On intuitionistic fuzzy sub-hyperquasigroups of hyperquasigroups

Wiesław A. Dudek^a, Bijan Davvaz^b, Young Bae Jun^{c,*}

 ^a Institute of Mathematics, Technical University, Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland
 ^b Department of Mathematics, Yazd University, Yazd, Iran
 ^c Department of Mathematics Educations, Gyeongsang National University, Chinju 660-701, Korea

Abstract

The notion of intuitionistic fuzzy sets was introduced by Atanassov as a generalization of the notion of fuzzy sets. In this paper, we consider the intuitionistic fuzzification of the concept of sub-hyperquasigroups in a hyperquasigroup and investigate some properties of such subhyperquasigroups. In particular, we investigate some natural equivalence relations on the set of all intuitionistic fuzzy sub-hyperquasigroups of a hyperquasigroup.

2000 Mathematics Subject Classification: 20N20, 20N25. Keywords: hyperquasigroup, fuzzy sub-hyperquasigroup, intuitionistic fuzzy sub-hyperquasigroup, quasigroup.

1 Introduction and preliminaries

The theory of hyperstructures which is a generalization of the concept of algebraic structures first was introduced by Marty [19] and then many researchers have been worked on this new field of modern algebra and developed it. A short review of the theory of hyperstructures appear in [6] and [23]. A recent book [5] contains a wealth of applications. There are applications to the following subjects: geometry, hypergraphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, combinatorics, codes, artificial intelligence, and probabilities. The theory of fuzzy sets proposed by Zadeh [24] has achieved a great success in various fields. Out of several higher order fuzzy sets, intuitionistic fuzzy sets introduced by Atanassov [1, 2, 3] have

E-mail address: dudek@im.pwr.wroc.pl (W. A. Dudek), davvaz@yazduni.ac.ir (B. Davvaz), ybjun@nongae.gsnu.ac.kr (Y. B. Jun) (hyper/His/INS394.tex)

been found to be highly useful to deal with vagueness. Gau and Buehrer [15] presented the concept of vague sets. But, Burillo and Bustince [4] showed that the notion of vague sets coincides with that of intuitionistic fuzzy sets. Szmidt and Kacprzyk [22] proposed a non-probabilistic-type entropy measure for intuitionistic fuzzy sets. De et al. [11] studied the Sanchez's approach for medical diagnosis and extended this concept with the notion of intuitionistic fuzzy set theory. Denging and Chuntian [12] introduced the concept of the degree of similarity between intuitionistic fuzzy sets, presented several new similarity measures for measuring the degree of similarity between intuitionistic fuzzy sets, which may be finite or continuous, and gave corresponding proofs of these similarity measures and discussed applications of the similarity measures between intuitionistic fuzzy sets to pattern reconnition problems. The notion of join space has been introduced by Prenowitz and used by him and afterwards together Jantosciak to build again several branches of geometry. A join space is a hypergroup with additional conditions. A generalization of join spaces for the point of view of independence, dimension etc., is that of cambiste hypergroups studied by Freni. Noticing that a hypergroup is a hyperquasigroup with the associative hyperoperation, the results of this paper will make a contribution to discuss a generalization of join spaces, to deal with several notions in geometries since there are deep relations between geometries and hypergroups (or, to say multigroups), and to develop the intuitionistic fuzzy theory in several algebraic structures.

A hypergroupoid (G, \circ) is a non-empty set G with a hyperoperation \circ defined on G, i.e., a mapping of $G \times G$ into the family of non-empty subsets of G. If $(x, y) \in G \times G$, its image under \circ is denoted by $x \circ y$. If $A, B \subseteq G$ then $A \circ B$ is given by $A \circ B = \bigcup \{x \circ y \mid x \in A, y \in B\}$. $x \circ A$ is used for $\{x\} \circ A$ and $A \circ x$ for $A \circ \{x\}$.

Definition 1.1. A hypergroupoid (G, \circ) is called a *hypergroup* if for all $x, y, z \in G$ the following two conditions hold:

- (i) $x \circ (y \circ z) = (x \circ y) \circ z$,
- (ii) $x \circ G = G \circ x = G$.

The second condition, called the *reproducibility condition*, means that for any $x, y \in G$ there exist $u, v \in G$ such that $y \in x \circ u$ and $y \in v \circ x$.

A hypergroupoid satisfying this condition is called a *hyperquasigroup*. Thus a hypergroup is a hyperquasigroup with the associative hyperoperation.

A non-empty subset K of a hyperquasigroup (G, \circ) is called a *sub-hyperquasigroup* if (K, \circ) is a hyperquasigroup.

The concept of fuzzy sets was introduced by Zadeh [24] in 1965. A mapping $\mu: X \to [0,1]$, where X is an arbitrary non-empty set, is called a *fuzzy set* in X. The *complement* of μ , denoted by μ^c , is the fuzzy set in X given by $\mu^c(x) = 1 - \mu(x)$ for all $x \in X$.

For any fuzzy set μ in X and any $t \in [0, 1]$ we define two sets

 $U(\mu; t) = \{ x \in X \mid \mu(x) \ge t \} \text{ and } L(\mu; t) = \{ x \in X \mid \mu(x) \le t \},\$

which are called an *upper* and *lower t-level cut* of μ and can be used to the characterization of μ .

In 1971, Rosenfeld [21] applied the concept of fuzzy sets to the theory of groups and studied fuzzy subgroups of a group. Davvaz applied in [8] fuzzy sets to the theory of algebraic hyperstructures and studied their fundamental properties. Further investigations are contained in [7], [9] and [10].

Definition 1.2. (cf. [8]) Let (G, \circ) be a hypergroup (resp. hyperquasigroup) and let μ be a fuzzy set in G. Then μ is said to be a *fuzzy sub-hypergroup* (resp. *fuzzy sub-hyperquasigroup*) of G if the following axioms hold:

- (1) $\min\{\mu(x), \mu(y)\} \le \inf\{\mu(z) \mid z \in x \circ y\}$ for all $x, y \in G$,
- (2) for all $x, a \in G$ there exists $y \in G$ such that $x \in a \circ y$ and

$$\min\{\mu(a), \mu(x)\} \le \mu(y),$$

(3) for all $x, a \in G$ there exists $z \in G$ such that $x \in z \circ a$ and

$$\min\{\mu(a), \mu(x)\} \le \mu(z).$$

As an important generalization of the notion of fuzzy sets in X, Atanassov [1] introduced the concept of *intuitionistic fuzzy sets* defined on a non-empty set X as objects having the form

$$A = \{ (x, \mu_A(x), \lambda_A(x)) \mid x \in X \},\$$

where the functions $\mu_A : X \to [0, 1]$ and $\lambda_A : X \to [0, 1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\lambda_A(x)$) of each element $x \in X$ to the set A respectively, and $0 \le \mu_A(x) + \lambda_A(x) \le 1$ for all $x \in X$.

Such defined objects are studied by many authors (see for example two journals: 1. *Fuzzy Sets and Systems* and 2. *Notes on Intuitionistic Fuzzy Sets*) and have many interesting applications not only in mathematics (see Chapter 5 in the book [3]). In particular, Kim, Dudek and Jun in [16] introduced the notion of an intuitionistic fuzzy subquasigroup of a quasigroup. Also in [17], Kim and Jun introduced the concept of intuitionistic fuzzy ideals of semigroups.

For every two intuitionistic fuzzy sets A and B in X we define (cf. [2]):

- (1) $A \subseteq B$ iff $\mu_A(x) \le \mu_B(x)$ and $\lambda_A(x) \ge \lambda_B(x)$ for all $x \in X$,
- (2) $A^c = \{(x, \lambda_A(x), \mu_A(x)) \mid x \in X\},\$

- (3) $A \cap B = \{(x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\lambda_A(x), \lambda_B(x)\}) \mid x \in X\},\$
- (4) $A \cup B = \{(x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\lambda_A(x), \lambda_B(x)\}) \mid x \in X\},\$

(5)
$$\Box A = \{(x, \mu_A(x), \mu_A^c(x)) \mid x \in X\},\$$

(6) $\Diamond A = \{(x, \lambda_A^c(x), \lambda_A(x)) \mid x \in X\}.$

2 Intuitionistic fuzzy sub-hyperquasigroups

For the sake of simplicity, we shall use the symbol $A = (\mu_A, \lambda_A)$ for the intuitionistic fuzzy set $A = \{(x, \mu_A(x), \lambda_A(x) \mid x \in X)\}$.

In what follows, let G denote a hyperquasigroup, and we start by defining the notion of intuitionistic fuzzy sub-hyperquasigroups.

Based on [16], we can extend the concept of the intuitionistic fuzzy subquasigroup to the concept of intuitionistic fuzzy sub-hyperquasigroups in the following way:

Definition 2.1. An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in G is called an *intuitionistic fuzzy sub-hyperquasigroup* of G (IFSH of G for short) if

- (1) $\min\{\mu_A(x), \mu_A(y)\} \le \inf\{\mu_A(z) \mid z \in x \circ y\}$ for all $x, y \in G$,
- (2) for all $x, a \in G$ there exist $y, z \in G$ such that $x \in (a \circ y) \cap (z \circ a)$ and

$$\min\{\mu_A(a), \mu_A(x)\} \le \min\{\mu_A(y), \mu_A(z)\},\$$

- (3) $\sup\{\lambda_A(z) \mid z \in x \circ y\} \le \max\{\lambda_A(x), \lambda_A(y)\}$ for all $x, y \in G$,
- (4) for all $x, a \in G$ there exist $y, z \in G$ such that $x \in (a \circ y) \cap (z \circ a)$ and

$$\max\{\lambda_A(y), \lambda_A(z)\} \le \max\{\lambda_A(a), \lambda_A(x)\}.$$

Lemma 2.2. If $A = (\mu_A, \lambda_A)$ is an IFSH of G, then so is $\Box A = (\mu_A, \mu_A^c)$.

Proof. It is sufficient to show that μ_A^c satisfies the third and fourth conditions of Definition 2.1. For $x, y \in G$ we have

$$\min\{\mu_A(x), \mu_A(y)\} \le \inf\{\mu_A(z) \mid z \in x \circ y\}$$

and so

$$\min\{1 - \mu_A^c(x), 1 - \mu_A^c(y)\} \le \inf\{1 - \mu_A^c(z) \mid z \in x \circ y\}.$$

Hence

$$\min\{1 - \mu_A^c(x), 1 - \mu_A^c(y)\} \le 1 - \sup\{\mu_A^c(z) \mid z \in x \circ y\}$$

which implies

$$\sup\{\mu_A^c(z) \mid z \in x \circ y\} \le 1 - \min\{1 - \mu_A^c(x), 1 - \mu_A^c(y)\}.$$

Therefore

$$\sup\{\mu_A^c(z) \mid z \in x \circ y\} \le \max\{\mu_A^c(x), \mu_A^c(y)\}.$$

Hence the third condition of Definition 2.1 is verified.

Now, let $a, x \in G$. Then there exist $y, z \in G$ such that $x \in a \circ y, x \in z \circ a$ and

$$\min\{\mu_A(a), \mu_A(x)\} \le \min\{\mu_A(y), \mu_A(z)\}.$$

 So

$$\min\{1 - \mu_A^c(a), 1 - \mu_A^c(x)\} \le \min\{1 - \mu_A^c(y), 1 - \mu_A^c(z)\}.$$

Hence

$$\max\{\mu_{A}^{c}(y), \mu_{A}^{c}(z)\} \le \max\{\mu_{A}^{c}(a), \mu_{A}^{c}(x)\},\$$

and the fourth condition of Definition 2.1 is satisfied.

Lemma 2.3. If $A = (\mu_A, \lambda_A)$ is an IFSH of G, then so is $\Diamond A = (\lambda_A^c, \lambda_A)$.

Proof. The proof is similar to the proof of Lemma 2.2.

Combining the above two lemmas it is not difficult to see that the following theorem is valid.

Theorem 2.4. $A = (\mu_A, \lambda_A)$ is an IFSH of G if and only if $\Box A$ and $\Diamond A$ are IFSHs of G.

Corollary 2.5. $A = (\mu_A, \lambda_A)$ is an IFSH of G if and only if μ_A and λ_A^c are fuzzy sub-hyperquasigroups of G.

Theorem 2.6. If $A = (\mu_A, \lambda_A)$ is an IFSH of G then the upper t-level cut $U(\mu_A; t)$ of μ_A and the lower t-level cut $L(\lambda_A; t)$ of λ_A are sub-hyperquasigroups of G for every $t \in Im(\mu_A) \cap Im(\lambda_A)$.

Proof. Let $t \in Im(\mu_A) \cap Im(\lambda_A) \subseteq [0,1]$ and let $x, y \in U(\mu_A; t)$. Then $\mu_A(x) \ge t$ and $\mu_A(y) \ge t$ and so $\min\{\mu_A(x), \mu_A(y)\} \ge t$. It follows from the first condition of Definition 2.1 that $\inf\{\mu_A(z) \mid z \in x \circ y\} \ge t$. Therefore for all $z \in x \circ y$ we have $z \in U(\mu_A; t)$, so $x \circ y \subseteq U(\mu_A; t)$. Hence for all $a \in U(\mu_A; t)$ we have $a \circ U(\mu_A; t) \subseteq U(\mu_A; t)$ and $U(\mu_A; t) \circ a \subseteq U(\mu_A; t)$. Now, let $x \in U(\mu_A; t)$ then there exist $y, z \in G$ such that $x \in a \circ y, x \in z \circ a$ and $\min\{\mu_A(x), \mu_A(a)\} \le \min\{\mu(y), \mu(z)\}$. Since $x, a \in U(\mu_A; t)$, we have $t \le \min\{\mu_A(x), \mu_A(a)\}$ and so $t \le \min\{\mu_A(y), \mu_A(z)\}$ which implies $y \in U(\mu_A; t), z \in U(\mu_A; t)$ and these prove that $U(\mu_A; t) \subseteq a \circ U(\mu_A; t)$ and $U(\mu_A; t) \subseteq U(\mu_A; t) \circ a$.

Now let $x, y \in L(\lambda_A; t)$. Then $\lambda_A(x) \leq t$, $\lambda_A(y) \leq t$ and, consequently, $\max\{\lambda_A(x), \lambda_A(y)\} \leq t$. It follows from the third condition of Definition

2.1 that $\sup\{\lambda_A(z) \mid z \in x \circ y\} \leq t$. Therefore for all $z \in x \circ y$ we have $z \in L(\lambda_A; t)$, so $x \circ y \subseteq L(\lambda_A; t)$. Hence for all $a \in L(\lambda_A; t)$ we have $a \circ L(\lambda_A; t) \subseteq L(\lambda_A; t)$ and $L(\lambda_A; t) \circ a \subseteq L(\lambda_A; t)$. Now, let $x \in L(\lambda_A; t)$. Then there exist $y, z \in G$ such that $x \in a \circ y, x \in z \circ a$ and $\max\{\lambda_A(y), \lambda_A(z)\} \leq \max\{\lambda(a), \lambda(x)\}$. Since $x, a \in L(\lambda_A; t)$, we have $\max\{\lambda_A(a), \lambda_A(x)\} \leq t$ and so $\max\{\lambda_A(y), \lambda_A(z)\} \leq t$ which implies $y \in L(\lambda_A; t), z \in L(\lambda_A; t)$ and these prove that $L(\lambda_A; t) \subseteq a \circ L(\lambda_A; t)$ and $L(\lambda_A; t) \subseteq L(\lambda_A; t) \circ a$. Thus $a \circ L(\lambda_A; t) = L(\lambda_A; t) = L(\lambda_A; t) \circ a$.

Theorem 2.7. If $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy set in G such that the non-empty sets $U(\mu_A; t)$ and $L(\lambda_A; t)$ are sub-hyperquasigroups of G for all $t \in [0, 1]$, then $A = (\mu_A, \lambda_A)$ is an IFSH of G.

Proof. For $t \in [0,1]$, assume that $U(\mu_A;t) \neq \emptyset$ and $L(\lambda_A;t) \neq \emptyset$ are subhyperquasigroups of G. We must show that $A = (\mu_A, \lambda_A)$ satisfies the all conditions in Definition 2.1. Let $x, y \in G$, we put $t_0 = \min\{\mu_A(x), \mu_A(y)\}$ and $t_1 = \max\{\lambda_A(x), \lambda_A(y)\}$. Then $x, y \in U(\mu_A; t_0)$ and $x, y \in L(\lambda_A; t_1)$. So $x \circ y \subseteq U(\mu_A; t_0)$ and $x \circ y \subseteq L(\lambda_A; t_1)$. Therefore for all $z \in x \circ y$ we have $\mu_A(z) \ge t_0$ and $\lambda_A(z) \le t_1$ which imply

$$\inf\{\mu_A(z) \mid z \in x \circ y\} \ge \min\{\mu_A(x), \mu_A(y)\}$$

and

$$\sup\{\lambda_A(z) \mid z \in x \circ y\} \le \max\{\lambda_A(x), \lambda_A(y)\}\$$

The conditions (1) and (3) of Definition 2.1 are verified.

Now, let $x, a \in G$. If $t_2 = \min\{\mu_A(a), \mu_A(x)\}$, then $a, x \in U(\mu_A; t_2)$. So there exist $y_1, z_1 \in U(\mu_A; t_2)$ such that $x \in a \circ y_1$ and $x \in z_1 \circ a$. Also we have $t_2 \leq \min\{\mu_A(y_1), \mu_A(z_1)\}$. Therefore the condition (2) of Definition 2.1 is verified. If we put $t_3 = \max\{\lambda_A(a), \lambda_A(x)\}$, then $a, x \in L(\lambda_A; t_3)$. So there exist $y_2, z_2 \in L(\lambda_A; t_3)$ such that $x \in a \circ y_2$ and $x \in z_2 \circ a$ and we have $\max\{\lambda_A(y_2), \lambda_A(y_2)\} \leq t_3$, and so the condition (4) of Definition 2.1 is verified. This completes the proof.

Corollary 2.8. Let K be a sub-hyperquasigroup of a hyperquasigroup (G, \circ) . If fuzzy sets μ and λ are defined on G by

$$\mu(x) = \begin{cases} \alpha_0 & \text{if } x \in K, \\ \alpha_1 & \text{if } x \in G \setminus K, \end{cases} \qquad \lambda(x) = \begin{cases} \beta_0 & \text{if } x \in K, \\ \beta_1 & \text{if } x \in G \setminus K, \end{cases}$$

where $0 \leq \alpha_1 < \alpha_0$, $0 \leq \beta_0 < \beta_1$ and $\alpha_i + \beta_i \leq 1$ for i = 0, 1, then $A = (\mu, \lambda)$ is an IFSH of G and $U(\mu; \alpha_0) = K = L(\lambda; \beta_0)$.

Corollary 2.9. Let χ_K be the characteristic function of a sub-hyperquasigroup K of (G, \circ) . Then $K = (\chi_K, \chi_K^c)$ is an IFSH of G.

Theorem 2.10. If $A = (\mu_A, \lambda_A)$ is an IFSH of G, then for all $x \in G$ we have

$$\mu_A(x) = \sup\{\alpha \in [0,1] \mid x \in U(\mu_A;\alpha)\}$$

and

$$\lambda_A(x) = \inf\{\alpha \in [0,1] \mid x \in L(\lambda_A;\alpha)\}.$$

Proof. Let $\delta = \sup\{\alpha \in [0,1] \mid x \in U(\mu_A; \alpha)\}$ and let $\varepsilon > 0$ be given. Then $\delta - \varepsilon < \alpha$ for some $\alpha \in [0,1]$ such that $x \in U(\mu_A; \alpha)$. This means that $\delta - \varepsilon < \mu_A(x)$ so that $\delta \le \mu_A(x)$ since ε is arbitrary.

We now show that $\mu_A(x) \leq \delta$. If $\mu_A(x) = \beta$, then $x \in U(\mu_A; \beta)$ and so

$$\beta \in \{\alpha \in [0,1] \mid x \in U(\mu_A; \alpha)\}.$$

Hence

$$\mu_A(x) = \beta \le \sup\{\alpha \in [0,1] \mid x \in U(\mu_A;\alpha)\} = \delta.$$

Therefore

$$\mu_A(x) = \delta = \sup\{\alpha \in [0,1] \mid x \in U(\mu_A;\alpha)\}.$$

Now let $\eta = \inf \{ \alpha \in [0, 1] \mid x \in L(\lambda_A; \alpha) \}$. Then

$$\inf\{\alpha \in [0,1] \mid x \in L(\lambda_A;\alpha)\} < \eta + \varepsilon$$

for any $\varepsilon > 0$, and so $\alpha < \eta + \varepsilon$ for some $\alpha \in [0,1]$ with $x \in L(\lambda_A; \alpha)$. Since $\lambda_A(x) \leq \alpha$ and ε is arbitrary, it follows that $\lambda_A(x) \leq \eta$.

To prove $\lambda_A(x) \geq \eta$, let $\lambda_A(x) = \zeta$. Then $x \in L(\lambda_A; \zeta)$ and thus $\zeta \in \{\alpha \in [0, 1] \mid x \in L(\lambda_A; \alpha)\}$. Hence

$$\inf\{\alpha \in [0,1] \mid x \in L(\lambda_A; \alpha)\} \le \zeta,$$

i.e. $\eta \leq \zeta = \lambda_A(x)$. Consequently

$$\lambda_A(x) = \eta = \inf\{\alpha \in [0,1] \mid x \in L(\lambda_A; \alpha)\},\$$

which completes the proof.

Theorem 2.11. Let Ω be a non-empty finite subset of [0,1]. If $\{K_{\alpha} \mid \alpha \in \Omega\}$ is a collection of sub-hyperquasigroups of G such that

(i) G = ⋃_{α∈Ω} K_α,
(ii) α > β ⇔ K_α ⊂ K_β for all α, β ∈ Ω,

then an intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ defined on G by

 $\mu_A(x) = \sup\{\alpha \in \Omega \mid x \in K_\alpha\} \quad and \quad \lambda_A(x) = \inf\{\alpha \in \Omega \mid x \in K_\alpha\}$ is an IFSH of G.

Proof. According to Theorem 2.7, it is sufficient to show that the non-empty sets $U(\mu_A; \alpha)$ and $L(\lambda_A; \beta)$ are sub-hyperquasigroups of G. We show that $U(\mu_A; \alpha) = K_{\alpha}$. This holds, since

$$x \in U(\mu_A; \alpha) \iff \mu_A(x) \ge \alpha$$
$$\iff \sup\{\gamma \in \Omega \mid x \in K_\gamma\} \ge \alpha$$
$$\iff \exists \gamma_0 \in \Omega, \ x \in K_{\gamma_0}, \ \gamma_0 \ge \alpha$$
$$\iff x \in K_\alpha \quad (\text{since } K_{\gamma_0} \subseteq K_\alpha).$$

Now, we prove that $L(\lambda; \beta) \neq \emptyset$ is a sub-hyperquasigroup of G. We have

$$x \in L(\lambda_A; \beta) \iff \lambda_A(x) \le \beta$$
$$\iff \inf\{\gamma \in \Omega \mid x \in K_\gamma\} \le \beta$$
$$\iff \exists \gamma_0 \in \Omega, \ x \in K_{\gamma_0}, \ \gamma_0 \le \beta$$
$$\iff x \in \bigcup_{\gamma \le \beta} K_\gamma$$

and hence $L(\lambda_A; \beta) = \bigcup_{\gamma \leq \beta} K_{\gamma}$. It is not difficult to see that the union of any family of increasing sub-hyperquasigroups of a given hyperquasigroup is a sub-hyperquasigroup. This completes the proof.

3 Relations

Let $\alpha \in [0,1]$ be fixed and let IFSH(G) be the family of all intuitionistic fuzzy sub-hyperquasigroups of a hyperquasigroup G. For any $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ from IFSH(G) we define two binary relations \mathfrak{U}^{α} and \mathfrak{L}^{α} on IFSH(G) as follows:

$$(A, B) \in \mathfrak{U}^{\alpha} \iff U(\mu_A; \alpha) = U(\mu_B; \alpha)$$

and

$$(A, B) \in \mathfrak{L}^{\alpha} \iff L(\lambda_A; \alpha) = L(\lambda_B; \alpha).$$

These two relations \mathfrak{U}^{α} and \mathfrak{L}^{α} are equivalence relations. Hence IFSH(G)can be divided into the equivalence classes of \mathfrak{U}^{α} and \mathfrak{L}^{α} , denoted by $[A]_{\mathfrak{U}^{\alpha}}$ and $[A]_{\mathfrak{L}^{\alpha}}$ for any $A = (\mu_A, \lambda_A) \in IFSH(G)$, respectively. The corresponding quotient sets will be denoted by $IFSH(G)/\mathfrak{U}^{\alpha}$ and $IFSH(G)/\mathfrak{L}^{\alpha}$, respectively.

For the family S(G) of all sub-hyperquasigroups of G we define two maps U_{α} and L_{α} from IFSH(G) to $S(G) \cup \{\emptyset\}$ by putting

$$U_{\alpha}(A) = U(\mu_A; \alpha)$$
 and $L_{\alpha}(A) = L(\lambda_A; \alpha)$

for each $A = (\mu_A, \lambda_A) \in IFSH(G)$.

It is not difficult to see that these maps are well-defined.

Lemma 3.1. For any $\alpha \in (0,1)$ the maps U_{α} and L_{α} are surjective.

Proof. Let **0** and **1** be fuzzy sets in *G* defined by $\mathbf{0}(x) = 0$ and $\mathbf{1}(x) = 1$ for all $x \in G$. Then $\mathbf{0}_{\sim} = (\mathbf{0}, \mathbf{1}) \in IFSH(G)$ and $U_{\alpha}(\mathbf{0}_{\sim}) = L_{\alpha}(\mathbf{0}_{\sim}) = \emptyset$ for any $\alpha \in (0, 1)$. Moreover for any $K \in S(G)$ we have $K_{\sim} = (\chi_{K}, \chi_{K}^{c}) \in IFSH(G), U_{\alpha}(K_{\sim}) = U(\chi_{K}; \alpha) = K$ and $L_{\alpha}(K_{\sim}) = L(\chi_{K}^{c}; \alpha) = K$. Hence U_{α} and L_{α} are surjective.

Theorem 3.2. For any $\alpha \in (0,1)$ the sets $IFSH(G)/\mathfrak{U}^{\alpha}$ and $IFSH(G)/\mathfrak{L}^{\alpha}$ are equipotent to $S(G) \cup \{\emptyset\}$.

Proof. Let $\alpha \in (0,1)$. Putting $U^*_{\alpha}([A]_{\mathfrak{U}^{\alpha}}) = U_{\alpha}(A)$ and $L^*_{\alpha}([A]_{\mathfrak{L}^{\alpha}}) = L_{\alpha}(A)$ for any $A = (\mu_A, \lambda_A) \in IFSH(G)$, we obtain two maps

 $U^*_\alpha: IFSH(G)/\mathfrak{U}^\alpha \to S(G) \cup \{ \emptyset \} \ \, \text{and} \ \ L^*_\alpha: IFSH(G)/\mathfrak{L}^\alpha \to S(G) \cup \{ \emptyset \}.$

If $U(\mu_A; \alpha) = U(\mu_B; \alpha)$ and $L(\lambda_A; \alpha) = L(\lambda_B; \alpha)$ for some $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ from IFSH(G), then $(A, B) \in \mathfrak{U}^{\alpha}$ and $(A, B) \in \mathfrak{L}^{\alpha}$, whence $[A]_{\mathfrak{U}^{\alpha}} = [B]_{\mathfrak{U}^{\alpha}}$ and $[A]_{\mathfrak{L}^{\alpha}} = [B]_{\mathfrak{L}^{\alpha}}$, which means that $U_{*\alpha}$ and L^*_{α} are injective.

To show that the maps U_{α}^* and L_{α} are surjective, let $K \in S(G)$. Then for $K_{\sim} = (\chi_K, \chi_K^c) \in IFSH(G)$ we have $U_{\alpha}^*([K_{\sim}]_{\mathfrak{U}^{\alpha}}) = U(\chi_K; \alpha) = K$ and $L_{\alpha}^*([K_{\sim}]_{\mathfrak{L}^{\alpha}}) = L(\chi_K^c; \alpha) = K$. Also $\mathbf{0}_{\sim} = (\mathbf{0}, \mathbf{1}) \in IFSH(G)$. Moreover $U_{\alpha}^*([\mathbf{0}_{\sim}]_{\mathfrak{U}^{\alpha}}) = U(\mathbf{0}; \alpha) = \emptyset$ and $L_{\alpha}^*([\mathbf{0}_{\sim}]_{\mathfrak{L}^{\alpha}}) = L(\mathbf{1}; \alpha) = \emptyset$. Hence U_{α}^* and L_{α}^* are surjective.

Now for any $\alpha \in [0,1]$ we define a new relation \mathfrak{R}^{α} on IFSH(G) by putting:

$$(A,B) \in \mathfrak{R}^{\alpha} \Longleftrightarrow U(\mu_A;\alpha) \cap L(\lambda_A;\alpha) = U(\mu_B;\alpha) \cap L(\lambda_B;\alpha),$$

where $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$. Obviously \mathfrak{R}^{α} is an equivalence relation.

Lemma 3.3. The map $I_{\alpha}: IFSH(G) \to S(G) \cup \{\emptyset\}$ defined by

$$I_{\alpha}(A) = U(\mu_A; \alpha) \cap L(\lambda_A; \alpha),$$

where $A = (\mu_A, \lambda_A)$, is surjective for any $\alpha \in (0, 1)$.

Proof. If $\alpha \in (0,1)$ is fixed, then for $\mathbf{0}_{\sim} = (\mathbf{0},\mathbf{1}) \in IFSH(G)$ we have

$$I_{\alpha}(\mathbf{0}_{\sim}) = U(\mathbf{0}; \alpha) \cap L(\mathbf{1}; \alpha) = \emptyset$$

and for any $K \in S(G)$ there exists $K_{\sim} = (\chi_K, \chi_K^c) \in IFSH(G)$ such that $I_{\alpha}(K_{\sim}) = U(\chi_K; \alpha) \cap L(\chi_K^c; \alpha) = K.$

Theorem 3.4. For any $\alpha \in (0,1)$ the quotient set $IFSH(G)/\Re^{\alpha}$ is equipotent to $S(G) \cup \{\emptyset\}$.

Proof. Let $I_{\alpha}^* : IFSH(G)/\mathfrak{R}^{\alpha} \to S(G) \cup \{\emptyset\}$, where $\alpha \in (0,1)$, be defined by the formula:

$$I^*_{\alpha}([A]_{\mathfrak{R}^{\alpha}}) = I_{\alpha}(A)$$
 for each $[A]_{\mathfrak{R}^{\alpha}} \in IFSH(G)/\mathfrak{R}^{\alpha}$.

If $I^*_{\alpha}([A]_{\mathfrak{R}^{\alpha}}) = I^*_{\alpha}([B]_{\mathfrak{R}^{\alpha}})$ for some $[A]_{\mathfrak{R}^{\alpha}}, [B]_{\mathfrak{R}^{\alpha}} \in IFSH(G)/\mathfrak{R}^{\alpha}$, then

$$U(\mu_A; \alpha) \cap L(\lambda_A; \alpha) = U(\mu_B; \alpha) \cap L(\lambda_B; \alpha),$$

which implies $(A, B) \in \mathfrak{R}^{\alpha}$ and, in the consequence, $[A]_{\mathfrak{R}^{\alpha}} = [B]_{\mathfrak{R}^{\alpha}}$. Thus I_{α}^{*} is injective.

It is also onto because $I_{\alpha}^{*}(\mathbf{0}_{\sim}) = I_{\alpha}(\mathbf{0}_{\sim}) = \emptyset$ for $\mathbf{0}_{\sim} = (\mathbf{0}, \mathbf{1}) \in IFSH(G)$, and $I_{\alpha}^{*}(K_{\sim}) = I_{\alpha}(K) = K$ for $K \in S(G)$ and $K_{\sim} = (\chi_{K}, \chi_{K}^{c}) \in IFSH(G)$.

4 Connections with binary quasigroups

A groupoid (Q, \cdot) is called a (*binary*) quasigroup if each of the equations ax = b and ya = b has a unique solution for any $a, b \in Q$. Since a nonempty subset of Q closed with respect to this operation is not in general a quasigroup we must use the another equivalent definition of a quasigroup. A quasigroup (Q, \cdot) can be defined (cf. [20]) as an algebra $(Q, \cdot, \backslash, /)$ with three binary operation such that (Q, \cdot) is a quasigroup in the above sense and

$$x \setminus y = z \Leftrightarrow xz = y$$
 and $x/y = z \Leftrightarrow zy = x$

for all $x, y, z \in Q$. In this case a non-empty subset of Q is a subquasigroup of (Q, \cdot) (and $(Q, \cdot, \backslash, /)$) if and only if it is closed with respect to these three operations. This gives the possibility to the introduction of a good definition of intuitionistic fuzzy subquasigroups of binary quasigroups [16].

Definition 4.1. Let (Q, \cdot) be a quasigroup. An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in Q is called an *intuitionistic fuzzy subquasigroup* of Q if

- (i) $\min\{\mu_A(x), \mu_A(y)\} \le \mu_A(x * y)$
- (ii) $\lambda_A(x * y) \le \max\{\lambda_A(x), \lambda_A(y)\}$

hold for all $x, y \in Q$ and $* \in \{\cdot, \backslash, /\}$.

In this case an intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy subquasigroup of $(Q, \cdot, \backslash, /)$ if and only if all non-empty $U(\mu; t)$ and $L(\mu; t)$ are subquasigroups of $(Q, \cdot, \backslash, /)$ (cf. [16]).

A hyperquasigroup (G, \circ) is called *regular* if

$$x \in y \circ z$$
 implies $y \in x \circ z$ and $z \in y \circ x$

for all $x, y, z \in G$. Let (G, \circ) be a regular hyperquasigroup. The relation β^* is the smallest equivalence relation on G such that the quotient G/β^* , the set of all equivalence classes, is a quasigroup. β^* is called the fundamental equivalence relation on G and G/β^* is called the fundamental quasigroup. The equivalence relation β^* was introduced by Koskas [18] and studied mainly by Corsini [6] and Freni [13], [14] concerning hypergroups and Vougiouklis [23] concerning H_v -groups.

Let us denote by \mathcal{U} the set of all finite products of elements of G as follows:

 $x\beta y$ if and only if $\{x, y\} \subseteq u$ for some $u \in \mathcal{U}$.

The fundamental relation β^* is the transitive closure of the relation β (see Theorem 1.2.2 in [23]). Suppose $\beta^*(a)$ is the equivalence class containing $a \in G$. Then the product "." on G/β^* is defined as follows:

 $\beta^*(a) \cdot \beta^*(b) = \beta^*(c)$ for all $c \in \beta^*(a) \circ \beta^*(b)$.

In this case, each of the equations $\beta^*(a) \cdot \beta^*(x) = \beta^*(b)$ and $\beta^*(y) \cdot \beta^*(a) = \beta^*(b)$ has a unique solution for any $\beta^*(a), \beta^*(b) \in G/\beta^*$. The quasigroup $(G/\beta^*, \cdot, \cdot, \cdot)$ corresponds to quasigroup $(G/\beta^*, \cdot)$, where

$$\begin{array}{ll} \beta^*(x) \setminus \beta^*(y) = \beta^*(z) & \Longleftrightarrow & \beta^*(x) \cdot \beta^*(z) = \beta^*(y), \\ \beta^*(x) / \beta^*(y) = \beta^*(z) & \Longleftrightarrow & \beta^*(z) \cdot \beta^*(y) = \beta^*(x). \end{array}$$

Let μ be a fuzzy set in G. The fuzzy set μ_{β^*} in G/β^* is defined as follows:

$$\mu_{\beta^*}: G/\beta^* \to [0,1], \quad \beta^*(x) \mapsto \sup\{\mu(a) \mid a \in \beta^*(x)\}.$$

Now, we have

Theorem 4.2. Let G be a regular hyperquasigroup and $A = (\mu_A, \lambda_A)$ an intuitionistic fuzzy sub-hyperquasigroup of G. Then $A/\beta^* = (\mu_{\beta^*}, \lambda_{\beta^*})$ is an intuitionistic fuzzy subquasigroup of the fundamental quasigroup G/β^* .

Acknowledgements. The authors are highly grateful to the referees for their valuable comments and suggestions for improving the paper.

References

- K. T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1986), 87 - 96.
- K. T. Atanassov, New operations defined over the intuitionistic fuzzy sets, Fuzzy Sets and Systems 61 (1994), 137 - 142.
- [3] K. T. Atanassov, Intuitionistic fuzzy sets. Theory and applications, Studies in Fuzziness and Soft Computing, 35. Heidelberg; Physica-Verlag 1999.

- [4] P. Burillo and H. Bustince, Vague sets are intuitionistic fuzzy sets, Fuzzy Sets and Systems 79 (1996), 403–405.
- [5] P. Corsini and V. Leoreanu, Applications of hyperstructures theory, Advanced in Mathematics, Kluwer Academic Publishers, 2003.
- [6] P. Corsini, Prolegomena of hypergroup theory, Second Edition, Aviani Editor, 1993.
- [7] B. Davvaz, T_H and S_H -interval valued fuzzy subhypergroups, Indian J. Pure Appl. Math. (to appear).
- [8] B. Davvaz, *Fuzzy H_v-groups*, Fuzzy Sets and Systems **101** (1999), 191-195.
- [9] B. Davvaz, Product of fuzzy H_v -subgroups, J. Fuzzy Math. 8(1) (2000), 43-51.
- [10] B. Davvaz, Interval-valued fuzzy subhypergroups, Korean J. Comput. Appl. Math. 6(1) (1999), 197 - 202.
- [11] S. K. De, R. Biswas and A. R. Roy, An application of intuitionistic fuzzy sets in medical diagnosis, Fuzzy Sets and Systems 117 (2001), 209–213.
- [12] L. Dengfeng and C. Chuntian, New similarity measures of intuitionistic fuzzy sets and application to pattern recognitions, Pattern Recognition Letters 23 (2002), 221–225.
- [13] D. Freni, Una nota sul cuore di un ipergruppo e sulla chiusura transitive β^* di β , Rivista Mat. Pura Appl. 8 (1991), 153 156.
- [14] D. Freni, A new characterization of the derived hypergroup via strongly regular equivalences, Commun. Algebra 30 (2002), 3977 – 3989.
- [15] W. L. Gau and D. J. Buehrer, Vague sets, IEEE Trans. Systems Man Cybernet 23 (1993), 610–614.
- [16] K. H. Kim, W. A. Dudek and Y. B. Jun, On intuitionistic fuzzy subquasigroups of quasigroups, Quasigroups and Related Systems 7 (2000), 15-28.
- [17] K. H. Kim and Y. B. Jun, Intuitionistic fuzzy ideals of semigroups, Indian J. Pure Appl. Math. 33(4) (2002), 443 – 449.
- [18] M. Koskas, Groupoids, demi-hypergroupes et hypergroupes, J. Math. Pure Appl. 49 (1970), no. 9, 155 - 192.
- [19] F. Marty, Sur une généralization de la notion de group, 8th Congress Math. Scandenaves, Stockholm 1934, 45 – 49.

- [20] H. Pflugfelder, Quasigroups and loops. Introduction, Helderman-Verlag 1990.
- [21] A. Rosenfeld, Fuzzy groups, J. Math. Anal. Appl. 35 (1971), 512-517.
- [22] E. Szmidt and J. Kacprzyk, Entropy for intuitionistic fuzzy sets, Fuzzy Sets and Systems 118 (2001), 467–477.
- [23] T. Vougiouklis, Hyperstructures and their representations, Hadronic Press, Inc, 115, Palm Harber, USA 1994.
- [24] L. A. Zadeh, *Fuzzy sets*, Inform. Control 8 (1965), 338 353.