Two Approaches to Partial-nodes-based State Estimation for Delayed Complex Networks with Intermittent Measurement Transmissions

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***Abstract*—This paper is concerned with the state estimation problem for delayed complex dynamic networks with non-identical local dynamical systems. The state estimation is conducted based on constrained information of the measurement outputs. Specifically, the network outputs are available only from a portion of network nodes, and such outputs are transmitted from the network nodes to the estimator in an intermittent way. By utilizing the Halanay inequality method as well as the average dwell-time approach, two sets of sufficient conditions are established that ensure the error dynamics of the state estimation to converge to zero exponentially, and explicit expressions of the estimator gains are further characterized. Finally, a numerical example is presented to demonstrate the effectiveness of the proposed approaches.**

***Index Terms*—State estimation; complex networks; partial-nodes-based measurement; intermittent transmission.**

I. Introduction

In the past few decades, complex networks have attracted increasing attention from both science and engineering fields. Mathematically, a network is represented as a graph that consists of nodes (or vertices) representing the objects or agents in the network, and a set of edges (links or connections) representing the interactions (or relations) of the nodes [1], [5], [17], [18], [28], [29]. Common examples of networks include the Internet, the World Wide Web, and social networks. Due to complicated links and interactions between nodes, complex networks exhibit abundant dynamical behaviors (e.g. synchronization and spatiotemporal chaos) that have received a constant research interest [2], [8], [16], [31], [46].

In practical applications, the knowledge of the states of a system is crucially important for certain tasks or purposes. Unfortunately, the system states are not always easy to be measured (or observed) directly. Instead of the system states, in most cases, what can be accessed are system outputs/measurements. Thus, the state estimation problem arises out of the desire to estimate unmeasurable system states by utilizing the available output information. Generally speaking, the *Luenberger observer* and the *recursive estimator* are two common structures of the state estimators [3], [4], [15], [19], [40]–[42]. The former requires constructing an observer to track system states, while the latter gives optimal state estimates. In recent years, state estimation problems for complex networks have aroused considerable research interest [14], [20], [21], [25]–[27], [44], and inspiring results have been reported for various complex networks, e.g. delayed complex networks [25], [47], stochastic complex networks [9], [37], [38], and complex networks subject to network-induced phenomena [10].

It is worth noting that, in the aforementioned work, an implicitly assumption is that the measurement outputs are available from *all* network nodes for the state estimation tasks. This assumption might hold for low-dimensional systems or complex networks with small amount of nodes, where the measurement outputs are easily accessible from economic reasons. Unfortunately, for complex networks of large scale with an excessive number of nodes, measuring the outputs of all network nodes can be expensive and sometimes impractical. In fact, it is often the case that we can only acquire network outputs from just a small portion of nodes. Thus, it is desirable to estimate all network states via measurements of a fraction of nodes, which results in the so-called partial-nodes-based (PNB) state estimation, and a number of results have been available in the literature [7], [20], [21], [26], [39].

On the other hand, communication constraint is ubiquitous in control systems. In the implementation of system control or state estimation, a common case is that the signal is transmitted in an *intermittent* way, which is either for the sake of saving resources or due to hardware limitations [12], [32]–[35]. Quite a few results on control/estimation with intermittent measurements have been published in the literature on various topics such as synchronization control [24], [44], [45] and Kalman filtering [11], [14], [22], [30], [48]. Nevertheless, to the best of our knowledge, little progress has been made on the PNB estimation problem for complex networks with communication constraints, and this constitutes our main motivation.

The main contribution of this paper can be highlighted as follows.

1) A novel state estimation framework is developed where the estimator is constructed based on the outputs just form a fraction of network nodes (rather than all the nodes), and the output signals are transmitted to the estimator in an intermittent way.

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2) The desired estimators are designed by separately using the Halanay inequality approach and the average dwell-time (ADT) approach. In both cases, sufficient conditions are derived to ensure the exponential stability of the corresponding error dynamics.
3) A comparison is made between sufficient conditions derived from the Halanay inequality approach and the ADT approach.

II. Problem Formulation

Consider the following delayed dynamical complex network with N nodes described by

$$\frac{dx_i(t)}{dt} = A_i x_i(t) + f(x_i(t)) + g(x_i(t-\tau)) - c\sum_{l=1}^{N} l_{ij}\Gamma x_j(t), \quad i = 1, 2, \cdots, N, \tag{1}$$

where $x_i(t) = \left(x_{i1}(t), x_{i2}(t), \cdots, x_{in}(t)\right)^T$ denotes the n-dimensional state vector of the ith node; A_i is a real constant matrix; $f, g : \mathbb{R}^n \to \mathbb{R}^n$ are continuous nonlinear vector-valued functions; τ is the time delay; and $L \triangleq \left(l_{i,j}\right)$ is the Laplacian matrix of the network (1) with $l_{i,j} \le 0$ $(j \ne i)$ and $l_{i,i} = -\sum_{j\ne i} l_{i,j}$. It is well-known that $l_{i,j} < 0$ if there is a directed edge from node j to node i, otherwise $l_{i,j} = 0$. $\Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \cdots, \gamma_n) \ge 0$ is an inner-coupling diagonal matrix linking the jth state variable of each node if $\gamma_j \ne 0$.

Remark 1: Notice that the individual local dynamical nodes in network (1) are heterogeneous. Therefore, in this paper, the state estimation problem will be coped with in a more general scenario. Our goal is to estimate states of network (1) via measurements from a portion of network nodes.

Assume that the outputs of the first l_0 nodes are available. To reduce unnecessary consumption of limited communication resource or for other reasons, it is also assumed that the output signals are transmitted in an *intermittent* way. To be specific, the output of node i $(1 \le i \le l_0)$ is described as

$$y_i(t) = \begin{cases} C_i x_i(t), & t \in [sT, sT + T_1),\ 1 \le i \le l_0; \\ 0 & t \in [sT + T_1, (s+1)T),\ 1 \le i \le l_0; \\ 0, & t \in [sT, (s+1)T),\ l_0 + 1 \le i \le N, \end{cases} \tag{2}$$

where $y_i(t) = (y_{i1}(t), y_{i2}(t), \cdots, y_{im_i}(t)) \in \mathbb{R}^{m_i}$ $(1 \le m_i \le n)$ is the output of the ith node, $C_i \in \mathbb{R}^{m_i \times n}$ is known, T stands for the output transmission period, T_1 is the duration time of output transmission over a period, and $T - T_1$ denoted by T_2 is the transmission intermission.

For network (1) and measurement (2), we construct the following state estimator.

$$\frac{d\hat{x}_i(t)}{dt} = A_i\hat{x}_i(t) + f(\hat{x}_i(t)) + g(\hat{x}_i(t-\tau)) - c\sum_{l=1}^{N} l_{ij}\Gamma\hat{x}_j(t) + K_i(y_i(t) - C_i\hat{x}_i(t)), \quad t \in [sT, sT + T_1],\ 1 \le i \le l_0, \tag{3a}$$

$$\frac{d\hat{x}_i(t)}{dt} = A_i\hat{x}_i(t) + f(\hat{x}_i(t)) + g(\hat{x}_i(t-\tau)) - c\sum_{l=1}^{N} l_{ij}\Gamma\hat{x}_j(t), \quad t \in [sT + T_1, (s+1)T],\ 1 \le i \le l_0, \tag{3b}$$

$$\frac{d\hat{x}_i(t)}{dt} = A_i\hat{x}_i(t) + f(\hat{x}_i(t)) + g(\hat{x}_i(t-\tau)) - c\sum_{l=1}^{N} l_{ij}\Gamma\hat{x}_j(t), \quad l_0 + 1 \le i \le N, \tag{3c}$$

where $\hat{x}_i(t)$ is the estimate of $x_i(t)$, and $K_i \in \mathbb{R}^{n\times m_i}$ is the estimator gain matrix to be designed.

Denote by $\varepsilon_i(t) = \hat{x}_i(t) - x_i(t)$ the estimation error. Then, from (1) and (3), it is clear that the estimation error $\varepsilon_i(t)$ satisfies

$$\frac{d\varepsilon_i(t)}{dt} = (A_i - K_iC_i)\varepsilon_i(t) + \tilde{f}(\varepsilon_i(t)) + \tilde{g}(\varepsilon_i(t-\tau)) - c\sum_{l=1}^{N} l_{ij}\Gamma\varepsilon_j(t),\ t \in [sT, sT + T_1],\ 1 \le i \le l_0, \tag{4a}$$

$$\frac{d\varepsilon_i(t)}{dt} = A_i\varepsilon_i(t) + \tilde{f}(\varepsilon_i(t)) + \tilde{g}(\varepsilon_i(t-\tau)) - c\sum_{l=1}^{N} l_{ij}\Gamma\varepsilon_j(t), \quad t \in [sT + T_1, (s+1)T],\ 1 \le i \le l_0, \tag{4b}$$

$$\frac{d\varepsilon_i(t)}{dt} = A_i\varepsilon_i(t) + \tilde{f}(\varepsilon_i(t)) + \tilde{g}(\varepsilon_i(t-\tau)) - c\sum_{l=1}^{N} l_{ij}\Gamma\varepsilon_j(t), \quad i = l_0 + 1 \le i \le N, \tag{4c}$$

where $\tilde{f}(\varepsilon_i(t)) = f(\hat{x}_i(t)) - f(x_i(t))$, and $\tilde{g}(\varepsilon_i(t-\tau)) = g(\hat{x}_i(t-\tau)) - f(x_i(t-\tau))$.

For (4), we denote $\mathcal{E}(t) = \left(\varepsilon_1^T(t), \varepsilon_2^T(t), \cdots, \varepsilon_N^T(t)\right)^T$.

Definition 1: Estimator (3) is said to be an *exponential state estimator* of the complex network (1) if there exist constants $M_0 > 0$, $\mu > 0$ such that (4) satisfies

$$\lim_{t\to\infty} |\mathcal{E}(t)| \le M_0 \exp(-\mu t).$$

In this paper, our objective is to design the exponential state estimator for network (1). All the measurement information available to us is just from a fraction of network nodes. Besides, the measurement outputs are received in an intermittent way. By means of the Halanay inequality method and the ADT approach, respectively, the sufficient conditions are established such that the error dynamical system (4) is exponentially stable. Furthermore, the corresponding gain matrices are given explicitly.

III. Main Results and Proofs

In this section, we shall deal with the PNB estimation problem of network (1) by the Halanay inequality method and the ADT approach, respectively.

For a vector-valued function φ defined over the interval $[t-\tau, t]$, denote φ_t by $\varphi_t(s) = \varphi(t+s), -\tau \le s \le 0$, with the norm $|\varphi_t| \triangleq \sup_{-\tau\le s\le 0} |\varphi(t+s)|$.

The following two lemmas are used in the derivation of the main results.

Lemma 1 ([13], [45]): Let $t \ge t_0$ and $u\colon [t_0 - \tau, +\infty) \to [0, +\infty)$ be a continuous function. Suppose that $p > q \ge 0$ and u satisfy the following scalar differential inequality:

$$\frac{du(t)}{dt} \le -pu(t) + q|u_t|, \tag{5}$$

then

$$u(t)\leq|u_{t_0}|\exp(-\gamma(t-t_0)),\ t\geq t_0, \tag{6}$$

where γ is the unique positive solution of the equation

$$\gamma-p+q\exp(\gamma\tau)=0.$$

Lemma 2 ([13], [45]): Let $t\geq t_0$ and $u\colon[t_0-\tau,+\infty)\to[0,+\infty)$ be a continuous function. Suppose that $p>0,q>0$ and u satisfy the following scalar differential inequality:

$$\frac{du(t)}{dt}\leq pu(t)+q|u_t|, \tag{7}$$

then

$$u(t)\leq u_{t_0}\exp((p+q)(t-t_0)),\ t\geq t_0. \tag{8}$$

Assumption 1: The nonlinear vector-valued functions f and g are continuous and satisfy

$$|f(x)-f(y)|\leq\vartheta_1|x-y|,\quad \forall x,\ y\in\mathbb{R}^n, \tag{9}$$
$$|g(x)-g(y)|\leq\vartheta_2|x-y|,\quad \forall x,\ y\in\mathbb{R}^n, \tag{10}$$

where ϑ_1 and ϑ_2 are known constant scalars.

A. Halanay inequality method

In this subsection, we will discuss the existence of the exponential state estimator based on the Halanay inequality method.

In the sequel, for notational convenience, we denote

$$\begin{aligned}\mathcal{A}=&\mathrm{diag}(A_1,A_2,\cdots,A_N),\\ \Omega=&\mathrm{diag}(\omega_1,\omega_2,\cdots,\omega_{l_0}),\\ K=&\mathrm{diag}(K_1,K_2,\cdots,K_{l_0}),\\ C=&\mathrm{diag}(C_1,C_2,\cdots,C_{l_0}),\\ \bar{K}=&\begin{bmatrix}K\\0\end{bmatrix},\\ \mathcal{A}_K=&\mathcal{A}-\bar{K}C.\end{aligned}$$

Theorem 1: Assume that $\tau\leq\min\{T_1,T-T_1\}$. Then, under Assumption 1, estimator (3) is an exponential state estimator for network (1) if there exist a matrix K, scalar parameters $\vartheta_0>0$, and $\bar{\vartheta}_0>0$ such that

$$\begin{aligned}&\mathcal{A}_K+\mathcal{A}_K^T-c(L\otimes\Gamma)\\ &\quad-c(L^T\otimes\Gamma)+\big(\vartheta_0+2\vartheta_1+\vartheta_2\big)I<0, &(11)\\ &\mathcal{A}+\mathcal{A}^T-c(L\otimes\Gamma)-c(L^T\otimes\Gamma)\\ &\quad+\big(-\bar{\vartheta}_0+2\vartheta_1+\vartheta_2\big)I<0, &(12)\\ &\gamma(T_1-\tau)-\varrho(T-T_1)>0, &(13)\end{aligned}$$

where γ is the unique positive solution of the equation

$$\gamma-\vartheta_0+\vartheta_2\exp(\gamma\tau)=0, \tag{14}$$

and

$$\varrho=\bar{\vartheta}_0+\vartheta_2. \tag{15}$$

Proof: To begin with, choose the following Lyapunov function

$$\mathcal{V}(t)=|\mathcal{E}(t)|^2=\sum_{i=1}^{N}\varepsilon_i(t)\varepsilon_i^T(t). \tag{16}$$

Then, the time derivative of $\mathcal{V}(t)$ along the trajectory of (4) can be piecewisely calculated as follows.

1) When $t\in[sT,sT+T_1]$, one has

$$\begin{aligned}\dot{\mathcal{V}}(t)=&2\sum_{i=1}^{l_0}\varepsilon_i^T(t)\dot{\varepsilon}_i(t)+2\sum_{i=l_0+1}^{N}\varepsilon_i^T(t)\dot{\varepsilon}_i(t)\\ =&2\sum_{i=1}^{l_0}\varepsilon_i^T(t)\Big[(A_i-K_iC_i)\varepsilon_i(t)+\tilde{f}(\varepsilon_i(t))\\ &+\tilde{g}(\varepsilon_i(t-\tau))-c\sum_{l=1}^{N}l_{ij}\Gamma\varepsilon_j(t)\Big]\\ &+2\sum_{i=l_0+1}^{N}\varepsilon_i^T(t)\Big[A_i\varepsilon_i(t)+\tilde{f}(\varepsilon_i(t))\\ &+\tilde{g}(\varepsilon_i(t-\tau))-c\sum_{l=1}^{N}l_{ij}\Gamma\varepsilon_j(t)\Big]\\ =&2\sum_{i=1}^{l_0}\varepsilon_i^T(t)(A_i-K_iC_i)\varepsilon_i(t)+2\sum_{i=l_0+1}^{N}\varepsilon_i^T(t)A_i\varepsilon_i(t)\\ &+2\sum_{i=1}^{N}\varepsilon_i^T(t)\big[\tilde{f}(\varepsilon_i(t))+\tilde{g}(\varepsilon_i(t-\tau))\big]\\ &-2c\sum_{i=1}^{N}\varepsilon_i^T(t)\sum_{l=1}^{N}l_{ij}\Gamma\varepsilon_j(t).\end{aligned} \tag{17}$$

Utilizing the Cauchy inequality and Assumption 1, we obtain

$$2\varepsilon_i^T(t)\tilde{f}(\varepsilon_i(t))\leq2|\varepsilon_i(t)|\big|\tilde{f}(\varepsilon_i(t))\big|\leq2\vartheta_1|\varepsilon_i(t)|^2, \tag{18}$$

and

$$\begin{aligned}2\varepsilon_i^T(t)\tilde{g}(\varepsilon_i(t-\tau))\leq&2|\varepsilon_i(t)|\big|\tilde{g}(\varepsilon_i(t-\tau))\big|\\ \leq&2\vartheta_2|\varepsilon_i(t)||\varepsilon_i(t-\tau)|\\ \leq&\vartheta_2(|\varepsilon_i(t)|^2+|\varepsilon_i(t-\tau)|^2).\end{aligned} \tag{19}$$

Therefore, it follows that

$$\begin{aligned}&2\varepsilon_i^T(t)\big[\tilde{f}(\varepsilon_i(t))+\tilde{g}(\varepsilon_i(t-\tau))\big]\\ \leq&\big(2\vartheta_1+\vartheta_2\big)\varepsilon_i^T(t)\varepsilon_i(t)+\vartheta_2\varepsilon_i^T(t-\tau)\varepsilon_i(t-\tau).\end{aligned} \tag{20}$$

Substituting (20) into (17) yields

$$\begin{aligned}\dot{\mathcal{V}}(t)\leq&2\sum_{i=1}^{l_0}\varepsilon_i^T(t)(A_i-K_iC_i)\varepsilon_i(t)+2\sum_{i=l_0+1}^{N}\varepsilon_i^T(t)A_i\varepsilon_i(t)\\ &+\sum_{i=1}^{N}\Big[\big(2\vartheta_1+\vartheta_2\big)\varepsilon_i^T(t)\varepsilon_i(t)+\vartheta_2\varepsilon_i^T(t-\tau)\varepsilon_i(t-\tau)\Big]\\ &-2c\sum_{i=1}^{N}\varepsilon_i^T(t)\sum_{l=1}^{N}l_{ij}\Gamma\varepsilon_j(t).\end{aligned} \tag{21}$$

Note that

$$\sum_{i=1}^{N}\varepsilon_i^T(t)\sum_{l=1}^{N}l_{ij}\Gamma\varepsilon_j(t)=\mathcal{E}^T(t)(L\otimes\Gamma)\mathcal{E}(t),$$

then (21) can be rewritten as

$$\begin{aligned}\dot{\mathcal{V}}(t)\le\ &\mathcal{E}^T(t)\big[\mathcal{A}_K+\mathcal{A}_K^T-c(L\otimes\Gamma)-c(L^T\otimes\Gamma)\\&+\big(\vartheta_0+2\vartheta_1+\vartheta_2\big)I\big]\mathcal{E}(t)\\&-\vartheta_0\mathcal{E}^T(t)\mathcal{E}(t)+\vartheta_2\mathcal{E}^T(t-\tau)\mathcal{E}(t-\tau).\end{aligned}\tag{22}$$

Applying the condition (11) to (22) leads to

$$\dot{\mathcal{V}}(t)\le-\vartheta_0\mathcal{E}^T(t)\mathcal{E}(t)+\vartheta_2\mathcal{E}^T(t-\tau)\mathcal{E}(t-\tau),$$

which implies

$$\dot{\mathcal{V}}(t)\le-\vartheta_0\mathcal{V}(t)+\vartheta_2\mathcal{V}(t-\tau).\tag{23}$$

2) When $t\in[sT+T_1,(s+1)T]$, similar to previous steps, we have

$$\begin{aligned}\dot{\mathcal{V}}(t)&=2\sum_{i=1}^{N}\varepsilon_i^T(t)\dot{\varepsilon}_i(t)\\&=2\sum_{i=1}^{N}\varepsilon_i^T(t)\Big[A_i\varepsilon_i(t)+\tilde{f}(\varepsilon_i(t),\varepsilon_i(t-\tau))\\&\quad-c\sum_{l=1}^{N}l_{ij}\Gamma\varepsilon_j(t)\Big]\\&\le2\sum_{i=1}^{N}\varepsilon_i^T(t)A_i\varepsilon_i(t)+\sum_{i=1}^{N}\Big[\big(2\vartheta_1+\vartheta_2\big)\varepsilon_i^T(t)\varepsilon_i(t)\\&\quad+\vartheta_2\varepsilon_i^T(t-\tau)\varepsilon_i(t-\tau)\Big]-2c\sum_{i=1}^{N}\varepsilon_i^T(t)\sum_{l=1}^{N}l_{ij}\Gamma\varepsilon_j(t)\\&=\mathcal{E}^T(t)\big[\mathcal{A}+\mathcal{A}^T-c(L\otimes\Gamma)-c(L^T\otimes\Gamma)\\&\quad+\big(-\bar{\vartheta}_0+2\vartheta_1+\vartheta_2\big)I\big]\mathcal{E}(t)\\&\quad+\bar{\vartheta}_0\mathcal{E}^T(t)\mathcal{E}(t)+\vartheta_2\mathcal{E}^T(t-\tau)\mathcal{E}(t-\tau).\end{aligned}\tag{24}$$

Substituting the condition (12) to (24) yields

$$\dot{\mathcal{V}}(t)\le\bar{\vartheta}_0\mathcal{E}^T(t)\mathcal{E}(t)+\vartheta_2\mathcal{E}^T(t-\tau)\mathcal{E}(t-\tau),$$

which implies

$$\dot{\mathcal{V}}(t)\le\bar{\vartheta}_0\mathcal{V}(t)+\vartheta_2\mathcal{V}(t-\tau).\tag{25}$$

Now we can proceed with the estimate on the asymptotical behavior of $\mathcal{V}(t)$ based on the results of the steps 1) and 2).

First, noting $\vartheta_0>\vartheta_2$ and using Lemma 1, one has from (23) that

$$\mathcal{V}(t)\le\mathcal{V}_{t_0}\exp(-\gamma(t-t_0)),\ t\ge t_0\ge0,\tag{26}$$

where γ is an unique positive solution to

$$\gamma-\vartheta_0+\vartheta_2\exp(\gamma\tau)=0.$$

Also, from (25) and Lemma 2, we obtain

$$\mathcal{V}(t)\le\mathcal{V}_{t_0}\exp(\varrho(t-t_0)),\ t\ge t_0\ge0,\tag{27}$$

where $\varrho=\bar{\vartheta}_0+\vartheta_2$. Then, as in [13], [45], the evolution law of $\mathcal{V}(t)$ can be obtained by means of the mathematical induction.

In fact, it follows from (26) that

$$\mathcal{V}(t)\le|\mathcal{V}_0|\exp(-\gamma t),\ \text{for } t\in[0,\ T_1].\tag{28}$$

When $t\in[\,T_1,\ T]$, from (27) and (28), it is also clear that,

$$\begin{aligned}\mathcal{V}(t)&\le|\mathcal{V}_{T_1}|\exp(\varrho(t-T_1))\\&\le|\mathcal{V}_0|\exp(\varrho(t-T_1)-\gamma(T_1-\tau)).\end{aligned}\tag{29}$$

In general, it can be derived by induction that, for $t\in[sT,\ sT+T_1]$,

$$\mathcal{V}(t)\le|\mathcal{V}_0|\exp(-\gamma(t-sT)-s\gamma(T_1-\tau)+s\varrho(T-T_1)),\tag{30}$$

and for $t\in[sT+T_1,\ (s+1)T]$,

$$\begin{aligned}\mathcal{V}(t)\le|\mathcal{V}_0|\exp(&\varrho(t-sT-T_1)\\&-(s+1)\gamma(T_1-\tau)+s\varrho(T-T_1)).\end{aligned}\tag{31}$$

For all $t\in[0,\ +\infty)$, one infers from (30) and (31) that

$$\begin{aligned}\mathcal{V}(t)\le|\mathcal{V}_0|\exp\Big(&-\frac{\gamma(T_1-\tau)-\varrho(T-T_1)}{T}t\\&+\frac{\gamma T_1(T_1-\tau)}{T}\Big).\end{aligned}\tag{32}$$

Noticing that $\mathcal{V}(t)=|\mathcal{E}(t)|^2$, we arrive at

$$\begin{aligned}|\mathcal{E}(t)|&\le|\mathcal{E}_0|\exp\Big(-\frac{\gamma(T_1-\tau)-\varrho(T-T_1)}{2T}t\\&\qquad+\frac{\gamma T_1(T_1-\tau)}{2T}\Big)\\&=M_0\exp\Big(-\frac{\gamma(T_1-\tau)-\varrho(T-T_1)}{2T}t\Big),\end{aligned}\tag{33}$$

where $M_0=|\mathcal{E}_0|\exp\Big(\frac{\gamma T_1(T_1-\tau)}{2T}\Big)$. ■

Remark 2: In Theorem 1, sufficient conditions for the existence of an exponential estimator are derived based on a given Lyapunov function. Such an Lyapunov function might cause some conservatism. Therefore, how to reduce such conservatism is a topic of practical significance that is worth further study.

Remark 3: In Theorem 1, condition (13) can be rewritten as $\gamma(\frac{T_1}{T}-\frac{\tau}{T})-\varrho(1-\frac{T_1}{T})>0$, from which we can see that condition (13) requires the time of the output transmission to take a great proportion, which is in agreement of the common sense.

B. *Average Dwell-Time approach*

In the previous subsection, the existence of an exponential estimator is discussed via the Halanay inequality method. In Theorem 1, a restricted condition ($\tau\le\min\{T_1,T-T_1\}$) is imposed on time delays, which might limit the application of the results in Theorem 1. This restriction is lifted in the following theorem based on the ADT approach.

Theorem 2: Let matrices $K_i(1\le i\le3)$ and constant scalar $\alpha>0$ be given. Then, under Assumption 1, estimator (3) is an exponential estimator of network (1) if there exist two sets of positive matrices $\mathcal{P}_i=\mathrm{diag}\big(P_{1i},P_{2i},...,P_{Ni}\big)$ and $\mathcal{Q}_i=\mathrm{diag}\big(Q_{1i},Q_{2i},...,Q_{Ni}\big)$, two positive diagonal matrices $\Delta=\mathrm{diag}(\delta_1,\delta_2,...,\delta_N)$ and $\Sigma=\mathrm{diag}(\sigma_1,\sigma_2,...,\sigma_N)$, and scalar constants $\kappa_1 > 0$, $\kappa_2 > 0$, $\varpi > 0$, $\mu > 1$ such that the following inequalities hold:

$$\mathcal{Q}_1 \leq \kappa_1 I,\ \mathcal{Q}_2 \leq \kappa_2 I, \tag{34}$$

$$\mathcal{P}_i \leq \mu\mathcal{P}_j,\ \mathcal{Q}_i \leq \mu\mathcal{Q}_j,\ \text{for } i,j=1,2, \tag{35}$$

$$\Phi \triangleq \begin{bmatrix} \Pi & \mathcal{P}_1 \\ \mathcal{P}_1 & -e^{\alpha\tau}\mathcal{Q}_1 \end{bmatrix} \leq 0, \tag{36}$$

$$\bar{\Phi} \triangleq \begin{bmatrix} \bar{\Pi} & \mathcal{P}_2 \\ \mathcal{P}_2 & -e^{\alpha\tau}\mathcal{Q}_2 \end{bmatrix} \leq 0, \tag{37}$$

$$\frac{\alpha T_1}{2T} - \frac{\ln\mu}{T} - \frac{\beta}{2}(1-\frac{T_1}{T}) > 0, \tag{38}$$

where

$$\begin{aligned}
\Pi =& \mathcal{P}_1\mathcal{A}_K + \mathcal{A}_K^T\mathcal{P}_1 + \alpha\mathcal{P}_1 + \mathcal{P}_1(\Delta^{-1}\otimes I)\mathcal{P}_1 \\
&+ \vartheta_1^2(\Delta\otimes I) + \kappa_1\vartheta_2^2 I - c\mathcal{P}_1(L\otimes\Gamma) - c(L^T\otimes\Gamma)\mathcal{P}_1, \\
\bar{\Pi} =& \mathcal{P}_2\mathcal{A} + \mathcal{A}^T\mathcal{P}_2 + (\alpha-\varpi)\mathcal{P}_2 + \mathcal{P}_2(\Sigma^{-1}\otimes I)\mathcal{P}_2 \\
&+ \vartheta_1^2(\Sigma\otimes I) + \kappa_2\vartheta_2^2 I - c\mathcal{P}_2(L\otimes\Gamma) - c(L^T\otimes\Gamma)\mathcal{P}_2, \\
\beta =& -\alpha + \varpi,
\end{aligned}$$

and $K, \mathcal{A}_K$ are defined as before.

Proof: Rewrite (4) in the following compact form:

$$\begin{aligned}
\frac{d\mathcal{E}(t)}{dt} =& \mathcal{A}_K\mathcal{E}(t) + \tilde{F}(\mathcal{E}(t)) + \tilde{G}(\mathcal{E}(t-\tau)) \\
&- c(L\otimes\Gamma)\mathcal{E}(t),\ t\in[sT, sT+T_1),
\end{aligned} \tag{39a}$$

$$\begin{aligned}
\frac{d\mathcal{E}(t)}{dt} =& \mathcal{A}_K\mathcal{E}(t) + \tilde{F}(\mathcal{E}(t)) + \tilde{G}(\mathcal{E}(t-\tau)) \\
&- c(L\otimes\Gamma)\mathcal{E}(t),\ t\in[sT+T_1, (s+1)T),
\end{aligned} \tag{39b}$$

where $\tilde{F}(\mathcal{E}(t)) = \left(\tilde{f}^T(\varepsilon_1(t)), \tilde{f}^T(\varepsilon_2(t)), ..., \tilde{f}^T(\varepsilon_N(t))\right)^T$, and $\tilde{G}(\mathcal{E}(t)) = \left(\tilde{g}^T(\varepsilon_1(t)), \tilde{g}^T(\varepsilon_2(t)), ..., \tilde{g}^T(\varepsilon_N(t))\right)^T$.

Different from the Lyapunov function in Theorem 1, construct the following piecewise Lyapunov-Krasovskii functional

$$\hat{\mathcal{V}}(t) = \hat{\mathcal{V}}_1(t) + \hat{\mathcal{V}}_2(t) \tag{40}$$

where, for $t\in[sT, sT+T_1)$,

$$\hat{\mathcal{V}}_1(t) = \mathcal{E}^T(t)\mathcal{P}_1\mathcal{E}(t) = \sum_{i=1}^N \varepsilon_i^T(t)P_{i1}\varepsilon_i(t), \tag{41}$$

$$\hat{\mathcal{V}}_2(t) = \int_{t-\tau}^t e^{\alpha(\theta-t)}\tilde{G}^T(\mathcal{E}(\theta))\mathcal{Q}_1\tilde{G}(\mathcal{E}(\theta))d\theta, \tag{42}$$

and for $t\in[sT+T_1, (s+1)T)$,

$$\hat{\mathcal{V}}_1(t) = \mathcal{E}^T(t)\mathcal{P}_2\mathcal{E}(t) = \sum_{i=1}^N \varepsilon_i^T(t)P_{i2}\varepsilon_i(t), \tag{43}$$

$$\hat{\mathcal{V}}_2(t) = \int_{t-\tau}^t e^{\alpha(\theta-t)}\tilde{G}^T(\mathcal{E}(\theta))\mathcal{Q}_2\tilde{G}(\mathcal{E}(\theta))d\theta, \tag{44}$$

Then, the time derivative of $\mathcal{V}(t)$ along the trajectory of (39) can be piecewisely calculated as

1) When $t\in[sT, sT+T_1)$, one has

$$\dot{\hat{\mathcal{V}}}(t) = \dot{\hat{\mathcal{V}}}_1(t) + \dot{\hat{\mathcal{V}}}_2(t), \tag{45}$$

where

$$\begin{aligned}
\dot{\hat{\mathcal{V}}}_1(t) =& 2\mathcal{E}^T(t)\mathcal{P}_1\dot{\mathcal{E}}(t) \\
=& 2\mathcal{E}^T(t)\mathcal{P}_1\Big[\mathcal{A}_K\mathcal{E}(t) + \tilde{F}(\mathcal{E}(t)) + \tilde{G}(\mathcal{E}(t-\tau)) \\
&- c(L\otimes\Gamma)\mathcal{E}(t)\Big],
\end{aligned} \tag{46}$$

and

$$\begin{aligned}
\dot{\hat{\mathcal{V}}}_2(t) =& -\alpha\int_{t-\tau}^t e^{\alpha(\theta-t)}\tilde{G}^T(\mathcal{E}(\theta))\mathcal{Q}_1\tilde{G}(\mathcal{E}(\theta))d\theta \\
&+ \tilde{G}^T(\mathcal{E}(t))\mathcal{Q}_1\tilde{G}(\mathcal{E}(t)) \\
&- e^{\alpha\tau}\tilde{G}^T(\mathcal{E}(t-\tau))\mathcal{Q}_1\tilde{G}(\mathcal{E}(t-\tau)) \\
=& -\alpha\hat{\mathcal{V}}_2(t) + \tilde{G}^T(\mathcal{E}(t))\mathcal{Q}_1\tilde{G}(\mathcal{E}(t)) \\
&- e^{\alpha\tau}\tilde{G}^T(\mathcal{E}(t-\tau))\mathcal{Q}_1\tilde{G}(\mathcal{E}(t-\tau)).
\end{aligned} \tag{47}$$

In (46), it is clear that

$$\begin{aligned}
& 2\mathcal{E}^T(t)\mathcal{P}_1\tilde{F}(\mathcal{E}(t)) \\
=& 2\sum_{i=1}^N \varepsilon_i^T(t)P_{i1}\tilde{f}(\varepsilon_i(t)) \\
\leq& \sum_{i=1}^N [\delta_i^{-1}\varepsilon_i^T(t)P_{i1}P_{i1}\varepsilon_i^T(t) + \delta_i\tilde{f}^T(\varepsilon_i(t))\tilde{f}(\varepsilon_i(t))] \\
\leq& \sum_{i=1}^N [\delta_i^{-1}\varepsilon_i^T(t)P_{i1}P_{i1}\varepsilon_i^T(t) + \vartheta_1^2\delta_i\varepsilon_i^T(t)\varepsilon_i(t)] \\
& \qquad \text{(with Lemma 1)} \\
=& \mathcal{E}^T(t)\mathcal{P}_1(\Delta^{-1}\otimes I)\mathcal{P}_1\mathcal{E}(t) + \vartheta_1^2\mathcal{E}^T(t)(\Delta\otimes I)\mathcal{E}(t).
\end{aligned} \tag{48}$$

Similarly, in (47), one has from $\mathcal{Q}_1 < \kappa_1 I$ that

$$\begin{aligned}
\tilde{G}^T(\mathcal{E}(t))\mathcal{Q}_1\tilde{G}(\mathcal{E}(t)) \leq& \kappa_1\tilde{G}^T(\mathcal{E}(t))\mathcal{Q}_1\tilde{G}(\mathcal{E}(t)) \\
\leq& \kappa_1\vartheta_2^2\mathcal{E}^T(t)\mathcal{E}(t).
\end{aligned} \tag{49}$$

Substituting (46)–(49) into (45) leads to

$$\begin{aligned}
& \dot{\hat{\mathcal{V}}}(t) + \alpha\hat{\mathcal{V}}(t) \\
=& \mathcal{E}^T(t)\big[\mathcal{P}_1\mathcal{A}_K + \mathcal{A}_K^T\mathcal{P}_1 + \alpha\mathcal{P}_1 + \mathcal{P}_1(\Delta^{-1}\otimes I)\mathcal{P}_1 \\
&+ \vartheta_1^2(\Delta\otimes I) + \kappa_1\vartheta_2^2 I - c\mathcal{P}_1(L\otimes\Gamma) \\
&- c(L^T\otimes\Gamma)\mathcal{P}_1\big]\mathcal{E}(t) + 2\mathcal{E}^T(t)\mathcal{P}_1\tilde{G}(\mathcal{E}(t-\tau)) \\
&- e^{\alpha\tau}\tilde{G}^T(\mathcal{E}(t-\tau))\mathcal{Q}_1\tilde{G}(\mathcal{E}(t-\tau)) \\
=& \xi^T(t)\Phi\xi(t) \leq 0,
\end{aligned} \tag{50}$$

where $\xi(t) = \left[\mathcal{E}^T(t),\ \tilde{G}^T(\mathcal{E}(t-\tau))\right]^T$.

2) When $t\in[sT+T_1, (s+1)T)$, the time derivative of $\mathcal{V}(t)$ along the trajectory of (39) can be calculated as follows:

$$\begin{aligned}
\dot{\hat{\mathcal{V}}}_1(t) =& 2\mathcal{E}^T(t)\mathcal{P}_2\Big[\mathcal{A}\mathcal{E}(t) + \tilde{F}(\mathcal{E}(t)) + \tilde{G}(\mathcal{E}(t-\tau)) \\
&- c(L\otimes\Gamma)\mathcal{E}(t)\Big],
\end{aligned} \tag{51}$$

and

$$\begin{aligned}
\dot{\hat{\mathcal{V}}}_2(t) =& -\alpha\hat{\mathcal{V}}_2(t) + \tilde{G}^T(\mathcal{E}(t))\mathcal{Q}_2\tilde{G}(\mathcal{E}(t)) \\
&- e^{\alpha\tau}\tilde{G}^T(\mathcal{E}(t-\tau))\mathcal{Q}_2\tilde{G}(\mathcal{E}(t-\tau)).
\end{aligned} \tag{52}$$

Similar to (48) and (49), we have

$$\begin{aligned}
& 2\mathcal{E}^T(t)\mathcal{P}_2\tilde{F}(\mathcal{E}(t)) \\
\leq& \mathcal{E}^T(t)\mathcal{P}_2(\Sigma^{-1}\otimes I)\mathcal{P}_2\mathcal{E}(t) + \vartheta_1^2\mathcal{E}^T(t)(\Sigma\otimes I)\mathcal{E}(t),
\end{aligned} \tag{53}$$

and

$$\tilde{G}^T(\mathcal{E}(t))\mathcal{Q}_2\tilde{G}(\mathcal{E}(t)) \leq \kappa_2\vartheta_2^2\mathcal{E}^T(t)\mathcal{E}(t). \quad (54)$$

From (51)–(54), one infers

$$\begin{aligned}
&\dot{\hat{\mathcal{V}}}(t) + (\alpha - \varpi)\hat{\mathcal{V}}(t) \\
&= \mathcal{E}^T(t)\big[\mathcal{P}_2\mathcal{A} + \mathcal{A}^T\mathcal{P}_2 + \alpha\mathcal{P}_2 + \mathcal{P}_2(\Sigma^{-1}\otimes I)\mathcal{P}_2 \\
&\quad + \vartheta_1^2(\Sigma\otimes I) + \kappa_2\vartheta_2^2 I - \varpi\mathcal{P}_2 - c\mathcal{P}_2(L\otimes\Gamma) \\
&\quad - c(L\otimes\Gamma)\mathcal{P}_2\big]\mathcal{E}(t) + 2\mathcal{E}^T(t)\mathcal{P}_2\tilde{G}(\mathcal{E}(t-\tau)) \\
&\quad - e^{\alpha\tau}\tilde{G}^T(\mathcal{E}(t-\tau))\mathcal{Q}_2\tilde{G}(\mathcal{E}(t-\tau)) \\
&= \xi^T(t)\bar{\Phi}\xi(t) \leq 0. \quad (55)
\end{aligned}$$

Now, we can analyze the convergence of $\mathcal{V}(t)$. First, from (50) and (55), we have

$$\dot{\hat{\mathcal{V}}}(t) \leq -\alpha\hat{\mathcal{V}}(t),\ t\in[sT,\ sT+T_1), \quad (56)$$

$$\dot{\hat{\mathcal{V}}}(t) \leq \beta\hat{\mathcal{V}}(t)\ t\in[sT+T_1,\ (s+1)T), \quad (57)$$

where $\beta = \varpi - \alpha$, which implies that

$$\hat{\mathcal{V}}(t) \leq \hat{\mathcal{V}}_{sT}\exp(-\alpha(t-sT)),\ t\in[sT,\ sT+T_1), \quad (58)$$

$$\begin{aligned}
\hat{\mathcal{V}}(t) &\leq \hat{\mathcal{V}}_{sT+T_1}\exp(\beta(t-sT-T_1)), \\
&\quad t\in[sT+T_1,\ (s+1)T). \quad (59)
\end{aligned}$$

Thus, when $t\in[0,\ T_1)$, one has

$$\mathcal{V}(t) \leq \hat{\mathcal{V}}_0\exp(-\alpha t). \quad (60)$$

Denote $\mathcal{V}_{t_0^-} = \lim_{t\to t_0^-}\mathcal{V}_t$. Noting (35), it is clear that

$$\mathcal{V}_{sT+T_1} \leq \mu\mathcal{V}_{sT+T_1^-}, \quad (61)$$

and

$$\mathcal{V}_{(s+1)T} \leq \mu\mathcal{V}_{(s+1)T^-}. \quad (62)$$

Then, when $t\in[T_1,\ T)$, it follows from (1) that

$$\begin{aligned}
\hat{\mathcal{V}}(t) &\leq \hat{\mathcal{V}}_{T_1}\exp(\beta(t-T_1)) \\
&\leq \mu\hat{\mathcal{V}}_{T_1^-}\exp(\beta(t-T_1)) \\
&\leq \mu\hat{\mathcal{V}}_0\exp(-\alpha T_1)\exp(\beta(t-T_1)) \\
&= \mu\hat{\mathcal{V}}_0\exp(-\alpha T_1 + \beta(t-T_1)). \quad (63)
\end{aligned}$$

Similarly, when $t\in[T,\ T+T_1)$,

$$\begin{aligned}
\hat{\mathcal{V}}(t) &\leq \hat{\mathcal{V}}_T\exp(-\alpha(t-T)) \\
&\leq \mu\hat{\mathcal{V}}_{T^-}\exp(-\alpha(t-T)) \\
&\leq \mu^2\hat{\mathcal{V}}_0\exp(-\alpha(t-T) - \alpha T_1 + \beta(T-T_1)), \quad (64)
\end{aligned}$$

and when $t\in[T+T_1,\ 2T)$,

$$\begin{aligned}
\hat{\mathcal{V}}(t) &\leq \hat{\mathcal{V}}_{T+T_1}\exp(\beta(t-T-T_1)) \\
&\leq \mu\hat{\mathcal{V}}_{T+T_1^-}\exp(\beta(t-T-T_1)) \\
&\leq \mu^3\hat{\mathcal{V}}_0\exp(\beta(t-T-T_1) - 2\alpha T_1 + \beta(T-T_1)). \quad (65)
\end{aligned}$$

By induction, we have the following results

$$\begin{aligned}
\hat{\mathcal{V}}(t) &\leq \mu^{2s}\hat{\mathcal{V}}_0\exp(-\alpha(t-sT) - s\alpha T_1 + s\beta(T-T_1)), \\
&\quad t\in[sT,\ sT+T_1), \quad (66)
\end{aligned}$$

$$\begin{aligned}
\hat{\mathcal{V}}(t) &\leq \mu^{2s+1}\hat{\mathcal{V}}_0\exp(\beta(t-sT-T_1) - (s+1)\alpha T_1 \\
&\quad + s\beta(T-T_1)),\ t\in[sT+T_1,\ (s+1)T) \quad (67)
\end{aligned}$$

From (66), it follows that, for $t\in[sT,\ sT+T_1)$,

$$\begin{aligned}
\hat{\mathcal{V}}(t) &\leq \hat{\mathcal{V}}_0\exp\Big(\frac{2\ln\mu}{T}t - sT\frac{\alpha T_1}{T} + sT\beta(1-\frac{T_1}{T})\Big) \\
&\leq \hat{\mathcal{V}}_0\exp\Big(-(\frac{\alpha T_1}{T} - \frac{2\ln\mu}{T} - \beta(1-\frac{T_1}{T}))t + \frac{\alpha T_1^2}{T}\Big). \quad (68)
\end{aligned}$$

Next, from (67), it is not difficult to see that, for $t\in[sT+T_1,\ (s+1)T)$,

$$\begin{aligned}
\hat{\mathcal{V}}(t) &\leq \mu^{2s+1}\hat{\mathcal{V}}_0\exp(\beta(t-sT-T_1) - (s+1)\alpha T_1 \\
&\quad + s\beta(T-T_1)) \\
&\leq \mu\hat{\mathcal{V}}_0\exp\Big(-(\frac{\alpha T_1}{T} - \frac{2\ln\mu}{T} - \beta(1-\frac{T_1}{T}))t\Big). \quad (69)
\end{aligned}$$

From (68) and (69), we obtain

$$\hat{\mathcal{V}}(t) \leq \mu\hat{\mathcal{V}}_0\exp\Big(-(\alpha\frac{T_1}{T} - \frac{2\ln\mu}{T} - \beta(1-\frac{T_1}{T}))t + \alpha\frac{T_1^2}{T}\Big). \quad (70)$$

Setting

$$M_0 = \sqrt{\frac{\mu\hat{\mathcal{V}}_0\exp(\alpha\frac{T_1^2}{T})}{\min\{\lambda_{\min}(\mathcal{P}_1), \lambda_{\min}(\mathcal{P}_2)\}}},$$

we can conclude that

$$\mathcal{E}(t) \leq M_0\exp\Big(-(\frac{\alpha T_1}{2T} - \frac{\ln\mu}{T} - \frac{\beta}{2}(1-\frac{T_1}{T}))t\Big). \quad (71)$$

Therefore, the estimation errors converge exponentially to zero, and the proof of this theorem is now complete. ■

In Theorem 2, the exponential convergence of the estimator has been analyzed with known gains. Now, let us turn to the design problem of estimator (3).

Theorem 3: Under Assumption 1, for a given constant scalar $\alpha > 0$, estimator (3) is an exponential state estimator of network (1) if there exist two sets of positive matrices $\mathcal{P}_i = \mathrm{diag}\big(P_{1i}, P_{2i}, ..., P_{Ni}\big)$, $\mathcal{Q}_i = \mathrm{diag}\big(Q_{1i}, Q_{2i}, ..., Q_{Ni}\big)$, two positive diagonal matrices $\Delta = \mathrm{diag}(\delta_1, \delta_2, ..., \delta_N)$ and $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, ..., \sigma_N)$, a matrix $X = \mathrm{diag}(X_1, X_2, ..., X_{l_0})$, constant scalars $\kappa_1 > 0$, $\kappa_2 > 0$, $\varpi > 0$, $\mu > 1$ such that the following inequalities hold:

$$\mathcal{Q}_1 \leq \kappa_1 I,\ \mathcal{Q}_2 \leq \kappa_2 I, \quad (72)$$

$$\mathcal{P}_i \leq \mu\mathcal{P}_j,\ \mathcal{Q}_i \leq \mu\mathcal{Q}_j,\ \text{for } i,j = 1,2, \quad (73)$$

$$\Psi \triangleq \begin{bmatrix} \Xi & \mathcal{P}_1 & \mathcal{P}_1 \\ \mathcal{P}_1 & -e^{\alpha\tau}\mathcal{Q}_1 & 0 \\ \mathcal{P}_1 & 0 & -\Delta\otimes I \end{bmatrix} \leq 0, \quad (74)$$

$$\bar{\Psi} \triangleq \begin{bmatrix} \bar{\Xi} & \mathcal{P}_2 & \mathcal{P}_2 \\ \mathcal{P}_2 & -e^{\alpha\tau}\mathcal{Q}_2 & 0 \\ \mathcal{P}_2 & 0 & -\Sigma\otimes I \end{bmatrix} \leq 0, \quad (75)$$

$$\frac{\alpha T_1}{2T} - \frac{\ln\mu}{T} - \frac{\beta}{2}(1-\frac{T_1}{T}) > 0, \quad (76)$$

where

$$\begin{aligned}
\Xi &= \mathcal{P}_X + \mathcal{P}_X^T + \alpha\mathcal{P}_1 + \vartheta_1^2(\Delta\otimes I) \\
&\quad + \kappa_1\vartheta_2^2 I - c\mathcal{P}_1(L\otimes\Gamma) - c(L^T\otimes\Gamma)\mathcal{P}_1,
\end{aligned}$$

$$\bar{\Xi} = \mathcal{P}_2\mathcal{A} + \mathcal{A}^T\mathcal{P}_2 + (\alpha - \varpi)\mathcal{P}_2 + \vartheta_1^2(\Sigma \otimes I) + \kappa_2\vartheta_2^2 I - c\mathcal{P}_2(L \otimes \Gamma) - c(L^T \otimes \Gamma)\mathcal{P}_2,$$
$$\beta = -\alpha + \varpi$$

with $\mathcal{P}_X = \mathcal{P}\mathcal{A} - \bar{X}C$, $\bar{X} \triangleq \begin{bmatrix} X \\ 0 \end{bmatrix}$. In this case, the estimator gain K_i is designed by

$$K_i = P_i^{-1}X_i. \tag{77}$$

Proof: The result follows readily from Theorem 2, and the proof is therefore omitted here. ■

Remark 4: In this paper, the state estimation problem has been investigated under the circumstance that the outputs are available only from a fraction of network nodes, and signals are transmitted in an intermittent way. Two approaches have been applied to establish the conditions ensuring the existence of an exponential estimator. It should be poi out that, though both conditions obtained are sufficient, don't contain each other, which is illustrated in the nume example later.

IV. A Numerical example

For the sake of simplicity, we consider a delayed com dynamical network with five nodes and non-identical loca namical systems, where outputs of three nodes are availabl the purpose of estimation, i.e., $l_0 = 3$. The other param are

$$A_1 = A_2 = A_3 = \begin{bmatrix} 3 & 8 & -2 \\ -8 & 3 & 3 \\ 1 & 2 & -5 \end{bmatrix},$$
$$A_4 = A_5 = \begin{bmatrix} -4.4 & -0.2 & 0.6 \\ -0.4 & -4 & 0.5 \\ -1 & -0.0667 & -4.3 \end{bmatrix},$$
$$c = 2,\ \tau = 3,\ \Gamma = I,$$
$$L = \begin{bmatrix} 2.5 & -1.5 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix},$$
$$C_1 = C_2 = C_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$
$$f(x) = g(x) = \big(0.3(|x_1 + 2| - |x_1 - 2|), 0.2x_2,\ 0.3x_3\big)^T.$$

For the functions given above, a direct calculation yields that $\vartheta_1 = \vartheta_2 = 0.6$.

With these parameters, we proceed with the numerical simulations to confirm the theoretical results based on the Halanay inequality method and the ADT approach, respectively.

(i) *Simulation via Halanay inequality method*

Solve the LMIs (11) and (12) to obtain the following feasible solutions:

$$K_1 = \begin{bmatrix} 8.5403 & -0.0403 \\ -0.0403 & 8.5940 \\ -1.0559 & 5.0605 \end{bmatrix},\ K_2 = \begin{bmatrix} 9.2243 & 0.2512 \\ -0.2254 & 9.2071 \\ -0.9851 & 4.9839 \end{bmatrix},$$

$$K_3 = \begin{bmatrix} 9.4155 & -0.0316 \\ -0.0472 & 9.4680 \\ -1.0462 & 5.0500 \end{bmatrix},\ \vartheta_0 = 3.8672,\ \bar{\vartheta}_0 = 7.6271.$$

By the Newton-Raphson method for solving transcendental equation (14), we obtain an approximation of the unique positive solution: $\gamma = 0.5682$. Take $T = 49, T_1 = 46, \tau = 3$. Then, it can be verified that condition (13) is satisfied. Therefore, from Theorem (1), the estimation error approaches to zero exponentially.

In fact, the numerical simulation is in agreement with the theoretical result perfectly. To be more specific, we randomly choose two sets of initial values for the network and the estimator, and then the evolution of the network state and its estimate are shown in Fig. 1 and Fig 2, respectively. Furthermore, estimation errors are depicted in Fig. 3, which show such errors converge to 0 exponentially.

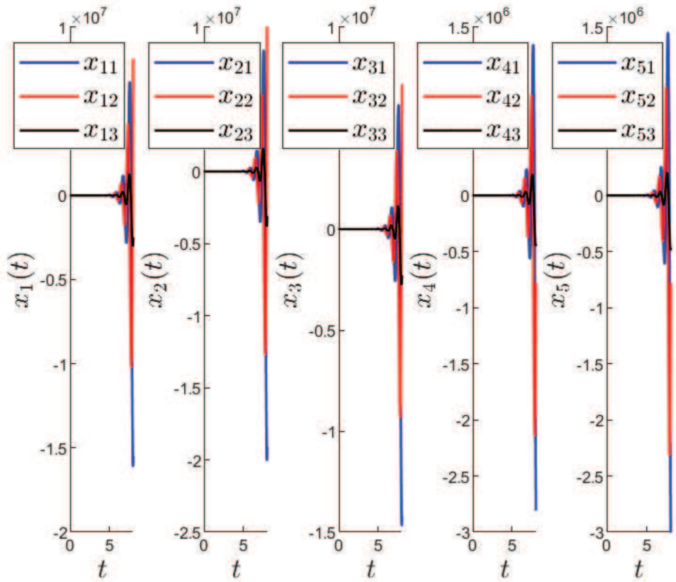

Fig. 1. Evolution of network states

(ii) *Simulation via ADT approach*

Setting $\alpha = 0.3$, $\mu = 3$, and $\varpi = 9.5$, and solving the LMIs (72)-(75), we have feasible solutions with estimator gains given as follows:

$$K_1 = \begin{bmatrix} 4.8949 & 0.4967 \\ 0.1599 & 5.4727 \\ 1.2388 & 1.5266 \end{bmatrix},\ K_2 = \begin{bmatrix} 6.8527 & 0.6352 \\ 0.2132 & 7.5351 \\ 1.7183 & 1.9758 \end{bmatrix},$$
$$K_3 = \begin{bmatrix} 5.6592 & 0.5286 \\ 0.1733 & 6.4753 \\ 1.4541 & 1.6957 \end{bmatrix}.$$

Noting $\beta = -\alpha + \varpi = 9.2$, we find that inequality (13) is violated if the parameters T and T_1 are the same as before. Now, we take $T = 20$ and $T_1 = 19.6$. Then, it can be verified that inequality (13) is satisfied with these new parameters. According to Theorem 3, the estimation errors converge to zero exponentially as t tends to infinity. The simulation result is shown in Fig 4, which is completely consistent with the developed theory.

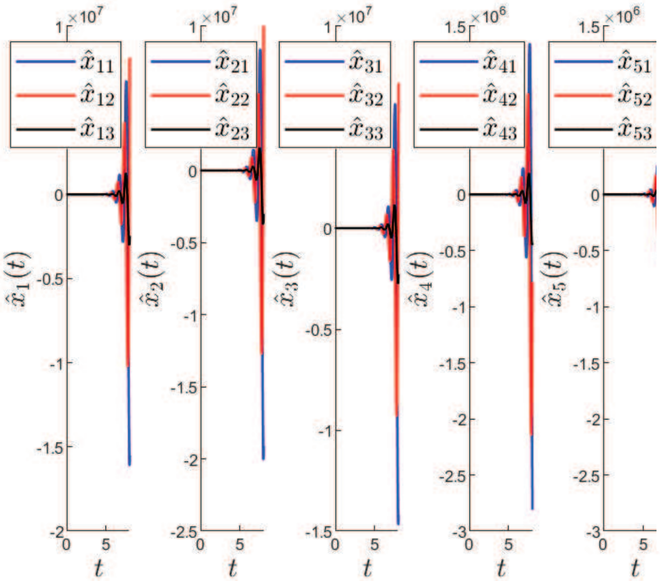

Fig. 2. Evolution of the state estimate

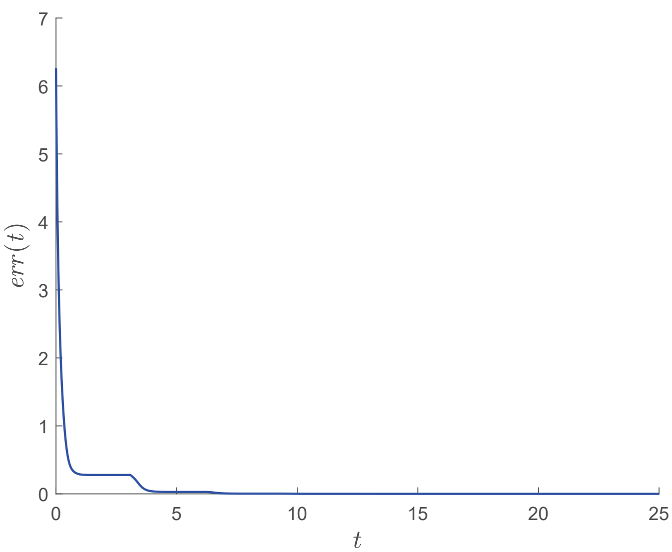

Fig. 3. Evolution of the state estimate

Remark 5: It should be mentioned that parameters in simulation (i) satisfies the conditions of Theorem 1, but do not satisfy all conditions of Theorem 3. However, it should also be pointed out that Theorem 3 is not applicable to the parameters in simulation (i) since $\tau \leq \min\{T_1, T - T_1\}$ does not hold any more. Therefore, Theorem 1 and Theorem 3 are applicable to different situations, and cannot be substituted with each other.

Remark 6: What is worth pointing out is that our approaches developed in this paper are *non-trivial*. In fact, as shown in Fig 1 and Fig 2, even though the concerned target network is unstable, the estimation errors are still convergent. The numerical simulation confirms the effectiveness of the theoretical results.

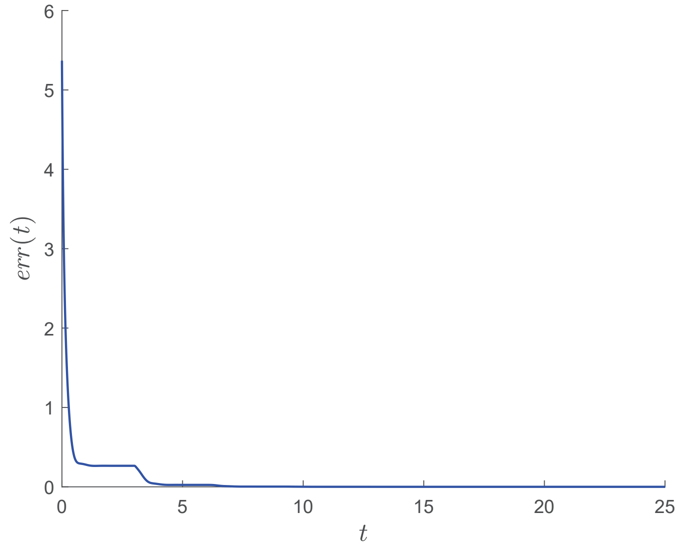

Fig. 4. Evolution of state estimation error (ii)

V. Conclusions

In this paper, we have investigated the estimation problem for delayed complex networks with non-identical nodes, where the network output information is only from a fraction of nodes, and the output signals are transmitted intermittently over communications channels. By exploiting the Halanay inequality method and the ADT approach, respectively, some sufficient criteria have been established to guarantee that the error dynamics is exponentially stable. Finally, a numerical example has been presented to demonstrate the effectiveness of the given estimator. In addition, related topics for further research work can be listed as follows. 1) The Kalman filtering problems for delayed systems with measurement missing, quantization and censoring; and 2) The H_∞ control problems for delayed systems with state saturations, measurement fading and sensor failures [6], [23], [36], [43].

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