An algorithm for group decision making using n-dimensional fuzzy sets, admissible orders and OWA operators

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Abstract

In this paper we propose an algorithm to solve group decision making problems using n-dimensional fuzzy sets, namely, sets in which the membership degree of each element to the set is given by an increasing tuple of n elements. The use of these sets has naturally led us to define admissible orders for n-dimensional fuzzy sets, to present a construction method for those orders and to study OWA operators for aggregating the tuples used to represent the membership degrees of the elements. In these conditions, we present an algorithm and apply it to a case study, in which we show that the exploitation phase which appears in many decision making methods can be omitted by just considering linear orders between tuples.

Keywords: fuzzy multisets, n-dimensional fuzzy sets, OWA operator, decision-making

1. Introduction

A multiple criteria group decision making problem consists in choosing a solution A_i out of a set of p ($p \geq 2$) alternatives according to the evaluations,

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given by n decision makers e_k ($k \in \{1, \ldots, n\}$), to each alternative with respect to q criteria. Thus, we have that:

1. The evaluations for the alternative A_i ($1 \leq i \leq p$) with respect to the criterion C_1 are given by the tuple $(d_{i1}^{e_1}, d_{i1}^{e_2}, \ldots, d_{i1}^{e_n})$, where $d_{i1}^{e_k} \in [0,1]$ represents the evaluation of the decision maker e_k for the alternative A_i ($1 \leq i \leq p$) with respect to the criterion C_1.
2. The evaluations for the alternative A_i ($1 \leq i \leq p$) with respect to the criterion C_2 are given by the tuple $(d_{i2}^{e_1}, d_{i2}^{e_2}, \ldots, d_{i2}^{e_n})$, where $d_{i2}^{e_k} \in [0,1]$ represents the evaluation of the decision maker e_k for the alternative A_i ($1 \leq i \leq p$) with respect to the criterion C_2.
3. We proceed analogously for every criteria.

In this manner, we can represent the problem using multisets (see [1, 2]), i.e., sets of this form:

$$D = \{D_{ij} = (d_{ij}^{e_1}, d_{ij}^{e_2}, \ldots, d_{ij}^{e_n}) \mid i \in \{1, \ldots, p\},\ j \in \{1, \ldots, q\}\},$$

where each element is a tuple D_{ij} of n elements consisting of the evaluations of the criteria.

The use of different generalizations of fuzzy sets is frequent to model the uncertainty inherent in many decision making and consensus problems [3, 4, 5, 6, 7]. Moreover, in most of these problems, the order in which the decision makers provide their evaluation does not have an impact in the election of the solution. Bearing that in mind, for our problem we can consider a particular case of multisets, the so-called n-dimensional fuzzy sets [8], where the membership of each element is given by a tuple of n numbers in $[0,1]$ increasingly ordered.

When solving decision making problems, a numerical value is usually associated to each alternative and the solution is taken as the alternative with the greatest value [9, 10, 11, 12, 13, 14, 15]. However, in the cases where the resolution uses interval-valued fuzzy sets or Atanassov's intuitionistic fuzzy sets [16, 17, 18], each alternative is associated to an interval or to a pair of numbers, respectively. In these cases, we are compelled to use linear orders for intervals or pairs of numbers (see [19, 20]), so that the solution is given by the greatest interval or pair of numbers. Since, in the selected context, n-tuples are used, we need to define a linear order to compare n-tuples.

With all previous considerations, our objectives for this work are:

1. To present the concept of admissible order for n-dimensional fuzzy sets.
2. To give a construction method for admissible orders using aggregation functions [21, 22, 23].
3. To extend to n-dimensional fuzzy sets the concept of OWA operators (which are always associated to a linear order).
4. To design a decision making algorithm using n-dimensional fuzzy sets and n-tuple OWA operators.
5. To justify our theoretical developments with an illustrative example applying the proposed algorithm.

Some of the most widely used methods for solving multiple criteria decision making problems consist of two phases [9, 10, 12, 13]: the aggregation phase and the exploitation phase. In these methods, each decision maker represents his/her evaluations by means of preference relations (matrices) whose inputs are the d_{ij} values. So we have as many preference relations as decision makers.

In the aggregation phase, an aggregation function is chosen in order to aggregate the n preference relations (matrices) to produce a single matrix: the *collective* matrix. This collective matrix has as many rows as alternatives and as many columns as considered criteria. In the exploitation phase, an aggregation function is also selected for aggregating the elements of the collective matrix row by row to get one single number for each row. In the final step of the exploitation phase, we get as many numbers as alternatives and we take as solution the alternative associated to the greatest of these numbers.

One advantage of the method that we propose in this work is that we may omit the exploitation phase. This is due to the fact that the aggregation of the collective matrix produces a tuple for each alternative, so it is enough to order these tuples in a decreasing way according to a linear order so that we can choose as solution the first ranked tuple, i.e., the greatest tuple with respect to the linear order.

The possibility of omitting the exploitation phase is very relevant due to the fact that we do not need to reduce the elements of each row of the collective matrix to a single value and, hence, we do not modify the original data provided by the decision makers. This means that our results are obtained more straightforwardly from the evaluations of the decision makers than in those methods where the two phases are considered. These considerations are further developed in the last section, devoted to the application of our algorithm.

The structure of the work is as follows: in Section 2 we recall some preliminary notions about admissible orders and the extensions of fuzzy sets. In Section 3 we study the theoretical concepts that are required for the development of our model. Firstly, we generalize the concept of admissible order for n-dimensional fuzzy sets and present a construction method. Secondly, we introduce the concept of MOWA operator, studying its monotonicity with respect to a certain admissible order. An algorithm for decision making problems that makes use of all previous concepts is presented in Section 4, while in Section 5, we apply this algorithm in an illustrative example in the context of a multiple criteria group decision making problem. We finish in Section 6 with some conclusions and directions for future research.

2. Preliminaries

We first introduce some theoretical notions in order to fix the notation for the subsequent sections. Let O^n be the set of increasing n-tuples on $[0,1]$, namely, the set

$$O^n = \{\mathbf{x} = (x_1, \ldots, x_n) \in [0,1]^n \mid x_1 \leq x_2 \leq \ldots \leq x_n\}.$$

We recall that there is a natural partial order $\preceq$ on $O^n \subseteq \mathbb{R}^n$ given by $(x_1, \ldots, x_n) \preceq (y_1, \ldots, y_n)$ if and only if $x_i \leq y_i$, $1 \leq i \leq n$. In this way, $(O^n, \preceq)$ is a complete lattice and $(0, \ldots, 0)$ and $(1, \ldots, 1)$ are the bottom and top elements of the partial order, respectively.

Fuzzy multisets are a generalization of fuzzy sets which were defined in [2] by Yager. Like many other generalizations, the aim of these sets lies on the formalization of a representation to deal with imprecision, inexactness, ambiguity, or uncertainty intrinsic to many problems. In particular, in the case of fuzzy multisets, a fixed number n of membership values is assigned to each element. Taking into account that in a group decision making problem we have as many evaluations as decision makers, fuzzy multisets are suitable models for these problems. In the case of fuzzy multisets, the different membership values are considered as a set, not as an n-tuple, since they are not necessarily ordered. If the values of the membership degree of each element are ordered in an increasing way, fuzzy multisets are called n-dimensional fuzzy sets.

Definition 1. *[8] Let U be a nonempty set usually called a universe. A n-dimensional fuzzy set A over U is given by*

$$A : U \mapsto O^n$$

where $A(u)$ denotes the membership degree of the element $u \in U$ to A.

Note that usual fuzzy sets are a specific example of a n-dimensional fuzzy set with $n = 1$. Analogously, interval-valued fuzzy sets [24] can be seen as an example of 2-dimensional fuzzy sets.

Given an element $u \in U$, we denote the n-dimensional membership tuple of the element u to the n-dimensional fuzzy set A by $A(u) \in O^n$. Moreover, it is worth mentioning that we recover fuzzy multisets when $[0, 1]^n$ is considered instead of O^n.

In this work, due to the selected context, anonymity is a key point in the implemented algorithm. We consider a multiple criteria group decision making problem where each decision maker gives a evaluation about each alternative with respect to each criterion in terms of a fuzzy membership degree. We select a context where all the decision makers' evaluations are valued equally, independently of their identity. In this way, the n decision maker's values are sorted producing a single n-dimensional fuzzy set.

As we have mentioned before, our construction method is underpinned in aggregation functions. These functions, which play a crucial role in both applied and theoretical fields, were originally defined in the unit interval $[0,1]$. However, they can be readily extended to any poset [25].

Definition 2. *An aggregation function M is a mapping $M : [0,1]^n \to [0,1]$ satisfying*

- $M(0,\ldots,0) = 0, \quad M(1,\ldots,1) = 1$, *and*
- *for all n-tuples $(x_1,\ldots,x_n),(y_1,\ldots,y_n) \in [0,1]^n$ such that $x_i \leq y_i$, for all $1 \leq i \leq n$ then $M(x_1,\ldots,x_n) \leq M(y_1,\ldots,y_n)$.*

The aim of this study is to generalize the concept of OWA operators to deal with n-dimensional fuzzy sets. Let us first recall their definition in $[0,1]$.

Definition 3. *[26] Let w be a weighting vector, i.e., $w = (w_1,\ldots,w_m) \in [0,1]^m$ such that $w_1+\ldots+w_m = 1$. The Ordered Weighted Averaging (OWA) operator associated to w is a mapping $OWA_w : [0,1]^m \longrightarrow [0,1]$ given by*

$$OWA_w(x_1,\ldots,x_m) = \sum_{i=1}^{m} w_i x_{(i)} \ ,$$

where $x_{(i)}$, denotes the i-th greatest component of the vector $(x_1,\ldots,x_m)$.

Note that although aggregation functions can be defined on a strict partially ordered set, OWA operators require all the elements to be comparable, i.e., OWA operators require a linear order to be properly defined. Nevertheless, recent studies in the literature have proposed definitions for these operators in more general lattices [27].

3. Admissible orders and OWA operators on fuzzy multisets

The notion of admissible order was first introduced in [16] for interval-valued fuzzy sets and later on in [28] for interval-valued Atanassov's intuitionistic fuzzy sets. In this section, we first generalize the notion of admissible order to the setting of n-dimensional fuzzy sets showing some particular examples. We also provide a construction method for these orders which makes use of appropriate aggregation functions on O^n.

We start defining admissible orders on O^n.

Definition 4. *A linear order $\leq_L$ on O^n is called admissible if for all $\mathbf{x}, \mathbf{y} \in O^n$ satisfying $x_i \leq y_i$ for all $1 \leq i \leq n$ then $\mathbf{x} \leq_L \mathbf{y}$.*

Example 1.

- *As a first example of admissible order on O^n for every $n \geq 1$ we consider the first lexicographical order (with respect to the first variable), $\mathbf{x} \leq_L \mathbf{y}$ if the i-th component of $\mathbf{x}$ is strictly less than the i-th component of $\mathbf{y}$ ($i \in \{1, \ldots, n\}$), whereas $x_j = y_j$ for every $j < i$.*

- *For $n = 2$ the Xu and Yager order ([29]) is defined by*

$$(x_1, x_2) \leq (y_1, y_2) \text{ if and only if } \quad \frac{x_1 + x_2}{2} < \frac{y_1 + y_2}{2} \text{ or}$$
$$\left(\frac{x_1 + x_2}{2} = \frac{y_1 + y_2}{2} \text{ and } x_2 - x_1 < y_2 - y_1 \right)$$

We are interested in those admissible orders which can be obtained by means of appropriate aggregation functions. In particular, we consider the following result.

Definition 5. *Let $\boldsymbol{M} = (M_1, \ldots, M_n)$ be a sequence of n aggregation functions $M_i : [0,1]^n \to [0,1]$. Given $\mathbf{x}, \mathbf{y} \in O^n$,*

- *$\mathbf{x} <_{\boldsymbol{M}} \mathbf{y}$ if and only if there exists k with $1 \leq k \leq n$ such that $M_j(\mathbf{x}) = M_j(\mathbf{y})$ for all $1 \leq j \leq k-1$ and $M_k(\mathbf{x}) < M_k(\mathbf{y})$.*

- *$\mathbf{x} \leq_{\boldsymbol{M}} \mathbf{y}$ if and only if $\mathbf{x} <_{\boldsymbol{M}} \mathbf{y}$ or $\mathbf{x} = \mathbf{y}$.*

Proposition 1. *Let $\boldsymbol{M} = (M_1, \ldots, M_n)$ be a sequence of n aggregation functions $M_i : [0,1]^n \to [0,1]$. The order relation $\mathbf{x} \leq_{\boldsymbol{M}} \mathbf{y}$ is an admissible order on O^n if and only if the functions M_i satisfy*

$$(M_i(\mathbf{x}) = M_i(\mathbf{y}), \text{ for all } 1 \leq i \leq n) \Leftrightarrow \mathbf{x} = \mathbf{y}. \tag{1}$$

Proof. It is a straightforward calculation. $\square$

Example 2. *The lexicographic orders can be constructed as before from the n projections given by $\pi_i(x_1, \ldots, x_n) = x_i$.*

For example, the first lexicographical order is generated taking $M_i = \pi_i$. But observe that, if we consider any permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ and we take $M_i = \pi_{\sigma(i)}$, then we get different examples of lexicographic orders which are different from each other.

In order to get examples of admissible orders on O^n we consider aggregation functions which are defined in terms of linear expressions, and, more specifically, in terms of weighted arithmetic means.

Proposition 2. *Let $\boldsymbol{M} = (M_1, \ldots, M_n)$ be a sequence of n aggregation functions given by*

$$M_i(x_1, \ldots, x_n) = \alpha_{i1}x_1 + \alpha_{i2}x_2 + \ldots + \alpha_{in}x_n\,, \quad 1 \leq i \leq n\,, \tag{2}$$

such that $\alpha_{i1} + \alpha_{i2} + \ldots + \alpha_{in} = 1$ with $\alpha_{ij} \in [0, 1]$ for all $1 \leq j \leq n$. The order $\leq_{\boldsymbol{M}}$ is an admissible order on O^n if and only if the $n \times n$ matrix A given by

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \ldots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \ldots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{2n} & \ldots & \alpha_{nn} \end{pmatrix}$$

is regular.

Proof. Notice that Eq. (1) can be rewritten as $A(\mathbf{x} - \mathbf{y}) = \mathbf{0}$ if and only if $\mathbf{x} = \mathbf{y}$, which is equivalent to A being regular. $\square$

Example 3. *Let $\leq_{\boldsymbol{M}}$ be the order generated by the following functions M_i:*

- $M_1(x_1, \ldots x_5) = \frac{1}{10}x_1 + \frac{1}{5}x_2 + \frac{1}{5}x_3 + \frac{1}{4}x_4 + \frac{1}{4}x_5,$
- $M_2(x_1, \ldots x_5) = \frac{3}{10}x_1 + \frac{1}{5}x_2 + \frac{1}{2}x_5,$
- $M_3(x_1, \ldots x_5) = \frac{3}{5}x_1 + \frac{1}{10}x_2 + \frac{1}{10}x_3 + \frac{1}{10}x_4 + \frac{1}{10}x_5,$

- $M_4(x_1, \dots x_5) = \frac{1}{5}x_2 + \frac{3}{10}x_3 + \frac{3}{10}x_4 + \frac{1}{5}x_5$,
- $M_5(x_1, \dots x_5) = \frac{1}{4}x_1 + \frac{1}{4}x_2 + \frac{1}{4}x_4 + \frac{1}{4}x_5$.

It is a simple calculation to see that the matrix A generated by the coefficients of the aggregation function is a regular matrix. Hence, the order relation $\leq_{\boldsymbol{M}}$ is an admissible order and we can compare, for instance, $\mathbf{x} = (0.2, 0.4, 0.9, 1, 1)$ and $\mathbf{y} = (0, 0.6, 0.8, 1, 1)$. In fact, $\mathbf{y} <_{\boldsymbol{M}} \mathbf{x}$ since $M_1(\mathbf{y}) = 0.78 = M_1(\mathbf{x})$ and $M_2(\mathbf{y}) = 0.62 < 0.64 = M_2(\mathbf{x})$.

Once we have introduced the concept of admissible orders on O^n, we can define OWA operators on this set. Firstly, we generalize the concept of aggregation function on O^n.

Definition 6. *Let $\leq_L$ be an admissible order on O^n. An aggregation function M on O^n, is a mapping $M : (O^n)^m \to O^n$ satisfying*

- $M(\mathbf{0}, \dots, \mathbf{0}) = \mathbf{0}, \quad M(\mathbf{1}, \dots, \mathbf{1}) = \mathbf{1}$*, and*
- *for all $(\mathbf{x_1}, \dots, \mathbf{x_m}), (\mathbf{y_1}, \dots, \mathbf{y_m}) \in (O^n)^m$ such that $\mathbf{x_1} \leq_L \mathbf{y_1}$, ..., $\mathbf{x_m} \leq_L \mathbf{y_m}$ then $M(\mathbf{x_1}, \dots, \mathbf{x_m}) \leq_L M(\mathbf{y_1}, \dots, \mathbf{y_m})$.*

Definition 7. *Let w be a weighting vector and let $\leq_L$ be an admissible order. The OWA operator associated to w and $\leq_L$ is a mapping $(O^n)^m \mapsto O^n$ defined by*

$$MOWA_{[w, \leq_L]}(\mathbf{x}_1, \dots, \mathbf{x}_m) = \sum_{i=1}^{m} w_i \mathbf{x}_{(i)}$$

where $\mathbf{x}_{(i)}$ denotes the i-th greatest n-dimensional fuzzy value of the inputs $(\mathbf{x}_1 \dots, \mathbf{x}_m)$ with respect to the order $\leq_L$ on O^n and $w_i\mathbf{x} = (w_i x_1, \dots, w_i x_n)$.

Example 4.

- *If we take $w = (1, 0, \dots, 0)$, then we recover the maximum operator.*
- *If we take $w = (0, 0, \dots, 0, 1)$, then we recover the minimum operator.*

Notice that the MOWA operator is well defined, namely, the image of m elements in O^n is a new element in O^n due to the increasingness of the weighted arithmetic mean.

In the usual fuzzy setting, OWA operators play a crucial role since their monotonicity enables to classify OWA operators as a particular class of aggregation functions. In the following, we study the monotonicity of MOWA operators with respect to an order generated as in Prop. 2.

Theorem 1. *Let $w = (w_1, \ldots, w_m)$ be a weighting vector such that $w_i > 0$ for all $1 \leq i \leq m$ and let $\leq_{\mathbf{M}}$ be an admissible order on O^n generated as in Prop. 2. Then the MOWA operator is an increasing function.*

Proof. Let us show that if $\mathbf{x}_i \leq_{\mathbf{M}} \mathbf{x}'_i$ then $MOWA_{[w,\leq_{\mathbf{M}}]}(\mathbf{x}_1, \ldots, \mathbf{x}_i, \ldots, \mathbf{x}_m) \leq MOWA_{[w,\leq_{\mathbf{M}}]}(\mathbf{x}_1, \ldots, \mathbf{x}'_i, \ldots, \mathbf{x}_m)$. It holds trivially if $\mathbf{x}_i = \mathbf{x}'_i$ so we only need to prove the case $\mathbf{x}_i <_{\mathbf{M}} \mathbf{x}'_i$.

If $\mathbf{x}_i <_{\mathbf{M}} \mathbf{x}'_i$ then there is an index $1 \leq j \leq n$ such that

$$M_k(\mathbf{x}_i) = M_k(\mathbf{x}'_i) \text{ for all } k \leq j-1 \text{ and } M_j(\mathbf{x}_i) < M_j(\mathbf{x}'_i). \tag{3}$$

Notice that if $j = 1$ the condition is reduced to $M_1(\mathbf{x}_i) < M_1(\mathbf{x}'_i)$.

Moreover, due to the fact that the functions which generate the order $\leq_{\mathbf{M}}$ are weighted arithmetic means, it holds that

$$M_k(MOWA_{[w,\leq_{\mathbf{M}}]}(\mathbf{x}_1, \ldots, \mathbf{x}'_i, \ldots, \mathbf{x}_m)) = \sum_{h=1}^{m} w_h M_k(\mathbf{x}_{(h)}) \,.$$

Without loss of generality, we suppose the n-dimensional fuzzy values are ordered in a decreasing way, i.e., $\mathbf{x}_1 \geq_{\mathbf{M}} \mathbf{x}_2 \geq_{\mathbf{M}} \ldots \geq_{\mathbf{M}} \mathbf{x}_m$.

We distinguish two different cases:

- If the n-dimensional fuzzy value $\mathbf{x}'_i$ has not altered the order of the n-dimensional fuzzy values, namely, $\mathbf{x}_{i-1} \geq_{\mathbf{M}} \mathbf{x}'_i \geq_{\mathbf{M}} \mathbf{x}_{i+1}$, then it holds that

$$\begin{aligned} &M_k(MOWA_{[w,\leq_{\mathbf{M}}]}(\mathbf{x}_1, \ldots, \mathbf{x}_i, \ldots \mathbf{x}_m)) - M_k(MOWA_{[w,\leq_{\mathbf{M}}]}(\mathbf{x}_1, \ldots, \mathbf{x}'_i, \ldots \mathbf{x}_m)) \\ &\qquad = \sum_{h=1}^{m} w_h M_k(\mathbf{x}_h) - \left(w_i M_k(\mathbf{x}'_i) + \sum_{h \neq i} w_h M_k(\mathbf{x}_h) \right) \\ &\qquad = w_i \left(M_k(\mathbf{x}_i) - M_k(\mathbf{x}'_i) \right) \,, \end{aligned}$$

which, due to the Eq. (3), equals to 0 for all $1 \leq k \leq j-1$ and is less than 0 for the index j. Hence,

$$MOWA_{[w,\leq_{\mathbf{M}}]}(\mathbf{x}_1, \ldots, \mathbf{x}_i, \ldots \mathbf{x}_m) < MOWA_{[w,\leq_{\mathbf{M}}]}(\mathbf{x}_1, \ldots, \mathbf{x}'_i, \ldots \mathbf{x}_m)\,.$$

- If the n-dimensional fuzzy value $\mathbf{x}'_i$ has altered the order of the n-dimensional fuzzy values in l positions, namely, $\mathbf{x}_{i-l-1} \geq_{\mathbf{M}} \mathbf{x}'_i \geq_{\mathbf{M}} \mathbf{x}_{i-l} \geq_{\mathbf{M}} \cdots \geq_{\mathbf{M}} \mathbf{x}_{i-1} \geq_{\mathbf{M}} \mathbf{x}_i \geq_{\mathbf{M}} \mathbf{x}_{i+1}$ for some $l \geq 1$, we find that

$$\begin{aligned}
&M_k(MOWA_{[w,\leq_{\mathbf{M}}]}(\mathbf{x}_1, \ldots, \mathbf{x}_i, \ldots \mathbf{x}_m)) - M_k(MOWA_{[w,\leq_{\mathbf{M}}]}(\mathbf{x}_1, \ldots, \mathbf{x}'_i, \ldots \mathbf{x}_m)) \\
&= \sum_{h=1}^{m} w_h M_k(\mathbf{x}_h) - \left(w_{i-l} M_k(\mathbf{x}'_i) + \sum_{h=i-l+1}^{i} w_h M_k(\mathbf{x}_{h-1}) + \sum_{\substack{h \leq i-l-1 \\ \text{or} \\ h \geq i+1}} w_h M_k(\mathbf{x}_h) \right) \\
&= w_{i-l}\left(M_k(\mathbf{x}_{i-l}) - M_k(\mathbf{x}'_i)\right) + \sum_{h=i-l+1}^{i} w_h \left(M_k(\mathbf{x}_h) - M_k(\mathbf{x}_{h-1})\right).
\end{aligned} \tag{4}$$

Further, since the n-dimensional fuzzy values are ordered in a decreasing way, it follows that $M_1(\mathbf{x}'_i) \geq M_1(\mathbf{x}_{i-l}) \geq M_1(\mathbf{x}_{i-l+1}) \ldots \geq M_1(\mathbf{x}_{i-1}) \geq M_1(\mathbf{x}_i)$. Moreover, due to Eq. (3), it holds that $M_1(\mathbf{x}_i) = M_1(\mathbf{x}'_i)$ and, hence, $M_1(\mathbf{x}'_i) = M_1(\mathbf{x}_{i-l}) = M_1(\mathbf{x}_{i-l+1}) = \ldots = M_1(\mathbf{x}_{i-1}) = M_1(\mathbf{x}_i)$.

Iteratively,

$$\begin{aligned}
&M_k(\mathbf{x}'_i) = M_k(\mathbf{x}_{i-l}) = M_k(\mathbf{x}_{i-l+1}) = \\
&\qquad \ldots = M_k(\mathbf{x}_{i-1}) = M_k(\mathbf{x}_i) \text{ for all } 1 \leq k \leq j-1.
\end{aligned} \tag{5}$$

Besides,

$$M_j(\mathbf{x}'_i) \geq M_j(\mathbf{x}_{i-l+1}) \geq \ldots \geq M_j(\mathbf{x}_{i-1}) \geq M_j(\mathbf{x}_i) \tag{6}$$

where at least one of the inequalities is strict since, by Eq. (3), it holds that $M_j(\mathbf{x}'_i) > M_j(\mathbf{x}_i)$.

Using Eqs. (5) and (6) in Eq. (4), we find that

$$M_k(MOWA_{[w,\leq_{\mathbf{M}}]}(\mathbf{x}_1,\ldots,\mathbf{x}_i,\ldots\mathbf{x}_m)) = M_k(MOWA_{[w,\leq_{\mathbf{M}}]}(\mathbf{x}_1,\ldots,\mathbf{x}'_i,\ldots\mathbf{x}_m))$$
$$\text{for all } 1 \leq k \leq j-1, \text{ and}$$

$$M_j(MOWA_{[w,\leq_{\mathbf{M}}]}(\mathbf{x}_1,\ldots,\mathbf{x}_i,\ldots\mathbf{x}_m)) < M_j(MOWA_{[w,\leq_{\mathbf{M}}]}(\mathbf{x}_1,\ldots,\mathbf{x}'_i,\ldots\mathbf{x}_m)).$$

Hence, $MOWA_{[w,\leq_{\mathbf{M}}]}(\mathbf{x}_1,\ldots,\mathbf{x}_i,\ldots\mathbf{x}_m) < MOWA_{[w,\leq_{\mathbf{M}}]}(\mathbf{x}_1,\ldots,\mathbf{x}'_i,\ldots\mathbf{x}_m)$.

□

Notice that since $MOWA_{[w,\leq_{\mathbf{M}}]}(\mathbf{0},\ldots,\mathbf{0}) = \sum_{i=1}^m w_i\mathbf{0} = \mathbf{0}$ and $MOWA_{[w,\leq_{\mathbf{M}}]}(\mathbf{1},\ldots,\mathbf{1}) = \sum_{i=1}^m w_i\mathbf{1} = \mathbf{1}$, MOWA operators are aggregation functions with respect to the order $\leq_{\mathbf{M}}$ generated as in Prop. 2.

4. An algorithm for group decision making using MOWA operators

Multiple criteria group decision making consists in choosing an alternative out of a given set $A = \{A_1,\ldots,A_p\}$ ($p \geq 2$) according to the evaluations given by a group of decision makers $E = \{e_1,\ldots,e_n\}$ ($n \geq 2$) with respect to some criteria $C = \{C_1,\ldots,C_q\}$ ($q \geq 2$). Thus, we can generate a matrix $D = (D_{ij})_{p\times q}$ of memberships of fuzzy multisets, where D_{ij} denotes the n-tuple of evaluations of the decision makers about alternative A_i under the criterion C_j.

Once the order and the weighting vector are set, the following algorithm, which is schematically represented in Figure 1, can be applied. Notice that this procedure maintains all the evaluations provided by the decision makers, as in [30].

Step 1. To generate the matrix D whose elements D_{ij} are n-dimensional fuzzy values; this step consists in generating an ordered tuple with the n evaluations of the decision makers.

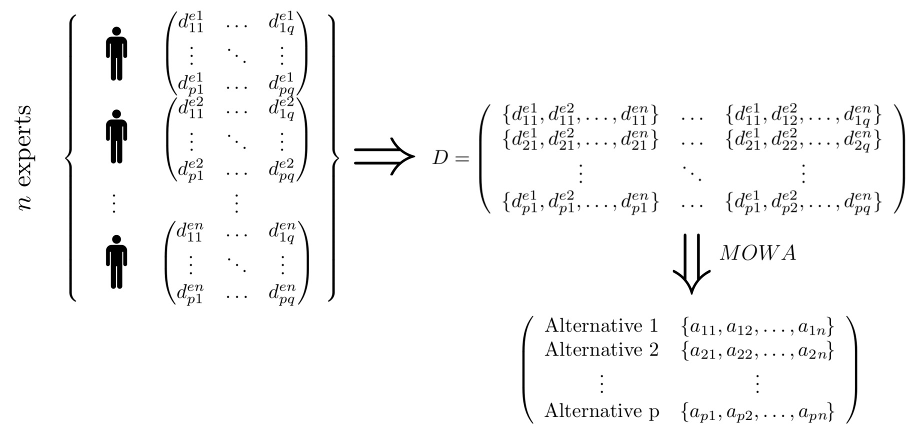

Figure 1: Schematic representation of Algorithm 1

Step 2. To generate an order $\leq_{\mathbf{M}}$, selecting a sequence of $(M_1, \ldots, M_n)$ of aggregation functions that satisfy the conditions in Prop. 2.

Step 3. To select the weighting vector w of q components; one for each criterion.

Step 4. To apply the $MOWA$ operator to each row of the matrix D using the order $\leq_{\mathbf{M}}$ in **Step 2** and the weighting vector w in **Step 3**.

Step 5. To select as the best alternative the greatest n-dimensional fuzzy value with respect to the order in **Step 2**.

Remark. The transformation of fuzzy multisets into n-dimensional fuzzy values ensures anonymity. In this manner, it does not matter which decision maker has provided each value of the fuzzy multiset and all of them are treated equally.

The output of Algorithm 1 can differ greatly depending upon Steps 2 and 3. The parameters with influence in such steps (that is, the aggregation functions used for the linear order and the weighing vector) become very relevant for the result of the algorithm; hence, their setting ought to be adapted depending on the specific problem.

It is worth mentioning that, in real scenarios, the assignment of non homogeneous weights to decision makers is rather common, and is simply done in order to weight their level of expertise or simply their relevance in the decision making process. In Algorithm 1, this cannot be directly done, as long as the weights are applied to the data according to their sorting, not to the relevance of the expert that provided them. For example, in a scenario in which experts tend to be optimistic, it seems appropriate to select weighing vectors empowering the lowest ranked elements, i.e., the first elements of the tuple. That is, using weighting vectors with decreasing values, so that the highest ranked (hence, more optimistic) evaluations receive less influence in the final decision.

5. Illustrative example

Ye et al. introduced in [31] a multiple criteria group decision making problem adapted from [13]. In this section, we show that Algorithm 1 is also a suitable option to solve that problem.

The practical example consists in determining the best company for investment. Four possible companies are considered: a car company A_1, a food company A_2, a computer company A_3 and an arm company A_4. Three decision makers are asked about their opinions with respect to three criteria: the risk analysis C_1, the growth analysis C_2 and the environmental impact analysis C_3. We take the same weighting vector as in [31], namely, $w = (0.35, 0.25, 0.4)$.

The main difference between our approach and Ye's [31] lies on the use of a different generalization of fuzzy sets. While in [31] dual hesitant fuzzy sets are considered, in our framework we make use of 3-dimensional fuzzy sets. The former have both membership and nonmembership degrees and the latter only membership degrees, so, for the practical example, we only consider the values of the membership degrees from [31]. Another difference is that dual hesitant fuzzy sets do not permit repeated membership values. Therefore, if some decision makers' evaluations coincide, the value is taken into account only once. Nevertheless, in our method the value can be repeated as many times as decision makers coincide.

We generate the decision makers' evaluations accordingly to the data in [31]:

$$\begin{array}{ccc} e_1 & e_2 & e_3 \\ \begin{pmatrix} 0.5 & 0.4 & 0.2 \\ 0.4 & 0.7 & 0.6 \\ 0.4 & 0.5 & 0.6 \\ 0.7 & 0.6 & 0.3 \end{pmatrix} & \begin{pmatrix} 0.4 & 0.6 & 0.1 \\ 0.6 & 0.6 & 0.7 \\ 0.6 & 0.5 & 0.5 \\ 0.6 & 0.6 & 0.4 \end{pmatrix} & \begin{pmatrix} 0.3 & 0.4 & 0.3 \\ 0.7 & 0.7 & 0.4 \\ 0.3 & 0.6 & 0.6 \\ 0.8 & 0.7 & 0.3 \end{pmatrix} \end{array}$$

We apply our method, described in Algorithm 1, to solve the problem.

Step 1. We generate the matrix D where the elements are 3-dimensional fuzzy values, namely, increasing 3-tuples:

$$D = \begin{pmatrix} \{0.3, 0.4, 0.5\} & \{0.4, 0.4, 0.6\} & \{0.1, 0.2, 0.3\} \\ \{0.4, 0.6, 0.7\} & \{0.6, 0.7, 0.7\} & \{0.4, 0.6, 0.7\} \\ \{0.3, 0.4, 0.6\} & \{0.5, 0.5, 0.6\} & \{0.5, 0.6, 0.6\} \\ \{0.6, 0.7, 0.8\} & \{0.6, 0.6, 0.7\} & \{0.3, 0.3, 0.4\} \end{pmatrix}.$$

Step 2. The order $\leq_{\mathbf{M}}$ considered is generated by

- $M_1(x_1, x_2, x_3) = \frac{1}{3}x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_3$;
- $M_2(x_1, x_2, x_3) = \frac{1}{2}x_1 + \frac{1}{2}x_2$;
- $M_3(x_1, x_2, x_3) = \frac{1}{4}x_1 + \frac{1}{2}x_2 + \frac{1}{4}x_3$;

which satisfy the conditions of Prop. 2.

Step 3. The selected weighting vector is $w = (0.35, 0.25, 0.4)$ (as in [31]).

Step 4. We apply the $MOWA$ operator associated to $\leq_{\mathbf{M}}$ and w whose result is

$$\begin{pmatrix} \text{Company 1} & \longrightarrow & \{0.255, 0.32, 0.455\} \\ \text{Company 2} & \longrightarrow & \{0.47, 0.635, 0.7\} \\ \text{Company 3} & \longrightarrow & \{0.42, 0.495, 0.6\} \\ \text{Company 4} & \longrightarrow & \{0.48, 0.515, 0.615\} \end{pmatrix}$$

Step 5. We order the alternatives with respect to the selected order $\leq_{\mathbf{M}}$:

$$\text{Company } 2 \geq_{\mathbf{M}} \text{Company } 4 \geq_{\mathbf{M}} \text{Company } 3 \geq_{\mathbf{M}} \text{Company } 1\,.$$

We select Company 2, which means that the best company to invest in is the car company.

Notice that, since the developed illustrative example is introduced in [31], we have a fixed weighting vector. We have no extra information about the problem and, hence, we set the aggregation functions (which generate the linear order of Step 2) arbitrarily. In some others studies, such as in [19, 28], a comparison between the different solutions using some different orders is made. However, since our sole intention is to show the validity of our proposal, we have only shown the results corresponding to one order.

The best alternative according to Algorithm 1 coincides with best alternative in [31]. However, the treatment of the data is different. Let us highlight the main advantages of the proposed algorithm.

On the one hand, using n-dimensional fuzzy sets the anonymity between the decision makers is assured. Values are the only relevant information, without taking into account the identities of the decision makers. Moreover, if some decision makers coincide in their evaluations, we are able to consider the repeated values avoiding the loss of information that some other systems suffer from. Besides, n-dimensional fuzzy sets do not need the duality membership/non membership degree and consequently, our algorithm derives the same result using less information.

On the other hand, most of the works that consider generalizations of fuzzy sets make use of partial orders. In this direction, novel studies are trying to generate linear orders in most of the generalizations of fuzzy sets, but they require a study of the monotonicity with respect to the considered linear order. A first study about OWA operators in n-dimensional fuzzy sets as well as the study of their monotonicity with respect to certain admissible orders is found as a theoretical base for the proposed algorithm.

Finally, we solve the problem with the standard aggregation and exploitation phases in order to show that the solutions coincide.

Aggregation phase: we take the OWA operator with weighting vector $w = (0.35, 0.25, 0.4)$ as in [31].

$$C = \begin{pmatrix} 0.395 & 0.47 & 0.195 \\ 0.55 & 0.66 & 0.55 \\ 0.43 & 0.535 & 0.56 \\ 0.695 & 0.635 & 0.333 \end{pmatrix}$$

Exploitation phase: we take the OWA operator with weighting vector $w = (0.35, 0.25, 0.4)$ as in [31].

$$\begin{pmatrix} \text{Company 1} & \longrightarrow & 0.341 \\ \text{Company 2} & \longrightarrow & 0.589 \\ \text{Company 3} & \longrightarrow & 0.502 \\ \text{Company 4} & \longrightarrow & 0.533 \end{pmatrix}$$

Consequently, $\quad \text{Company 2} \geq \text{Company 4} \geq \text{Company 3} \geq \text{Company 1}\,.$

It is clear that with our method, we actually do not need to carry the exploitation phase out so we need to modify the original data less than in the method which consists of both phases.

6. Conclusions

In order to resemble the behavior of membership degrees of fuzzy sets, novel studies generating linear orders for the generalization of fuzzy sets have been presented. However, the linearity of the orders compel us to revise the concept of aggregation functions studying their monotonicity.

In this direction, this work introduces the concept of admissible order for n-dimensional fuzzy sets as well as a construction method for these orders. It is worth mentioning that if the considered membership values are not ordered, the generalizations are also suitable for fuzzy multisets without noteworthy effort.

We also introduce some operators for n-dimensional fuzzy sets, denoted by MOWA, which resemble OWA operators on fuzzy sets. Moreover, we prove that they are increasing functions with respect to a particular class of admissible orders generated by weighted arithmetic means.

Finally, we present an algorithm for multiple criteria group decision making problem using n-dimensional fuzzy sets so that the election of the solution is made by taking the alternative associated to the greatest tuple with respect to the considered admissible order. In order to construct the solution tuple we use the previously introduced OWA operators. Another advantage of our proposal is that it allows to omit the exploitation phase in decision making problems, so the procedure to solve these problems becomes simpler.

For future work, linear orders modify the concept of increasingness in aggregation functions and, hence, a theoretical effort must be done to define and generalize this notion in the different generalizations of fuzzy sets.

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