Intuitionistic fuzzy H_v -submodules

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Abstract

After the introduction of fuzzy sets by Zadeh, there have been a number of generalizations of this fundamental concept. The notion of intuitionistic fuzzy sets introduced by Atanassov is one among them. In this paper, we apply the concept of an intuitionistic fuzzy set to H_v -modules. The notion of an intuitionistic fuzzy H_v -submodule of an H_v -module is introduced, and some related properties are investigated. Characterizations of intuitionistic fuzzy H_v -submodules are given.

2000 Mathematics Subject Classification: 16D99, 20N20, 20N25. *Keywords:* H_v -semigroup, H_v -group, H_v -ring, H_v -module, intuitionistic fuzzy H_v -submodule, sup property.

1 Introduction

The concept of hyperstructure was introduced in 1934 by Marty [14] at the 8th congress of Scandinavian Mathematicians. Hyperstructures have many applications to several branches of both pure and applied sciences [4, 5]. Vougiouklis [19] introduced a new class of hyperstructures so-called H_v -structure, and Davvaz [9] surveyed the theory of H_v -structures. After the introduction of fuzzy sets by Zadeh [21], there have been a number of generalizations of this fundamental concept. The notion of intuitionistic fuzzy sets introduced by Atanassov [1] is one among them. In [3], Biswas applied the concept of intuitionistic fuzzy sets to the theory of groups and studied intuitionistic fuzzy subgroups of a group. In [12], Kim, Dudek and

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Jun introduced the notion of intuitionistic fuzzy subquasigroups of a quasigroup. Also in [13], Kim and Jun introduced the concept of intuitionistic fuzzy ideals of a semigroup. Recently, Dudek, Davvaz and Jun [11] considered the intuitionistic fuzzification of the concept of sub-hyperquasigroups in a hyperquasigroup and investigated some properties of such hyperquasigroups. In this paper, we apply the concept of intuitionistic fuzzy sets to H_v -modules. We introduce the notion of intuitionistic fuzzy H_v -submodules of an H_v -module and investigate some related properties. We give characterizations of intuitionistic fuzzy H_v -submodules.

2 Fuzzy sets and intuitionistic fuzzy sets

The concept of a fuzzy set in a non-empty set was introduced by Zadeh [21] in 1965.

Let X be a non-empty set. A mapping $\mu : X \longrightarrow [0, 1]$ is called a *fuzzy* set in X. The complement of μ , denoted by μ^c , is the fuzzy set in X given by $\mu^c(x) = 1 - \mu(x)$ for all $x \in X$.

Definition 2.1. Let f be a mapping from a set X to a set Y. Let μ be a fuzzy set in X and λ be a fuzzy set in Y. Then the *inverse image* $f^{-1}(\lambda)$ of λ is a fuzzy set in X defined by

$$f^{-1}(\lambda)(x) = \lambda(f(x))$$
 for all $x \in X$.

The *image* $f(\mu)$ of μ is the fuzzy set in Y defined by

$$f(\mu)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \emptyset, \\ x \in f^{-1}(y) & \\ 0 & \text{otherwise,} \end{cases}$$

for all $y \in Y$. We have always

$$f(f^{-1}(\lambda)) \leq \lambda$$
 and $\mu \leq f^{-1}(f(\mu))$.

Rosenfeld [16] applied the concept of fuzzy sets to the theory of groups and defined the concept of fuzzy subgroups of a group. The concept of fuzzy modules was introduced by Negoita and Ralescu in [15].

Definition 2.2. (cf. Negoita and Ralescu [15]). Let M be a module over a ring R. A fuzzy set μ in M is called a *fuzzy submodule* of M if for every $x, y \in M$ and $r \in R$ the following conditions are satisfied:

- (1) $\mu(0) = 1$,
- (2) $\min\{\mu(x), \mu(y)\} \le \mu(x-y)$ for all $x, y \in M$,
- (3) $\mu(x) \leq \mu(rx)$ for all $x \in M$ and $r \in R$.

Definition 2.3. An *intuitionistic fuzzy set* A in a non-empty set X is an object having the form

$$A = \{ (x, \mu_A(x), \lambda_A(x)) \mid x \in X \},\$$

where the functions $\mu_A : X \longrightarrow [0, 1]$ and $\lambda_A : X \longrightarrow [0, 1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\lambda_A(x)$) of each element $x \in X$ to the set A respectively, and $0 \le \mu_A(x) +$ $\lambda_A(x) \le 1$ for all $x \in X$. For the sake of simplicity, we shall use the symbol $A = (\mu_A, \lambda_A)$ for the intuitionistic fuzzy set $A = \{(x, \mu_A(x), \lambda_A(x)) \mid x \in X\}$.

Definition 2.4. Let $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ be intuitionistic fuzzy sets in X. Then

- (1) $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x)$ and $\lambda_A(x) \geq \lambda_B(x)$ for all $x \in X$,
- (2) $A^c = \{(x, \lambda_A(x), \mu_A(x)) \mid x \in X\},\$
- (3) $A \cap B = \{(x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\lambda_A(x), \lambda_B(x)\}) \mid x \in X\},\$
- (4) $A \cup B = \{(x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\lambda_A(x), \lambda_B(x)\}) \mid x \in X\},\$

(5)
$$\Box A = \{(x, \mu_A(x), \mu_A^c(x)) \mid x \in X\}$$

(6) $\Diamond A = \{(x, \lambda_A^c(x), \lambda_A(x)) \mid x \in X\}.$

Now, we define an intuitionistic fuzzy submodule of a module.

Definition 2.5. Let M be a module over a ring R. An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in M is called an *intuitionistic fuzzy submodule* of M if

- (1) $\mu_A(0) = 1$,
- (2) $\min\{\mu_A(x), \mu_A(y)\} \le \mu_A(x-y)$ for all $x, y \in M$,
- (3) $\mu_A(x) \le \mu_A(r \cdot x)$ for all $x \in M$ and $r \in R$,
- (4) $\lambda_A(0) = 0$,
- (5) $\lambda_A(x-y) \le \max\{\lambda_A(x), \lambda_A(y)\}$ for all $x, y \in M$,
- (6) $\lambda_A(r \cdot x) \leq \lambda_A(x)$ for all $x \in M$ and $r \in R$.

3 H_v -structures

A hyperstructure is a non-empty set H together with a map $* : H \times H \to \mathcal{P}^*(H)$ which is called hyperoperation, where $\mathcal{P}^*(H)$ denotes the set of all

non-empty subsets of H. The image of pair (x, y) is denoted by x * y. If $x \in H$ and $A, B \subseteq H$, then by A * B, A * x and x * B we mean

$$A * B = \bigcup_{a \in A, b \in B} a * b, A * x = A * \{x\} \text{ and } x * B = \{x\} * B,$$

respectively. A hyperstructure (H, *) is called an H_v -semigroup if

$$(x * (y * z)) \cap ((x * y) * z) \neq \emptyset$$
 for all $x, y, z \in H$.

Definition 3.1. An H_v -ring is a system $(R, +, \cdot)$ with two hyperoperations satisfying the following axioms:

(i) (R, +) is an H_v -group, i.e.,

$$((x+y)+z) \cap (x+(y+z)) \neq \emptyset \quad \text{for all } x, y \in R, \\ a+R=R+a=R \quad \text{for all } a \in R;$$

- (ii) (R, \cdot) is an H_v -semigroup;
- (iii) " \cdot " is weak distributive with respect to "+", i.e., for all $x, y, z \in R$:

$$(x \cdot (y+z)) \cap (x \cdot y + x \cdot z) \neq \emptyset, ((x+y) \cdot z) \cap (x \cdot z + y \cdot z) \neq \emptyset.$$

Definition 3.2. (cf. Vougiouklis [20]). A non-empty set M is called an H_v -module over an H_v -ring R if (M, +) is a weak commutative H_v -group and there exists a map

$$\cdot : R \times M \longrightarrow \mathcal{P}^*(M), \, (r, x) \mapsto r \cdot x$$

such that for all $a, b \in R$ and $x, y \in M$, we have

$$\begin{array}{l} (a \cdot (x+y)) \cap (a \cdot x + a \cdot y) \neq \emptyset, \\ ((a+b) \cdot x) \cap (a \cdot x + b \cdot x) \neq \emptyset, \\ ((ab) \cdot x) \cap (a \cdot (b \cdot x)) \neq \emptyset. \end{array}$$

We note that an H_v -module is a generalization of a module. For more definitions, results and applications on H_v -modules, we refer the reader to [9, 18, 20]. Note that by using fuzzy sets, we can consider the structure of H_v -module on any ordinary module.

Example 3.3. (cf. Davvaz [6]). Let M be an ordinary module over an ordinary ring R, and let μ_A be a fuzzy set in M and μ_B be a fuzzy set in R. We define hyperoperations \circ , *, \oplus and \odot as follows:

$$a \circ b = \{t \in R \mid \mu_B(t) = \mu_B(a+b)\}$$
 for all $a, b \in R$,

$$a * b = \{t \in R \mid \mu_B(t) = \mu_B(ab)\} \text{ for all } a, b \in R,$$
$$x \oplus y = \{s \in M \mid \mu_A(s) = \mu_A(x+y)\} \text{ for all } x, y \in M,$$
$$r \odot x = \{s \in M \mid \mu_A(s) = \mu_A(r \cdot x)\} \text{ for all } r \in R \text{ and } x \in M.$$

respectively. Then

- (i) $(R, \circ, *)$ is an H_v -ring.
- (ii) (M, \oplus, \odot) is an H_v -module over the H_v -ring $(R, \circ, *)$.

Definition 3.4. Let M be an H_v -module over an H_v -ring R. A non-empty subset S of M is called an H_v -submodule of M if the following axioms hold:

- (i) (S, +) is an H_v -subgroup of (M, +),
- (ii) $R \cdot S \subseteq S$.

Definition 3.5. Let M_1 and M_2 be two H_v -modules over an H_v -ring R. A mapping f from M_1 into M_2 is called a *homomorphism* if for all $x, y \in M_1$ and $r \in R$,

$$f(x+y) = f(x) + f(y)$$
 and $f(r \cdot x) = r \cdot f(x)$.

The homomorphism f is said to be strong on the left if

$$f(z) \in f(x) + f(y) \implies \exists x' \in M_1 : f(x) = f(x') \text{ and } z \in x' + y.$$

Similarly, we can define a homomorphism which is strong on the right. If a homomorphism f is strong on the right and left, we say f is a strong homomorphism.

Proposition 3.6. Let M_1 and M_2 be two H_v -modules over an H_v -ring R and $f: M_1 \longrightarrow M_2$ a strong epimorphism. If N is an H_v -submodule of M_2 , then $f^{-1}(N)$ is an H_v -submodule of M_1 .

Proof. Assume that $x_1, x_2 \in f^{-1}(N)$. Then there exists $y_1, y_2 \in N$ such that $f(x_1) = y_1, f(x_2) = y_2$ and so $f(x_1 + x_2) = y_1 + y_2$. Hence for every $x \in x_1 + x_2$ we have $f(x) \in y_1 + y_2 \subseteq N$ which implies $x \in f^{-1}(N)$ and so $x_1 + x_2 \subseteq f^{-1}(N)$. Therefore $x_1 + f^{-1}(N) \subseteq f^{-1}(N)$ for every $x_1 \in f^{-1}(N)$. Now, we show that $f^{-1}(N) \subseteq x_1 + f^{-1}(N)$. For $z \in f^{-1}(N)$, there exists $y \in N$ such that f(z) = y. Since $y, y_1 \in N$ and N is an H_v -subgroup of M_2 , there exists $b \in N$ such that $y \in y_1 + b$. Since f is onto, there exists $a \in M_1$ such that f(a) = b or $a \in f^{-1}(b)$. Hence we have $f(z) \in f(x_1) + f(a)$. Since f is a strong homomorphism, there exists $a' \in M_1$ such that f(a) = f(a') and $z \in x_1 + a'$. Since $f(a') = b \in N$, we have $a' \in f^{-1}(N)$ and $z \in x_1 + f^{-1}(N)$, and so $f^{-1}(N) \subseteq x_1 + f^{-1}(N)$. Therefore we have $f^{-1}(N) = x_1 + f^{-1}(N)$. Similarly, we obtain $f^{-1}(N) = f^{-1}(N) + x_1$. Thus the condition (i) of

Definition 3.4 is satisfied.

For the condition (ii), let $r \in R$ and $x \in f^{-1}(N)$, then $f(x) \in N$, and so $r \cdot f(x) \subseteq N$ or $f(r \cdot x) \subseteq N$, which implies $r \cdot x \subseteq f^{-1}(N)$. Therefore the condition (ii) of Definition 3.4 is satisfied.

In [7], Davvaz applied the concept of fuzzy sets to the algebraic hyperstructures. In particular, he defined the concept of a fuzzy H_v -submodule of an H_v -module which is a generalization of the concept of a fuzzy submodule (see [6]), and he studied further properties in [8], [9] and [10].

Definition 3.7. (cf. Davvaz [6]). Let M be an H_v -module over an H_v -ring R and μ a fuzzy set in M. Then μ is said to be a fuzzy H_v -submodule of M if the following axioms hold:

- (1) $\min\{\mu(x), \mu(y)\} \le \inf\{\mu(z) \mid z \in x+y\} \quad \text{for all } x, y \in M,$
- (2) for all $x, a \in M$ there exists $y \in M$ such that $x \in a + y$ and

$$\min\{\mu(a), \mu(x)\} \le \mu(y)$$

(3) for all $x, a \in M$ there exists $z \in M$ such that $x \in z + a$ and

$$\min\{\mu(a), \mu(x)\} \le \mu(z),$$

(4) $\mu(x) \leq \inf\{\mu(z) \mid z \in r \cdot x\}$ for all $x \in M$ and $r \in R$.

4 Intuitionistic fuzzy H_v -submodules

In what follows, let M denote an H_v -module over an H_v -ring R unless otherwise specified. We start by defining the notion of intuitionistic fuzzy H_v -submodules.

Definition 4.1. An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in M is called an *intuitionistic fuzzy* H_v -submodule of M if

- (1) $\min\{\mu_A(x), \mu_A(y)\} \le \inf\{\mu_A(z) \mid z \in x + y\}$ for all $x, y \in M$,
- (2) for all $x, a \in M$ there exist $y, z \in M$ such that $x \in (a + y) \cap (z + a)$ and

 $\min\{\mu_A(a), \mu_A(x)\} \le \min\{\mu_A(y), \mu_A(z)\},\$

- (3) $\mu_A(x) \leq \inf\{\mu_A(z) \mid z \in r \cdot x\}$ for all $x \in M$ and $r \in R$,
- (4) $\sup\{\lambda_A(z) \mid z \in x+y\} \le \max\{\lambda_A(x), \lambda_A(y)\}$ for all $x, y \in M$,
- (5) for all $x, a \in M$ there exist $y, z \in M$ such that $x \in (a + y) \cap (z + a)$ and

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\max\{\lambda_A(y), \lambda_A(z)\} \le \max\{\lambda_A(a), \lambda_A(x)\},\
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(6) $\sup\{\lambda_A(z) \mid z \in r \cdot x\} \leq \lambda_A(x)$ for all $x \in M$ and $r \in R$.

Lemma 4.2. If $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy H_v -submodule of M, then so is $\Box A = (\mu_A, \mu_A^c)$.

Proof. It is sufficient to show that μ_A^c satisfies the conditions (4),(5), and (6) of Definition 4.1. For $x, y \in M$ we have

$$\min\{\mu_A(x), \mu_A(y)\} \le \inf\{\mu_A(z) \mid z \in x+y\}$$

and so $\min\{1 - \mu_A^c(x), 1 - \mu_A^c(y)\} \le \inf\{1 - \mu_A^c(z) \mid z \in x + y\}$. Hence

$$\min\{1 - \mu_A^c(x), 1 - \mu_A^c(y)\} \le 1 - \sup\{\mu_A^c(z) \mid z \in x + y\}$$

which implies $\sup\{\mu_A^c(z) \mid z \in x+y\} \le 1 - \min\{1 - \mu_A^c(x), 1 - \mu_A^c(y)\}$. Therefore

$$\sup\{\mu_{A}^{c}(z) \mid z \in x+y\} \le \max\{\mu_{A}^{c}(x), \mu_{A}^{c}(y)\},\$$

and thus the condition (4) of Definition 4.1 is valid.

Now, let $a, x \in M$. Then there exist $y, z \in M$ such that $x \in (a+y) \cap (z+a)$ and $\min\{\mu_A(a), \mu_A(x)\} \leq \min\{\mu_A(y), \mu_A(z)\}$. It follows that

$$\min\{1 - \mu_A^c(a), 1 - \mu_A^c(x)\} \le \min\{1 - \mu_A^c(y), 1 - \mu_A^c(z)\}\$$

so that

$$\max\{\mu_{A}^{c}(y), \mu_{A}^{c}(z)\} \le \max\{\mu_{A}^{c}(a), \mu_{A}^{c}(x)\}.$$

Hence the condition (5) of Definition 4.1 is satisfied.

For the condition (6), let $x \in M$ and $r \in R$. Since μ_A is a fuzzy H_v -submodule of M, we have

$$\mu_A(x) \le \inf\{\mu_A(z) \mid z \in r \cdot x\}$$

and so

$$1 - \mu_A^c(x) \le \inf\{1 - \mu_A^c(z) \mid z \in r \cdot x\},\$$

which implies

$$\sup\{\mu_A^c(z) \mid z \in r \cdot x\} \le \mu_A^c(x).$$

Therefore the condition (6) of Definition 4.1 is satisfied.

Lemma 4.3. If $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy H_v -submodule of M, then so is $\Diamond A = (\lambda_A^c, \lambda_A)$.

Proof. The proof is similar to the proof of Lemma 4.2. \Box

Combining the above two lemmas it is not difficult to verify that the following theorem is valid.

Theorem 4.4. $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy H_v -submodule of M if and only if $\Box A$ and $\Diamond A$ are intuitionistic fuzzy H_v -submodules of M. \Box

Corollary 4.5. $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy H_v -submodule of M if and only if μ_A and λ_A^c are fuzzy H_v -submodules of M.

Definition 4.6. For any $t \in [0, 1]$ and fuzzy set μ in M, the set

$$U(\mu; t) = \{x \in M \mid \mu(x) \ge t\} \text{ (resp. } L(\mu; t) = \{x \in M \mid \mu(x) \le t\})$$

is called an *upper* (resp. *lower*) *t*-*level* cut of μ .

Theorem 4.7. If $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy H_v -submodule of M, then the sets $U(\mu_A; t)$ and $L(\lambda_A; t)$ are H_v -submodules of M for every $t \in Im(\mu_A) \cap Im(\lambda_A)$.

Proof. Let $t \in Im(\mu_A) \cap Im(\lambda_A) \subseteq [0,1]$ and let $x, y \in U(\mu_A; t)$. Then $\mu_A(x) \ge t$ and $\mu_A(y) \ge t$ and so $\min\{\mu_A(x), \mu_A(y)\} \ge t$. It follows from the condition (1) of Definition 4.1 that $\inf\{\mu_A(z) \mid z \in x + y\} \ge t$. Therefore $z \in U(\mu_A; t)$ for all $z \in x + y$, and so $x + y \subseteq U(\mu_A; t)$. Hence $a + U(\mu_A; t) \subseteq U(\mu_A; t)$ and $U(\mu_A; t) + a \subseteq U(\mu_A; t)$ for all $a \in U(\mu_A; t)$. Now, let $x \in U(\mu_A; t)$. Then there exist $y, z \in M$ such that $x \in (a + y) \cap (z + a)$ and $\min\{\mu_A(x), \mu_A(a)\} \le \min\{\mu_A(y), \mu_A(z)\}$. Since $x, a \in U(\mu_A; t)$, we have $t \le \min\{\mu_A(x), \mu_A(a)\}$ and so $t \le \min\{\mu_A(y), \mu_A(z)\}$, which implies $y \in U(\mu_A; t)$ and $z \in U(\mu_A; t)$. This proves that $U(\mu_A; t) \subseteq a + U(\mu_A; t)$ and $U(\mu_A; t) + a$.

Now, for every $r \in R$ and $x \in U(\mu_A; t)$ we show that $r \cdot x \subseteq U(\mu_A; t)$. Since A is an intuitionistic fuzzy H_v -submodule of M, we have

$$t \le \mu_A(x) \le \inf\{\mu_A(z) \mid z \in r \cdot x\}.$$

Therefore, for every $z \in r \cdot x$ we get $\mu_A(z) \ge t$ which implies $z \in U(\mu_A; t)$, so $r \cdot x \subseteq U(\mu_A; t)$.

If $x, y \in L(\lambda_A; t)$, then $\max\{\lambda_A(x), \lambda_A(y)\} \leq t$. It follows from the condition (4) of Definition 4.1 that $\sup\{\lambda_A(z) \mid z \in x + y\} \leq t$. Therefore for all $z \in x + y$ we have $z \in L(\lambda_A; t)$, so $x + y \subseteq L(\lambda_A; t)$. Hence for all $a \in L(\lambda_A; t)$ we have $a + L(\lambda_A; t) \subseteq L(\lambda_A; t)$ and $L(\lambda_A; t) + a \subseteq L(\lambda_A; t)$. Now, let $x \in L(\lambda_A; t)$. Then there exist $y, z \in M$ such that $x \in (a + y) \cap (z + a)$ and $\max\{\lambda_A(a), \lambda_A(z)\} \leq \max\{\lambda_A(a), \lambda_A(x)\}$. Since $x, a \in L(\lambda_A; t)$, we have $\max\{\lambda_A(a), \lambda_A(x)\} \leq t$ and so $\max\{\lambda_A(y), \lambda_A(z)\} \leq t$. Thus $y \in L(\lambda_A; t) + a$. and $z \in L(\lambda_A; t)$. Hence $L(\lambda_A; t) \subseteq a + L(\lambda_A; t)$ and $L(\lambda_A; t) \subseteq L(\lambda_A; t) + a$.

Now, we show that $r \cdot x \subseteq L(\lambda_A; t)$ for every $r \in R$ and $x \in L(\lambda_A; t)$. Since A is an intuitionistic fuzzy H_v -submodule of M, we have

$$\sup\{\lambda_A(z) \mid z \in r \cdot x\} \le \lambda_A(x) \le t.$$

Therefore, for every $z \in r \cdot x$ we get $\lambda_A(z) \leq t$, which implies $z \in L(\lambda_A; t)$, so $r \cdot x \subseteq L(\lambda_A; t)$.

Theorem 4.8. If $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy set in M such that all non-empty level sets $U(\mu_A; t)$ and $L(\lambda_A; t)$ are H_v -submodules of M, then $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy H_v -submodule of M.

Proof. Assume that all non-empty level sets $U(\mu_A; t)$ and $L(\lambda_A; t)$ are H_v -submodules of M. If $t_0 = \min\{\mu_A(x), \mu_A(y)\}$ and $t_1 = \max\{\lambda_A(x), \lambda_A(y)\}$ for $x, y \in M$, then $x, y \in U(\mu_A; t_0)$ and $x, y \in L(\lambda_A; t_1)$. So $x+y \subseteq U(\mu_A; t_0)$ and $x + y \subseteq L(\lambda_A; t_1)$. Therefore for all $z \in x + y$ we have $\mu_A(z) \ge t_0$ and $\lambda_A(z) \le t_1$, i.e.,

$$\inf\{\mu_A(z) \mid z \in x + y\} \ge \min\{\mu_A(x), \mu_A(y)\}$$

and

$$\sup\{\lambda_A(z) \mid z \in x+y\} \le \max\{\lambda_A(x), \lambda_A(y)\},\$$

which verify the conditions (1) and (4) of Definition 4.1.

Now, if $t_2 = \min\{\mu_A(a), \mu_A(x)\}$ for $x, a \in M$, then $a, x \in U(\mu_A; t_2)$. So there exist $y_1, z_1 \in U(\mu_A; t_2)$ such that $x \in a + y_1$ and $x \in z_1 + a$. Also we have $t_2 \leq \min\{\mu_A(y_1), \mu_A(z_1)\}$. Therefore the condition (2) of Definition 4.1 is verified. If we put $t_3 = \max\{\lambda_A(a), \lambda_A(x)\}$ then $a, x \in L(\lambda_A; t_3)$. So there exist $y_2, z_2 \in L(\lambda_A; t_3)$ such that $x \in a + y_2$ and $x \in z_2 + a$ and we have $\max\{\lambda_A(y_2), \lambda_A(y_2)\} \leq t_3$, and so the condition (5) of Definition 4.1 is verified.

Now, we verify the conditions (3) and (6). Let $t_4 = \mu_A(x)$ and $t_5 = \lambda_A(x)$ for some $x \in M$ and let $r \in R$. Then $x \in U(\mu_A; t_4)$ and $x \in L(\lambda_A, t_5)$. Since $U(\mu_A; t_4)$ and $L(\lambda_A, t_5)$ are H_v -submodules of M, we get $r \cdot x \subseteq U(\mu_A; t_4)$ and $r \cdot x \subseteq L(\lambda_A, t_5)$. Therefore for every $z \in r \cdot x$ we have $z \in U(\mu_A; t_4)$ and $z \in L(\lambda_A, t_5)$ which imply $\mu_A(z) \ge t_4$ and $\lambda_A(z) \le t_5$. Hence

$$\inf\{\mu_A(z) \mid z \in r \cdot x\} \ge t_4 = \mu_A(x)$$

and

$$\sup\{\lambda_A(z) \mid z \in r \cdot x\} \le t_5 = \lambda_A(x).$$

This completes the proof.

Corollary 4.9. Let S be an H_v -submodule of an H_v -module M. If fuzzy sets μ and λ in M are defined by

$$\mu(x) = \begin{cases} \alpha_0 & \text{if } x \in S, \\ \alpha_1 & \text{if } x \in M \setminus S, \end{cases} \qquad \lambda(x) = \begin{cases} \beta_0 & \text{if } x \in S, \\ \beta_1 & \text{if } x \in M \setminus S, \end{cases}$$

where $0 \leq \alpha_1 < \alpha_0$, $0 \leq \beta_0 < \beta_1$ and $\alpha_i + \beta_i \leq 1$ for i = 0, 1. Then $A = (\mu, \lambda)$ is an intuitionistic fuzzy H_v -submodule of M and $U(\mu; \alpha_0) = S = L(\lambda; \beta_0)$.

Corollary 4.10. Let χ_s be the characteristic function of an H_v -submodule S of M. Then $A = (\chi_s, \chi_s^c)$ is an intuitionistic fuzzy H_v -submodule of M.

Theorem 4.11. If $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy H_v -submodule of M, then

$$\mu_A(x) = \sup\{\alpha \in [0,1] \mid x \in U(\mu_A;\alpha)\}$$

and

$$\lambda_A(x) = \inf\{\alpha \in [0,1] \mid x \in L(\lambda_A;\alpha)\}$$

for all $x \in M$.

Proof. Let $\delta = \sup\{\alpha \in [0,1] \mid x \in U(\mu_A; \alpha)\}$ and let $\varepsilon > 0$ be given. Then $\delta - \varepsilon < \alpha$ for some $\alpha \in [0,1]$ such that $x \in U(\mu_A; \alpha)$. This means that $\delta - \varepsilon < \mu_A(x)$ so that $\delta \le \mu_A(x)$ since ε is arbitrary.

We now show that $\mu_A(x) \leq \delta$. If $\mu_A(x) = \beta$, then $x \in U(\mu_A; \beta)$ and so

$$\beta \in \{ \alpha \in [0,1] \mid x \in U(\mu_A; \alpha) \}.$$

Hence

$$\mu_A(x) = \beta \le \sup\{\alpha \in [0,1] \mid x \in U(\mu_A;\alpha)\} = \delta.$$

Therefore

$$\mu_A(x) = \delta = \sup\{\alpha \in [0,1] \mid x \in U(\mu_A;\alpha)\}.$$

Now let $\eta = \inf\{\alpha \in [0,1] \mid x \in L(\lambda_A; \alpha)\}$. Then

$$\inf\{\alpha \in [0,1] \mid x \in L(\lambda_A;\alpha)\} < \eta + \varepsilon$$

for any $\varepsilon > 0$, and so $\alpha < \eta + \varepsilon$ for some $\alpha \in [0, 1]$ with $x \in L(\lambda_A; \alpha)$. Since $\lambda_A(x) \leq \alpha$ and ε is arbitrary, it follows that $\lambda_A(x) \leq \eta$.

To prove $\lambda_A(x) \geq \eta$, let $\lambda_A(x) = \zeta$. Then $x \in L(\lambda_A; \zeta)$ and thus $\zeta \in \{\alpha \in [0, 1] \mid x \in L(\lambda_A; \alpha)\}$. Hence

$$\inf\{\alpha \in [0,1] \mid x \in L(\lambda_A;\alpha)\} \le \zeta,\$$

i.e. $\eta \leq \zeta = \lambda_A(x)$. Consequently

$$\lambda_A(x) = \eta = \inf\{\alpha \in [0, 1] \mid x \in L(\lambda_A; \alpha)\},\$$

which completes the proof.

Definition 4.12. A fuzzy set μ in a set X is said to have *sup property* if for every non-empty subset S of X, there exists $x_0 \in S$ such that

$$\mu(x_0) = \sup_{x \in S} \{\mu(x)\}.$$

Proposition 4.13. Let M_1 and M_2 be two H_v -modules over an H_v -ring Rand $f: M_1 \longrightarrow M_2$ be a surjection. If $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy H_v -submodule of M_1 such that μ_A and λ_A have sup property, then

(i)
$$f(U(\mu_A; t)) = U(f(\mu_A); t),$$

(ii) $f(L(\lambda_A; t)) \subseteq L(f(\lambda_A); t)$

Proof. (i) We have

$$y \in U(f(\mu_A); t) \iff f(\mu_A)(y) \ge t$$

$$\iff \sup_{x \in f^{-1}(y)} \{\mu_A(x)\} \ge t$$

$$\iff \exists x_0 \in f^{-1}(y), \ \mu_A(x_0) \ge t$$

$$\iff \exists x_0 \in f^{-1}(y), \ x_0 \in U(\mu_A; t)$$

$$\iff f(x_0) = y, \ x_0 \in U(\mu_A; t)$$

$$\iff y \in f(U(\mu_A; t)).$$

(ii) We have

$$\begin{split} y \in L(f(\lambda_A);t) & \Longrightarrow f(\lambda_A)(y) \leq t \\ & \Longrightarrow \sup_{x \in f^{-1}(y)} \{\lambda_A(x)\} \leq t \\ & \Longrightarrow \lambda_A(x) \leq t \text{ for all } x \in f^{-1}(y) \\ & \Longrightarrow x \in L(\lambda_A;t) \text{ for all } x \in f^{-1}(y) \\ & \Longrightarrow y \in f(L(\lambda_A;t)). \end{split}$$

Proposition 4.14. Let M_1 and M_2 be two H_v -modules over an H_v -ring R and $f: M_1 \longrightarrow M_2$ be a map. If $B = (\mu_B, \lambda_B)$ is an intuitionistic fuzzy H_v -submodule of M_2 , then

(i) $f^{-1}(U(\mu_B; t)) = U(f^{-1}(\mu_B); t),$ (ii) $f^{-1}(L(\lambda_B; t)) = L(f^{-1}(\lambda_B); t)$

for every $t \in [0, 1]$.

Proof. (i) We have

$$\begin{aligned} x \in U(f^{-1}(\mu_B);t) & \iff f^{-1}(\mu_B)(x) \ge t \\ & \iff \mu_B(f(x)) \ge t \\ & \iff f(x) \in U(\mu_B;t) \\ & \iff x \in f^{-1}(U(\mu_B;t)). \end{aligned}$$

(ii) We have

$$\begin{aligned} x \in L(f^{-1}(\lambda_B);t) & \iff f^{-1}(\lambda_B)(x) \leq t \\ & \iff \lambda_B(f(x)) \leq t \\ & \iff f(x) \in L(\lambda_B;t) \\ & \iff x \in f^{-1}(L(\lambda_B;t)). \end{aligned}$$

 \Box

Definition 4.15. Let f be a map from a set X to a set Y. If $B = (\mu_B, \lambda_B)$ is an intuitionistic fuzzy set in Y, then the *inverse image* of B under f is defined by:

$$f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(\lambda_B)).$$

It is easy to see that $f^{-1}(B)$ is an intuitionistic fuzzy set in X.

Corollary 4.16. Let M_1 and M_2 be two H_v -modules over an H_v -ring Rand $f : M_1 \longrightarrow M_2$ be a strong epimorphism. If $B = (\mu_B, \lambda_B)$ is an intuitionistic fuzzy H_v -submodule of M_2 , then $f^{-1}(B)$ is an intuitionistic fuzzy H_v -submodule of M_1 .

Proof. Assume that $B = (\mu_B, \lambda_B)$ is an intuitionistic fuzzy H_v -submodule of M_2 . By Theorem 4.7, we know that the sets $U(\mu_B; t)$ and $L(\lambda_B; t)$ are H_v -submodules of M_2 for every $t \in Im(\mu_B) \cap Im(\lambda_B)$. It follows from Proposition 3.6 that $f^{-1}(U(\mu_B; t))$ and $f^{-1}(L(\lambda_B; t))$ are H_v -submodules of M_1 . Using Proposition 4.14, we have

$$f^{-1}(U(\mu_B;t)) = U(f^{-1}(\mu_B);t),$$

$$f^{-1}(L(\lambda_B;t)) = L(f^{-1}(\lambda_B);t).$$

Now by Theorem 4.8, the proof is completed.

5 On fundamental modules

The main tools in the theory of H_v -structures are the fundamental relations. Consider an H_v -module M over an H_v -ring R. If the relation γ^* is the smallest equivalence relation on R such that the quotient R/γ^* , the set of all equivalence classes, is a ring, we say that γ^* is the fundamental equivalence relation on R and R/γ^* is the fundamental ring (see [17, 19]). The fundamental relation ϵ^* on M over R is the smallest equivalence relation on M such that M/ϵ^* is a module over the ring R/γ^* . Let \mathcal{U} be the set of all expressions consisting of finite hyperoperations either on R and M or the external hyperoperation applied to finite sets of elements of R and M. We define the relation ϵ on M as follows:

 $a \in b$ if and only if $\{a, b\} \subseteq u$ for some $u \in \mathcal{U}$.

Let us denote $\hat{\epsilon}$ the transitive closure of ϵ . Then we can rewrite the definition of $\hat{\epsilon}$ on M as follows:

 $a \hat{\epsilon} b$ if and only if there exist $z_1, \ldots, z_{n+1} \in M$ with $z_1 = a, z_{n+1} = b$ and $u_1, \ldots, u_n \in \mathcal{U}$ such that

$$\{z_i, z_{i+1}\} \subseteq u_i \quad (i=1,\ldots,n).$$

Theorem 5.1. (cf. Vougiouklis [20]). The fundamental relation ϵ^* is the transitive closure of the relation ϵ .

Suppose $\gamma^*(r)$ is the equivalence class containing $r \in R$, and $\epsilon^*(x)$ the equivalence class containing $x \in M$. On M/ϵ^* , the sum \oplus and the external product \odot using the γ^* classes in R are defined as follows:

$$\epsilon^*(x) \oplus \epsilon^*(y) = \epsilon^*(c) \text{ for all } c \in \epsilon^*(x) + \epsilon^*(y),$$
$$\gamma^*(r) \odot \epsilon^*(x) = \epsilon^*(d) \text{ for all } d \in \gamma^*(r) \cdot \epsilon^*(x).$$

The kernel of the canonical map $\varphi : M \longrightarrow M/\epsilon^*$ is called the *core* of M and is denoted by ω_M . Here we also denote ω_M the zero element of M/ϵ . We have

$$\omega_M = \epsilon^*(0)$$
, and $\epsilon^*(-x) = -\epsilon^*(x)$ for all $x \in M$

Definition 5.2. Let M be an H_v -module over an H_v -ring R and let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy H_v -submodule of M. The intuitionistic fuzzy set $A/\epsilon^* = (\overline{\mu_A}^{\epsilon^*}, \underline{\lambda_A}_{\epsilon^*})$ is defined as follows:

$$\overline{\mu_A}^{\epsilon^*} : M/\epsilon^* \longrightarrow [0,1]$$

$$\overline{\mu_A}^{\epsilon^*}(\epsilon^*(x)) = \begin{cases} \sup_{a \in \epsilon^*(x)} \{\mu_A(a)\} & \text{if } \epsilon^*(x) \neq \omega_M \\ 1 & \text{if } \epsilon^*(x) = \omega_M \end{cases}$$

and

$$\frac{\underline{\lambda}_{A_{\epsilon^{*}}}: M/\epsilon^{*} \longrightarrow [0, 1]}{\underline{\lambda}_{A_{\epsilon^{*}}}(\epsilon^{*}(x))} = \begin{cases} \inf_{a \in \epsilon^{*}(x)} \{\lambda_{A}(a)\} & \text{if } \epsilon^{*}(x) \neq \omega_{M} \\ 0 & \text{if } \epsilon^{*}(x) = \omega_{M}. \end{cases}$$

In the following we show that

$$0 \le \overline{\mu_A}^{\epsilon^*}(\epsilon^*(x)) + \underline{\lambda_A}_{\epsilon^*}(\epsilon^*(x)) \le 1,$$

for all $\epsilon^*(x) \in M/\epsilon^*$.

If $\epsilon^*(x) = \omega_M$, then the above inequalities are clear. Assume that $x \in H$ and $\epsilon^*(x) \neq \omega_M$. Since $0 \leq \mu_A(a)$ and $0 \leq \lambda_A(a)$ for all $a \in \epsilon^*(x)$, we have

$$0 \le \sup_{a \in \epsilon^*(x)} \{\mu_A(a)\} + \inf_{a \in \epsilon^*(x)} \{\lambda_A(a)\}$$

or

$$0 \le \overline{\mu_A}^{\epsilon^*}(\epsilon^*(x)) + \underline{\lambda_A}_{\epsilon^*}(\epsilon^*(x)).$$

On the other hand, we have

$$\mu_A(a) + \lambda_A(a) \le 1$$
 or $\mu_A(a) \le 1 - \lambda_A(a)$,

for all $a \in \epsilon^*(x)$, and so

$$\overline{\mu_A}^{\epsilon^*}(\epsilon^*(x)) = \sup_{a \in \epsilon^*(x)} \{\mu_A(a)\}$$

$$\leq \sup_{a \in \epsilon^*(x)} \{1 - \lambda_A(a)\}$$

$$= 1 - \inf_{a \in \epsilon^*(x)} \{\lambda_A(a)\}$$

$$= 1 - \underline{\lambda_A}_{\epsilon^*}(\epsilon^*(x)).$$

Hence $\overline{\mu_A}^{\epsilon^*}(\epsilon^*(x)) + \underline{\lambda_A}_{\epsilon^*}(\epsilon^*(x)) \le 1.$

Theorem 5.3. (cf. Davvaz [6]). Let M be an H_v -module over an H_v -ring R and let μ be a fuzzy H_v -submodule of M. Then $\overline{\mu_A}^{\epsilon^*}$ is a fuzzy submodule of the module M/ϵ^* .

Lemma 5.4. We have

$$(\overline{\lambda_A^c}^{\epsilon^*})^c = \underline{\lambda_A}_{\epsilon^*}.$$

Proof. If $\epsilon^*(x) = \omega_M$, then

$$(\overline{\lambda_A^c}^{\epsilon^*})^c(\omega_H) = 1 - (\overline{\lambda_A^c}^{\epsilon^*})(\omega_M) = 0 = \underline{\lambda_A}_{\epsilon^*}(\omega_M).$$

Now, assume that $\epsilon^*(x) \neq \omega_M$. Then

$$\begin{aligned} (\overline{\lambda_A^c}^{\epsilon^*})^c(\epsilon^*(x)) &= 1 - (\overline{\lambda_A^c}^{\epsilon^*})(\epsilon^*(x)) \\ &= 1 - \sup_{a \in \epsilon^*(x)} \{\lambda_A^c(a)\} \\ &= 1 - \sup_{a \in \epsilon^*(x)} \{1 - \lambda_A(a)\} \\ &= \inf_{a \in \epsilon^*(x)} \{\lambda_A(a)\} \\ &= \underbrace{\lambda_A_{\epsilon^*}}(\epsilon^*(x)). \end{aligned}$$

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Theorem 5.5. Let M be an H_v -module over an H_v -ring R and let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy H_v -submodule of M. Then $A/\epsilon^* = (\overline{\mu_A}\epsilon^*, \underline{\lambda_A}_{\epsilon^*})$ is an intuitionistic fuzzy submodule of the fundamental module M/ϵ^* .

Proof. Suppose that $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy H_v -submodule of M. Using Lemma 4.3, λ_A^c is a fuzzy H_v -submodule of M and by Theorem 5.3, $\overline{\mu_A}^{\epsilon^*}$ and $\overline{\lambda_A^c}^{\epsilon^{\epsilon^*}}$ are fuzzy H_v -submodules of M/ϵ^* , and so $(\overline{\lambda_A^c}^{\epsilon^*})^c$ satisfies the conditions (4), (5), (6) of Definition 2.5. Hence by Lemma 5.4, $\underline{\lambda_A}_{\epsilon^*}$ satisfies the conditions (4),(5),(6) of Definition 2.5. Therefore $A/\epsilon^* = (\overline{\mu_A}^{\epsilon^*}, \underline{\lambda_A}_{\epsilon^*})$ is an intuitionistic fuzzy submodule of M/ϵ^* .

6 Conclusions

As a generalization of fuzzy sets, the notion of intuitionistic fuzzy sets was introduced by Atanassov [1], and applications of intuitionistic fuzzy concepts have already been done by Atanassov and others in algebra, topological space, knowledge engineering, natural language, and neural network etc. Biswas [3] have applied the concept of intuitionistic fuzzy sets to the theory of groups and studied intuitionistic fuzzy subgroups of a group. The notion of an intuitionistic fuzzy subquasigroup of a guasigroup was discussed by Kim, Dudek and Jun [12]. Also the concept of intuitionistic fuzzy ideals of semigroups was considered by Kim and Jun [13]. The concept of hyperstructure first was introduced by Marty [14]. Vougiouklis [19], in the fourth AHA congress (1990), introduced the notion of H_v -structures. Recently, present authors [11] have discussed the intuitionistic fuzzification of the concept of subhyperquasigroups in a hyperquasigroup. The aim of this paper is to introduce the notion of an intuitionistic fuzzy H_v -submodule of an H_{ν} -module, and to investigate related properties. Characterizations of intuitionistic fuzzy H_v -submodules are given. Our future work will focus on studying the intuitionistic fuzzy structure of H_v -nearring modules.

7 Acknowledgements

The authors are highly grateful to referees and Professor Witold Pedrycz, Editor-in-Chief, for their valuable comments and suggestions for improving the paper.

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