Modelling Argument Accrual with Possibilistic Uncertainty in a Logic Programming Setting

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Abstract

Argumentation frameworks have proven to be a successful approach to formalizing commonsense reasoning. Recently, some argumentation frameworks have emerged which incorporate the treatment of possibilistic uncertainty, notably Possibilistic Defeasible Logic Programming (P-DeLP). At the same time, modelling *argument accrual* has gained attention from the argumentation community. Even though some preliminary formalizations have been advanced, they do not take into account possibilistic uncertainty when accruing arguments. In this paper we present a novel approach to model argument accrual with possibilistic uncertainty in a constructive way. The formalization proposed uses P-DeLP's representation language and notion of argument as a basis.

Keywords: argumentation, possibilistic uncertainty, argument accrual

1. Introduction

Argumentation frameworks have proven to be a successful approach to formalizing qualitative, commonsense reasoning [23, 18]. Argumentation can be abstractly defined as the interaction of different arguments for and against some conclusion. Over the last few years, argumentation has been gaining increasing importance in multi-agent systems, mainly as a vehicle for facilitating "rational interaction" (i.e., interaction which involves the giving and receiving of reasons). This is because argumentation provides tools for designing, implementing and analysing sophisticated forms of interaction among rational agents. Argumentation has made solid contributions to the practice of multi-agent dialogues, including several application domains such as legal disputes, business negotiation, labor disputes, team formation, scientific inquiry, deliberative democracy, ontology reconciliation, risk analysis, scheduling, and logistics. In such a setting, a single agent may also use argumentation techniques to perform its individual reasoning because it needs to make decisions under complex preferences policies in a highly dynamic environment.

Recently, some argumentation frameworks have emerged which incorporate the treatment of possibilistic uncertainty. Amgoud et al. have developed argumentative approaches to decision making with uncertainty [5] and merging conflicting databases [4]). In recent work, Alsinet et al. formalized *Possibilistic Defeasible Logic Programming* (P-DeLP) [3, 2], an argumentation framework based on logic programming which incorporates the treatment of possibilistic uncertainty at the object-language level.

At the same time, the notion of argument accrual has received some attention from the argumentation community [28, 21, 17]. This notion is based on the intuitive idea that having more reasons or arguments for a given conclusion makes such a conclusion more credible or stronger. Let us consider the following example: Alice is looking for an apartment to rent, and when she is considering one of the candidate apartments she analyzes different arguments in favor and against renting it:

(A) the apartment is in very good location (according to her interests); therefore, she should rent it.

(B_1) the apartment is rather small; therefore, she should not rent it.

(B_2) The apartment seems to have humidity problems; therefore, she should not rent it.

(B_3) building tenants are mostly students, so disorders may happen to be usual; therefore, she should not rent it.

Suppose that Alice considers that location is more important than any of the other features considered individually (that is, that argument A is stronger than each of $B_1 - B_3$). However, Alice considers that the three arguments $B_1 - B_3$ together are stronger than A (see Fig. 1). Traditional argumentation systems (including those using possibilistic logic, such as P-DeLP [3, 2]) only consider individual arguments, and conflicting arguments

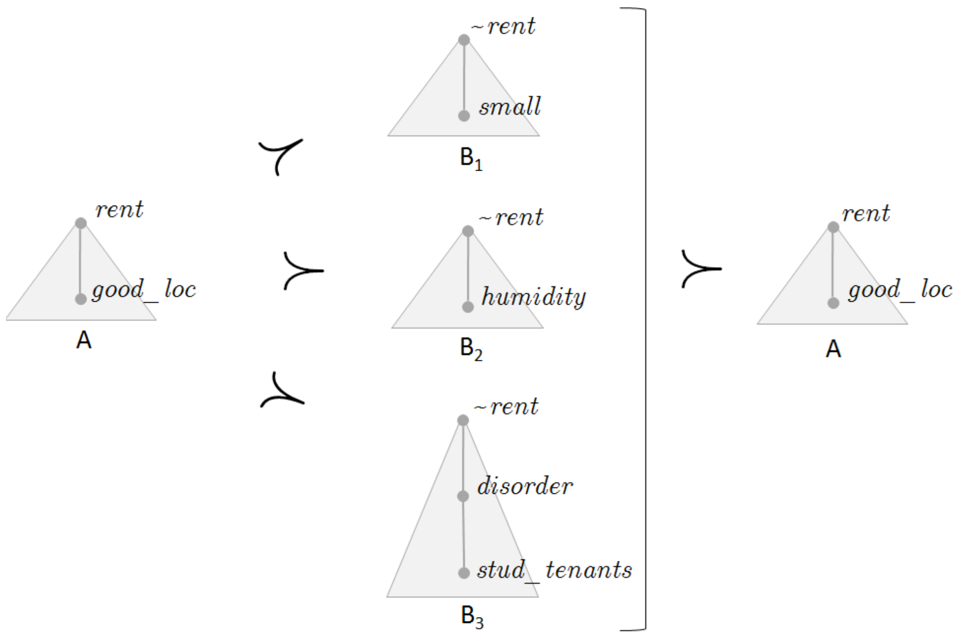


Figure 1: Apartment rental example

are compared pairwise, leading to counter-intuitive conclusions in situations like this. So, accrual of arguments must be explicitly taked into account.

Moreover, although some approaches modeling argument accrual exist, none of them proposes a concrete representation of argument strength, nor a mechanism to explicitly aggregate the strength of individual arguments supporting the same conclusion. As we will see later, possibilistic logic allows us to attach necessity degrees to logical formulas at object-level language. Such necessity degrees can be obtained from a variety of sources (e.g. in this particular example they could correspond to user-defined values when assessing the apartment for rent, or to ranking value obtained from apartment rental services, etc.). Necessity degrees just provide a weight expressing certainty or priority associated with a given logical formula.[1]

In this work we propose a formalization of argument accrual with possibilistic uncertainty in a logic programming setting, which is partly based on

[1]An in-depth account of possibilistic logic and their applications in knowledge representation and reasoning is presented in [8].

previous research work ([16, 17]). Our proposal will be obtained by instantiating and extending a generic formalization for skeptical argumentation very similar to those well-known abstract approaches presented in [30, 19, 26]. This generic formalization characterizes a dialectical proof procedure for determining which arguments are ultimately accepted or *warranted*. In order to define our approach, this generic formalization is firstly instantiated with P-DeLP's representation language and the associated notion of (weighted) argument, obtaining a characterization of skeptical argumentation in a logic programming setting with possibilistic uncertainty. Secondly, the resulting possibilistic instantiation is extended to incorporate accrual. Accrued arguments will be conceptualized as weighted structures accounting for different P-DeLP arguments supporting a given conclusion, and as we will see, defining the necessity measure associated with those accrued structures is not a simple task. On the one hand, we want to combine the propagation of necessity degrees when performing rule-based inference (as in P-DeLP) with a way of accumulating necessity values coming from different rules with the same conclusion. On the other hand, we do not want to commit ourselves to a specific way of aggregating necessity degrees; this will be abstracted away in terms of a user-defined function. Finally, the dialectical proof procedure of the generic formalization will also be adapted in order to take accrued structures into account, instead of individual arguments.

The rest of this paper is structured as follows. Section 2 presents a generic formalization for skeptical argumentation. Next, in Section 3 this generic formalization is instantiated using P-DeLP. Section 4 presents the notion of *accrued structure*, which plays a central role in our proposal. Based on this notion, we then formalize the notions of attack and defeat among accruals in Section 5, and present a dialectical proof procedure taking accrued structures into account in Section 6. Then, in Section 7 we discuss related work and describe some significant features of our approach contrasted with the other existing formalizations. Finally, in Section 8 we present the main conclusions obtained.

2. Argumentation: a generic formalization for skeptical semantics

In this section we present a generic formalization for *skeptical* argumentation. Our formalization will be based on the well-known abstract argumentation framework proposed by Dung [9], suitably adapted to capture different flavors of skeptical semantics. In particular, we will present a dialectical proof procedure for determining which arguments are ultimately accepted or *warranted*. This dialectical proof procedure is very similar to those defined in [30, 19, 26] for Dung's (skeptical) semantics, and the resulting formalization can be seen as generalizing other non-abstract argumentation frameworks also based on dialectical proof procedures (e.g. [22, 3]).

Definition 1 (Abstract Argumentation Framework [9]). *An abstract argumentation framework (AAF) is a pair* $(Args, \ def)$*, where* $Args$ *is a set of arguments and* $def \subseteq Args \times Args$ *is a binary relation representing a defeat relation between arguments.*

If $A_1 \ def \ A_2$, then we will say that A_1 is a *defeater* for A_2. In order to define the dialectical proof procedure we will distinguish *blocking* and *proper* defeaters. Formally:

Definition 2 (Proper and Blocking defeaters). *Let* $(Args, \ def)$ *be an AAF. Let* $A_1, \ A_2 \in Args$*. We will say that* A_1 *is a* blocking defeater *of* A_2 *iff* $A_1 \ def \ A_2$ *and* $A_2 \ def \ A_1$*. We will say that* A_1 *is a* proper defeater *of* A_2 *iff* $A_1 \ def \ A_2$ *and it is not the case that* $A_2 \ def \ A_1$*.*

Intuitively, we say that A_1 is a blocking defeater for A_2 when the defeat relationship def is symmetric for A_1 and A_2 (ie., both of them are in conflict, but none of them is "preferred" over the other).

A dialectical line represents a line of discussion about a given argument A, and is a sequence of arguments (starting with A) where each one deafeats the previous in the line. We will consider each element (argument) in a dialectical line as assuming one of two roles, determined by its position in the line: PRO (in favor of the initial argument) if it appears in an odd position, or CON (against the initial argument) if it appears in an even position.

Definition 3 (Dialectical Line). *Let* $(Args, \ def)$ *be an AAF and* $A \in Args$*. A* dialectical line about A *(or just dialectical line) is a finite nonempty sequence of arguments* $[A_1, A_2, .., A_n]$*, with* $A_1 = A$*, such that:*

(d1) $A_i \ def \ A_{i-1}$*,* $1 < i \leq n$*,*

(d2) *if* A_i *and* A_j *are* PRO *arguments,* $i \neq j$*, then* $A_i \neq A_j$*,*

(d3) *if* A_i *is a* PRO *argument,* $i \neq 1$*, then* A_i *is a proper defeater of* A_{i-1}*,* *where A_i is said to be a* PRO *argument of the line if i is odd, or a* CON *argument of the line if i is even.*

Condition d1 states that each argument in the line (except the first) defeats the previous one. Condition d2 avoids repetition of PRO arguments in the line. Finally, the purpose of d3 is to avoid PRO to introduce blocking defeaters.

A dialectical line about a given argument A ended with a PRO argument represents a line of discussion justifying A, while one ending with a CON argument represents a line of discussion not justifying A. In this context, condition d2 can be considered a skeptical solution to cope with cycles in the arguments graph (the graph of arguments and defeats associated with the AAF). A cycle in the arguments graph may lead to an infinite "dialogue line", and so to indecision about its result (justifying or not justifying the initial argument). By avoiding the repetition of PRO arguments (cond. d2) while allowing repetition of CON arguments, an infinite line of dialogue consequence of a cycle will prompt to a finite dialectical line ending with a CON argument, and then not justifying the initial argument. Thus, d2 causes that ties between PRO and CON consequence of graph cycles to be skeptically broken in favor of CON.

The purpose of condition d3 is analogous to that of d2, but for the specific case of two length cycles (blocking defeat situations): by not allowing PRO to introduce blocking defeaters, while CON is indeed allowed to introduce them, blocking cycles are broken in favor of CON. Although d2 is sufficient to skeptically break blocking cycle ties, condition d3 allows to detect and break them more efficiently, that is, without the necessity of repeating CON arguments. Fig. 2b shows a dialectical line about A with respect to the AAF in Fig. 2a according to definition 3 (requiring condition d3), whereas Fig. 2c shows the corresponding sequence of arguments if condition d3 would not be required.

The following proposition formally states that consecutive CON argument repetition never occurs in a dialectical line, and as can be realized from the associated proof, this is consequence of condition d3.

Proposition 1. *Let $[A_1, A_2, .., A_n]$ be a dialectical line. If A_i and A_{i+2} are* CON *arguments, then $A_i \neq A_{i+2}$.*

Proof. Suppose by contradiction that $A_i = A_{i+2}$. Consider the PRO argument A_{i+1}. As A_{i+1} defeats A_i and is defeated by $A_{i+2} = A_i$ (condition d1),

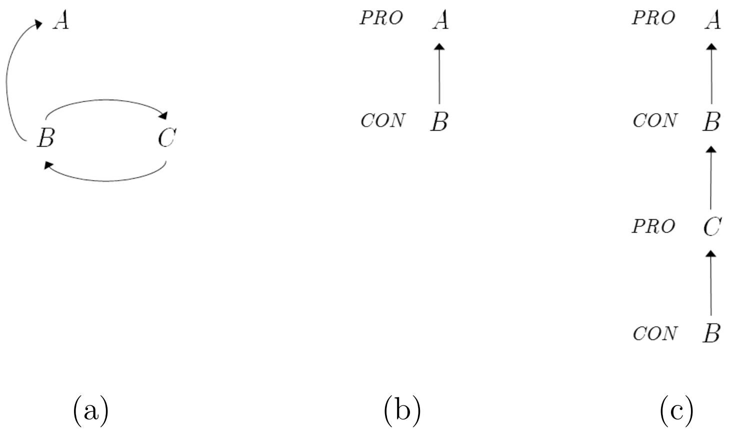


Figure 2: Effect of condition d3.

then A_{i+1} is a blocking defeater for A_i, violating condition d3 of definition 3. Therefore $A_i \neq A_{i+2}$ □

Definition 4 (Exhaustive Dialectical Line). *Let $\Lambda = [A_1, A_2, .., A_n]$ be a dialectical line. We will say that Λ is exhaustive if there exists no argument B such that $[A_1, A_2, .., A_n, B]$ is a dialectical line.*

The set of all exhaustive dialectical lines about a given argument A can be represented as a tree structure, where every line corresponds to a branch in the tree.

Definition 5 (Dialectical Tree). *Let $A \in Args$. A dialectical tree about A, denoted T_A, is defined as follows:*

1. *Nodes are labeled with arguments of $Args$.*
2. *$\Lambda = [A_1, A_2, .., A_n]$ is the sequence of labels of a branch of the tree (path from the root to a leave) iff Λ is an exhaustive dialectical line about A.*
3. *There exist no sibling nodes (children of the same node) M and M' in the tree labeled with the same argument.*

The purpose of condition 3 is to factor out as much as possible in the tree the common prefixes of dialectical lines about A.

Definition 6 (Evaluated Dialectical Tree). *Let T_A be a dialectical tree. The corresponding evaluated dialectical tree, denoted T^*_A, will be obtained by marking every node N in T_A as 'U' (undefeated) or 'D' (defeated) according to the following rules:*

1. *if N is a leaf, then N is marked as 'U'.*
2. *if all children M_i of N are marked as 'D', then N is marked as 'U'.*
3. *if there exists a child M_i of N marked as 'U', then N is marked as 'D'.*

Clearly, this procedure resembles the one applied for computing well-founded semantics in a logic programming setting ([13]).

Definition 7 (Warranted argument). *Let $\Gamma = (Args,\ def)$ be an AAF. Let $A \in Args$ such that the root of T^*_A is marked as 'U'. Then we say that A is a warranted argument w.r.t. Γ.*

3. An Instantiation using Necessity Degrees in a Logic Programming Setting

In this section we will present an instantiation of the argumentation framework given in the previous section in a logic programming setting. It must be noted that logic programming has proven to be a flexible formalization for argumentation systems (e.g. [22, 12], among others). In order to incorporate uncertainty in our formalization we will make use of possibilistic logic for incorporating necessity degrees attached to formulas at the object level. Our approach for knowledge representation is the same as the one used in Possibilistic DeLP ([3]).

3.1. Knowledge representation and argumentation in P-DeLP

A *weighted* clause is a pair (φ, α), where φ is a rule $q \leftarrow p_1 \wedge \ldots \wedge p_k$ or a fact q (i.e., a rule with empty antecedent), where $q, p_1, \ldots, p_k$ are literals, and $\alpha \in (0, 1]$ expresses a lower bound for the necessity degree of φ. We distinguish between *certain* and *uncertain* clauses. A clause (φ, α) is referred as certain if $\alpha = 1$ and uncertain, otherwise. Given a set Δ of weighted clauses we will distinguish the set $CC(\Delta)$ of all the certain clauses in Δ, and the set $UC(\Delta)$ of all the uncertain clauses in Δ.

A set of weighted clauses Δ will be deemed as *contradictory*, denoted $\Delta \vdash \perp$, if, for some atom a, $\Delta \vdash (a, \alpha)$ and $\Delta \vdash (\sim a, \beta)$, with $\alpha > 0$ and $\beta > 0$, where $\vdash$ stands for deduction by means of the following particular instance of the *generalized modus ponens rule*:

$$\frac{\begin{array}{l}(q \leftarrow p_1 \wedge \cdots \wedge p_k,\ \alpha)\\ (p_1, \beta_1), \ldots, (p_k, \beta_k)\end{array}}{(q,\ \min(\alpha, \beta_1, \ldots, \beta_k))}\ [GMP]$$

A *possibilistic knowledge base* (or just possibilistic KB) $\mathcal{K}$ is a finite set of weighted clauses such that $CC(\mathcal{K})$ is non-contradictory.

Definition 8 (Argument). *Let $\mathcal{K}$ be a possibilistic KB. We say that a set $\mathcal{A} \subseteq \mathcal{K}$ is an* argument *for a literal h with necessity degree $\alpha > 0$, denoted $\langle \mathcal{A}, h, \alpha \rangle$, iff:*

1. *$\mathcal{A} \vdash (h,\ \alpha)$,*
2. *$CC(\mathcal{K}) \cup \mathcal{A}$ is non-contradictory,*
3. *$\mathcal{A}$ is minimal w.r.t. set inclusion, i.e. there is no $\mathcal{A}_1 \subset \mathcal{A}$ s.t. $\mathcal{A}_1 \vdash (h,\ \alpha)$.*

Remark: Logic programs, and therefore defeasible logic programs, may contain unintended cycles in the rules, implying that from them it is possible to build cyclic derivations from particular programs. Nevertheless, the representation of arguments as sets together with the minimality condition contained in the definition above preclude that these cycles will come into play in the non-pathological cases; for these unwanted cases defeasible logic programs are in the same situation as standard logic programs [25].

Definition 9 (Subargument). *An argument $\langle \mathcal{S}, k, \gamma \rangle$ is a sub-argument of $\langle \mathcal{A}, h, \alpha \rangle$ iff $\mathcal{S} \subseteq \mathcal{A}$.*

In what follows, for a given literal h, we will write $\overline{h}$ to denote "$\sim a$" if $h \equiv a$, and "a" if $h \equiv \sim a$.

Definition 10 (Attack). *Let $\mathcal{K}$ be a possibilistic KB, and let $\langle \mathcal{A}, h, \alpha \rangle$ and $\langle \mathcal{B}, k, \beta \rangle$ be two arguments w.r.t. $\mathcal{K}$. We say that $\langle \mathcal{B}, k, \beta \rangle$ attacks $\langle \mathcal{A}, h, \alpha \rangle$ (at literal $\overline{k}$) iff there exists a subargument (called* disagreement subargument*) $\langle \mathcal{S}, \overline{k}, \gamma \rangle$ of $\langle \mathcal{A}, h, \alpha \rangle$.*

Definition 11 (Defeat). *Let $\langle \mathcal{A}, h, \alpha \rangle$ and $\langle \mathcal{B}, k, \beta \rangle$ be two arguments. Then we say that $\langle \mathcal{B}, k, \beta \rangle$ defeats $\langle \mathcal{A}, h, \alpha \rangle$ (or equivalently that $\langle \mathcal{B}, k, \beta \rangle$ is a defeater of $\langle \mathcal{A}, h, \alpha \rangle$) if $\langle \mathcal{B}, k, \beta \rangle$ attacks $\langle \mathcal{A}, h, \alpha \rangle$, where 1) $\langle \mathcal{S}, \overline{k}, \gamma \rangle$ is the disagreement subargument, and 2) $\beta \geq \gamma$.*

In order to define the warranted arguments w.r.t. a given possibilistic KB $\mathcal{K}$ we have just to instantiate the abstract argumentation framework presented in Section 2 by making *Args* be the set of all arguments w.r.t.

$\mathcal{K}$ and by defining *def* relation as follows: $\langle \mathcal{A}_1, h_1, \alpha_1 \rangle$ *def* $\langle \mathcal{A}_2, h_2, \alpha_2 \rangle$ iff $\langle \mathcal{A}_1, h_1, \alpha_1 \rangle$ defeats $\langle \mathcal{A}_2, h_2, \alpha_2 \rangle$.

Finally we introduce the notion of *warranted conclusion with necessity degree* α, which is based on that of warranted argument.

Definition 12 (Warranted conclusions). *Let $\mathcal{K}$ be a possibistic KB and let $\langle \mathcal{A}, h, \alpha \rangle$ be a warranted argument w.r.t. $\mathcal{K}$ such that there exist no other warranted argument $\langle \mathcal{A}', h, \alpha' \rangle$ where $\alpha' > \alpha$. Then we say that h is* warranted *w.r.t. $\mathcal{K}$ with necessity α.*

3.2. Refining the notion of warranted arguments: taking subarguments into account

Next we will refine the previous formalization by taking advantage of the notion of subargument associated with the definition of defeat. Recent research ([24]) has shown the importance of this notion when carrying out the dialectical analysis of the status of arguments.

The following definition refines the distinction between proper and blocking defeaters by considering that a defeater is blocking if it is reciprocally defeated by the disagreement subargument. This refinement is a natural consequence of instantiating the *def* relation with the particular notion of defeat introduced in definition 11, which states that a given attack constitutes or does not constitute a defeat depending on the comparison between the attacking argument and the disagreement subargument.

Definition 13 (Proper and Blocking Defeaters, refined). *Let $\langle \mathcal{A}, h, \alpha \rangle$ and $\langle \mathcal{B}, k, \beta \rangle$ be two arguments. We will say that $\langle \mathcal{B}, k, \beta \rangle$ is a* blocking defeater *of $\langle \mathcal{A}, h, \alpha \rangle$ if $\langle \mathcal{B}, k, \beta \rangle$ defeats $\langle \mathcal{A}, h, \alpha \rangle$, with $\langle \mathcal{S}, \overline{k}, \gamma \rangle$ as the associated disagreement subargument, and $\langle \mathcal{S}, \overline{k}, \gamma \rangle$ defeats $\langle \mathcal{B}, k, \beta \rangle$. We will say that $\langle \mathcal{B}, k, \beta \rangle$ is a* proper defeater *of $\langle \mathcal{A}, h, \alpha \rangle$ if $\langle \mathcal{B}, k, \beta \rangle$ defeats $\langle \mathcal{A}, h, \alpha \rangle$, with $\langle \mathcal{S}, \overline{k}, \gamma \rangle$ as the associated disagreement subargument, and it is not the case that $\langle \mathcal{S}, \overline{k}, \gamma \rangle$ defeats $\langle \mathcal{B}, k, \beta \rangle$*[2].

By considering this refined notion of blocking defeat (instead of the original one) in the context of condition d3, some blocking tie situations between PRO and CON are detected and resolved sooner, resulting in shorter dialectical

[2]Notice that, in particular, in this possibilistic instantiation of the *AAF* presented in Section 2 the defeater $\langle \mathcal{B}, k, \beta \rangle$ will be proper iff $\beta > \gamma$, and blocking iff $\beta = \gamma$.

lines. That is because the refined blocking defeat notion captures some defeats that are not blocking according to the original one, but which anticipate a reciprocal defeat between the defeater and the disagreement subargument. For the sake of example, consider the arguments and defeat relation shown in Fig. 3 for a given KB. The defeat of $\langle \mathcal{A}_3, h_3, \alpha_3 \rangle$ against $\langle \mathcal{A}_2, h_2, \alpha_2 \rangle$ is blocking according to the refined notion, but not according to the original. As consequence, the sequence $[\langle \mathcal{A}_1, h_1, \alpha_1 \rangle, \langle \mathcal{A}_2, h_2, \alpha_2 \rangle]$ is an exhaustive dialectical line if the refined blocking definition is considered in condition d3, whereas $[\langle \mathcal{A}_1, h_1, \alpha_1 \rangle, \langle \mathcal{A}_2, h_2, \alpha_2 \rangle, \langle \mathcal{A}_3, h_3, \alpha_3 \rangle, \langle \mathcal{S}, k, \gamma \rangle]$ is the corresponding exhaustive dialectical line if the original one is considered.

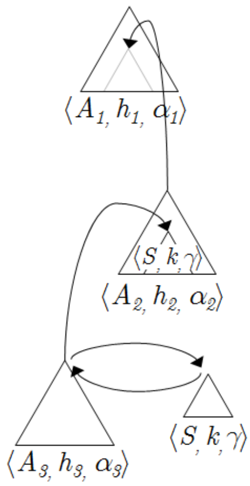


Figure 3: Graph of defeats and refined blocking defeat

Definition 14 (Dialectical Line, refined). *Let* $\langle \mathcal{A}, h, \alpha \rangle \in Args$*. A dialectical line about* $\langle \mathcal{A}, h, \alpha \rangle$ *(or just dialectical line) is a finite nonempty sequence of arguments* $[\langle \mathcal{A}_1, h_1, \alpha_1 \rangle, \langle \mathcal{A}_2, h_2, \alpha_2 \rangle, ..., \langle \mathcal{A}_n, h_n, \alpha_n \rangle]$*, with* $\langle \mathcal{A}_1, h_1, \alpha_1 \rangle = \langle \mathcal{A}, h, \alpha \rangle$*, such that:*

(d1') $\langle \mathcal{A}_i, h_i, \alpha_i \rangle$ *defeats* $\langle \mathcal{A}_{i-1}, h_{i-1}, \alpha_{i-1} \rangle$*,* $1 < i \leq n$*,*

(d2') *if* $\langle \mathcal{A}_i, h_i, \alpha_i \rangle$ *is* PRO *and* Σ *is the set of all disagreement subarguments associated with the attacks against* PRO *arguments appearing before* $\langle \mathcal{A}_i, h_i, \alpha_i \rangle$ *in the line, then no argument in* Σ *is a subargument of* $\langle \mathcal{A}_i, h_i, \alpha_i \rangle$*,*

(d3') *if* $\langle \mathcal{A}_i, h_i, \alpha_i \rangle$ *is a* PRO *argument and* $i \neq 1$*, then* $\langle \mathcal{A}_i, h_i, \alpha_i \rangle$ *is a (refined) proper defeater of* $\langle \mathcal{A}_{i-1}, h_{i-1}, \alpha_{i-1} \rangle$*,*

where $\langle \mathcal{A}_i, h_i, \alpha_i \rangle$ *is said to be a* PRO *argument in the line if i is odd, or a* CON *argument in the line if i is even*[3].

Notice that condition d1' is just an instantiation of condition d1 in definition 3. Condition d2' avoids the occurrence of a PRO argument $\langle \mathcal{A}_i, h_i, \alpha_i \rangle$ if it involves the disagreement subargument of a previous PRO argument $\langle \mathcal{A}_j, h_j, \alpha_j \rangle$ in the line ($j < i$). Notice that if such an argument $\langle \mathcal{A}_i, h_i, \alpha_i \rangle$ exists, then the defeater $\langle \mathcal{A}_{j+1}, h_{j+1}, \alpha_{j+1} \rangle$ (CON argument) of $\langle \mathcal{A}_j, h_j, \alpha_j \rangle$ will also be a defeater of $\langle \mathcal{A}_i, h_i, \alpha_i \rangle$, implying a cycle in the graph of defeats. By avoiding $\langle \mathcal{A}_i, h_i, \alpha_i \rangle$, the cycle is broken in favor of CON, and without introducing $\langle \mathcal{A}_i, h_i, \alpha_i \rangle$, nor reintroducing $\langle \mathcal{A}_{j+1}, h_{j+1}, \alpha_{j+1} \rangle$. It can also be noted that d2' avoids PRO repetition, as d2 does, just by taking $\langle \mathcal{A}_j, h_j, \alpha_j \rangle$ as the previous occurrence of $\langle \mathcal{A}_i, h_i, \alpha_i \rangle$ in the line. Finally, condition d3' is analogous to d3 but considers refined definition of proper defeater.
Observation: In a possibilistic context (like this), condition d2' (/d2) is redundant in presence of condition d3' (/d3). It can be proven that all the arguments and disagreement subarguments involved in a given graph cycle will have the same associated necessity value, and then all the defeats involved in the cycle will be blocking, causing cycles to be always broken by condition d3' before condition d2' is violated. However, we include condition d2' for two reasons. First, in a general (non-possibilistic) rule based formalization, condition d2' is necessary despite the existence of d3'. Second, and

[3]Although this definition is different from the one presented in [2] for P-DeLP, both capture the same notion, as we will try to intuitively expose next. Firstly, dialectical line definition in [2] requires concordance among the arguments proposed by each player, thus avoiding players to advance conflicting arguments. Although this is a natural policy to adopt, in our approach (and Dung's one) an alternative (also natural) strategy is taken: a player (PRO or CON) is indeed allowed to advance an argument $\langle \mathcal{A}_j, h_j, \alpha_j \rangle$ first, and later in the line an argument $\langle \mathcal{A}_k, h_k, \alpha_k \rangle$ that is in conflict with $\langle \mathcal{A}_j, h_j, \alpha_j \rangle$, in other words, a player is allowed to be incoherent, but then its opponent will put this incoherence in evidence by advancing $\langle \mathcal{A}_j, h_j, \alpha_j \rangle$ against $\langle \mathcal{A}_k, h_k, \alpha_k \rangle$, and finally winning the dispute (line). Secondly, definition in [2] requires the line to be "progressive", that is that every blocking defeater must be defeated by a proper one, forcing the line to progress towards greater valued arguments, and thus in particular to avoid cycles. In our formalization, condition d3' in Def. 14 requires PRO arguments to be always proper, making in particular the line to be progressive (the intuition behind this condition was explained in Section 2).

most important, condition d2' (with some refinements) is also necessary for our formalization of accrual.

Proposition 2. *Let* $[\langle \mathcal{A}_1, h_1, \alpha_1\rangle, \langle \mathcal{A}_2, h_2, \alpha_2\rangle, .., \langle \mathcal{A}_n, h_n, \alpha_n\rangle]$ *be a (refined) dialectical line. If* $\langle \mathcal{A}_i, h_i, \alpha_i\rangle$ *and* $\langle \mathcal{A}_{i+2}, h_{i+2}, \alpha_{i+2}\rangle$ *are* CON *arguments, and* $\langle \mathcal{S}, k, \gamma\rangle$ *is the disagreement subargument associated with the attack of* $\langle \mathcal{A}_{i+1}, h_{i+1}, \alpha_{i+1}\rangle$ *against* $\langle \mathcal{A}_i, h_i, \alpha_i\rangle$*, then* $\langle \mathcal{S}, k, \gamma\rangle \neq \langle \mathcal{A}_{i+2}, h_{i+2}, \alpha_{i+2}\rangle$*.*

Proof. Suppose by contradiction that $\langle \mathcal{S}, k, \gamma\rangle = \langle \mathcal{A}_{i+2}, h_{i+2}, \alpha_{i+2}\rangle$. Consider the PRO argument $\langle \mathcal{A}_{i+1}, h_{i+1}, \alpha_{i+1}\rangle$. As $\langle \mathcal{A}_{i+1}, h_{i+1}, \alpha_{i+1}\rangle$ defeats $\langle \mathcal{A}_i, h_i, \alpha_i\rangle$, with disagreement subargument $\langle \mathcal{S}, k, \gamma\rangle$, and is defeated by $\langle \mathcal{A}_{i+2}, h_{i+2}, \alpha_{i+2}\rangle = \langle \mathcal{S}, k, \gamma\rangle$ (condition d1), then $\langle \mathcal{A}_{i+1}, h_{i+1}, \alpha_{i+1}\rangle$ is a blocking defeater for $\langle \mathcal{A}_i, h_i, \alpha_i\rangle$, violating condition d3'. Therefore $\langle \mathcal{S}, k, \gamma\rangle \neq \langle \mathcal{A}_{i+2}, h_{i+2}, \alpha_{i+2}\rangle$.

4. Modelling Argument Accrual with Possibilistic Uncertainty

As stated in the introduction, our goal is to model argument accrual in a possibilistic setting taking into account several issues. In P-DeLP, the *GMP* inference rule allows us to propagate necessity degrees; however, given different arguments supporting the same conclusion, we want to be able to accumulate their strength in terms of possibilistic values. To do this we will define the notion of *accrued structure*, which will account for several arguments supporting the same conclusion, and whose necessity degree is defined in terms of two mutually recursive functions: $f_{\Phi}^{MP}(\cdot)$ (which propagates necessity degrees through individual rules, as *GMP*) and $f_{\Phi}^{+}(\cdot)$ (the accruing function, combining the necessity degrees of individual reasons supporting the same conclusion). As we do not want to commit ourselves to a specific way of calculating the aggregated value from a given set of necessity degrees we will assume that $f_{\Phi}^{+}(\cdot)$ is parameterized w.r.t. a user-specified function $ACC : [0,1]^2 \longrightarrow [0,1]$, satisfying the following (desirable) conditions[4]:

1. $ACC(x, y) = ACC(y, x)$ (commutativity)
2. $ACC(x, ACC(y, z)) = ACC(ACC(x, y), z)$ (associativity)
3. $ACC(x, y) \leq ACC(x, z)$ whenever $y \leq z$ (monotonicity)

[4]We consider that these conditions are necessary to ensure a sound election of the accruing function

4. $ACC(x, 0) = x$ (boundary condition)
5. $ACC(x, y) \geq max(x, y)$ (non-depreciation[5])
6. $ACC(x, y) = 1$ only if $x = 1$ or $y = 1$ (maximality)

Intuitively, we require commutativity and associativity since we want a unique way of aggregating a given set of possibilistic values, monotonicity since otherwise it would be unnatural that $ACC(x, y) < ACC(x, z)$ with $y > z$, boundary condition since $ACC(x, 0) > 0$ would allow for artificial accruals, non-depreciation since by accruing two arguments we should never obtain a weaker argument, and maximality since we consider that total certainty (necessity value 1) cannot be reached by accruing two uncertainty arguments. Moreover, it is interesting to note that conditions 1-4 characterize a well known class of aggregation functions known as triangular co-norms ([15]).

Definition 15 (Accrued Structure). *Let $\mathcal{K}$ be a possibilistic KB, and let Ω be a set of arguments in $\mathcal{K}$ supporting the same conclusion h, i.e., $\Omega = \{\langle \mathcal{A}_1, h, \alpha_1\rangle, ..., \langle \mathcal{A}_n, h, \alpha_n\rangle\}$[6]. We define the accrued structure for h (or just a-structure) from the set Ω (denoted $Accrual(\Omega)$) as a 3-uple $[\Phi, h, \alpha]$, where $\Phi = \mathcal{A}_1 \cup ... \cup \mathcal{A}_n$ and α is obtained using two mutually recursive functions, $f^+_\Phi(\cdot)$ and $f^{MP}_\Phi(\cdot)$, defined as follows. Let q be a literal appearing in Φ and let $(\varphi_1,\ \beta_1), ..., (\varphi_n,\ \beta_n)$ be all the weighted clauses in Φ with head q. Then*

$$f^+_\Phi(q) =_{def} ACC(f^{MP}_\Phi(\varphi_1),\ \ ..., ACC(f^{MP}_\Phi(\varphi_{n-1}),\ f^{MP}_\Phi(\varphi_n))...)$$

Let $(\varphi,\ \beta)$ be a weighted clause in Φ and let $\Phi' = \Phi \setminus \{(\varphi,\ \beta)\}$. Then

$$f^{MP}_\Phi(\varphi) =_{def} \begin{cases} \beta & if\ \varphi\ is\ a\ fact\ q; \\ \\ min(f^+_{\Phi'}(p_1), ..., f^+_{\Phi'}(p_n), \beta) & if\ \varphi = q \leftarrow p_1, ..., p_n \end{cases}$$

Finally, $\alpha = f^+_\Phi(h)$. When $\Omega = \emptyset$ we get the special accrued structure $[\emptyset, \epsilon, 0]$, representing the accrual of no argument[7].

[5]Non-depreciation is implied by conditions 1, 3 and 4, nonetheless we consider valuable to make this condition explicit.

[6]P-DeLP Possibilistic Knowledge Bases, as presented in Section 3, are finite and ground (do not involve variables), so that the number of subsets of a given possibilistic KB is also finite, as well as the number of arguments supporting a given conclusion h.

[7]Notice that this sentence is not a remark, but states a particular case of the definition

Although by definition the support set of an argument cannot be a recursive logic program, the support set of an accrual, defined as the union of the support sets of individual arguments, may indeed turn out to be recursive. To avoid a cyclic definition of the necessity measure α, the function $f_{\Phi}^{MP}(\varphi)$ considers the set $\Phi' = \Phi \setminus \{(\varphi,\ \beta)\}$ for the (indirectly) recursive case, avoiding in this way to subsequently reconsider the current rule φ.

Next we will present the function $ACC_{1'}$ (one-complement accrual) as a possible instantiation for ACC, which will be used in the examples that follow. Formally:

$$ACC_{1'}(\alpha_1, \alpha_2) = 1 - ((1 - \alpha_1) * (1 - \alpha_2))$$

It can be shown that $ACC_{1'}$ satisfies conditions 1-6.

Example 1. *Consider a possibilistic KB $\mathcal{K}$ where:*

$$\mathcal{K} = \left\{ \begin{array}{llll} (x \leftarrow z, k,\ 0.7) & (w \leftarrow r,\ 0.5) & ({\sim}o \leftarrow d,\ 0.9) & (p,\ 1) \\ (x \leftarrow y,\ 1) & (s \leftarrow p,\ 0.7) & ({\sim}y \leftarrow e,\ 1) & (q,\ 1) \\ (z \leftarrow t,\ 0.6) & ({\sim}w \leftarrow u,\ 0.75) & (e \leftarrow f,\ 0.4) & (r,\ 1) \\ (z \leftarrow v,\ 0.5) & ({\sim}w \leftarrow z,\ 1) & ({\sim}e \leftarrow d,\ 0.2) & (f,\ 1) \\ (y \leftarrow u,\ 0.3) & ({\sim}s \leftarrow o,\ 1) & ({\sim}x \leftarrow q,\ 0.45) & (d,\ 1) \\ ({\sim}z \leftarrow w,\ 0.4) & ({\sim}s \leftarrow t,\ 0.7) & (t,\ 1) \quad (u,\ 1) & (k,\ 1) \\ ({\sim}z \leftarrow s,\ 0.65) & (o \leftarrow q,\ 0.6) & (v,\ 1) \quad (w,\ 1) & \end{array} \right\}$$

Let $\langle \mathcal{A}_1, x, 0.6\rangle = \langle\{(x \leftarrow z, k,\ 0.7), (z \leftarrow t,\ 0.6), (t,\ 1), (k,\ 1)\}, x, 0.6\rangle$, $\langle \mathcal{A}_2, x, 0.5\rangle = \langle\{(x \leftarrow z, k,\ 0.7), (z \leftarrow v,\ 0.5), (v,\ 1), (k,\ 1)\}, x, 0.5\rangle$ *and* $\langle \mathcal{A}_3, x, 0.3\rangle = \langle\{(x \leftarrow y,\ 1), (y \leftarrow u,\ 0.3), (u,\ 1)\}, x, 0.3\rangle$ *be arguments in* $\mathcal{K}$. *Then* $Accrual(\{\langle \mathcal{A}_1, x, 0.6\rangle, \langle \mathcal{A}_3, x, 0.3\rangle\}) = [\Phi_1, z, 0.72]$ *where* $\Phi_1 = \{(x \leftarrow z, k,\ 0.7), (z \leftarrow t,\ 0.6), (t,\ 1), (k,\ 1), (x \leftarrow y,\ 1), (y \leftarrow u,\ 0.3), (u,\ 1)\}$ *(Fig. 4a)*

$Accrual(\{\langle \mathcal{A}_1, x, 0.6\rangle, \langle \mathcal{A}_2, x, 0.5\rangle\}) = [\Phi_2, x, 0.7]$ *where* $\Phi_2 = \{(x \leftarrow z, k,\ 0.7), (z \leftarrow t,\ 0.6), (t,\ 1), (k,\ 1), (z \leftarrow v,\ 0.5), (v,\ 1)\}$ *(Fig. 4b)*

of a-structure, that is, when $\Omega = \emptyset$. $[\emptyset, \epsilon, 0]$ is a special accrued structure representing the accrual of no argument and was introduced for uniformity purposes, as will be appreciated later on in the article. The reason to use a distinguished special conclusion ϵ is that since no argument is accrued, then no particular conclusion is involved. Although from the definition it happens that $\alpha > 0$ for any "none empty" accrued structure, we considered natural to adopt $\alpha = 0$ for the particular case of the a-structure representing the accrual of the empty set of arguments.

Accrual$(\{\langle \mathcal{A}_1, x, 0.6\rangle, \langle \mathcal{A}_2, x, 0.5\rangle, \langle \mathcal{A}_3, x, 0.3\rangle\}) = [\Phi_3, x, 0.79]$ *where* $\Phi_3 = \{(x \leftarrow z, k,\ 0.7), (z \leftarrow t,\ 0.6), (t,\ 1), (k,\ 1), (z \leftarrow v,\ 0.5), (v,\ 1), (x \leftarrow y,\ 1), (y \leftarrow u,\ 0.3), (u,\ 1)\}$ *(Fig. 4c).*

An a-structure for a conclusion h can be seen as a special kind of argument which subsumes different chains of reasoning which provide support for h. For instance, the a-structure $[\Phi_1, x, 0.72]$ (see Fig. 4a) provides two alternative chains of reasoning supporting x, both coming from each of the arguments accrued. The case of $[\Phi_2, x, 0.7]$ in Ex. 1 (see Fig. 4b) illustrates a situation similar to the previous one, but in this case the arguments involved share their topmost parts (more precisely the weighted clause $(x \leftarrow z, k,\ 0.7)$), differing in the reasons supporting the (shared) intermediate conclusion z. Figure 4 also shows how the possibilistic values associated with the depicted a-structures are obtained from the weighted clauses conforming them, using the functions $f^+_\Phi(\cdot)$ and $f^{MP}_\Phi(\cdot)$. Notice that weighted clauses were represented as black arrows labeled with their associated necessity measures. The values in gray ovals are computed using the mutually recursive functions.

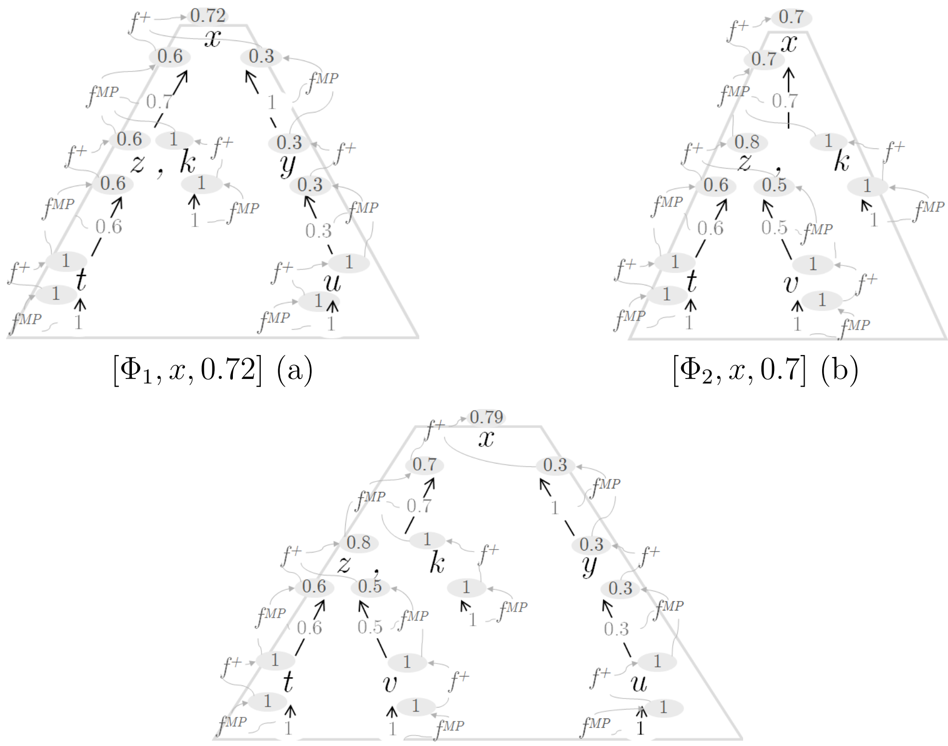
An important question that naturally emerges when considering the way we accrue arguments is what happens if we accrue two arguments that are in conflict (for instance because they have contradictory intermediate conclusions.) This issue will be dealt at the end of Section 6, where some consistency results regarding accrual acceptability are presented.

Definition 16. *Let* $[\Phi, h, \alpha]$ *be an a-structure. Then the set of* arguments in $[\Phi, h, \alpha]$*, denoted as* $Args([\Phi, h, \alpha])$*, is the set of all arguments* $\langle \mathcal{A}_i, h, \alpha_i\rangle$ *s.t.* $\mathcal{A}_i \subseteq \Phi$*. Note that* $Args([\emptyset, \epsilon, 0]) = \emptyset$*.*

Example 2. *Consider the arguments and a-structures presented in Ex. 1. Then* $Args([\Phi_1, x, 0.72]) = \{\langle \mathcal{A}_1, x, 0.6\rangle, \langle \mathcal{A}_3, x, 0.3\rangle\}$ *and*
$Args([\Phi_3, x, 0.79]) = \{\langle \mathcal{A}_1, x, 0.6\rangle, \langle \mathcal{A}_2, x, 0.5\rangle, \langle \mathcal{A}_3, x, 0.3\rangle\}$.

Among all the a-structures w.r.t. a KB $\mathcal{K}$, we will distinguish maximal a-structures, which are those (non-empty) a-structures obtained by accruing all the arguments supporting a given conclusion.

Definition 17 (Maximal a-structure). *Let* $\mathcal{K}$ *be a possibilistic KB and let* Ω_h *be the set of all arguments in* $\mathcal{K}$ *supporting a given conclusion* h*,* $\Omega_h \neq \emptyset$*. We say that the a-structure* $[\Phi, h, \alpha] = Accrual(\Omega_h)$ *is a* maximal *a-structure.*



$[\Phi_1, x, 0.72]$ (a) $[\Phi_2, x, 0.7]$ (b)

$[\Phi_3, x, 0.79]$ (c)

Figure 4: Accrued Structures

From the definition of maximal a-structure it trivially follows that, given a literal h, there exist at most one maximal a-structure supporting h. The set of individual arguments accounted by the maximal a-structure is finite (this trivially follows from the analysis in footnote 6), and then the support set Φ is also finite.

Example 3. *Consider the possibilistic KB $\mathcal{K}$ and the a-structures in Ex. 1. Then $[\Phi_3, x, 0.79]$ is a maximal a-structure in $\mathcal{K}$, whereas $[\Phi_1, x, 0.72]$ and $[\Phi_2, x, 0.7]$ are not.*

Next we will introduce the notion of *narrowing* of an a-structure, which is analogous to the notion of narrowing in [28]. Intuitively, a narrowing of an a-structure $[\Phi, h, \alpha]$ is an a-structure $[\Theta, h, \beta]$ accounting for a subset of $Args([\Phi, h, \alpha])$.

Definition 18 (Narrowing of an a-structure). *Let $[\Phi, h, \alpha]$ and $[\Theta, h, \beta]$ be two a-structures. We say that $[\Theta, h, \beta]$ is a* narrowing *of $[\Phi, h, \alpha]$, denoted as $[\Theta, h, \beta] \sqsubseteq [\Phi, h, \alpha]$, iff $Args([\Theta, h, \beta]) \subseteq Args([\Phi, h, \alpha])$.*

Example 4. *Consider the a-structures in Ex. 1. Then $[\Phi_1, x, 0.72]$, $[\Phi_2, x, 0.7]$ and $[\Phi_3, x, 0.79]$ are narrowings of $[\Phi_3, x, 0.79]$.*

Next we introduce the notion of accrued sub-structure, that is analogous to the notion of subargument but for a-structures. Intuitively, an accrued sub-structure of an a-structure $[\Phi, h, \alpha]$ is an a-structure supporting an intermediate conclusion k of $[\Phi, h, \alpha]$ and accounting for a subset of the reasons that support k in $[\Phi, h, \alpha]$. The one that accounts for all the reasons supporting k in $[\Phi, h, \alpha]$ is called *complete*.

Definition 19 (a-substructure and complete a-substructure). *Let $[\Phi, h, \alpha]$ and $[\Theta, k, \gamma]$ be two a-structures. Then we say that $[\Theta, k, \gamma]$ is an* accrued sub-structure *(or a-substructure) of $[\Phi, h, \alpha]$ iff $\Theta \subseteq \Phi$. We also say that $[\Theta, k, \gamma]$ is a* complete a-substructure *of $[\Phi, h, \alpha]$ iff for any other a-substructure $[\Theta', k, \gamma']$ of $[\Phi, h, \alpha]$ it holds that $\Theta' \subset \Theta$.*

Example 5. *Consider the a-structure $[\Phi_2, x, 0.7]$ in Ex. 1. Then the a-structures $[\{(z \leftarrow t,\ 0.6), (t,\ 1), (z \leftarrow v,\ 0.5), (v,\ 1)\}, z, 0.8]$, $[\{(z \leftarrow t,\ 0.6), (t,\ 1)\}, z, 0.6]$, and $[\Phi_2, x, 0.7]$ itself are a-substructures of $[\Phi_2, x, 0.7]$. Moreover, the two latter are complete.*

5. Modelling Conflict and Defeat in Accrued Structures

Next we will formalize the notion of attack between a-structures, which differs from the notion of attack in argumentation frameworks in several respects. First, an a-structure $[\Phi, h, \alpha]$ stands for (possibly) several chains of reasoning (arguments) supporting the conclusion h. Besides, some intermediate conclusions in $[\Phi, h, \alpha]$ could be shared by some, but not necessarily all the arguments in $[\Phi, h, \alpha]$. Thus, given two a-structures $[\Phi, h, \alpha]$ and $[\Psi, k, \beta]$, if $[\Phi, h, \alpha]$ involves $\overline{k}$ as intermediate conclusion, and then $[\Psi, k, \beta]$ contradicts $[\Phi, h, \alpha]$ at this literal, only those arguments in $Args([\Phi, h, \alpha])$ involving $\overline{k}$ will be affected by the conflict.

Next we will define the notion of *partial attack*, where the attacking a-structure generally affects only a narrowing of the attacked one (that one containing exactly the arguments in the attacked a-structure affected by the conflict), and we will refer to this narrowing as the *attacked narrowing.*

Definition 20 (Partial Attack and Attacked Narrowing). *Let* $[\Phi, h, \alpha]$ *and* $[\Psi, k, \beta]$ *be two a-structures. We say that* $[\Psi, k, \beta]$ partially attacks $[\Phi, h, \alpha]$ *(at literal* $\overline{k}$*), iff there exists a complete a-substructure* $[\Theta, \overline{k}, \gamma]$ *of* $[\Phi, h, \alpha]$*. The a-substructure* $[\Theta, \overline{k}, \gamma]$ *will be called the* disagreement a-substructure*. We will also say that* $[\Lambda, h, \delta] \sqsubseteq [\Phi, h, \alpha]$ *is the* attacked narrowing *of* $[\Phi, h, \alpha]$ *associated with the partial attack iff* $[\Lambda, h, \delta] = Accrual(\{\langle \mathcal{A}, h, \alpha_i \rangle \in Args([\Phi, h, \alpha]) \mid \textit{there exists a subargument } \langle \mathcal{S}, \overline{k}, \gamma_i \rangle \textit{ of } \langle \mathcal{A}, h, \alpha_i \rangle\})$ [8].

Example 6. *Consider the a-structures* $[\Phi_3, x, 0.79]$ *and* $[\Psi_1, \sim z, 0.82]$ *in Fig. 5. Then* $[\Psi_1, \sim z, 0.82]$ *partially attacks* $[\Phi_3, x, 0.79]$ *with disagreement a-substructure* $[\Theta, z, 0.8] = [\{(z \leftarrow t,\ 0.6), (t,\ 1), (z \leftarrow v,\ 0.5), (v,\ 1)\}, z, 0.8]$*. The attacked narrowing of* $[\Phi_3, x, 0.79]$ *is* $[\{(x \leftarrow z, k,\ 0.7), (z \leftarrow t,\ 0.6), (t,\ 1), (z \leftarrow v,\ 0.5), (v,\ 1)\}, x, 0.7]$*. Graphically, this partial attack relation will be depicted with a dotted arrow (see Fig. 5).*

5.1. Accrued Structures: Evaluation and Defeat

As in P-DeLP, we will use the necessity measures associated with a-structures in order to decide if a partial attack really succeeds and constitutes a defeat.

[8] For simplicity, we will often say just attack instead of partial attack when it is clear we are referring to a-structures.

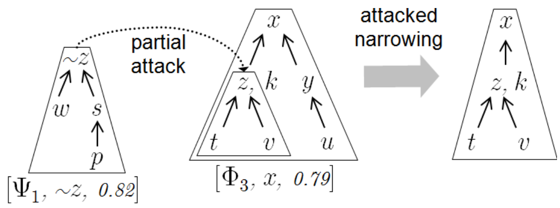


Figure 5: Partial Attack

Definition 21 (Partial Defeater). *Let* $[\Phi, h, \alpha]$ *and* $[\Psi, k, \beta]$ *be two a-structures. Then we say that* $[\Psi, k, \beta]$ *is a* partial defeater *of* $[\Phi, h, \alpha]$ *(or equivalently that* $[\Psi, k, \beta]$ *is a successful attack on* $[\Phi, h, \alpha]$*) iff 1)* $[\Psi, k, \beta]$ *partially attacks* $[\Phi, h, \alpha]$*, where* $[\Theta, \overline{k}, \gamma]$ *is the disagreement a-substructure, and 2)* $\beta \geq \gamma$*.*

Example 7. *Consider the partial attack from* $[\Psi_1, {\sim}z, 0.82]$ *against* $[\Phi_3, x, 0.79]$ *with disagreement a-substructure* $[\Theta, z, 0.8]$ *in Ex. 6 (Fig. 5). As the necessity measure associated with the attacking a-structure (0.82) is greater than the one associated with the disagreement a-substructure (0.8), then the attack succeeds, constituting a defeat. Graphically, this defeat relation will be depicted with a continuous arrow (see Fig. 6).*

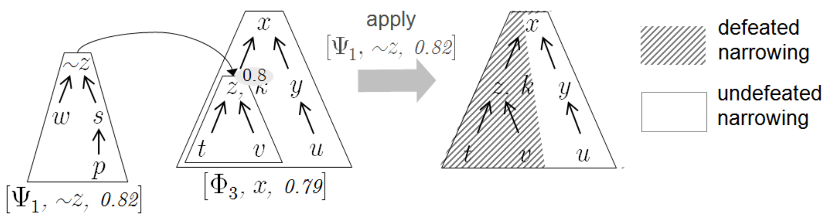


Figure 6: Defeated and Undefeated Narrowings

Given an attack relation, we will identify two complementary narrowings associated with the attacked a-structure: the narrowing that becomes defeated as a consequence of the attack, and the narrowing that remains undefeated.

Definition 22 (U/D-Narrowings). *Let* $[\Phi, h, \alpha]$ *and* $[\Psi, k, \beta]$ *be two a-structures such that* $[\Psi, k, \beta]$ *attacks* $[\Phi, h, \alpha]$*. Let* $[\Lambda, h, \delta]$ *be the attacked narrowing of* $[\Phi, h, \alpha]$*. Then the D-narrowing (*defeated narrowing*) of* $[\Phi, h, \alpha]$ *associated with the attack, denoted as* $DNwg([\Phi, h, \alpha], [\Psi, k, \beta])$*, is defined by cases as follows:*

- $DNwg([\Phi, h, \alpha], [\Psi, k, \beta]) =_{def} [\Lambda, h, \delta]$*, if* $[\Psi, k, \beta]$ *is a partial defeater of* $[\Phi, h, \alpha]$*, or*

- $DNwg([\Phi, h, \alpha], [\Psi, k, \beta]) =_{def} [\emptyset, \epsilon, 0]$, *otherwise.*

*The U-narrowing (*undefeated narrowing*) of* $[\Phi, h, \alpha]$ *associated with the attack, denoted as* $UNwg([\Phi, h, \alpha], [\Psi, k, \beta])$*, is the a-structure* $Accrual(Args([\Phi, h, \alpha]) \setminus Args(DNwg([\Phi, h, \alpha], [\Psi, k, \beta])))$.

Example 8. *Fig. 6 illustrates a successful attack from* $[\Psi_1, \sim z, 0.82]$ *against* $[\Phi_3, x, 0.79]$*, as well as the associated defeated and undefeated narrowings of* $[\Phi_3, x, 0.79]$*. As another example, consider the attack from* $[\Psi_2, \sim x, 0.45] = [\{(\sim x \leftarrow q,\ 0.45), (q,\ 1)\}, \sim x, 0.45]$ *against* $[\Phi_3, x, 0.79]$*, with* $[\Phi_3, x, 0.79]$ *itself as disagreement a-substructure. In this case the attack does not succeed, and then* $[\emptyset, \epsilon, 0]$ *is the defeated narrowing and* $[\Phi_3, x, 0.79]$ *is the undefeated narrowing.*

Definition 23 (Proper and Blocking partial defeaters). *Let* $[\Phi, h, \alpha]$ *and* $[\Psi, k, \beta]$ *be two a-structures. We will say that* $[\Psi, k, \beta]$ *is a* blocking (partial) defeater *of* $[\Phi, h, \alpha]$ *if* $[\Psi, k, \beta]$ *partially defeats* $[\Phi, h, \alpha]$*, with* $[\Theta, \overline{k}, \gamma]$ *as the associated disagreement a-substructure, and* $[\Theta, \overline{k}, \gamma]$ *(partially) defeats* $[\Psi, k, \beta]$*. We will say that* $[\Psi, k, \beta]$ *is a* proper (partial) defeater *of* $[\Phi, h, \alpha]$ *if* $[\Psi, k, \beta]$ *partially defeats* $[\Phi, h, \alpha]$*, but it is not the case that* $[\Theta, \overline{k}, \gamma]$ *(partially) defeats* $[\Psi, k, \beta]$[9]*.*

5.2. Combined Attack

Until now we have considered only *single* attacks. When a single attack succeeds, a nonempty narrowing of the attacked a-structure becomes defeated. But two or more a-structures could simultaneously attack another, possibly affecting different narrowings of the target a-structure, and thus causing a bigger narrowing to become defeated (compared with the defeated narrowings associated with the individual attacks). Fig. 7a illustrates a *combined* attack from the a-structures $[\Psi_1, \sim z, 0.82]$ and $[\Psi_3, \sim y, 0.4]$ against $[\Phi_3, x, 0.79]$. Even though each attacking a-structure defeats only a proper narrowing of $[\Phi_3, x, 0.79]$, the whole $[\Phi_3, x, 0.79]$ becomes defeated after applying *both* attacks.

Consider now the combined attack against $[\Phi_3, x, 0.79]$ shown in Fig. 7b. One of the attacking a-structures ($[\Psi_1, \sim z, 0.82]$) defeats a narrowing of

[9] Equivalently, the partial defeater $[\Psi, k, \beta]$ will be proper iff $\beta > \gamma$, and blocking iff $\beta = \gamma$.

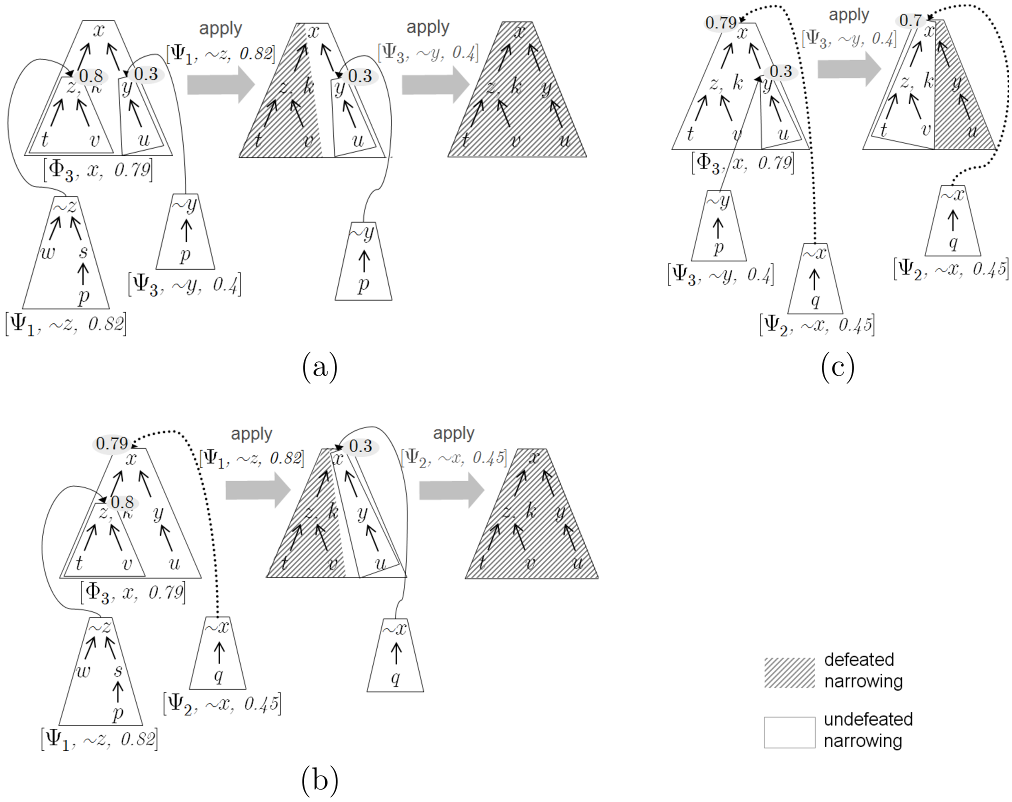


Figure 7: Combined Defeat

$[\Phi_3, x, 0.79]$ on its own, whereas the other ($[\Psi_2, \sim x, 0.45]$) only attacks $[\Phi_3, x, 0.79]$. Note also that, although $[\Phi_3, x, 0.79]$ is stronger than $[\Psi_2, \sim x, 0.45]$, $[\Psi_2, \sim x, 0.45]$ is stronger than $[\Phi', x, 0.3] = [\{(x \leftarrow y,\ 1), (y \leftarrow u,\ 0.3), (u,\ 1)\}, x, 0.3]$, a proper narrowing of $[\Phi_3, x, 0.79]$. Then, as shown in Fig. 7b, when the a-structures $[\Psi_1, \sim z, 0.82]$ and $[\Psi_2, \sim x, 0.45]$ combine their attacks, they cause the whole $[\Phi_3, x, 0.79]$ to become defeated. The reason is that the successful attack of $[\Psi_1, \sim z, 0.82]$ weakens the target a-structure, allowing the attack of $[\Psi_2, \sim x, 0.45]$ to succeed. Figure 7c illustrates a combined attack from $[\Psi_2, \sim x, 0.45]$ and $[\Psi_3, \sim y, 0.4]$ against $[\Phi_3, x, 0.79]$. In this case, a nonempty narrowing of the attacked a-structure remains undefeated.

Figures 7a, 7b and 7c suggests the following algorithmic procedure for computing the undefeated narrowing associated with a combined attack from a set Σ against an a-structure $[\Phi, h, \alpha]$:

1. Pick a defeater in Σ of $[\Phi, h, \alpha]$ (if any) and apply it, obtaining an undefeated narrowing $[\Theta, h, \beta]$ of $[\Phi, h, \alpha]$.
2. Repeat step 1 taking the resulting a-structure $[\Theta, h, \beta]$ as the new target for defeaters, until there is no more defeaters for $[\Theta, h, \beta]$ in Σ.

Notice that, according to this procedure, defeaters of $[\Phi, h, \alpha]$ are applied in sequence (in some order), and each defeater application causes a narrowing of the target a-structure to become defeated. In other words, the a-structure $[\Phi, h, \alpha]$ is sequentially degraded through defeater applications. The following definitions provide a formal characterization of the procedure presented.

Definition 24 (Sequential Degradation). *Let $[\Phi, h, \alpha]$ be an a-structure and let Σ be a set of a-structures attacking $[\Phi, h, \alpha]$. A* sequential degradation *of $[\Phi, h, \alpha]$ associated with the combined attack of the a-structures in Σ, consists of a finite sequence Υ of narrowings of $[\Phi, h, \alpha]$:*

$$\Upsilon = [\Phi_1, h, \alpha_1],\ [\Phi_2, h, \alpha_2],\ \ldots, [\Phi_{m+1}, h, \alpha_{m+1}]$$

provided there exists a finite sequence of a-structures in Σ:

$$[\Psi_1, k_1, \beta_1],\ [\Psi_2, k_2, \beta_2],\ \ldots, [\Psi_m, k_m, \beta_m]$$

where $[\Phi_1, h, \alpha_1] = [\Phi, h, \alpha]$, for each i, $1 \leq i \leq m$, $[\Psi_i, k_i, \beta_i]$ is a (partial) defeater of $[\Phi_i, h, \alpha_i]$ with associated undefeated narrowing $[\Phi_{i+1}, h, \alpha_{i+1}]$ and $[\Phi_{m+1}, h, \alpha_{m+1}]$ has no defeaters in Σ.

Given a combined attack against an a-structure $[\Phi, h, \alpha]$, there could exist several possible orders of defeater applications, and hence, more than one sequential degradation associated with the combined attack. Interestingly, it can be shown that all sequential degradations associated with a given combined attack converge to the same a-structure, provided that the function ACC satisfies non-depreciation.

Theorem 1 (Convergence). *Let $[\Phi, h, \alpha]$ be an a-structure and let Σ be a set of a-structures attacking $[\Phi, h, \alpha]$. Let $\Upsilon = [\Phi_1, h, \alpha_1], \ldots, [\Phi_m, h, \alpha_m]$ and $\Upsilon' = [\Phi'_1, h, \alpha'_1], \ldots, [\Phi'_n, h, \alpha'_n]$ be two sequential degradations of $[\Phi, h, \alpha]$ associated with the combined attack of the a-structures in Σ. Then $[\Phi_m, h, \alpha_m] = [\Phi'_n, h, \alpha'_n]$, provided that the ACC function satisfies non-depreciation* [10].

[10]Proofs of theorems and lemmas are included in the appendix.

Definition 25 (Narrowings associated with a Combined Attack). *Let* $[\Phi, h, \alpha]$ *be an a-structure and let* Σ *be a set of a-structures attacking* $[\Phi, h, \alpha]$. *Let* $[\Phi_1, h, \alpha_1], ..., [\Phi_{m+1}, h, \alpha_{m+1}]$ *be a sequential degradation of* $[\Phi, h, \alpha]$ *associated with the combined attack of the a-structures in* Σ. *Then* $[\Phi_{m+1}, h, \alpha_{m+1}]$ *is the U-narrowing of* $[\Phi, h, \alpha]$ *(noted* $UNwg([\Phi, h, \alpha],\ \Sigma)$*) associated with the combined attack, and* $Accrual(Args([\Phi, h, \alpha]) \setminus Args([\Phi_{m+1}, h, \alpha_{m+1}]))$ *is its D-narrowing (noted* $DNwg([\Phi, h, \alpha],\ \Sigma)$*).*

Example 9. *Consider the combined attack of* $[\Psi_1, {\sim}z, 0.82]$ *and* $[\Psi_2, {\sim}x, 0.45]$ *against* $[\Phi_3, x, 0.79]$ *(Fig. 7b). The associated undefeated narrowing of* $[\Phi_3, x, 0.79]$ *is* $[\emptyset, \epsilon, 0]$, *i.e., the whole* $[\Phi_3, x, 0.79]$ *results defeated. On the other hand, when* $[\Psi_2, {\sim}x, 0.45]$ *and* $[\Psi_3, {\sim}y, 0.4]$ *attack* $[\Phi_3, x, 0.79]$ *(Fig. 7c), its associated undefeated narrowing is* $[\{(x \leftarrow z, k,\ 0.7), (z \leftarrow t,\ 0.6), (t,\ 1), (z \leftarrow v,\ 0.5), (v,\ 1)\}, x, 0.7]$.

5.2.1. Proper-defeating U-narrowings

According to definitions 24 and 25, both kinds of defeaters (proper and blocking) are taken into account in order to obtain the U-narrowing of a given a-structure. The following definitions are "proper-defeating" versions of definitions 24 and 25, *i.e.*, where only proper defeaters are considered. These notions will be used to impose, in our formalization of accrual, a skeptical restriction similar to those for the abstract and possibilistic frameworks, requiring proper defeat for PRO.

Definition 26 (proper-defeating Sequential Degradation). *Let* $[\Phi, h, \alpha]$ *be an a-structure and let* Σ *be a set of a-structures attacking* $[\Phi, h, \alpha]$. *A* proper-defeating sequential degradation *of* $[\Phi, h, \alpha]$ *associated with the combined attack of the a-structures in* Σ*, consists of a finite sequence* Υ *of narrowings of* $[\Phi, h, \alpha]$*:*

$$\Upsilon = [\Phi_1, h, \alpha_1],\ [\Phi_2, h, \alpha_2],\ \ldots, [\Phi_{m+1}, h, \alpha_{m+1}]$$

provided there exists a finite sequence of a-structures in Σ*:*

$$[\Psi_1, k_1, \beta_1],\ [\Psi_2, k_2, \beta_2],\ \ldots, [\Psi_m, k_m, \beta_m]$$

where $[\Phi_1, h, \alpha_1] = [\Phi, h, \alpha]$*, for each* i*,* $1 \leq i \leq m$*,* $[\Psi_i, k_i, \beta_i]$ *is a (partial) proper defeater of* $[\Phi_i, h, \alpha_i]$ *with associated undefeated narrowing* $[\Phi_{i+1}, h, \alpha_{i+1}]$ *and* $[\Phi_{m+1}, h, \alpha_{m+1}]$ *has no proper defeaters in* Σ.

Definition 27 (proper-defeating Narrowings for Combined Attack). *Let* $[\Phi, h, \alpha]$ *be an a-structure and let* Σ *be a set of a-structures attacking* $[\Phi, h, \alpha]$. *Let* $[\Phi_1, h, \alpha_1], ..., [\Phi_{m+1}, h, \alpha_{m+1}]$ *be a proper defeating sequential degradation of* $[\Phi, h, \alpha]$ *associated with the combined attack of the a-structures in* Σ. *Then* $[\Phi_{m+1}, h, \alpha_{m+1}]$ *is the proper defeating U-narrowing of* $[\Phi, h, \alpha]$ *(noted* $UNwgP([\Phi, h, \alpha],\ \Sigma)$*) associated with the combined attack, and* $Accrual(Args([\Phi, h, \alpha]) \setminus Args([\Phi_{m+1}, h, \alpha_{m+1}]))$ *is its proper defeating D-narrowing (noted* $DNwgP([\Phi, h, \alpha],\ \Sigma)$*).*

It must be remarked that convergence (theorem 1) also holds for proper-defeating sequential degradations. The proof is the same as for sequential degradation, but replacing the notions of defeat and defeater by proper defeat and proper defeater, respectively [11]. The following theorem states that the undefeated narrowing emerging as a result of a sequential degradation is always a narrowing of the one emerging of a proper-defeating sequential degradation, and the reason is that blocking defeaters (if any) are applied only in the former, causing a 'smaller' narrowing to remain undefeated.

Theorem 2. *Let* $[\Phi, h, \alpha]$ *be an a-structure and let* Σ *be a set of a-structures attacking* $[\Phi, h, \alpha]$. *Then* $UNwg([\Phi, h, \alpha],\ \Sigma) \sqsubseteq UNwgP([\Phi, h, \alpha],\ \Sigma)$

6. Dialectical Analysis for Accrued Structures

Given a possibilistic KB $\mathcal{K}$ and a literal h, we are interested in determining if h is ultimately accepted (or warranted), and if so, with which necessity degree, but this time, by taking accrual of arguments into account. With this purpose, in this section we present a dialectical analysis that has a strong correspondence with the one presented in section 3.2 involving *individual* (possibilistic) arguments. This analysis is also formalized through a dialectical tree, but where nodes stand for a-structures instead of individual arguments. Next we present the notion of accrued dialectical line, which is a sequence of a-structures with certain restrictions, each one related with one of the conditions of the notion of dialectical line presented in section 3.2.

[11] In addition, a new proper-defeating version of lemma 1 (see section Appendix A) of theorem 1 will be needed, and the associated proof can be obtained by changing defeat by proper defeat and '$\geq$' by '$>$' in the original proof.

Definition 28 (Accrued Dialectical Line). *Let* $[\Phi, h, \alpha] \in Args$. *An* accrued dialectical line about $[\Phi, h, \alpha]$ *(or just accrued dialectical line) is a finite nonempty sequence of a-structures* $[\ [\Phi_1, h_1, \alpha_1],\ [\Phi_2, h_2, \alpha_2],\ \ldots,\ [\Phi_n, h_n, \alpha_n]\ ]$, *with* $[\Phi_1, h_1, \alpha_1] = [\Phi, h, \alpha]$, *such that:*

(a1) $[\Phi_i, h_i, \alpha_i]$ *partially attacks* $[\Phi_{i-1}, h_{i-1}, \alpha_{i-1}]$, $1 < i \leq n$,

(a2) *if* $[\Phi_i, h_i, \alpha_i]$ *is* PRO *and* Σ *is the set of all disagreement a-substructures associated with the attacks against* PRO *a-structures appearing before* $[\Phi_i, h_i, \alpha_i]$ *in the line, then* $[\Phi_i, h_i, \alpha_i]$ *is maximal (w.r.t.* $\sqsubseteq$*) verifying that no narrowing of an a-structure in* Σ *is an a-substructure of* $[\Phi_i, h_i, \alpha_i]$,

(a3) *if* $[\Phi_i, h_i, \alpha_i]$ *is* CON, *then it is a maximal a-structure,*

(a4) *If* $[\Phi_i, h_i, \alpha_i]$ *and* $[\Phi_{i+2}, h_{i+2}, \alpha_{i+2}]$ *are* CON *a-structures, and* $[\Theta, k, \beta]$ *is the disagreement a-substructure associated with the attack of* $[\Phi_{i+1}, h_{i+1}, \alpha_{i+1}]$ *against* $[\Phi_i, h_i, \alpha_i]$, *then* $[\Theta, k, \beta] \neq [\Phi_{i+2}, h_{i+2}, \alpha_{i+2}]$,

where $[\Phi_i, h_i, \alpha_i]$ *is said to be a* PRO *a-structure of the line if i is odd, or a* CON *a-structure of the line if i is even.*

Observation: as every a-structure $[\Phi_i, h_i, \alpha_i]$ in the line attacks or is attacked (or both) by another a-structure in the line (condition a1), then $[\Phi_i, h_i, \alpha_i] \neq [\emptyset, \epsilon, 0]$.

Next we will analyze each condition a1-a4 in definition 28, comparing them with conditions d1'-d3' in definition 14. Whereas condition d1' requires each argument in the line to defeat the argument appearing immediately before, condition a1 requires each a-structure in a line to *partially attack* its previous a-structure. The reason to consider partial attacks, and not just partial defeats, is that as shown in section 5.2, an a-structure that partially attacks a given target a-structure $[\Phi, h, \alpha]$, but does not defeat it, may become a defeater when considering other a-structures attacking $[\Phi, h, \alpha]$ (combined defeat).

Condition a2 is the natural adaptation for a-structures of d2'. Condition d2' avoids the inclusion of a given PRO argument when it contains a disagreement subargument of a previous PRO argument. Since a given a-structure generally stands for several arguments for a given conclusion, each of them involving different subarguments, we want to include in the line the maximal PRO a-structure such that none of the arguments it accounts for has as a subargument (an element of) a previous PRO disagreement a-substructure.

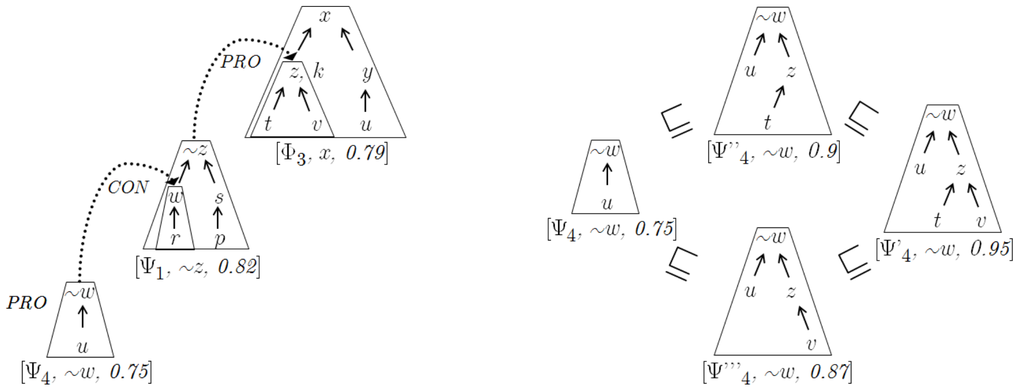
Condition a3, which has not corresponding condition in definition 14, states that CON a-structures are maximal, reflecting that CON has no restriction regarding other CON a-structures appearing before in the line.

Since partial attacks may become defeats when considering combined attacks, when defining accrued dialectical lines we do not know if a given attack will become a defeat, and even less if it will be proper or blocking. For this reason condition d3' has not corresponding condition in definition 28, but its purpose is fulfilled in the dialectical evaluation analysis we present next, where defeats are revealed.

Finally, the purpose of condition a4 is to retain the indirect effect of conditions d3 and d3' (as specified by propositions 1 and 2, respectively) of avoiding consecutive CON repetition, despite the impossibility of enforcing a condition equivalent to d3 or d3' when defining accrued dialectical line.

Example 10. *Figure 8a shows an accrued dialectical line about* $[\Phi_3, x, 0.79]$*, distinguishing* PRO *and* CON *a-structures. It is easy to see that conditions a1 and a3 are satisfied. Consider now condition a2. The first a-structure (*PRO*) in the sequence,* $[\Phi_3, x, 0.79]$*, is a maximal a-structure w.r.t.* $\mathcal{K}$*. According to condition a2, the first a-structure in a given accrued dialectical line is always a maximal a-structure, since* $\Sigma = \emptyset$*. Let us see that* $[\Psi_4, {\sim}w, 0.75]$ *(the other* PRO *a-structure in the line) satisfies a2. Consider* $[\Theta, z, 0.8]$*, with* $\Theta = \{(z \leftarrow t,\ 0.6), (z \leftarrow v,\ 0.5), (t,\ 1), (v,\ 1)\}$*, the unique disagreement a-substructure of a* PRO *argument appearing before* $[\Psi_4, {\sim}w, 0.75]$ *(i.e.,* $\Sigma = \{[\Theta, z, 0.8]\}$*). First, no narrowing of* $[\Theta, z, 0.8]$ *is an a-substructure of* $[\Psi_4, {\sim}w, 0.75]$*. Second,* $[\Psi_4, {\sim}w, 0.75]$ *is maximal verifying such a restriction, since all the a-structures in* $\mathcal{K}$ *that are "greater" (according to '*$\sqsubseteq$*') than* $[\Psi_4, {\sim}w, 0.75]$ *(*$[\Psi'_4, {\sim}w, 0.95]$*,* $[\Psi''_4, {\sim}w, 0.9]$ *and* $[\Psi^{(3)}_4, {\sim}w, 0.87]$ *in Fig. 8b) have a narrowing of* $[\Theta, z, 0.8]$ *as an a-substructure. Finally, note that a4 is also satisfied, since no disagreement a-substructure of a* CON *a-structure appears in the next consecutive* CON *position in the line.*

Figure 9 shows two sequences of a-structures that are not accrued dialectical lines. The sequence in Figure 9a illustrates a general restriction of dialectical lines consequence of condition a2: a disagreement a-substructure of a given PRO *argument (in this case* $[\Theta, z, 0.8]$*) cannot be introduced later in the sequence. In this particular case, note that the last a-structure in the*



(a) (b)

Figure 8: Accrued Dialectical Line

sequence, $[\Theta, z, 0.8]$, violates condition a2, since it has as an a-substructure a narrowing of a previous PRO *disagreement a-substructure ($[\Theta, z, 0.8]$ itself). The sequence of a-structures in Fig. 9b violates condition a4, since the disagreement a-substructure $[\{(w \leftarrow r,\ 0.5), (r,\ 1)\}, w, 0.5]$ of* CON *a-structure $[\Psi_1, \sim\!z, 0.82]$ appears as the* CON *a-structure that consecutively follows $[\Psi_1, \sim\!z, 0.82]$ in the sequence.*

Next we introduce the notions of *exhaustive accrued dialectical line* and *accrued dialectical tree*, which are direct instantiations of the corresponding notions for the *AAF* presented in Section 2.

Definition 29 (Exhaustive Accrued Dialectical Line). *Let $\Lambda = [\ [\Phi_1, h_1, \alpha_1], [\Phi_2, h_2, \alpha_2], ..., [\Phi_n, h_n, \alpha_n]\ ]$ be an accrued dialectical line. We will say that Λ is exhaustive if there exist no a-structure $[\Theta, q, \delta]$ such that $[\ [\Phi_1, h_1, \alpha_1], [\Phi_2, h_2, \alpha_2], ..., [\Phi_n, h_n, \alpha_n], [\Theta, q, \delta]\ ]$ is an accrued dialectical line.*

Definition 30 (Accrued Dialectical Tree). *Let $[\Phi, h, \alpha]$ be an a-structure. An accrued dialectical tree (of just ADT) for $[\Phi, h, \alpha]$, denoted $\mathrm{T}_{[\Phi,h,\alpha]}$, is defined as follows:*

1. *Nodes are labeled with a-structures.*
2. *$\Lambda = [\ [\Phi_1, h_1, \alpha_1], [\Phi_2, h_2, \alpha_2], .., [\Phi_n, h_n, \alpha_n]\ ]$ is the sequence of labels of a branch of the tree (path from the root to a leave) iff Λ is an exhaustive accrued dialectical line about $[\Phi, h, \alpha]$.*

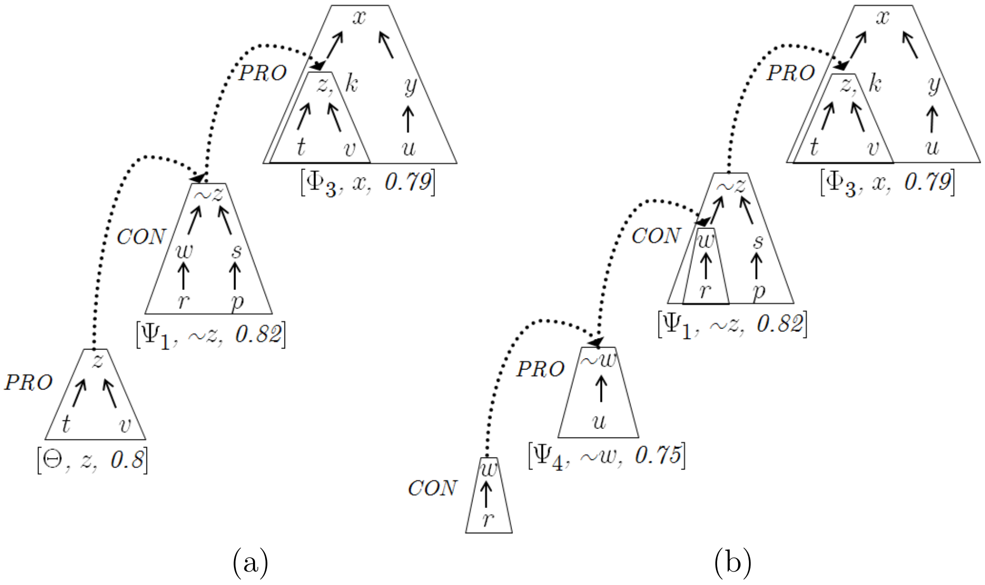


Figure 9: Sequences that are not Accrued Dialectical Lines

3. *There exist no sibling nodes (children of the same node) M and M' in the tree labeled with the same a-structure.*

Once the *ADT* has been constructed, it must be evaluated in order to determine the final undefeated narrowing of its root. Concretely, each combined attack is analyzed, from the deepest ones to the one against the root, in order to determine the undefeated narrowing of each node in the tree.

Definition 31 (Evaluated Accrued Dialectical Tree). *Let $\mathrm{T}_{[\Phi,h,\alpha]}$ be an ADT. The corresponding evaluated ADT, denoted $\mathrm{T}^*_{[\Phi,h,\alpha]}$, will be obtained by associating to every node N in $\mathrm{T}_{[\Phi,h,\alpha]}$, labeled with an a-structure $[\Theta, k, \beta]$, a (possibly empty) narrowing of $[\Theta, k, \beta]$ that will be called the U-narrowing of N (written $UNwg(N)$), and is defined as follows:*

1. *If N is a leaf, then $UNwg(N) = [\Theta, k, \beta]$.*
2. *Otherwise, let $M_1,\ldots,M_n$ be the children of N, and let $\Sigma = \{UNwg(M_i) \mid UNwg(M_i) \neq [\emptyset, \epsilon, 0],\ 1 \leq i \leq n\}$. If N is* CON *(i.e., $M_1,\ldots,M_n$ are* PRO*), then $UNwg(N) = UNwgP([\Theta, k, \beta],\ \Sigma)$. If N is* PRO *(i.e., $M_1,\ldots,M_n$ are* CON*), then $UNwg(N) = UNwg([\Theta, k, \beta],\ \Sigma)$.*

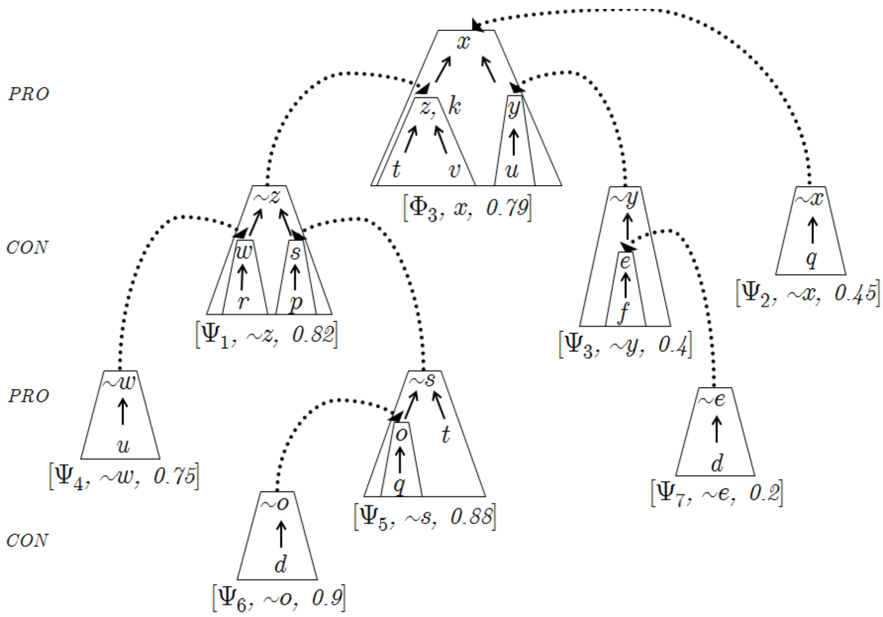


Figure 10: Accrued Dialectical Tree

Note that the purpose of condition d3 for the AAF, requiring PRO defeats to be proper, is fulfilled in the accrual framework by using $UNwgP([\Theta, k, \beta],\ \Sigma)$ when evaluating PRO partial attacks.

Example 11. *Fig. 10 shows the ADT for* $[\Phi_3, x, 0.79]$ *w.r.t. knowledge base* $\mathcal{K}$ *(Ex. 1). Fig. 11 shows the evaluated ADT for* $[\Phi_3, x, 0.79]$*, where the undefeated narrowings of each node are highlighted.*

Definition 32 (Warranted a-structures and conclusions). *Let* $\mathcal{K}$ *be a possibilistic* KB*. Let* $[\Phi, h, \alpha]$ *be a maximal a-structure of* $\mathcal{K}$ *such that the undefeated narrowing of the root of* $\mathrm{T}^*_{[\Phi,h,\alpha]}$ *is a nonempty a-structure* $[\Phi', h, \alpha']$*. Then we say that* $[\Phi', h, \alpha']$ *is a* warranted a-structure *and that* h *is* warranted *with necessity* α' *w.r.t.* $\mathcal{K}$*.*

Returning to our motivating example, we specify next a P-DeLP program modeling the different reasons posed by Alice in favor and against renting an apartment. The program considers additional information with respect to the original example, viz. the testimony of a neighbor of the building, called John (j), stating that there are not disorders. For this particular

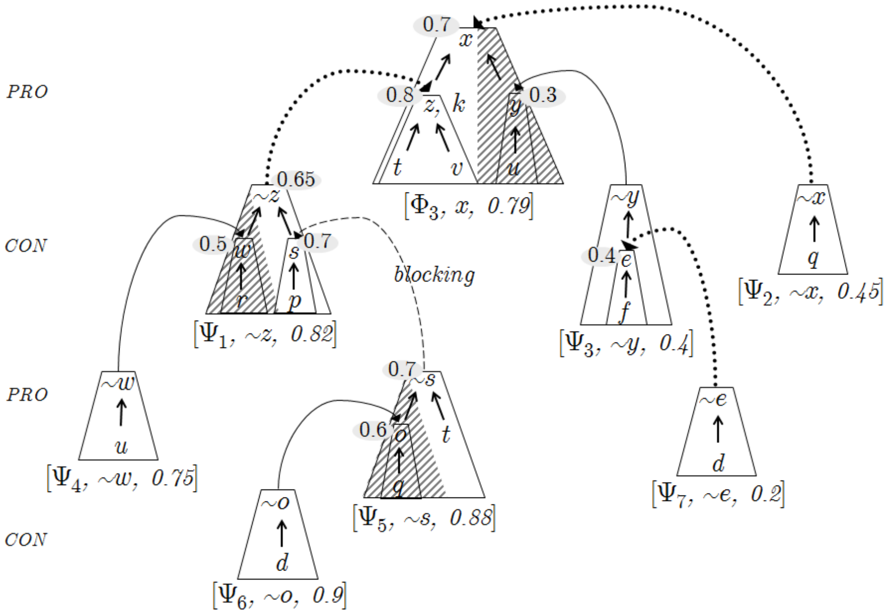


Figure 11: Evaluated Accrued Dialectical Tree

example, the possibilistic value associated to each program rule represents a quantitative measure of the strength, according to Alice criteria, with which the rule's premisses entails its conclusion.

$(rent \leftarrow g\_loc,\ 1)$
$({\sim}rent \leftarrow small,\ 0.5)$
$({\sim}rent \leftarrow humidity,\ 0.2)$
$({\sim}rent \leftarrow disorder,\ 0.4)$
$(disorder \leftarrow stud\_tenants,\ 0.8)$
$({\sim}disorder \leftarrow test_j,\ 0.7)$

$(g\_loc,\ 1)$
$(small,\ 1)$
$(humidity,\ 1)$
$(stud\_tenants,\ 1)$
$(test_j,\ 1)$

Fig. 12 shows the evaluated ADT for the maximal a-structure supporting $rent$ ($[\Phi, rent, 0.7]$). Notice that the testimony of John supporting the absence of disorders (0.7) is not strong enough to defeat the a-substructure of $[\Psi, {\sim}rent, 0.76]$ stating that there exist disorders given that there are student tenants (0.8). Then $[\Psi, {\sim}rent, 0.76]$ remains completely undefeated, making the a-structure for $rent$ to become defeated, and so, the conclusion $rent$ is not warranted. The evaluated ADT in Fig. 13 corresponds to the new situation where a testimony of another neighbor (Paul) is considered,

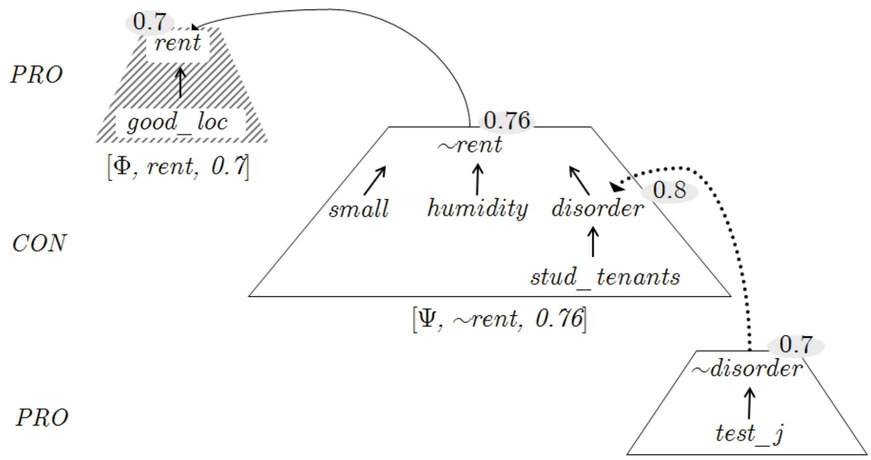


Figure 12: Evaluated Accrued Dialectical Tree

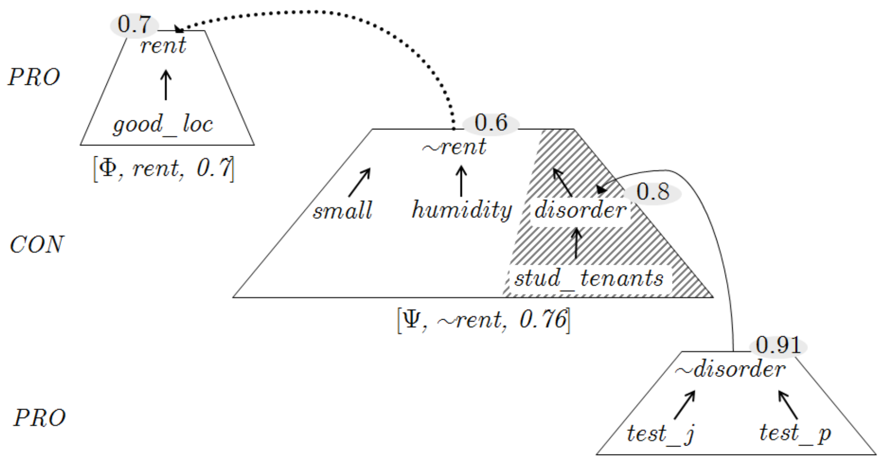


Figure 13: Evaluated Accrued Dialectical Tree

stating also that there are not disorders in the building (either with necessity degree 0.7). In this case, the accrual of both testimonies defeats the a-substructure of $[\Psi, \sim rent, 0.76]$ supporting *disorder*. Then the remaining undefeated narrowing of $[\Psi, \sim rent, 0.76]$ (with necessity 0.6) is not able to defeat $[\Phi, rent, 0.7]$, and so the conclusion *rent* results warranted with necessity degree 0.7.

Let us now consider some desirable results about our formalization. The U-narrowings associated with PRO nodes in an evaluated *ADT* can be seen as the a-structures effectively supporting (or defending) the warranted a-structure labeling the root of the tree. Therefore, it is desirable that those supporting a-structures do not conflict with each other. The following theorem states that the U-narrowings associated with the PRO nodes of a given evaluated *ADT* are concordant pairwise.

Theorem 3 (Pairwise pro concordance). *Let $\mathcal{K}$ be a possibilistic KB, and let $\mathrm{T}_{[\Phi,h,\alpha]}$ be an ADT for a given a-structure $[\Phi, h, \alpha]$ of $\mathcal{K}$. Let N_1 and N_2 be* PRO*-nodes in $\mathrm{T}_{[\Phi,h,\alpha]}$. Then there exist no literal q such that q appears in $UNwg(N_1)$ and $\overline{q}$ appears in $UNwg(N_2)$.*

Corollary 1 (Internal pro concordance). *Let $\mathcal{K}$ be a possibilistic KB, and let $\mathrm{T}_{[\Phi,h,\alpha]}$ be an ADT for a given a-structure $[\Phi, h, \alpha]$ of $\mathcal{K}$. Let N be* PRO*-node in $\mathrm{T}_{[\Phi,h,\alpha]}$. Then there exist no literal q such that q and $\overline{q}$ appear in $UNwg(N)$.*

The following corollary establishes that the a-structures emerging as a result of the dialectical process presented in this section cannot involve contradictory literals.

Corollary 2 (Internal concordance for Warranted a-structures). *Let $\mathcal{K}$ be a possibilistic KB, and let $[\Phi, h, \alpha]$ be a warranted a-structure w.r.t. $\mathcal{K}$. Then there exist no literal q such that q and $\overline{q}$ appear in $[\Phi, h, \alpha]$.*

Finally we present a pseudocode algorithm (the *UNarrowing* function) allowing to determine if a given conclusion h is warranted, and if so, the associated necessity degree α', according to Def. 32. Firstly the maximal a-structure $[\Phi, h, \alpha]$ for h must be obtained[12]. Then the proposed function

[12]A procedure for constructing a P-DeLP argument for a given conclusion a was already developed as part of the implementation [1]. Then, the maximal a-structure for h can be obtained by considering the set of all arguments for h as the set Ω in Def. 15 of accrued structure.

UNarrowing must be called, with $([\Phi, h, \alpha], \text{PRO})$ as N (representing the root node of the *ADT* we are going to construct), the $\emptyset$ as *OutSetPRO* and $[\emptyset, \epsilon, 0]$ as *OutCON*. The function result is the U-narrowing $[\Phi', h, \alpha']$ of N, which if none empty constitutes a warranted a-structure, and in this case a is warranted with necessity degree α'. Next is the *UNarrowing* function:

---

function *UNarrowing*

Arguments:

N**:** a node, represented as a pair $([\Phi, h, \alpha], Role)$, where $Role \in \{\text{PRO}, \text{CON}\}$.

*OutSetPRO***:** the set of disagreement a-substructures of PRO nodes in the path from the root to N (to check condition a2 in Def. 28).

*OutCON***:** the disagreement a-substructure of the last CON node previous to N (to check condition a4 in Def. 28).

Result: The undefeated narrowing of N.

`\* Obtain the child nodes of N *\`

$Childs \leftarrow \emptyset$

for each $[\Psi, k, \beta] \in partialAttackersOf([\Phi, h, \alpha])$ **do**

  if $RoleN = CON$ **then** `\*child will be PRO*\`

    $[\Psi', k, \beta'] \leftarrow Accrual(\{\langle B, k, \beta_i\rangle \in Args([\Psi, k, \beta]) \mid \nexists \langle C, q, \delta_j\rangle \in Args([\Theta, q, \delta])$, with $[\Theta, q, \delta] \in OutSetPRO$, such that $\langle C, q, \delta_j\rangle$ is a subargument of $\langle B, k, \beta_i\rangle\})$ `\*enforcing condition a2*\`

    $Childs \leftarrow Childs \cup \{([\Psi', k, \beta'], PRO)\}$

    $NewOutSetPRO \leftarrow OutSetPRO$

    $NewOutCON \leftarrow [\Phi, h, \alpha]$

  else if $[\Psi, k, \beta] \neq OutCON$ **then** `\*child CON and verifies a4*\`

    $Childs \leftarrow Childs \cup \{([\Psi, k, \beta], CON)\}$

    $NewOutSetPRO \leftarrow OutSetPRO \cup \{disagreement([\Phi, h, \alpha], [\Psi, k, \beta])\}$

    $NewOutCON \leftarrow OutCON$

```
    end if
end for each

\* Obtain the UNwg of N from (recursively) the UNwgs of its childs*\

if Childs = ∅ then                  \* base case *\
    UNwgN ← [Φ, h, α]
else                                \* recursive case *\
    UNwgsOfChilds ← ∅
    for each M ∈ Childs do
        UNwgsOfChilds ← UNwgsOfChilds ∪
                        {UNarrowing(M, NewOutSetPRO, NewOutCON)}
    end for each
    if Role = PRO then
        UNwgN ← UNwg([Φ, h, α], UNwgsOfChilds)      \* Def. 25 *\
    else
        UNwgN ← UNwgP([Φ, h, α], UNwgsOfChilds)     \* Def. 27 *\
    end if
end if

return UNwgN
```

---

7. Related work and salient features of our approach: discussion

There has been some previous research in argumentation concerning the treatment of accrual of reasons. In [21], Prakken enunciates three desirable principles that "any formal treatment of accrual should satisfy". The first principle says that *"accruals are sometimes weaker than their elements"* due to the possibility of accruing reasons are not independent. The second principle states that *"any 'larger' accrual that applies, makes all its 'lesser' versions inapplicable"*. Intuitively, that means that we should always accrue as many arguments as possible, even if in the end the accrual is outweighed by a conflicting accrual. The third principle states that *"flawed reasons or* *arguments may not accrue".* That means that when an individual argument turns out to be flawed, it should not take part in the accrual. This principle induces a two-stage argumentation process. First all the individual reasons for a certain claim are tested to see whether they may enter the accrual, then all the reasons that pass this test are accrued and compared to all accruing reasons for the opposite claim.

Next we will analyze the three principles in the context of our formalization. The first principle is not verified. Indeed, we explicitly require the possibilistic accruing function (ACC) to satisfy non-depreciation in order to prove convergence of sequential degradations. We recognize that in very rare situations (like the one presented by Prakken in [21]) accruals can be weaker than their elements, and therefore we consider an extension of our approach in this direction as one of the future lines to be pursued. The second principle is trivially verified since the dialectical acceptance analysis only considers maximal a-structures. Although this principle is vacuous if accruals are never weaker than their elements (as stated by Prakken), it would become valuable if eventually the framework is extended to satisfy the first principle. Finally, the third principle is also verified. The idea behind this principle is that when evaluating two accruals supporting opposite conclusions in order to determine which one prevails (*i.e.*, is accepted), the conflicting accruals evaluated should not contain flawed reasons. In our framework, this evaluation and comparison of conflicting accruals occurs when applying the notion of sequential degradation in order to determine the U-narrowing of a given node in an ADT, given the U-narrowings of its children (conflicting a-structures). First, notice that U-narrowings of child nodes considered in the sequential degradation do not contain flawed reasons, since flawed reasons were already defeated by their own children. Second, notice that in the sequential degradation, in order to determine if a given attack constitutes a defeat, the attacking a-structure is compared with the associated disagreement a-substructure, which is the "accrual supporting the opposite conclusion" we mentioned before. Finally, although defeaters are applied at any time in a sequential degradation, even if the disagreement a-substructure associated with the defeat contains flawed reasons, we can ensure that those defeat applications are safe. That is because due to non-depreciation of the evaluation function (ACC), the disagreement a-substructure cannot be weaker than the corresponding a-substructure not containing the flawed reasons, and then the defeat would be applied anyway.

In [21], Prakken also presents a formalization of accrual associated with the principles enunciated, that adapts the way of modeling accrual of reasons in Reason-Based Logic [14] to an argument-based setting. This formalization is based on a combination of two widely recognized argument-based logics: Dung's abstract approach to argumentation [9] instantiated with Pollock's approach to the structure of arguments [20]. Prakken defines accrued arguments (or just accruals) as a special kind of defeasible derivations involving labels. For the acceptability analysis, all the accruals are constructed (not only maximal) and conflicting accruals are evaluated according to a selected evaluation criterion to determine defeat relation. A special kind of 'constructions' called *accrual undercutters* are introduced as additional nodes in the graph, stating that when a given set of reasons for the same conclusion accrues, no proper subset accrues. The resulting *graph of defeats* is then analyzed under the selected Dung's semantics in order to determine the status of accruals.

Fig. 14 shows a KB, together with some labeled derivations and the associated graph of defeats according to Prakken's approach for this KB. For the comparison criterion it was assumed that the accrual for $\sim b$ is preferred to the accrual for b, and then the attacks of the former constitute defeats. For this graph, Dung's grounded extension coincides with the unique preferred extension, and it is the set containing the accrual for $\sim b$ and the accrual for a involving only A_2.

$$
\begin{array}{llll}
r_1 : b \Rightarrow a & r_3 : c \Rightarrow b & c & f \\
r_2 : d \Rightarrow a & r_4 : f \Rightarrow \sim b & d &
\end{array}
$$

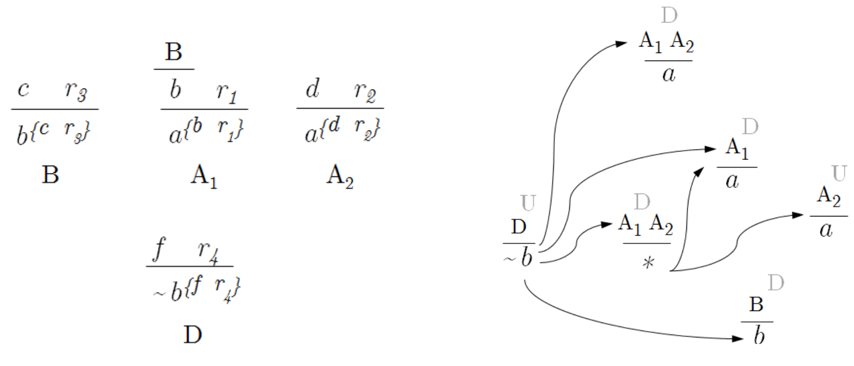


Figure 14: Prakken's graph of defeats

In Verheij's CumulA system [28, 27], arguments are tree like (recursive) structures, similar to Pollock's tree-based approach, but which can represent coordination (accrual) of reasons. Arguments are constructed from rules and individual sentences defined over an unstructured language. Conflicts among arguments is modeled through the notion of *compound defeat*, which in its more general version allows to state that a certain set of arguments defeats another set of arguments. The defeat relation is explicitly specified through a compound defeater construct with the following syntax:

$$A_1, ..., A_n[B_1, ..., B_m],$$

representing that if arguments $A_1, ..., A_n$ (challenging arguments) are undefeated, they cause arguments $B_1, ..., B_m$ (challenged arguments) to be collectively defeated.

$A_1; A_2 \rightarrow a\ [C \rightarrow no\_a]$
$C \rightarrow no\_a\ [A_1 \rightarrow a]$
$C \rightarrow no\_a\ [A_2 \rightarrow a]$

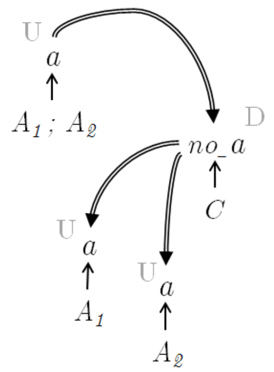


Figure 15: CumulA status assignment

Consider the set of defeaters and the associated status assignment in Fig. 15 (notice that the defeated and undefeated arguments are indicated with marks 'D' and 'U', respectively). This case models the situation where, although each of the individual arguments $A_1 \rightarrow a$ and $A_2 \rightarrow a$ are on their own defeated by $C \rightarrow no\_a$, the accrual $A_1; A_2 \rightarrow a$ defeats $C \rightarrow no\_a$. As $A_1; A_2 \rightarrow a$ has no defeaters, then it is trivially undefeated, causing the defeat of $C \rightarrow no\_a$ through defeater $A_1; A_2 \rightarrow a[C \rightarrow no\_a]$. As $C \rightarrow no\_a$ becomes defeated, then the defeaters with $C \rightarrow no\_a$ on their left-hand side are not triggered, causing $A_1 \rightarrow a$ and $A_2 \rightarrow a$ to remain undefeated.

Next we will summarize the most valuable features of our approach, contrasting it with Prakken's and Verheij's approaches.

7.1. Accrual evaluation: determining defeat relation

Prakken's formalization abstracts from the evaluation of accrual, assuming they are compared in some way in order to determine if a given conflict constitutes a defeat. Verheij's CumulA also abstracts from accrual evaluation, requiring defeat relation to be explicitly specified, instead of defining it in terms of a notion of conflict together with an evaluation phase. Our formalization of accrual advances in this sense by proposing a mechanism that takes possibilistic information into account to explicitly deal with the evaluation and comparison of accrual.

7.2. Complexity of the argumentative analysis

In the literature, complexity analysis for different argument-based dialectical proof procedures has been performed (see e.g. [11, 10]). In particular, Cecchi et al.[6] analyzed different complexity issues in DeLP, which resembles in many aspects the possibilistic approach to skeptical argumentation presented in Section 2,[13] establishing that the proof procedure of DeLP is NP. Although a detailed complexity analysis of our formalization will be left as future work, we will outline next why this complexity is not harder than the one associated with traditional argumentation systems. Concerning complexity issues, dialectical analysis for accruals differs from the one used traditionally in DeLP in two respects: 1) the structures labeling tree nodes, being single arguments for dialectical trees (DT/s) and maximal a-structures for ADT/s, and 2) the dialectical line conditions, that must be tested each time a tree node is added (those for accrual being direct extensions of those for arguments). With respect to 1, although constructing a maximal accrual for a given conclusion a implies obtaining all the arguments supporting a, only one accrual for a will be constructed in a given ADT line, whereas potentially all the arguments for a could appear in the same line and/or as sibling nodes (giving rise to alternative lines) in a DT, resulting in deeper and broader trees. Moreover, as a particular case, only one ADT must be constructed for the entire acceptability analysis of a, whereas possibly several DT/s for the same analysis (one per argument supporting a). With respect to 2 (dialectical conditions), the extra computation required to test ADT conditions (given that they are applied to accruals, encompassing sets of arguments) is partially compensated by the larger amount of DT lines and nodes on each DT line, where in fact DT/s can be seen as unfolded versions of the corresponding ADT.

[13]The argument construction procedure for P-DeLP is similar to its counterpart in DeLP, where no necessity degrees are taken into account.

Let us now consider Prakken' s formalization. Unlike our approach, in Prakken's not only maximal accruals, but also all the lesser ones are considered. Then, if there exist n different individual reasons for a given conclusion a, Prakken's system will construct one accrual supporting a for each nonempty subset of this n reasons. As consequence, the number of accrual undercutters is also exponential on the number of individual reasons for the associated conclusion, and the number of defeats (arrows) grows considerably. With respect to the acceptability analysis, Prakken's approach defines the status of accruals by using Dung's semantics on the graphs of defeats constructed, then any dialectical proof procedure of those defined in [26] can be used to determine if a given argument A (node in the graph) is accepted. However, they must be used in a rather inefficient way in order to obtain a useful answer, and this is even more evident when a skeptical semantics is selected, as will be explained next. According to Prakken's formalization all the accruals for a given conclusion a are considered as alternative potential justifications for a, that is, at most one of the accruals for a can be present in a given acceptability extension (the 'largest' not containing flawed reasons according to the extension). Therefore, although at most one of the accruals for a will be accepted (if a skeptical semantics is used), in the worst case, we have to apply the dialectical proof procedure for each accrual supporting a in order to discover which is the one (if any) that is accepted.

Similar to Prakken's approach, in CumulA system all possible accruals for each conclusion (and not only maximal) need to be considered, and then the number of defeats (either explicit or implicit) that must be taken into account when analyzing the status of arguments becomes also considerable. Finally, no proof procedure was presented associated with CumulA system.

7.3. Explanations of answers

A very valuable advantage of argumentation based frameworks is the idea of explanation of answers obtained. Most argument based frameworks generate structures or define representations in order to formalize the argumentative analysis that determines the status of arguments (the most popular are the graphs of arguments and defeats and dialectical trees/proofs). As humans are generally very familiar with the notion of argumentation, those representations constructed, resembling in some way the argumentative analysis performed, can be considered as human understandable explanations (or justifications) of the obtained answers.

Although both Prakken's and our approaches exhibits the previously mentioned feature, the complexity of the graph of defeats defined by Prakken limits its use as a human readable explanation of the answers obtained. Notice that Prakken's graphs not only considers all accruals for each conclusion (which also increases the number of defeats), but also includes a considerable number of artificial arguments called accrual undercutters. In the case of CumulA, no structure (like a graph or a dialectical tree) is defined in order to formalize the status of arguments. Although in this work we graphically represent CumulA's analysis as graphs, that is not enough to represent compound defeat.

8. Conclusions

In this paper we have proposed a novel approach to model argument accrual in a possibilistic setting. Our proposal was based on an generic formalization for skeptical semantics (Section 2), in which a dialectical proof procedure was characterized for determining which arguments are ultimately accepted (warranted). We instantiated this framework using P-DeLP (Section 3): necessity degrees were attached to formulas at the object-level, and propagated to arguments through the GMP inference rule. The warrant status of P-DeLP arguments was defined as an instance of our dialectical proof procedure. Thus, a logic programming setting for argumentation with possibilistic uncertainty was characterized.

In Sections 4 and 5 we presented our proposal for modelling *argument accrual* in the context of this setting. We defined our notion of *accrued structure*, which accounts for different P-DeLP arguments supporting a given conclusion, and associates a necessity measure with its conclusion obtained as an aggregation of the necessity measures of the individual arguments it accounts for. We have shown how accrued structures can be in conflict in terms of the notion of partial attack, and how possibilistic information is used to determine if a given attack succeeds, becoming a defeat. The notions of combined attack and sequential degradation were also defined, allowing us to characterize a dialectical process (Section 6) in which all accrued structures in favor and against a given conclusion are taken into account in order to determine if the conclusion is warranted, and if so, with which necessity degree. Additionally, we provided formal results characterizing convergence for sequential degradation of a-structures (theorem 1), as well as consistency properties associated with the computation of accrued dialectical trees (theorem 3).

As stated in Section 1, the formalization presented in this paper is partly based on previous research work in similar directions ([16] and [17]). However, our proposal presents significant improvements concerning the dialectical analysis of accrual in argumentation, namely: 1) direct correspondence with traditional, broadly recognized, Dung's skeptical argumentation semantics, which intuitively supports the soundness of our formalization; 2) consistency properties of the dialectical analysis, and 3) a refinement of the notion of accrued dialectical line, which takes advantage of the notion of a-substructure to shorten the dialectical analysis.

As discussed in Section 7, our approach presents several valuable features when contrasted with Prakken's [21] and Verheij's [27, 28], such as the incorporation of explicit treatment of possibilistic uncertainty in order to evaluate and compare accruals, the simplicity of the formalization, suggesting accrued dialectical trees as human understandable explanations (or justifications) of the obtained answers, and the operational conceptualization of our approach, which leads to directly implementable and efficient computation. Additionally, our formalization satisfies two of the three principles of accrual proposed by Prakken in [21], and also satisfies an interesting property (corollary 2) which suggests an additional principle: *accrued structures which are ultimately accepted as justified should not involve conflicting arguments.*

In order to test the applicability of our proposal we are developing an implementation of our formalization using the DeLP system [29, 12] as a basis. A detailed complexity analysis of our formalization is left as future work, and we are studying different theoretical results emerging from our proposal which could help to speed up the computation of accrued dialectical trees, in a similar fashion as done in [7]. Research in this direction is currently being pursued.

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Appendix A. Appendix: Proofs of Lemmas and Theorems

Lemma 1. *Let $[\Phi, h, \alpha]$ be an a-structure and let Σ be a set of a-structures attacking $[\Phi, h, \alpha]$. Let $\Upsilon = [\Phi_1, h, \alpha_1], \ldots, [\Phi_{m+1}, h, \alpha_{m+1}]$ be a sequential degradation of $[\Phi, h, \alpha]$ associated with the combined attack of the a-structures in Σ, and let $[\Psi_1, k_1, \beta_1]$, $[\Psi_2, k_2, \beta_2]$, $\ldots, [\Psi_m, k_m, \beta_m]$ be the sequence of defeaters associated with Υ by the definition of sequential degradation. Let $[\Phi', h, \alpha'] \sqsubseteq [\Phi, h, \alpha]$ such that $[\Phi', h, \alpha'] \not\sqsubseteq [\Phi_{m+1}, h, \alpha_{m+1}]$. Then $[\Psi_j, k_j, \beta_j]$ is a partial defeater of $[\Phi', h, \alpha']$ for some j, $1 \leq j \leq m$, provided that the function ACC satisfies non-depreciation.*

Proof. Let us show first that $[\Psi_i, k_i, \beta_i]$ attacks $[\Phi', h, \alpha']$ for some i, $1 \leq i \leq n$. As $[\Phi', h, \alpha'] \sqsubseteq [\Phi, h, \alpha]$ and $[\Phi', h, \alpha'] \not\sqsubseteq [\Phi_{m+1}, h, \alpha_{m+1}]$ (hypothesis), then there exists $\langle A, h, \delta\rangle \in Args([\Phi, h, \alpha])$ such that $\langle A, h, \delta\rangle \in Args([\Phi', h, \alpha'])$ and $\langle A, h, \delta\rangle \notin Args([\Phi_{m+1}, h, \alpha_{m+1}])$. As $[\Phi, h, \alpha] = [\Phi_1, h, \alpha_1] \sqsupseteq ... \sqsupseteq [\Phi_{m+1}, h, \alpha_{m+1}]$, then there exists i, $1 \leq i \leq m$, such that $\langle A, h, \delta\rangle \in Args([\Phi_i, h, \alpha_i])$ and $\langle A, h, \delta\rangle \notin Args([\Phi_{i+1}, h, \alpha_{i+1}])$. (that is, $[\Phi_i, h, \alpha_i]$ is the last narrowing of $[\Phi, h, \alpha]$ in Υ containing $\langle A, h, \delta\rangle$). Consider the a-structure (defeater) $[\Psi_i, k_i, \beta_i]$ ($[\Psi_i, k_i, \beta_i]$ is the defeater responsible for the ultimate defeat of $\langle A, h, \delta\rangle$ in Υ). Then we can ensure that $\overline{k_i}$ is an intermediate conclusion in $\langle A, h, \delta\rangle$. Moreover, as $\langle A, h, \delta\rangle \in Args([\Phi', h, \alpha'])$, $\overline{k_i}$ will be also an intermediate conclusion of $[\Phi', h, \alpha']$, and hence, $[\Phi', h, \alpha']$ will be attacked by $[\Psi_i, k_i, \beta_i]$.

Consider now the smaller j, $1 \leq j \leq m$, such that $[\Psi_j, k_j, \beta_j]$ attacks $[\Phi', h, \alpha']$. We will prove that $[\Psi_j, k_j, \beta_j]$ is a defeater for $[\Phi', h, \alpha']$. Let $[\Lambda, \overline{k_j}, \gamma]$ $[[\Lambda', \overline{k_j}, \gamma']]$ be the disagreement a-substructure associated with the attack of $[\Psi_j, k_j, \beta_j]$ against $[\Phi_j, h, \alpha_j]$ $[[\Phi', h, \alpha']]$. Let us prove that $[\Lambda', \overline{k_j}, \gamma'] \sqsubseteq [\Lambda, \overline{k_j}, \gamma]$.

Suppose by contradiction that $[\Lambda', \overline{k_j}, \gamma'] \not\sqsubseteq [\Lambda, \overline{k_j}, \gamma]$. Then as $[\Lambda', \overline{k_j}, \gamma']$ is an a-substructure of $[\Phi', h, \alpha']$ and $[\Lambda, \overline{k_j}, \gamma]$ is an a-substructure of $[\Phi_j, h, \alpha_j]$, it holds that $[\Phi', h, \alpha'] \not\sqsubseteq [\Phi_j, h, \alpha_j]$. Then there exists $\langle B, h, \mu\rangle \in Args([\Phi, h, \alpha])$ such that $\langle B, h, \mu\rangle \in Args([\Phi', h, \alpha'])$ and $\langle B, h, \mu\rangle \notin Args([\Phi_j, h, \alpha_j])$. Analogous to the analysis involving $\langle A, h, \delta\rangle$ and i, we can ensure that there exists r, $1 \leq r \leq j$, such that $\langle B, h, \mu\rangle \in Args([\Phi_r, h, \alpha_r])$ and $\langle B, h, \mu\rangle \notin Args([\Phi_{r+1}, h, \alpha_{r+1}])$. Moreover, it holds that the defeater $[\Psi_r, k_r, \beta_r]$, responsible for the ultimate defeat of $\langle B, h, \mu\rangle$ in Υ, attacks $[\Phi', h, \alpha']$ (which involves $\langle B, h, \mu\rangle$). We arrived to a contradiction, since $r < j$ and we had previously considered the subindex j as the smallest one such that $[\Psi_j, k_j, \beta_j]$ attacks $[\Phi', h, \alpha']$. Therefore $[\Lambda', \overline{k_j}, \gamma'] \sqsubseteq [\Lambda, \overline{k_j}, \gamma]$.

As the attack of $[\Psi_j, k_j, \beta_j]$ against $[\Phi_j, h, \alpha_j]$ with disagreement a-substructure $[\Lambda, \overline{k_j}, \gamma]$ constitutes a defeat, then $\beta_j \geq \gamma$. As the disagreement a-substructure $[\Lambda', \overline{k_j}, \gamma']$ associated with the attack of $[\Psi_j, k_j, \beta_j]$ against $[\Phi', h, \alpha']$ is a narrowing of $[\Lambda, \overline{k_j}, \gamma]$, and the *ACC* function satisfies non-depreciation, then $\gamma \geq \gamma'$. Finally $\beta_j \geq \gamma'$ and therefore $[\Psi_j, k_j, \beta_j]$ is a defeater of $[\Phi', h, \alpha']$ $\square$

Theorem 1 (Convergence). Let $[\Phi, h, \alpha]$ be an a-structure and let Σ be a set of a-structures attacking $[\Phi, h, \alpha]$. Let $\Upsilon = [\Phi_1, h, \alpha_1], \ldots, [\Phi_m, h, \alpha_m]$ and $\Upsilon' = [\Phi'_1, h, \alpha'_1], \ldots, [\Phi'_n, h, \alpha'_n]$ be two sequential degradations of $[\Phi, h, \alpha]$ associated with the combined attack of the a-structures in Σ. Then $[\Phi_m, h, \alpha_m] = [\Phi'_n, h, \alpha'_n]$, provided that the ACC function satisfies non-depreciation.

Proof. Suppose by contradiction that $[\Phi_m, h, \alpha_m] \neq [\Phi'_n, h, \alpha'_n]$. Then, either $[\Phi_m, h, \alpha_m] \not\sqsubseteq [\Phi'_n, h, \alpha'_n]$ or $[\Phi'_n, h, \alpha'_n] \not\sqsubseteq [\Phi_m, h, \alpha_m]$, or both. Let us suppose, without loss of generality, that $[\Phi'_n, h, \alpha'_n] \not\sqsubseteq [\Phi_m, h, \alpha_m]$. Let $[\Psi_1, k_1, \beta_1]$, $[\Psi_2, k_2, \beta_2]$, $\ldots, [\Psi_m, k_m, \beta_m]$ be the sequence of defeaters associated with Υ by the definition of sequential degradation. Then, by lemma 1, $[\Psi_j, k_j, \beta_j]$ is a partial defeater of $[\Phi'_n, h, \alpha'_n]$ for some j, $1 \leq j \leq m$. But by definition of sequential degradation, $[\Phi'_n, h, \alpha'_n]$ does not have defeaters in Σ, arriving to a contradiction. Therefore $[\Phi_m, h, \alpha_m] = [\Phi'_n, h, \alpha'_n]$ $\square$

Theorem 2. *Let $[\Phi, h, \alpha]$ be an a-structure and let Σ be a set of a-structures attacking $[\Phi, h, \alpha]$. Then $UNwg([\Phi, h, \alpha],\ \Sigma) \sqsubseteq UNwgP([\Phi, h, \alpha],\ \Sigma)$*

Proof. Let $\Upsilon = [\Phi_1, h, \alpha_1], \ldots, [\Phi_{m+1}, h, \alpha_{m+1}] = UNwgP([\Phi, h, \alpha],\ \Sigma)$ be a proper-defeating sequential degradation associated with the attack of Σ against $[\Phi, h, \alpha]$. We will consider two cases. If $[\Phi_{m+1}, h, \alpha_{m+1}]$ has no defeaters in Σ (case 1), then Υ is also a sequential degradation, and then $UNwg([\Phi, h, \alpha],\ \Sigma) = [\Phi_{m+1}, h, \alpha_{m+1}] = UNwgP([\Phi, h, \alpha],\ \Sigma)$. On the other hand, if $[\Phi_{m+1}, h, \alpha_{m+1}]$ has at least one defeater in Σ (case 2), then Υ can be extended by applying defeaters as much as possible, arriving to a sequence $\Upsilon' = [\Phi_1, h, \alpha_1],\ \ldots, [\Phi_{m+1}, h, \alpha_{m+1}],\ \ldots, [\Phi_{m+n}, h, \alpha_{m+n}]$, with $n > 1$, where $[\Phi_{m+n}, h, \alpha_{m+n}]$ has no defeaters in Σ. Then Υ' constitutes a sequential degradation associated with the attack of Σ against $[\Phi, h, \alpha]$, and so $[\Phi_{m+n}, h, \alpha_{m+n}] = UNwg([\Phi, h, \alpha],\ \Sigma)$. Since $[\Phi_{m+n}, h, \alpha_{m+n}]$ was obtained from $[\Phi_{m+1}, h, \alpha_{m+1}]$ by applying defeaters (at least one), then $[\Phi_{m+n}, h, \alpha_{m+n}] \sqsubseteq [\Phi_{m+1}, h, \alpha_{m+1}]$. Finally, in both cases (1 and 2) it holds that $UNwg([\Phi, h, \alpha],\ \Sigma) \sqsubseteq UNwgP([\Phi, h, \alpha],\ \Sigma)$ $\square$

Lemma 2. *Let $\mathcal{K}$ be a possibilistic KB, and let $\mathrm{T}_{[\Phi,\, h,\, \alpha]}$ be an ADT for a given a-structure $[\Phi, h, \alpha]$ of $\mathcal{K}$. Let N and N' be nodes in $\mathrm{T}_{[\Phi,\, h,\, \alpha]}$, where N is* PRO *and is labelled with $[\Theta, q, \delta]$, and N' is* CON *and is labelled with $[\Theta', q, \delta']$. Let M'*

be a child of N' labelled with $[\Psi', k, \beta']$, such that $[\Psi', k, \beta']$ attacks $[\Theta, q, \delta]$. Then there exists a child M of N labelled with an a-structure $[\Psi, k, \beta]$.

Proof. Suppose by contradiction that such a node M does not exist. Let $[\Psi, k, \beta]$ be the maximal a-structure for k in $\mathcal{K}$. Then, as N is PRO, the only reason causing that there does not exist a node M labeled with $[\Psi, k, \beta]$ must be condition (a5) in definition 28. That is, there exists a parent (CON) node P of N labelled with $[\Theta_0, q_0, \delta_0]$ such that $[\Psi, k, \beta]$ is the disagreement a-substructure associated with the attack of $[\Theta, q, \delta]$ against $[\Theta_0, q_0, \delta_0]$. Then $k = \overline{q}$. Consider the parent (PRO) node P' of N' (P' always exists since a CON node always has a parent). Note that the disagreement a-structure associated with the attack of $[\Theta', q, \delta']$ (the a-structure labelling N') against the a-structure labelling P' is an a-structure $[\Psi'', k, \beta'']$ supporting k ($= \overline{q}$). But then the existence of PRO node M', labelled with $[\Psi', k, \beta']$, violates condition $a4$, since $[\Psi'', k, \beta''] \in \Sigma$, arriving to a contradiction. Therefore there exists a child M of N labelled with an a-structure $[\Psi, k, \beta]$ $\square$

Lemma 3. *Let $[\Theta, q, \delta]$ and $[\Theta', q, \delta']$ be a-structures such that $[\Theta, q, \delta] \sqsubseteq [\Theta', q, \delta']$. Let Σ [Σ'] be a set of a-structures attacking $[\Theta, q, \delta]$ [$[\Theta', q, \delta']$], s.t. for every $[\Psi', k, \beta'] \in \Sigma'$, it holds that either: 1) $[\Psi', k, \beta']$ does not attack $[\Theta, q, \delta]$, or 2) there exists $[\Psi, k, \beta] \in \Sigma$ such that $[\Psi', k, \beta'] \sqsubseteq [\Psi, k, \beta]$. Then $UNwg([\Theta, q, \delta], \Sigma) \sqsubseteq UNwg([\Theta', q, \delta'], \Sigma')$.*

Proof. Suppose by contradiction that $UNwg([\Theta, q, \delta], \Sigma) \not\sqsubseteq UNwg([\Theta', q, \delta'], \Sigma')$.

Let $\Upsilon' = [\Theta'_1, q, \delta'_1], \ldots, [\Theta'_{m+1}, q, \delta'_{m+1}]$ ($= UNwg([\Theta', q, \delta'], \Sigma')$) be a sequential degradation of $[\Theta', q, \delta']$ associated with the combined attack of the a-structures in Σ', and let $[\Psi'_1, k_1, \beta'_1]$, $[\Psi'_2, k_2, \beta'_2]$, $\ldots, [\Psi'_m, k_m, \beta'_m]$ be the sequence of defeaters associated with Υ' by the definition of sequential degradation. As $UNwg([\Theta, q, \delta], \Sigma) \sqsubseteq [\Theta, q, \delta]$ (def. of undefeated narrowing) and $[\Theta, q, \delta] \sqsubseteq [\Theta', q, \delta']$ (hypothesis), then $UNwg([\Theta, q, \delta], \Sigma) \sqsubseteq [\Theta', q, \delta']$. Then by lemma 1 it holds that $[\Psi'_j, k_j, \beta'_j]$ is a partial defeater of $UNwg([\Theta, q, \delta], \Sigma)$ for some j, $1 \leq j \leq m$ (taking $[\Theta', q, \delta']$ as $[\Phi, h, \alpha]$ and $UNwg([\Theta, q, \delta], \Sigma)$ as $[\Phi', h, \alpha']$ in lemma 1).

In particular, since $UNwg([\Theta, q, \delta], \Sigma) \sqsubseteq [\Theta, q, \delta]$, we can ensure that $[\Psi'_j, k_j, \beta'_j]$ attacks $[\Theta, q, \delta]$. Then, by hypothesis, (as $[\Psi'_j, k_j, \beta'_j] \in \Sigma'$ and $[\Psi'_j, k_j, \beta'_j]$ attacks $[\Theta, q, \delta]$), there exists $[\Psi, k_j, \beta] \in \Sigma$ such that $[\Psi'_j, k_j, \beta'_j] \sqsubseteq [\Psi, k_j, \beta]$. Finally, as ACC satisfies non-depreciation, $\beta \geq \beta'_j$, and then $[\Psi, k_j, \beta]$ is also a partial defeater for $UNwg([\Theta, q, \delta], \Sigma)$. We arrived to a contradiction, since by definition of sequential degradation of an a-structure $[\Theta, q, \delta]$ associated with the attack of a-structures in Σ, the last element in the sequence $(UNwg([\Theta, q, \delta], \Sigma))$ has no defeaters in Σ. Therefore $UNwg([\Theta, q, \delta], \Sigma) \sqsubseteq UNwg([\Theta', q, \delta'], \Sigma')$ $\square$

Lemma 4. *Let $\mathcal{K}$ be a possibilistic KB, and let $\mathrm{T}_{[\Phi, h, \alpha]}$ be an ADT for a given a-structure $[\Phi, h, \alpha]$ of $\mathcal{K}$. Let N and N' be nodes in $\mathrm{T}_{[\Phi, h, \alpha]}$, where N is* PRO *and is labelled with $[\Theta, q, \delta]$, and N' is* CON *and is labelled with $[\Theta', q, \delta']$. Then $UNwg(N) \sqsubseteq UNwg(N')$.*

Proof. Let $n = min(HN, HN')$, where HN and HN' are the heights of N and N' in $\mathrm{T}_{[\Phi, h, \alpha]}$, respectively. We will prove that $UNwg(N) \sqsubseteq UNwg(N')$ by induction on n.

Basis Case: $n = 0$.

Case 1: $n = HN = 0$. Then N has not children, and hence, by definition of evaluated ADT, it holds that $UNwg(N) = [\Theta, q, \delta]$. As N' is CON, then $[\Theta', q, \delta']$ is maximal, and then $[\Theta, q, \delta] \sqsubseteq [\Theta', q, \delta']$. Finally, as N has not children and by lemma 2, we can ensure that none of the a-structures labelling the children of N' attacks $[\Theta, q, \delta]$ (since if there would exist a child M' of N' attacking $[\Theta, q, \delta]$, lemma 2 ensures that there exists a child M of N). Therefore we can ensure that $UNwg(N) = [\Theta, q, \delta] \sqsubseteq UNwg(N')$.

Case 2: $n = HN' = 0$.

As N' is CON, then the a-structure labelling N', $[\Theta', q, \delta']$, is maximal. Since $HN' = 0$, and by definition of evaluated ADT, it holds that $UNwg(N') = [\Theta', q, \delta']$. Finally, $UNwg(N) \sqsubseteq UNwg(N')$.

Inductive Case: $n > 0$.

Assume that the property holds for $i < n$ (Inductive Hypothesis).

According to def. 31, $UNwg(N) = UNwg([\Theta, q, \delta], \Sigma)$, where $\Sigma = \{UNwg(M) \mid M \textit{ is a child of } \mathrm{N}, \textit{ and } UNwg(M) \neq [\emptyset, \epsilon, 0]\}$, and $UNwg(N') = UNwgP([\Theta', q, \delta'], \Sigma')$, where $\Sigma' = \{UNwg(M') \mid M' \textit{ is a child of } N', \textit{ and } UNwg(M) \neq [\emptyset, \epsilon, 0]\}$. We will apply lemma 3 in order to prove that $UNwg([\Theta, q, \delta], \Sigma) \sqsubseteq UNwg([\Theta', q, \delta'], \Sigma')$. In order to apply lemma 3 we need first to show that $[\Theta, q, \delta] \sqsubseteq [\Theta', q, \delta']$. That holds trivially, since $[\Theta', q, \delta']$ is the a-structure labelling the CON-node N', and then $[\Theta', q, \delta']$ is maximal. Additionally, we need to show that for each element $UNwg(M') \in \Sigma'$ (where M' is a child of N') it holds that either: 1) $UNwg(M')$ does not attack $[\Theta, q, \delta]$, or 2) there exists $UNwg(M) \in \Sigma$ (where M is a child of N) such that $UNwg(M') \sqsubseteq UNwg(M)$. Suppose that $UNwg(M')$ does attack $[\Theta, q, \delta]$. Then the a-structure labelling M', $[\Psi', k, \beta']$, also attacks $[\Theta, q, \delta]$. By applying lemma 2 we can ensure that there exists a child node M (CON) of N labelled with an a-structure $[\Psi, k, \beta]$.

As M (CON-node) and M' (PRO-node) are children of N and N', respectively, then $min(HM, HM') < min(HN, HN') = n$. Therefore, by inductive hypothesis, $UNwg(M') \sqsubseteq UNwg(M)$. Now we can apply lemma 3, obtaining that $UNwg([\Theta, q, \delta], \Sigma)$ $(= UNwg(N)) \sqsubseteq UNwg([\Theta', q, \delta'], \Sigma')$. Finally, by proposition 2, $UNwg([\Theta', q, \delta'], \Sigma') \sqsubseteq UNwgP([\Theta', q, \delta'], \Sigma')$ $(= UNwg(N'))$, and then $UNwg(N) \sqsubseteq UNwg(N')$ $\Box$

Lemma 5. *Let $\mathcal{K}$ be a possibilistic KB, and let $\mathrm{T}_{[\Phi, h, \alpha]}$ be an ADT for a given a-structure $[\Phi, h, \alpha]$ of $\mathcal{K}$. Let N_1 and N_2 be nodes in $\mathrm{T}_{[\Phi, h, \alpha]}$, where N_1 is* PRO *and there exists an a-structure $[\Theta_1, q, \delta_1]$ that is a complete a-substructure of the a-structure labelling N_1, and N_2 is* CON *and is labelled with $[\Theta_2, q, \delta_2]$. Let $[\Theta'_1, q, \delta'_1]$ be the complete a-substructure of $UNwg(N_1)$ supporting q. Then $[\Theta'_1, q, \delta'_1] \sqsubseteq UNwg(N_2)$.*

Proof. Suppose by contrary that $[\Theta'_1, q, \delta'_1] \not\sqsubseteq UNwg(N_2)$. By definition of undefeated narrowing of a given node, $UNwg(N_2) = UNwgP([\Theta_2, q, \delta_2], \Sigma)$, where $\Sigma = \{UNwg(M) \mid M \text{ is a child of } N_2, \text{ and } UNwg(M) \neq [\emptyset, \epsilon, 0]\}$,

Let $\Upsilon = [\Theta'_{21}, q, \delta'_{21}], \ldots, [\Theta'_{2m+1}, q, \delta'_{2m+1}]$ $(= UNwg(N_2))$ be a sequential degradation of $[\Theta_2, q, \delta_2]$ associated with the combined attack of the a-structures in Σ, and let $[\Psi_1, k_1, \beta_1]$, $[\Psi_2, k_2, \beta_2]$, $\ldots, [\Psi_m, k_m, \beta_m]$ be the sequence of defeaters associated with Υ by the definition of sequential degradation. As N_2 is CON, then its label, $[\Theta_2, q, \delta_2]$, is maximal, and then it holds that $[\Theta'_1, q, \delta'_1] \sqsubseteq [\Theta_2, q, \delta_2]$. It also holds that $[\Theta'_1, q, \delta'_1] \not\sqsubseteq UNwg(N_2)$ (hypothesis). Then by lemma 1 it holds that $[\Psi_j, k_j, \beta_j]$ is a partial defeater of $[\Theta'_1, q, \delta'_1]$ for some j, $1 \leq j \leq m$ (taking $[\Theta_2, q, \delta_2]$ as $[\Phi, h, \alpha]$ in lemma 1 and $[\Theta'_1, q, \delta'_1]$ as $[\Phi', h, \alpha']$ in lemma 1). As $[\Psi_j, k_j, \beta_j] \in \Sigma$, then there exists a child node M_2 (PRO) of N_2 such that $[\Psi_j, k_j, \beta_j] = UNwg(M_2)$. Since $[\Psi_j, k_j, \beta_j] = UNwg(M_2)$ is a partial defeater of $[\Theta'_1, q, \delta'_1]$, in particular the a-structure $[\Psi'_j, k_j, \beta'_j]$ labelling M_2 does attack $[\Theta'_1, q, \delta'_1]$, and since $[\Theta'_1, q, \delta'_1]$ is a complete a-substructure of $UNwg(N_1)$ (hypothesis), then $[\Psi'_j, k_j, \beta'_j]$ also attacks the a-structure labelling N_1 (lets call it $[\Theta, q, \delta]$). By applying lemma 2 we can ensure that there exists a (CON) child node M_1 of N_1 labelled with an a-structure $[\Psi, k_j, \beta]$ (supporting k_j). Finally, by applying lemma 4 we get that $UNwg(M_2) \sqsubseteq UNwg(M_1)$ (taking M_2 as N in the lemma, and M_1 as the N' in the lemma). As $UNwg(M_2)$ is a partial defeater of $[\Theta'_1, q, \delta'_1]$ and ACC function satisfies non-depreciation, then $UNwg(M_1)$ is also a defeater for $[\Theta'_1, q, \delta'_1]$, and hence for $UNwg(N_1)$. We arrived to a contradiction, since the undefeated narrowing of a given node N cannot have defeaters among the undefeated narrowings of its children (by definition of undefeated narrowing of a node and of sequential degradation) $\Box$

Theorem 3 (Pairwise pro concordancy). *Let $\mathcal{K}$ be a possibilistic KB, and let $\mathrm{T}_{[\Phi,h,\alpha]}$ be an ADT for a given a-structure $[\Phi, h, \alpha]$ of $\mathcal{K}$. Let N_1 and N_2 be* PRO*-nodes in $\mathrm{T}_{[\Phi,h,\alpha]}$. Then there exist no literal q such that q appears in $UNwg(N_1)$ and $\overline{q}$ appears in $UNwg(N_2)$.*

Proof. Suppose by contrary that there exists such a literal q. Let $[\Theta_1, q, \delta_1]$ be the complete a-substructure of N_1 supporting q. Let $[\Lambda_2, \overline{q}, \gamma_2]$ be the complete a-substructure of N_2 supporting $\overline{q}$. We can ensure that there exists a CON child node M_1 of N_1 labelled with a maximal a-structure $[\Lambda_1, \overline{q}, \gamma_1]$, and that there exists a CON child node M_2 of N_2 labelled with a maximal a-structure $[\Theta_2, q, \delta_2]$. Let $[\Theta'_1, q, \delta'_1]$ be the complete a-substructure of $UNwg(N_1)$ supporting q. Let $[\Lambda'_2, \overline{q}, \gamma'_2]$ be the complete a-substructure of $UNwg(N_2)$ supporting $\overline{q}$. By applying lemma 5 for nodes N_1 and M_2 we get that $[\Theta'_1, q, \delta'_1] \sqsubseteq UNwg(M_2) = [\Theta'_2, q, \delta'_2]$. Similarly, by applying lemma 5 for nodes N_2 and M_1 we get that $[\Lambda'_2, \overline{q}, \gamma'_2] \sqsubseteq UNwg(M_1) = [\Lambda'_1, \overline{q}, \gamma'_1]$. As ACC function satisfies non-depreciation, it holds that $\delta'_2 \geq \delta'_1$ and that $\gamma'_1 \geq \gamma'_2$. Finally, note that (according to definition of partial attack) $UNwg(M_1) = [\Lambda'_1, \overline{q}, \gamma'_1]$ partially attacks $UNwg(N_1)$ with attacked narrowing $[\Theta'_1, q, \delta'_1]$. However, such an attack does not constitute a defeat (since M_1 is a child of N_1, and by definition of undefeated narrowing of a node and sequential degradation), and then we can ensure that $\delta'_1 > \gamma'_1$. Similarly, $UNwg(M_2) = [\Theta'_2, q, \delta'_2]$ partially attacks $UNwg(N_2)$ with attacked narrowing $[\Lambda'_2, \overline{q}, \gamma'_2]$, and such an attack does not constitute a defeat, concluding that $\gamma'_2 > \delta'_2$. Since we have shown that $\delta'_2 \geq \delta'_1 > \gamma'_1 \geq \gamma'_2 > \delta'_2$, we get that $\delta'_2 > \delta'_2$, arriving to a contradiction. Therefore, there exist no literal q such that q appears in $UNwg(N_1)$ and $\overline{q}$ appears in $UNwg(N_2)$ $\Box$