$, then $\frac{|X \cap [\bar{x}]_{\mathcal{R}}|}{|[\bar{x}]_{\mathcal{R}}|} \leq \beta'$ from Definition 1 (ii). By hypothesis, $\beta' < \alpha_2^{Ev}$. Thus, by Definition 6 (ii), \bar{x} cannot belong to $POS_{(\alpha,\beta)}^{Ev}(X)$. So, by (10) and the hypothesis $BND_{(\alpha,\beta)}^{Ev}(X) = \emptyset$, we can deduce that $\bar{x} \in NEG_{(\alpha,\beta)}^{Ev}(X)$.

$(BND_{(\alpha,\beta)}^{Ev}(X) = BND_{(\alpha',\beta')}(X))$. This equality follows from $POS_{(\alpha,\beta)}^{Ev}(X) = POS_{(\alpha',\beta')}(X)$ and $NEG_{(\alpha,\beta)}^{Ev}(X) = NEG_{(\alpha',\beta')}(X)$, considering that the sets $POS_{(\alpha,\beta)}^{Ev}(X)$, $NEG_{(\alpha,\beta)}^{Ev}(X)$, and $BND_{(\alpha,\beta)}^{Ev}(X)$ (as well as $POS_{(\alpha',\beta')}(X)$, $NEG_{(\alpha',\beta')}(X)$, and $BND_{(\alpha',\beta')}(X)$) cover the universe U (see (3) and (10)).

(\Leftrightarrow) . Let $\mathcal{T}_{(\alpha,\beta)}^{Ev}(X) = \mathcal{T}_{(\alpha',\beta')}(X)$, we intend to prove that $\beta_1^{Ev} \leq \beta'$ and $\alpha' \leq \alpha_2^{Ev}$.

$(\beta_1^{Ev} \leq \beta')$. Let $x_1 \in U$ such that $\beta_1^{Ev} = \frac{|X \cap [x_1]_{\mathcal{R}}|}{|[x_1]_{\mathcal{R}}|}$. Of course, $x_1 \in NEG_{(\alpha,\beta)}^{Ev}(X)$ from Definition 6 (iii).

It is clear that the inequality $\beta_1^{Ev} > \beta'$ leads to a contradiction:

- if $\beta_1^{Ev} > \beta'$, then $\bar{x} \notin NEG_{(\alpha',\beta')}(X)$ from Definition 6 (iii);
- but, this contradicts that $NEG_{(\alpha,\beta)}^{Ev}(X) = NEG_{(\alpha',\beta')}(X)$.

So, $\beta_1^{Ev} \leq \beta'$ must hold.

$(\alpha' \leq \alpha_2^{Ev})$. Let $x_2 \in U$ such that $\alpha_2^{Ev} = \frac{|X \cap [x_2]_{\mathcal{R}}|}{|[x_2]_{\mathcal{R}}|}$. Then, $x_2 \in POS_{(\alpha,\beta)}^{Ev}(X)$ from Definition 6 (ii).

It is clear that the inequality $\alpha' > \alpha_2^{Ev}$ leads to a contradiction:

- if $\alpha' > \alpha_2^{Ev}$, then $x_2 \notin POS_{(\alpha',\beta')}(X)$ from Definition 6 (ii);
- but, this contradicts that $POS_{(\alpha,\beta)}^{Ev}(X) = POS_{(\alpha',\beta')}(X)$.

Finally, $\alpha' \leq \alpha_2^{Ev}$ must hold.

□

Examples of evaluative expressions satisfying the hypothesis of Theorem 2 can be obtained from the class defined by (8). Indeed, let $t \in [0, 1]$, Δ_t is trivially an increasing function (i.e. $\Delta_t \in \mathcal{E}^+$) and the boundary region determined by Δ_t is always empty as shown by the following proposition. In addition, in Proposition 2, the formula of the three regions that are related to Δ_t is rewritten so that the thresholds α and β do not appear in it.

Proposition 2. *Let $t \in [0, 1]$ and let $\alpha, \beta \in [0, 1]$ such that $\beta < \alpha$, then*

- $POS_{(\alpha,\beta)}^{\Delta_t}(X) = \left\{ x \in U \mid \frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \geq t \right\};$
- $NEG_{(\alpha,\beta)}^{\Delta_t}(X) = \left\{ x \in U \mid \frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} < t \right\};$

(c) $BND_{(\alpha,\beta)}^{\Delta_t}(X) = \emptyset$.

Proof. (a). Let $\bar{x} \in U$. Thus, $\bar{x} \in POS_{(\alpha,\beta)}^{Ev}(X)$ if and only if

$$\Delta_t \left(\frac{|X \cap [\bar{x}]_{\mathcal{R}}|}{|[\bar{x}]_{\mathcal{R}}|} \right) \geq \alpha \quad (13)$$

from Definition 4 (i).

By (8), the inequality (13) is true if and only if $\frac{|X \cap [\bar{x}]_{\mathcal{R}}|}{|[\bar{x}]_{\mathcal{R}}|} \geq t$.

(b). Let $\bar{x} \in U$. Then, $\bar{x} \in NEG_{(\alpha,\beta)}^{Ev}(X)$ if and only if

$$\Delta_t \left(\frac{|X \cap [\bar{x}]_{\mathcal{R}}|}{|[\bar{x}]_{\mathcal{R}}|} \right) \leq \beta \quad (14)$$

from Definition 4 (ii).

By (8), the inequality (14) is true if and only if $\frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} < t$.

(c). Notice that $\{x \in U \mid \frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \geq t\} \cup \{x \in U \mid \frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} < t\} = U$.

Moreover, we have proved that $POS_{(\alpha,\beta)}^{Ev}(X) = \{x \in U \mid \frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \geq t\}$ and $NEG_{(\alpha,\beta)}^{Ev}(X) = \{x \in U \mid \frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} < t\}$. Hence, according to (10), $BND_{(\alpha,\beta)}^{Ev}(X)$ must be empty. \square

Example 8. Let us focus on $\mathcal{T}_{(\alpha,\beta)}^{\Delta_{0.5}}(X)$, where U , X , and \mathcal{R} are defined in Example 7. By Proposition 2, it is easy to verify that $POS_{(\alpha,\beta)}^{\Delta_{0.5}}(X) = C_2 \cup C_3$, $NEG_{(\alpha,\beta)}^{\Delta_{0.5}}(X) = C_1$, and $BND_{(\alpha,\beta)}^{\Delta_{0.5}}(X) = \emptyset$ for each $\alpha, \beta \in [0, 1]$ with $\beta < \alpha$. Furthermore, according to Theorem 2, $POS_{(\alpha',\beta')}(X) = C_2 \cup C_3$, $NEG_{(\alpha',\beta')}(X) = C_1$, and $BND_{(\alpha',\beta')}(X) = \emptyset$, for each $\alpha', \beta' \in [0, 1]$ such that $\beta_1^{\Delta_{0.5}} \leq \beta' < \alpha' \leq \alpha_2^{\Delta_{0.5}}$, where $\beta_1^{\Delta_{0.5}} = 0$, $\alpha_2^{\Delta_{0.5}} = 0.9$. For example, if we choose $\alpha' = 0.2$ and $\beta' = 0.7$, we obtain $POS_{(0.7,0.2)}(X) = C_2 \cup C_3$, $NEG_{(0.7,0.2)}(X) = C_1$, and $BND_{(0.7,0.2)}(X) = \emptyset$.

Now, let us suppose that the negative region is empty.

Theorem 3. Let $Ev \in \mathcal{E}^+$ such that $NEG_{(\alpha,\beta)}^{Ev} = \emptyset$ and $POS_{(\alpha,\beta)}^{Ev}, BND_{(\alpha,\beta)}^{Ev} \neq \emptyset$. Let $\alpha', \beta' \in [0, 1]$ such that $\beta' < \alpha'$. Then,

$$\mathcal{T}_{(\alpha,\beta)}^{Ev}(X) = \mathcal{T}_{(\alpha',\beta')}(X) \text{ if and only if } \beta' \in [0, \beta_2^{Ev}) \text{ and } \alpha' \in (\alpha_1^{Ev}, \alpha_2^{Ev}].$$

Proof. The proof is similar to that of Theorems 1 and 2. So, it is omitted. \square

Example 9. Let $U = \{u_1, \dots, u_{30}\}$. We supposed that U is divided into the following equivalence classes: $C_1 = \{u_1, \dots, u_5\}$, $C_2 = \{u_6, \dots, u_{10}\}$, and $C_3 = \{u_{11}, \dots, u_{30}\}$.

Also, let $X = \{u_1, \dots, u_{28}\}$, we are interested in $\mathcal{T}_{(0.8,0.4)}^{BiVe}(X)$. Then,

$$\frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} = \begin{cases} 0.9 & \text{if } x \in C_3, \\ 1 & \text{if } x \in C_1 \cup C_2. \end{cases}$$

and

$$BiVe \left(\frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \right) = \begin{cases} 0.58 & \text{if } x \in C_3, \\ 1 & \text{if } x \in C_1 \cup C_2. \end{cases}$$

Thus, by Definition 4, $POS_{(0.8,0.4)}^{BiVe}(X) = C_1 \cup C_2$, $BND_{(0.8,0.4)}^{BiVe}(X) = C_3$, and $NEG_{(0.8,0.4)}^{BiVe}(X) = \emptyset$. Moreover, $\beta_2^{Ev} = 0.9$, $\alpha_1^{BiVe} = 0.9$, and $\alpha_2^{BiVe} = 1$. So, according to Theorem 3, $POS_{(\alpha',\beta')}(X) = C_1 \cup C_2$, $BND_{(\alpha',\beta')}(X) = C_3$, and $NEG_{(\alpha',\beta')}(X) = C_3$ for each $\beta' \in [0, 0.9)$ and $\alpha' \in (0.9, 1]$. For example, if $\alpha' = 0.95$ and $\beta' = 0.8$, then $POS_{(0.8,0.95)}(X) = C_1 \cup C_2$, $BND_{(0.8,0.95)}(X) = C_3$, and $NEG_{(0.8,0.95)}(X) = \emptyset$.

Finally, the case of an empty positive region.

Theorem 4. Let $Ev \in \mathcal{E}^+$ such that $POS_{(\alpha,\beta)}^{Ev} = \emptyset$ and $NEG_{(\alpha,\beta)}^{Ev}, BND_{(\alpha,\beta)}^{Ev} \neq \emptyset$. Let $\alpha', \beta' \in [0, 1]$ such that $\beta' < \alpha'$. Then,

$$\mathcal{T}_{(\alpha,\beta)}^{Ev}(X) = \mathcal{T}_{(\alpha',\beta')}(X) \text{ if and only if } \beta' \in [\beta_1^{Ev}, \beta_2^{Ev}) \text{ and } \alpha' \in (\alpha_1^{Ev}, 1].$$

Proof. The proof is similar to that of Theorems 1 and 2. So, it is omitted. \square

Example 10. Consider the universe U and the equivalence classes C_1, C_2 , and C_3 , which are defined by Example 9. Let $X = \{u_1, u_6, u_{11}, \dots, u_{28}\}$, we focus on $\mathcal{T}_{(0.7,0.2)}^{BiVe}(X)$. Then,

$$\frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} = \begin{cases} 0.5 & \text{if } x \in C_1 \cup C_2, \\ 0.9 & \text{if } x \in C_3. \end{cases}$$

and

$$BiVe \left(\frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \right) = \begin{cases} 0.58 & \text{if } x \in C_3, \\ 0 & \text{if } x \in C_1 \cup C_2. \end{cases}$$

By Definition 4, $NEG_{(0.7,0.2)}^{BiVe}(X) = C_1 \cup C_2$, $BND_{(0.7,0.2)}^{BiVe}(X) = C_3$, and $POS_{(0.7,0.2)}^{BiVe}(X) = \emptyset$. Also, $\beta_1^{Ev} = 0.5$, $\beta_2^{Ev} = 0.9$, $\alpha_1^{Ev} = 0.9$. Thus, according to Theorem 4, $NEG_{(\alpha',\beta')}(X) = C_1 \cup C_2$, $BND_{(\alpha',\beta')}(X) = C_3$, and $POS_{(\alpha',\beta')}(X) = \emptyset$ for each $\beta' \in [0.5, 0.9)$ and $\alpha' \in (0.9, 1]$. For example, let $(\alpha', \beta') = (0.95, 0.6)$, we can easily verify that $NEG_{(0.95,0.6)}(X) = C_1 \cup C_2$, $BND_{(0.95,0.6)}(X) = C_3$, and $POS_{(0.95,0.6)}(X) = \emptyset$.

5 Conclusions and future directions

This work proposes a novel model for three-way decisions based on the concept of evaluative linguistic expressions. Thus, a new way is provided to divide the initial universe into three regions with the corresponding decisions rules. Moreover, our results allow decision-makers to give a linguistic interpretation to the regions already obtained using the probabilistic approach. Let us indicate some possible directions to continue this work. Firstly, we need to extend the results of Section 4 to the evaluative expressions that are not necessarily represented by increasing functions. Then, we want to deepen the study of linguistic-regions by comparing our methods with those presented in [21]. In addition, we intend to understand how the decisions about the elements change using different evaluative expressions. Finally, we could analyze the logical relations between the linguistic regions determined by a given evaluative expression and investigate their consequences in terms of decisions by constructing an hexagon of opposition.

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