Distributed H_∞ State Estimation for Stochastic Delayed 2-D Systems with Randomly Varying Nonlinearities over Saturated Sensor Networks

Jinling Liang, Zidong Wang, Tasawar Hayat and Ahmed Alsaedi

Abstract

In this paper, the distributed H_∞ state estimation problem is investigated for the two-dimensional (2-D) time-delay systems. The target plant is characterized by the generalized Fornasini-Marchesini 2-D equations where both stochastic disturbances and randomly varying nonlinearities (RVNs) are considered. The sensor measurement outputs are subject to saturation restrictions due to the physical limitations of the sensors. Based on the available measurement outputs from each individual sensor and its neighboring sensors, the main purpose of this paper is to design distributed state estimators such that not only the states of the target plant are estimated but also the prescribed H_∞ disturbance attenuation performance is guaranteed. By defining an energy-like function and utilizing the stochastic analysis as well as the inequality techniques, sufficient conditions are established under which the augmented estimation error system is globally asymptotically stable in the mean square and the prescribed H_∞ performance index is satisfied. Furthermore, the explicit expressions of the individual estimators are also derived. Finally, numerical example is exploited to demonstrate the effectiveness of the results obtained in this paper.

Index Terms

Two-dimensional (2-D) systems, distributed state estimation, H_∞ index, randomly varying nonlinearities (RVNs), sensor saturation.

I. Introduction

The last decade has seen a rapid surge of research interest in both the theoretical development and practical applications of sensor networks that are capable of distributed sensing, computing and communication. So far, sensor networks have found countless successful applications in areas such as environment and habitat monitoring, health care applications, traffic control, distributed robotics, and industrial & manufacturing automation [7], [11], [19]–[21]. In a sensor network, the spatially distributed sensor nodes collaboratively process a limited amount of data for the purpose of sensing, tracking or detecting the target. Through efficient coordination between the densely deployed sensors, the overall sensor network is able to monitor, detect and estimate the real states of a physical plant under certain possibly harsh environments such as the battle-filed surveillance [2], [34]. A distinguished

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feature of the signal processing over a sensor network is its collaborative manner when the large amount of sensors work together to achieve certain state estimation tasks. This is called the distributed state estimation (or filtering) problem where each individual sensor in a sensor network locally estimates the system state by utilizing both its own measurement and its neighboring sensors' measurements according to the given topology [22], [23], [35].

The smooth operation of a sensor network relies heavily on the communications between the sensor nodes. With the ever-increasing number of sensor nodes and size of the sensor field, the limited communication resources would become a major concern. For example, most nodes are now battery-powered and most of the communication is carried out through wireless channels of limited bandwidth. As such, the resulting communication constraints would unavoidably deteriorate the performance (e.g. for distributed state estimation) of the sensor networks. Such network-induced problems include, but are not limited to, packet dropout, communication delays, sensor saturation and nonlinear disturbances. Due to the random variation of the network load and monitoring conditions, the network-induced phenomena often occur in a probabilistic way [8], [9], [43], [46]. So far, the problems of randomly occurring packet dropout and communication delays have gained much research interest, see [13] for a survey. Nevertheless, the randomly varying nonlinearities and the sensor saturations have received relatively less research attention despite their importance in practical engineering, and the relevant results have been scattered. For example, the distributed average set-membership filtering problem has been investigated in [48] over sensor networks with sensor saturation, where the estimation error is required to achieve the bounded consensus. The random nature of the sensor saturations has been examined in [12] for the distributed filtering problem where the issue of successive packet dropouts has also been addressed.

On another research forefront, due primarily to their theoretical significance and practical insights, the two-dimensional (2-D) discrete systems have been stirring a recurring research interest in the past few decades [1], [3], [25], [37], [38], [45]. As discussed in [39], 2-D systems have been playing an increasingly important role in mathematical modeling in many areas such as image processing, seismographic data processing, thermal processes and water stream heating. A variety of 2-D state-space models have been studied, among which the Fornasini-Marchesini (FM) first and second models as well as the Roesser model have proven to be most popular. Up to now, almost all fundamental behaviors of 2-D systems have been investigated and a rich body of literature has appeared that contributes largely to the better understanding of how 2-D systems are controlled. For example, some earlier results can be found in [18], [30] for the stability analysis problem, for 2-D systems has been investigated in [18], [30], in [14], [15], [32], [40], [41] for the controller/filter design problems and in [17] for the model approximation problem. Recently, in [26], [27], the state estimation problem has been extensively tackled for 2-D systems subject to network-induced phenomena including missing measurements, sensor saturation, sensor delays and randomly occurring nonlinearities.

In some sensor network applications such as geographical data processing, power transmission lines and electromagnetic wave propagation, the 2-D system plays an irreplaceable role when it comes to the modeling issue. For example, in [47], the spatial-temporal, geographical and environmental factors have been examined for wireless sensor networks for utilizing the intermittent recharging opportunities to support low-rate data services. In [49], the 2-D system has been used for modeling the ad hoc networks with two-dimensional lattices and the percolation theory has been employed for the connectivity study. As such, four seemingly natural yet interrelated questions arise as follows. 1) How do we deal with the distributed state estimation problem for the target plant modeled by a 2-D system over a sensor network? 2) How do we examine the impact of the network-induced phenomena (e.g., randomly varying nonlinearites and sensor saturations) on the estimation performance of the sensor networks? 3) What if the target plant is further subject to time-delays, exogenous and stochastic disturbances? 4) Can we attenuate the effect from exogenous disturbances on the estimation accuracy through a prespecified H_∞ performance constraint? Unfortunately, a literature review has revealed that these four questions have remained unanswered till now due probably to the mathematical difficulties complicated by the topology structure of the sensor networks, the stochastic analysis as well as the estimation performance specifications. It is, therefore, the main motivation of the present research to deal with the aforementioned questions.

In this paper, we aim to deal with the distributed H_∞ state estimation problem for a class of stochastic 2-D systems with RVNs and time-varying delays. We are interested in designing distributed state estimators and then deriving sufficient criteria under which such kind of estimators do exist. *The main contribution of this paper is threefold: 1) distributed state estimators are designed firstly for the general 2-D target plant such that the states of the system are estimated in a distributed way, in other words, each sensor estimates the states of the stochastic 2-D system based on the measurement outputs not only from the sensor itself but also from its neighboring sensors; 2) an H_∞ index is also introduced in the process of state estimation to further characterize the attenuation level of the estimated output signals against the exogenous disturbances; and 3) a comprehensive 2-D model is proposed where the RVNs are introduced in the target plant and the sensors saturation case is also considered in the sensor measurement equations, both of which make the system under consideration more realistic.*

The rest of this paper is outlined as follows. In Section II, the distributed H_∞ state estimation problem addressed is formulated and some preliminaries are introduced. In Section III, the global asymptotic stability in the mean square is investigated for the augmented estimation error system, and the H_∞ performance constraint is analyzed. Furthermore, explicit design schemes are given for the estimator gain matrices. In Section IV, the effectiveness of the obtained results are demonstrated by an illustrative numerical example. Finally, conclusions are drawn in Section V.

Notation. The notation used here is fairly standard except where otherwise stated. $\mathbb{Z}_+$ is used to be the set $\{0, 1, 2 \ldots\}$. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote the n-dimensional Euclidean space and the set of all $n \times m$ real matrices, respectively. For integers m and n with $m \le n$, $\lfloor m, n\rfloor$ represents the integers set $\{m, m+1, \ldots, n\}$ and $\lfloor m, \infty)$ means the integers set $\{m, m+1, m+2, \ldots\}$. I and 0 stand for the identity matrix and the zero matrix with appropriate dimensions, respectively. For matrix $A \in \mathbb{R}^{n\times n}$, $\mathrm{Sym}(A)$ denotes the matrix $(A + A^T)/2$ and '$*$' in a matrix is used to denote the term which is induced by symmetry. The notation $X > 0$ means that matrix X is real, symmetric and positive definite. 1_n stands for the vector in $\mathbb{R}^n$ with all elements being 1s and the Kronecker product of matrices A and B is represented as $A \otimes B$. The shorthand $\mathrm{diag}(A_1, A_2, \ldots, A_n)$ means a block diagonal matrix with diagonal blocks being the matrices $A_1, A_2, \ldots, A_n$, and $\mathrm{col}(A_i)_{i=1}^n = \mathrm{col}(A_1, A_2, \ldots, A_n)$ represents the column-wise concatenation of the matrices $A_1, A_2, \ldots, A_n$. For a complete probability space $(\Omega, \mathscr{F}, \mathrm{Prob})$, $\mathbb{E}\{\alpha\}$ and $\mathbb{E}\{\alpha|\beta\}$ denote, respectively, the mathematical expectation of the stochastic variable α and the expectation of α conditional on β with respect to the given probability measure Prob which has total mass 1. $\|\cdot\|$ refers to the Euclidean vector norm and for $\nu \in l_2(\mathbb{Z}_+ \times \mathbb{Z}_+, \mathbb{R}^n)$, define $\|\nu\|_{l_2}^2 = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\mathbb{E}\{\|\nu(k,h)\|^2\} - \frac{1}{2}\sum_{k=0}^{\infty}\mathbb{E}\{\|\nu(k,0)\|^2\} - \frac{1}{2}\sum_{h=0}^{\infty}\mathbb{E}\{\|\nu(0,h)\|^2\}$ which has also been used in [14]. Matrices without explicit specification are assumed to have compatible dimensions.

II. Problem formulation

Consider a discrete system along two directions described by the general Fornasini-Marchesini state-space model [16] with time-varying delays and stochastic disturbances of the following form:

$$
\begin{aligned}
x(k+1,h+1) =& A_1x(k+1,h) + A_2x(k,h+1) + D_1x(k+1,h-\sigma(h)) + D_2x(k-\tau(k),h+1)\\
&+ \alpha(k,h)B_1f_1(x(k+1,h),x(k,h+1))\\
&+ (1-\alpha(k,h))B_2f_2(x(k+1,h-\sigma(h)),x(k-\tau(k),h+1))\\
&+ E_1\nu(k+1,h) + E_2\nu(k,h+1) + \hbar(x(k+1,h),x(k,h+1))\omega(k,h) \qquad (1)
\end{aligned}
$$

with output

$$z(k,h) = M_0 x(k,h), \tag{2}$$

where $k,\ h \in \mathbb{Z}_+$; $x(k,h) \in \mathbb{R}^n$ is the state vector of the target plant and $z(k,h) \in \mathbb{R}^q$ is the output of the state combination to be estimated; A_i, D_i, E_i, B_i $(i = 1,2)$ and M_0 are system matrices with compatible dimensions; the exogenous disturbance input $\nu(\cdot,\cdot) \in l_2(\mathbb{Z}_+ \times \mathbb{Z}_+, \mathbb{R}^p)$. $\tau(k)$ and $\sigma(h)$ are time-varying positive integers representing, respectively, the delays along the horizontal direction and the delays along the vertical direction, which satisfy

$$\underline{\tau} \le \tau(k) \le \overline{\tau}, \qquad \underline{\sigma} \le \sigma(h) \le \overline{\sigma}; \qquad \forall k,h \in \mathbb{Z}_+ \tag{3}$$

where $\underline{\tau}$, $\overline{\tau}$, $\underline{\sigma}$ and $\overline{\sigma}$ are known positive integers being the lower and the upper bounds of the time-varying delays. $\omega(k,h)$ is a standard random scalar signal on the probability space $(\Omega, \mathscr{F}, \mathrm{Prob})$ with

$$\mathbb{E}\{\omega(k,h)\} = 0, \qquad \mathbb{E}\{\omega(k,h)\omega(k',h')\} = \begin{cases} 1, & \text{if } (k,h) = (k',h') \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Let $(\Omega, \mathscr{F}, \{\mathscr{F}_l\}_{l\in\mathbb{Z}_+}, \mathrm{Prob})$ be a filtered probability space where $\{\mathscr{F}_l\}_{l\in\mathbb{Z}_+}$ is the family of sub σ-algebras of $\mathscr{F}$ generated by $\{\omega(i,j)\}_{i,j\in\mathbb{Z}_+}$. Specifically, $\mathscr{F}_l$ is the minimal σ-algebra generated by $\{\omega(i,j)\}_{0\le i+j\le l-1}$, while $\mathscr{F}_0$ is assumed to be some given sub σ-algebra of $\mathscr{F}$ independent of $\mathscr{F}_l$ for all $l > 0$.

Moreover, $\hbar(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is the noise intensity function which is assumed to satisfy the following condition

$$\hbar^T(u,v)\hbar(u,v) \le \left\| H \begin{pmatrix} u \\ v \end{pmatrix} \right\|^2, \tag{5}$$

where $u,\ v \in \mathbb{R}^n$ and H is a known constant matrix with appropriate dimensions.

The nonlinear functions $f_i(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ $(i = 1,2)$ are subject to the condition $f_i(0,0) = 0$ and the following sector-bounded condition [29]

$$\big(f_i(u,v) - f_i(\tilde{u},\tilde{v}) - F_1^{(i)}\varsigma\big)^T \big(f_i(u,v) - f_i(\tilde{u},\tilde{v}) - F_2^{(i)}\varsigma\big) \le 0 \tag{6}$$

with u, v, $\tilde{u}$ and $\tilde{v} \in \mathbb{R}^n$, $\varsigma = \big((u-\tilde{u})^T \quad (v-\tilde{v})^T\big)^T$ and $F_1^{(i)} = [F_{i11}\ F_{i12}]$, $F_2^{(i)} = [F_{i21}\ F_{i22}] \in \mathbb{R}^{n\times 2n}$ are known constant matrices.

In (1), $\alpha(k,h) \in \mathbb{R}$ is a Bernoulli distributed white sequence which takes values of either 1 or 0 with

$$\mathrm{Prob}\{\alpha(k,h) = 1\} = \bar{\alpha}, \qquad \mathrm{Prob}\{\alpha(k,h) = 0\} = 1 - \bar{\alpha}, \tag{7}$$

where $\bar{\alpha} \in [0,1]$ is a known constant. Obviously, for all $k,\ h \in \mathbb{Z}_+$, the stochastic variable $\alpha(k,h)$ has the variance $\bar{\alpha}(1-\bar{\alpha})$. It is further assumed that in this paper $\omega(k,h)$ and $\alpha(k',h')$ are mutually independent for all k, h, k', $h' \in \mathbb{Z}_+$.

Remark 1: In the discrete 2-D target plant equation (1), random variable $\alpha(k,h)$ is introduced to account for the phenomena of nonlinearities varying in a random way induced by, for instance, asynchronous multiplexed data communication. The concept of RVNs, accounting for the binary switch between two nonlinear functions, has been firstly proposed in [36] to investigate the synchronization problem for the delayed complex networks, which might reflect more realistic characteristics in complex networks. Such an idea was originated from [44] where stabilizing control laws have been found for the linear systems with randomly varying distributed delays. Thereafter, such kind of characterizations has been extensively utilized in literature for references. For example, the fault detection problem has been discussed in [10] for the discrete-time Markovian jump systems with incomplete knowledge of transition probabilities, and the state estimation problem has been addressed in [6] for the discrete time-delay nonlinear complex networks with randomly occurring sensor saturations and randomly varying sensor delays. It should be noted that in all the references mentioned above, the systems under consideration are all 1-D, when referring to the 2-D systems, to the best of the authors' knowledge, this might be the first few attempts [26].

In this paper, suppose there are N sensors locating spatially around the target plant and let $\mathscr{G}=(\mathscr{V},\mathscr{E})$ be the directed graph formed by the N sensors, where $\mathscr{V}=\{1,2,\ldots,N\}$ denotes the set of labeled sensors, $\mathscr{E}\subseteq\mathscr{V}\times\mathscr{V}$ is the set of edges and each edge is represented by an ordered pair (i,j), which means that there is information transmission from sensor j to sensor i. Associated with the graph $\mathscr{G}$ is the nonnegative adjacency matrix $\mathcal{L}=[l_{ij}]$, which characterizes the interconnection topology of the sensors and is defined as follows: $l_{ij}>0$ if $(i,j)\in\mathscr{E}$; $l_{ij}=0$ otherwise. Sensor j is called one of the neighbors of sensor i if $(i,j)\in\mathscr{E}$. For all $i\in\mathscr{V}$, denote $\mathcal{N}_i=\{j\in\mathscr{V}|(i,j)\in\mathscr{E}\}$. Moreover, it is assumed that the graph $\mathscr{G}$ discussed in this paper is self-connected, i.e., $l_{ii}=1$ for all $i\in\mathscr{V}$; and the dynamics of sensor i is of the form

$$y_i(k,h)=g(C_ix(k,h))+W_i\nu(k,h),\quad i=1,2,\ldots,N \tag{8}$$

where $y_i(k,h)\in\mathbb{R}^m$ is the measured output vector from the ith sensor on the target plant, C_i and W_i are known constant real matrices with appropriate dimensions, the nonlinear saturated function $g(\cdot):\mathbb{R}^m\rightarrow\mathbb{R}^m$ has the following form

$$g(u)=\begin{bmatrix} g_1(u_1) & g_2(u_2) & \cdots & g_m(u_m)\end{bmatrix}^T \tag{9}$$

with $u=(u_1,u_2,\ldots,u_m)^T\in\mathbb{R}^m$ and, for $l=1,2,\ldots,m$, $g_l(u_l)=\mathrm{sign}(u_l)\min\{|u_l|,u_{l,\max}\}$ where $u_{l,\max}$ is the lth element of the saturation level vector $u_{\max}$.

To facilitate the analysis of the problem discussed in this paper, similar as the technique employed in [24], [42], it is assumed that there exist two diagonal matrices S_1, $S_2\in\mathbb{R}^{m\times m}$ such that $0\leq S_1<I\leq S_2$ and the saturation function $g(\cdot)$ in (9) is rewritten as

$$g(u)=S_1u+\tilde{g}(u), \tag{10}$$

where the nonlinear function $\tilde{g}(\cdot):\mathbb{R}^m\rightarrow\mathbb{R}^m$ satisfies the sector condition [5]: $\tilde{g}^T(u)(\tilde{g}(u)-Su)\leq 0$ with $S=S_2-S_1$.

The initial boundary condition associated with the discrete 2-D target plant (1) is taken as

$$x(k,h)=\begin{cases}\varphi(k,h), & (k,h)\in\lfloor-\overline{\tau},0\rfloor\times\lfloor 0,\kappa_1\rfloor\\ \phi(k,h), & (k,h)\in\lfloor 0,\kappa_2\rfloor\times\lfloor-\overline{\sigma},0\rfloor\\ \quad 0 & (k,h)\in\lfloor-\overline{\tau},0\rfloor\times\lfloor\kappa_1+1,\infty)\ \text{or}\ \lfloor\kappa_2+1,\infty)\times\lfloor-\overline{\sigma},0\rfloor\end{cases} \tag{11}$$

with $\varphi(0,0)=\phi(0,0)$, where κ_1 and κ_2 are two finite positive integers, $\varphi(k,h)$ and $\phi(k,h)$ are vectors with elements in $\mathscr{F}_0$.

The aim of the H_∞ state estimation problem addressed in this paper is to estimate the states and the output signals of the target plant (1). Illuminated by the novel distributed ideas employed in [31], [34], here we construct the distributed state estimator for sensor i as follows:

$$\begin{aligned}\hat{x}_i(k+1,h+1)=&A_1\hat{x}_i(k+1,h)+A_2\hat{x}_i(k,h+1)+\bar{\alpha}B_1f_1(\hat{x}_i(k+1,h),\hat{x}_i(k,h+1))\\&+(1-\bar{\alpha})B_2f_2(\hat{x}_i(k+1,h-\sigma(h)),\hat{x}_i(k-\tau(k),h+1))\\&+\sum_{j\in\mathcal{N}_i}l_{ij}K_{1ij}\big(y_j(k+1,h)-S_1C_j\hat{x}_j(k+1,h)\big)\\&+\sum_{j\in\mathcal{N}_i}l_{ij}K_{2ij}\big(y_j(k,h+1)-S_1C_j\hat{x}_j(k,h+1)\big)\end{aligned} \tag{12}$$

with

$$\hat{z}_i(k,h) = M_i\hat{x}_i(k,h), \qquad i = 1,2,\dots,N \tag{13}$$

where $\hat{x}_i(k,h) \in \mathbb{R}^n$ is the estimate of the target plant state $x(k,h)$ and $\hat{z}_i(k,h) \in \mathbb{R}^q$ is the estimate of the output signal $z(k,h)$ on sensor i; K_{1ij}, $K_{2ij} \in \mathbb{R}^{n\times m}$ and $M_i \in \mathbb{R}^{q\times n}$ $(i = 1,2,\dots,N; j \in \mathcal{N}_i)$ are the estimator gain matrices to be designed. The initial boundary condition for estimator (12) is taken to be $\hat{x}_i(k,h) \equiv 0$ for $k \in \lfloor -\overline{\tau}, 0\rfloor$ or $h \in \lfloor -\overline{\sigma}, 0\rfloor$.

Remark 2: The states and the output signals of the target plant (1) are estimated in a distributed way as shown in (12). To be more specific, the sensor i estimates the states of system (1) based on the measurements not only from the sensor i itself but also from its neighboring sensors $j \in \mathcal{N}_i$ according to the given graph topology. Such kind of original distributed ideas has been proposed in [31] to solve the data fusion problem where an average consensus based distributed filter has been utilized to track the average of N sensor measurements. More recently, by using a stochastic sampled-data approach, the problem of distributed filtering has been investigated in [34] for sensor networks. It will be further demonstrated later in the example section that compared with the usual estimation method, such kind of distributed ideas will make the H_∞ attenuation level γ^* be much smaller.

By setting $\tilde{x}(k,h) = (\tilde{x}_1^T(k,h), \tilde{x}_2^T(k,h), \dots, \tilde{x}_N^T(k,h))^T$ with $\tilde{x}_i(k,h) = x(k,h) - \hat{x}_i(k,h)$ $(i = 1,2,\dots,N)$ and resorting to the Kronecker product, the state estimation error dynamics can be obtained from (1), (8), (10) and (12) as follows:

$$\begin{aligned}\tilde{x}(k+1,h+1) =& \big(I_N \otimes A_1 - \mathscr{K}_1(I_N \otimes S_1)\overline{\mathscr{C}}\big)\tilde{x}(k+1,h) + 1_N \otimes D_1 x(k+1,h-\sigma(h))\\ &+ \big(I_N \otimes A_2 - \mathscr{K}_2(I_N \otimes S_1)\overline{\mathscr{C}}\big)\tilde{x}(k,h+1) + 1_N \otimes D_2 x(k-\tau(k),h+1)\\ &+ \bar{\alpha} I_N \otimes B_1 \mathbb{F}_1(k,h) + (1-\bar{\alpha}) I_N \otimes B_2 \mathbb{F}_2(k,h) + (1_N \otimes E_1 - \mathscr{K}_1\widetilde{\mathscr{W}})\nu(k+1,h)\\ &+ (\alpha(k,h) - \bar{\alpha})\big(1_N \otimes B_1 f_1(k,h) - 1_N \otimes B_2 f_2(k,h)\big) + (1_N \otimes E_2 - \mathscr{K}_2\widetilde{\mathscr{W}})\nu(k,h+1)\\ &- \mathscr{K}_1\mathbb{G}(k+1,h) - \mathscr{K}_2\mathbb{G}(k,h+1) + 1_N \otimes \hbar(x(k+1,h), x(k,h+1))\omega(k,h),\end{aligned} \tag{14}$$

where $\overline{\mathscr{C}} = \mathrm{diag}(C_1, C_2, \dots, C_N)$, $\widetilde{\mathscr{W}} = \mathrm{col}(W_i)_{i=1}^N$, $\mathbb{G}(k,h) = \mathrm{col}(\tilde{g}(C_i x(k,h)))_{i=1}^N$; $\mathbb{F}_l(k,h) = \mathrm{col}(\tilde{f}_{li}(k,h))_{i=1}^N$ $(l = 1,2)$ with

$$\begin{aligned}\tilde{f}_{1i}(k,h) =& f_1(k,h) - f_1(\hat{x}_i(k+1,h), \hat{x}_i(k,h+1)),\\ \tilde{f}_{2i}(k,h) =& f_2(k,h) - f_2(\hat{x}_i(k+1,h-\sigma(h)), \hat{x}_i(k-\tau(k),h+1)),\\ f_1(k,h) =& f_1(x(k+1,h), x(k,h+1)), \qquad f_2(k,h) = f_2(x(k+1,h-\sigma(h)), x(k-\tau(k),h+1));\end{aligned}$$

$\mathscr{K}_1 = (l_{ij}K_{1ij})_{N\times N}$ and $\mathscr{K}_2 = (l_{ij}K_{2ij})_{N\times N} \in \mathcal{W}_{n\times m}$ with $\mathcal{W}_{n\times m}$ being defined as

$$\mathcal{W}_{n\times m} = \{\bar{U} = [U_{ij}] \in \mathbb{R}^{nN\times mN} \mid U_{ij} \in \mathbb{R}^{n\times m},\ U_{ij} = 0 \text{ if } j \notin \mathcal{N}_i\}. \tag{15}$$

For simplicity, by denoting $\tilde{z}(k,h) = \mathrm{col}(\tilde{z}_i(k,h))_{i=1}^N$ with $\tilde{z}_i(k,h) = z(k,h) - \hat{z}_i(k,h)$, the output estimation error dynamics can be derived from (2) and (13) that

$$\tilde{z}(k,h) = \mathcal{M}\eta(k,h), \tag{16}$$

where $\mathcal{M} = [1_N \otimes M_0 - \widetilde{\mathscr{M}},\ \bar{\mathscr{M}}]$ with $\bar{\mathscr{M}} = \mathrm{diag}(M_1, M_2, \dots, M_N)$ and $\widetilde{\mathscr{M}} = \mathrm{col}(M_i)_{i=1}^N$, and $\eta(k,h) = (x^T(k,h), \tilde{x}^T(k,h))^T$ is the augmented state estimation error satisfying

$$\begin{aligned}\eta(k+1,h+1) =& \mathcal{A}_1\eta(k+1,h) + \mathcal{A}_2\eta(k,h+1) + \mathcal{D}_1\eta(k+1,h-\sigma(h)) + \mathcal{D}_2\eta(k-\tau(k),h+1) + \mathcal{B}_1\mathcal{F}(k,h)\\ &+ \mathcal{E}_1\nu(k+1,h) + \mathcal{E}_2\nu(k,h+1) + (\alpha(k,h) - \bar{\alpha})\mathcal{B}_2\mathcal{F}(k,h) + \mathcal{H}(k,h)\omega(k,h),\end{aligned} \tag{17}$$

where $\mathcal{D}_1 = (1_{N+1} \otimes D_1)\mathscr{L}_1$ and $\mathcal{D}_2 = (1_{N+1} \otimes D_2)\mathscr{L}_1$ with $\mathscr{L}_1 = [I_n,\ 0_{n\times nN}]$;

$$\mathcal{A}_1 = \mathrm{diag}(A_1, I_N \otimes A_1 - \mathscr{K}_1(I_N \otimes S_1)\overline{\mathscr{C}}), \qquad \mathcal{A}_2 = \mathrm{diag}(A_2, I_N \otimes A_2 - \mathscr{K}_2(I_N \otimes S_1)\overline{\mathscr{C}});$$

$$\mathcal{B}_1 = \begin{bmatrix} \bar{\alpha}B_1 & (1-\bar{\alpha})B_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{\alpha}(I_N \otimes B_1) & (1-\bar{\alpha})(I_N \otimes B_2) & -\mathscr{K}_1 & -\mathscr{K}_2 \end{bmatrix}, \quad \mathcal{E}_1 = \begin{bmatrix} E_1 \\ 1_N \otimes E_1 - \mathscr{K}_1\widetilde{\mathscr{W}} \end{bmatrix};$$

$$\mathcal{B}_2 = \begin{bmatrix} B_1 & -B_2 & 0 & 0 & 0 & 0 \\ 1_N \otimes B_1 & -1_N \otimes B_2 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathcal{E}_2 = \begin{bmatrix} E_2 \\ 1_N \otimes E_2 - \mathscr{K}_2\widetilde{\mathscr{W}} \end{bmatrix};$$

$$\mathcal{F}(k,h) = \mathrm{col}(f_1(k,h), f_2(k,h), \mathbb{F}_1(k,h), \mathbb{F}_2(k,h), \mathbb{G}(k+1,h), \mathbb{G}(k,h+1));$$

$$\mathcal{H}(k,h) = 1_{N+1} \otimes \hbar(\mathscr{L}_1\eta(k+1,h), \mathscr{L}_1\eta(k,h+1)).$$

To proceed, the following definition for the distributed H_∞ state estimation is introduced.

Definition 1: For all $i = 1,2,\ldots,N$, the system in (12)-(13) is said to be a distributed H_∞ state estimator on sensor i for the target plant (1)-(2) with output measurements (8) if the following two statements hold:

(1) for every initial boundary condition in (11), system (14) is globally asymptotically stable in the mean square in the case of $\nu(k,h) \equiv 0$, i.e., the trivial solution of (14) is stable in the mean square (in the sense of Lyapunov) and $\lim_{k+h\to\infty} \mathbb{E}\{\|\tilde{x}(k,h)\|\} = 0$;

(2) for the given scalar $\gamma > 0$, under zero-initial condition, i.e., $\phi(k,h) = \varphi(k,h) \equiv 0$, the output estimation error system (16) satisfies the H_∞ performance constraint, i.e., $\|\tilde{z}\|_{l_2}^2 \le \gamma^2\|\nu\|_{l_2}^2$.

The objective of this paper is to find the matrices K_{lij} and M_i $(i = 1,2,\ldots,N; j \in \mathcal{N}_i; l = 1,2)$ of the distributed state estimator in (12)-(13) for the stochastic 2-D target plant in (1)-(2) with N sensor measurement outputs (8) such that the state estimation error system (14) is globally asymptotically stable in the mean square and the H_∞ performance constraint is satisfied for the output estimation error system (16).

III. Main results

In this section, we deal with the distributed H_∞ state estimation problem formulated in the previous section for the discrete 2-D system (1)-(2) with N sensor measurement outputs (8).

For brevity, introduce the notations

$$\mathcal{T}_1 = \begin{bmatrix} I_n & 0_{n\times(n+2(m+n)N)} \end{bmatrix}, \qquad \mathcal{T}_2 = \begin{bmatrix} 0_{n\times n} & I_n & 0_{n\times 2(n+m)N} \end{bmatrix},$$
$$\mathcal{T}_3 = \begin{bmatrix} 0_{nN\times 2n} & I_{nN} & 0_{nN\times(n+2m)N} \end{bmatrix}, \qquad \mathcal{T}_4 = \begin{bmatrix} 0_{nN\times(2+N)n} & I_{nN} & 0_{nN\times 2mN} \end{bmatrix},$$
$$\mathcal{T}_5 = \begin{bmatrix} 0_{mN\times 2n(1+N)} & I_{mN} & 0_{mN\times mN} \end{bmatrix}, \qquad \mathcal{T}_6 = \begin{bmatrix} 0_{mN\times(2n+(2n+m)N)} & I_{mN} \end{bmatrix}.$$

From the representation of function $\mathcal{F}(k,h)$ defined in (17), it is easy to see that the following equalities hold:

$$\begin{aligned} f_1(k,h) = \mathcal{T}_1\mathcal{F}(k,h), \qquad f_2(k,h) = \mathcal{T}_2\mathcal{F}(k,h); \qquad \mathbb{F}_1(k,h) = \mathcal{T}_3\mathcal{F}(k,h), \\ \mathbb{F}_2(k,h) = \mathcal{T}_4\mathcal{F}(k,h); \qquad \mathbb{G}(k+1,h) = \mathcal{T}_5\mathcal{F}(k,h), \qquad \mathbb{G}(k,h+1) = \mathcal{T}_6\mathcal{F}(k,h). \end{aligned} \tag{18}$$

First, the distributed H_∞ state estimation problem is analyzed, and the following theorem provides a key role in the derivation of our main results.

Theorem 1: Let the scalar $\gamma > 0$ and the estimation gain matrices K_{lij} and M_i $(i = 1,2,\ldots,N; j \in \mathcal{N}_i; l = 1,2)$ be given. For all $i = 1,2,\ldots,N$, the system in (12)-(13) is a distributed H_∞ state estimator on sensor i for the target plant (1)-(2) with output measurements (8) if there exist matrices $\mathcal{P}_l > 0$ and $\mathcal{Q}_l > 0$, positive diagonal matrices $\vec{\delta}_l$, $\vec{\varepsilon}_l$ and $\vec{\theta} = \mathrm{diag}(\theta_0, \theta_1, \ldots, \theta_N)$, positive scalars $\varepsilon_0^{(l)}$ $(l = 1, 2)$ such that the following matrix inequality holds:

$$\vec{\Xi} = \begin{bmatrix} \Psi & 0 & \Gamma \\ * & -\gamma^2 I_{2p} & \vec{\mathcal{E}}^T(\mathcal{P}_1 + \mathcal{P}_2) \\ * & * & -(\mathcal{P}_1 + \mathcal{P}_2) \end{bmatrix} < 0 \tag{19}$$

where $\vec{\mathcal{E}} = [\mathcal{E}_1,\ \mathcal{E}_2]$,

$$\Psi = \begin{bmatrix} \Psi_{11} & 0 & \Psi_{13} & 0 \\ * & \Psi_{22} & \Psi_{23} & 0 \\ * & * & \Psi_{33} & 0 \\ * & * & * & (\mathcal{P}_1 + \mathcal{P}_2) - \vec{\theta} \otimes I_n \end{bmatrix}, \quad \Gamma = \begin{bmatrix} \vec{\mathcal{A}}^T \\ \vec{\mathcal{D}}^T \\ \mathcal{B}_1^T \\ 0 \end{bmatrix} (\mathcal{P}_1 + \mathcal{P}_2)$$

with $\vec{\mathcal{A}} = [\mathcal{A}_1,\ \mathcal{A}_2]$, $\vec{\mathcal{D}} = [\mathcal{D}_1,\ \mathcal{D}_2]$,

$$\begin{aligned}
\Psi_{11} =& \mathrm{diag}((\overline{\sigma} - \underline{\sigma} + 1)\mathcal{Q}_1 - \mathcal{P}_1 + \mathcal{M}^T\mathcal{M}, (\overline{\tau} - \underline{\tau} + 1)\mathcal{Q}_2 - \mathcal{P}_2 + \mathcal{M}^T\mathcal{M}) - (I_2 \otimes \mathscr{L}_2^T)\mathscr{U}_2^{(1)}(I_2 \otimes \mathscr{L}_2) \\
& - \varepsilon_0^{(1)}(I_2 \otimes \mathscr{L}_1^T)\mathrm{Sym}((F_1^{(1)})^T F_2^{(1)})(I_2 \otimes \mathscr{L}_1) + \sum_{i=0}^{N} \theta_i (I_2 \otimes \mathscr{L}_1^T) H^T H (I_2 \otimes \mathscr{L}_1), \\
\Psi_{22} =& -\mathrm{diag}(\mathcal{Q}_1, \mathcal{Q}_2) - \varepsilon_0^{(2)}(I_2 \otimes \mathscr{L}_1^T)\mathrm{Sym}((F_1^{(2)})^T F_2^{(2)})(I_2 \otimes \mathscr{L}_1) - (I_2 \otimes \mathscr{L}_2^T)\mathscr{U}_2^{(2)}(I_2 \otimes \mathscr{L}_2), \\
\Psi_{13} =& \frac{\varepsilon_0^{(1)}}{2}(I_2 \otimes \mathscr{L}_1^T)(F_1^{(1)} + F_2^{(1)})^T \mathcal{T}_1 + (I_2 \otimes \mathscr{L}_2^T)\mathscr{U}_1^{(1)}\mathcal{T}_3 + \frac{1}{2}\mathcal{S}, \\
\Psi_{23} =& \frac{\varepsilon_0^{(2)}}{2}(I_2 \otimes \mathscr{L}_1^T)(F_1^{(2)} + F_2^{(2)})^T \mathcal{T}_2 + (I_2 \otimes \mathscr{L}_2^T)\mathscr{U}_1^{(2)}\mathcal{T}_4, \\
\Psi_{33} =& \bar{\alpha}(1 - \bar{\alpha})\mathcal{B}_2^T(\mathcal{P}_1 + \mathcal{P}_2)\mathcal{B}_2 - \varepsilon_0^{(1)}\mathcal{T}_1^T\mathcal{T}_1 - \varepsilon_0^{(2)}\mathcal{T}_2^T\mathcal{T}_2 - \mathcal{T}_3^T(\vec{\varepsilon}_1 \otimes I_n)\mathcal{T}_3 \\
& - \mathcal{T}_4^T(\vec{\varepsilon}_2 \otimes I_n)\mathcal{T}_4 - \mathcal{T}_5^T(\vec{\delta}_1 \otimes I_m)\mathcal{T}_5 - \mathcal{T}_6^T(\vec{\delta}_2 \otimes I_m)\mathcal{T}_6; \\
\mathscr{L}_2 =& [0_{nN \times n},\ I_{nN}], \qquad \mathcal{S} = \mathrm{col}(\mathscr{L}_1^T \widetilde{\mathscr{C}}^T(\vec{\delta}_1 \otimes S)\mathcal{T}_5, \mathscr{L}_1^T \widetilde{\mathscr{C}}^T(\vec{\delta}_2 \otimes S)\mathcal{T}_6), \qquad \widetilde{\mathscr{C}} = \mathrm{col}(C_i)_{i=1}^N;
\end{aligned}$$

$$\mathscr{U}_1^{(1)} = \begin{bmatrix} \vec{\varepsilon}_1 \otimes \frac{F_{111}^T + F_{121}^T}{2} \\ \vec{\varepsilon}_1 \otimes \frac{F_{112}^T + F_{122}^T}{2} \end{bmatrix}, \qquad \mathscr{U}_2^{(1)} = \begin{bmatrix} \vec{\varepsilon}_1 \otimes \mathrm{Sym}(F_{111}^T F_{121}) & \vec{\varepsilon}_1 \otimes \frac{F_{111}^T F_{122} + F_{121}^T F_{112}}{2} \\ * & \vec{\varepsilon}_1 \otimes \mathrm{Sym}(F_{112}^T F_{122}) \end{bmatrix};$$

$$\mathscr{U}_1^{(2)} = \begin{bmatrix} \vec{\varepsilon}_2 \otimes \frac{F_{211}^T + F_{221}^T}{2} \\ \vec{\varepsilon}_2 \otimes \frac{F_{212}^T + F_{222}^T}{2} \end{bmatrix}, \qquad \mathscr{U}_2^{(2)} = \begin{bmatrix} \vec{\varepsilon}_2 \otimes \mathrm{Sym}(F_{211}^T F_{221}) & \vec{\varepsilon}_2 \otimes \frac{F_{211}^T F_{222} + F_{221}^T F_{212}}{2} \\ * & \vec{\varepsilon}_2 \otimes \mathrm{Sym}(F_{212}^T F_{222}) \end{bmatrix}.$$

Proof: The notation of function $\mathcal{H}(k, h)$ given in (17) and the constraint condition (5) on the noise intensity function $\hbar(\cdot, \cdot)$ guarantee the validity of the following inequality:

$$\mathcal{H}^T(k, h)(\vec{\theta} \otimes I_n)\mathcal{H}(k, h) \leq \sum_{i=0}^{N} \theta_i \xi_1^T(k, h)(I_2 \otimes \mathscr{L}_1^T) H^T H (I_2 \otimes \mathscr{L}_1)\xi_1(k, h), \tag{20}$$

where $\xi_1(k, h) = \mathrm{col}(\eta(k + 1, h), \eta(k, h + 1))$ and the matrix $\mathscr{L}_1$ is defined in (17).

From the definition of function $\mathbb{G}(k, h)$ defined in (14) and the treatment for function $\tilde{g}(\cdot)$ shown in (10), one knows that for any positive diagonal matrix $\vec{\delta} = \mathrm{diag}(\delta_1, \delta_2, \ldots, \delta_N)$, the following inequality holds:

$$\begin{aligned}
\mathbb{G}^T(k, h)(\vec{\delta} \times I_m)\mathbb{G}(k, h) =& \sum_{i=1}^{N} \delta_i \tilde{g}^T(C_i x(k, h))\tilde{g}(C_i x(k, h)) \\
\leq& \sum_{i=1}^{N} \delta_i \tilde{g}^T(C_i x(k, h)) S C_i x(k, h) = \mathbb{G}^T(k, h)(\vec{\delta} \times S)\widetilde{\mathscr{C}} x(k, h),
\end{aligned} \tag{21}$$

which ensures the validity of the two inequalities given below:

$$\mathcal{F}^T(k,h)\mathcal{T}_5^T(\vec{\delta}_1 \times I_m)\mathcal{T}_5\mathcal{F}(k,h) \le \mathcal{F}^T(k,h)\mathcal{T}_5^T(\vec{\delta}_1 \times S)\widetilde{\mathscr{C}}\mathscr{L}_1\eta(k+1,h),$$
$$\mathcal{F}^T(k,h)\mathcal{T}_6^T(\vec{\delta}_2 \times I_m)\mathcal{T}_6\mathcal{F}(k,h) \le \mathcal{F}^T(k,h)\mathcal{T}_6^T(\vec{\delta}_2 \times S)\widetilde{\mathscr{C}}\mathscr{L}_1\eta(k,h+1);$$

where the last two relationships in (18) have been utilized and the matrices $\vec{\delta}_1$ and $\vec{\delta}_2$ are the solution for matrix inequality (19). In a compact form, the above two inequalities can be unified into the following one

$$\begin{aligned}&\mathcal{F}^T(k,h)\big(\mathcal{T}_5^T(\vec{\delta}_1 \otimes I_m)\mathcal{T}_5 + \mathcal{T}_6^T(\vec{\delta}_2 \otimes I_m)\mathcal{T}_6\big)\mathcal{F}(k,h)\\ &\quad\le \mathcal{F}^T(k,h)\big(\mathcal{T}_5^T(\vec{\delta}_1 \otimes S)\widetilde{\mathscr{C}}\mathscr{L}_1\eta(k+1,h) + \mathcal{T}_6^T(\vec{\delta}_2 \otimes S)\widetilde{\mathscr{C}}\mathscr{L}_1\eta(k,h+1)\big)\\ &\quad= \xi_1^T(k,h)\mathcal{S}\mathcal{F}(k,h).\end{aligned} \tag{22}$$

Let $\aleph(k,h) =: \{\eta(k+1,h), \eta(k+1,h-1), \ldots, \eta(k+1,h-\overline{\sigma}), \eta(k,h+1), \eta(k-1,h+1), \ldots, \eta(k-\overline{\tau},h+1)\}$ and consider the following energy-like function

$$V(k,h) =: V_1(k,h) + V_2(k,h) = \sum_{i=1}^{3}\big(V_{1i}(k,h) + V_{2i}(k,h)\big) \tag{23}$$

with

$$\begin{aligned}
V_{11}(k,h) &= \eta^T(k,h)\mathcal{P}_1\eta(k,h), & V_{12}(k,h) &= \sum_{i=h-\sigma(h)}^{h-1}\eta^T(k,i)\mathcal{Q}_1\eta(k,i),\\
V_{13}(k,h) &= \sum_{i=h-\overline{\sigma}+1}^{h-\underline{\sigma}}\sum_{j=i}^{h-1}\eta^T(k,j)\mathcal{Q}_1\eta(k,j); & V_{21}(k,h) &= \eta^T(k,h)\mathcal{P}_2\eta(k,h),\\
V_{22}(k,h) &= \sum_{j=k-\tau(k)}^{k-1}\eta^T(j,h)\mathcal{Q}_2\eta(j,h), & V_{23}(k,h) &= \sum_{j=k-\overline{\tau}+1}^{k-\underline{\tau}}\sum_{i=j}^{k-1}\eta^T(i,h)\mathcal{Q}_2\eta(i,h);
\end{aligned}$$

where k, $h \in \mathbb{Z}_+$ and positive definite matrices $\mathcal{P}_l$ and $\mathcal{Q}_l$ $(l = 1,2)$ are the solution to the matrix inequality (19).

First, we investigate the stochastic asymptotic stability case (i.e., $\nu(k,h) \equiv 0$ for $k,h \in \mathbb{Z}_+$). Define the index $\mathscr{J}$ as follows:

$$\begin{aligned}\mathscr{J} =:& \mathbb{E}\Big\{\big(V(k+1,h+1) - V_1(k+1,h) - V_2(k,h+1)\big)|\aleph(k,h)\Big\}\\ =& \mathbb{E}\Big\{\sum_{s=1}^{3}(\Delta V_{1s}(k,h) + \Delta V_{2s}(k,h))|\aleph(k,h)\Big\}\end{aligned} \tag{24}$$

with $\Delta V_{1s}(k,h) = V_{1s}(k+1,h+1) - V_{1s}(k+1,h)$ and $\Delta V_{2s}(k,h) = V_{2s}(k+1,h+1) - V_{2s}(k,h+1)$. Then calculating (24) along the trajectories of the augmented state estimation system (17), one has

$$\mathbb{E}\{\Delta V_{11}(k,h)|\aleph(k,h)\} = \mathbb{E}\Big\{\big(\eta^T(k+1,h+1)\mathcal{P}_1\eta(k+1,h+1) - \eta^T(k+1,h)\mathcal{P}_1\eta(k+1,h)\big)|\aleph(k,h)\Big\}, \tag{25}$$

$$\begin{aligned}\mathbb{E}\{\Delta V_{12}(k,h)|\aleph(k,h)\} \le& \mathbb{E}\Big\{\big(\eta^T(k+1,h)\mathcal{Q}_1\eta(k+1,h) - \eta^T(k+1,h-\sigma(h))\mathcal{Q}_1\eta(k+1,h-\sigma(h))\\ &+ \sum_{i=h+1-\sigma(h+1)}^{h-\underline{\sigma}}\eta^T(k+1,i)\mathcal{Q}_1\eta(k+1,i)\big)|\aleph(k,h)\Big\}\\ \le& \mathbb{E}\Big\{\big(\eta^T(k+1,h)\mathcal{Q}_1\eta(k+1,h) - \eta^T(k+1,h-\sigma(h))\mathcal{Q}_1\eta(k+1,h-\sigma(h))\\ &+ \sum_{i=h+1-\overline{\sigma}}^{h-\underline{\sigma}}\eta^T(k+1,i)\mathcal{Q}_1\eta(k+1,i)\big)|\aleph(k,h)\Big\},\end{aligned} \tag{26}$$

$$\begin{aligned}\mathbb{E}\{\Delta V_{13}(k,h)|\aleph(k,h)\} =&\mathbb{E}\Big\{\big(\sum_{i=h+2-\overline{\sigma}}^{h+1-\underline{\sigma}}\sum_{j=i}^{h}-\sum_{i=h+1-\overline{\sigma}}^{h-\underline{\sigma}}\sum_{j=i}^{h-1}\big)\eta^T(k+1,j)\mathcal{Q}_1\eta(k+1,j)|\aleph(k,h)\Big\}\\ =&\mathbb{E}\Big\{\big((\overline{\sigma}-\underline{\sigma})\eta^T(k+1,h)\mathcal{Q}_1\eta(k+1,h)\\ &-\sum_{j=h-\overline{\sigma}+1}^{h-\underline{\sigma}}\eta^T(k+1,j)\mathcal{Q}_1\eta(k+1,j)\big)|\aleph(k,h)\Big\},\end{aligned}\tag{27}$$

where condition (3) has been utilized to obtain inequality (26). Similarly, we have that

$$\mathbb{E}\{\Delta V_{21}(k,h)|\aleph(k,h)\}=\mathbb{E}\Big\{\big(\eta^T(k+1,h+1)\mathcal{P}_2\eta(k+1,h+1)-\eta^T(k,h+1)\mathcal{P}_2\eta(k,h+1)\big)|\aleph(k,h)\Big\},\tag{28}$$

$$\begin{aligned}\mathbb{E}\{\Delta V_{22}(k,h)|\aleph(k,h)\}\leq&\mathbb{E}\Big\{\big(\eta^T(k,h+1)\mathcal{Q}_2\eta(k,h+1)-\eta^T(k-\tau(k),h+1)\mathcal{Q}_2\eta(k-\tau(k),h+1)\\ &+\sum_{j=k+1-\overline{\tau}}^{k-\underline{\tau}}\eta^T(j,h+1)\mathcal{Q}_2\eta(j,h+1)\big)|\aleph(k,h)\Big\},\end{aligned}\tag{29}$$

$$\begin{aligned}\mathbb{E}\{\Delta V_{23}(k,h)|\aleph(k,h)\} =&\mathbb{E}\Big\{\big((\overline{\tau}-\underline{\tau})\eta^T(k,h+1)\mathcal{Q}_2\eta(k,h+1)\\ &-\sum_{i=k+1-\overline{\tau}}^{k-\underline{\tau}}\eta^T(i,h+1)\mathcal{Q}_2\eta(i,h+1)\big)|\aleph(k,h)\Big\}.\end{aligned}\tag{30}$$

Substituting equalities/inequalities from (25)-(30) into (24), one obtains

$$\begin{aligned}\mathscr{J}\leq&\mathbb{E}\Big\{\big[\eta^T(k+1,h+1)(\mathcal{P}_1+\mathcal{P}_2)\eta(k+1,h+1)+\eta^T(k+1,h)((\overline{\sigma}-\underline{\sigma}+1)\mathcal{Q}_1-\mathcal{P}_1)\eta(k+1,h)\\ &+\eta^T(k,h+1)((\overline{\tau}-\underline{\tau}+1)\mathcal{Q}_2-\mathcal{P}_2)\eta(k,h+1)-\eta^T(k+1,h-\sigma(h))\mathcal{Q}_1\eta(k+1,h-\sigma(h))\\ &-\eta^T(k-\tau(k),h+1)\mathcal{Q}_2\eta(k-\tau(k),h+1)\big]|\aleph(k,h)\Big\}.\end{aligned}\tag{31}$$

Furthermore, it follows from (17) that

$$\eta(k+1,h+1)=\vec{\mathcal{A}}\xi_1(k,h)+\vec{\mathcal{D}}\xi_2(k,h)+\mathcal{B}_1\mathcal{F}(k,h)+(\alpha(k,h)-\bar{\alpha})\mathcal{B}_2\mathcal{F}(k,h)+\mathcal{H}(k,h)\omega(k,h),\tag{32}$$

where $\xi_2(k,h)=\mathrm{col}(\eta(k+1,h-\sigma(h)),\eta(k-\tau(k),h+1))$ and matrices $\vec{\mathcal{A}}$ and $\vec{\mathcal{D}}$ are defined in (19), which immediately infers that

$$\begin{aligned}&\mathbb{E}\Big\{\eta^T(k+1,h+1)(\mathcal{P}_1+\mathcal{P}_2)\eta(k+1,h+1)|\aleph(k,h)\Big\}\\ &=\mathbb{E}\Big\{\Big[\xi_1^T(k,h)\vec{\mathcal{A}}^T(\mathcal{P}_1+\mathcal{P}_2)\vec{\mathcal{A}}\xi_1(k,h)+\xi_2^T(k,h)\vec{\mathcal{D}}^T(\mathcal{P}_1+\mathcal{P}_2)\vec{\mathcal{D}}\xi_2(k,h)+\mathcal{F}^T(k,h)\mathcal{B}_1^T(\mathcal{P}_1+\mathcal{P}_2)\mathcal{B}_1\mathcal{F}(k,h)\\ &\quad+2\xi_1^T(k,h)\vec{\mathcal{A}}^T(\mathcal{P}_1+\mathcal{P}_2)\big(\vec{\mathcal{D}}\xi_2(k,h)+\mathcal{B}_1\mathcal{F}(k,h)\big)+2\xi_2^T(k,h)\vec{\mathcal{D}}^T(\mathcal{P}_1+\mathcal{P}_2)\mathcal{B}_1\mathcal{F}(k,h)\\ &\quad+\bar{\alpha}(1-\bar{\alpha})\mathcal{F}^T(k,h)\mathcal{B}_2^T(\mathcal{P}_1+\mathcal{P}_2)\mathcal{B}_2\mathcal{F}(k,h)+\mathcal{H}^T(k,h)(\mathcal{P}_1+\mathcal{P}_2)\mathcal{H}(k,h)\Big]|\aleph(k,h)\Big\},\end{aligned}\tag{33}$$

where conditions (4) and (7) have been utilized when deriving the above equality.

On the other hand, it follows from condition (6) that for any given scalars $\varepsilon_0^{(1)}>0$ and $\varepsilon_i^{(1)}>0$ $(i=1,2,\dots,N)$, the following inequalities hold:

$$\begin{aligned}&\varepsilon_0^{(1)}\mathcal{F}^T(k,h)\mathcal{T}_1^T\mathcal{T}_1\mathcal{F}(k,h)-\varepsilon_0^{(1)}\xi_1^T(k,h)(I_2\otimes\mathscr{L}_1^T)(F_1^{(1)}+F_2^{(1)})^T\mathcal{T}_1\mathcal{F}(k,h)\\ &\qquad+\varepsilon_0^{(1)}\xi_1^T(k,h)(I_2\otimes\mathscr{L}_1^T)\mathrm{Sym}((F_1^{(1)})^TF_2^{(1)})(I_2\otimes\mathscr{L}_1)\xi_1(k,h)\leq0,\end{aligned}\tag{34}$$

$$\begin{aligned}&\varepsilon_i^{(1)}\tilde{f}_{1i}^T(k,h)\tilde{f}_{1i}(k,h)-\varepsilon_i^{(1)}(\mathrm{col}(\tilde{x}_i(k+1,h),\tilde{x}_i(k,h+1)))^T(F_1^{(1)}+F_2^{(1)})^T\tilde{f}_{1i}(k,h)\\ &\qquad+\varepsilon_i^{(1)}(\mathrm{col}(\tilde{x}_i(k+1,h),\tilde{x}_i(k,h+1)))^T\mathrm{Sym}((F_1^{(1)})^TF_2^{(1)})\mathrm{col}(\tilde{x}_i(k+1,h),\tilde{x}_i(k,h+1))\leq0,\end{aligned}\tag{35}$$

where function $\tilde{f}_{1i}(k,h)$ is defined in (14) and the first relationship in (18) has been utilized. Rewrite the N inequalities expressed in (35) into a compact form and one obtains

$$\begin{aligned}&\mathbb{F}_1^T(k,h)(\vec{\varepsilon}_1\otimes I_n)\mathbb{F}_1(k,h)-2(\mathrm{col}(\tilde{x}(k+1,h),\tilde{x}(k,h+1)))^T\mathscr{U}_1^{(1)}\mathbb{F}_1(k,h)\\&\qquad+(\mathrm{col}(\tilde{x}(k+1,h),\tilde{x}(k,h+1)))^T\mathscr{U}_2^{(1)}\mathrm{col}(\tilde{x}(k+1,h),\tilde{x}(k,h+1))\le 0\end{aligned}\tag{36}$$

or in an equivalent form

$$\begin{aligned}&\mathcal{F}^T(k,h)\mathcal{T}_3^T(\vec{\varepsilon}_1\otimes I_n)\mathcal{T}_3\mathcal{F}(k,h)-2\xi_1^T(k,h)(I_2\otimes\mathscr{L}_2^T)\mathscr{U}_1^{(1)}\mathcal{T}_3\mathcal{F}(k,h)\\&\qquad+\xi_1^T(k,h)(I_2\otimes\mathscr{L}_2^T)\mathscr{U}_2^{(1)}(I_2\otimes\mathscr{L}_2)\xi_1(k,h)\le 0\end{aligned}\tag{37}$$

where function $\mathbb{F}_1(k,h)$ and matrix $\mathscr{L}_2$ are defined, respectively, in (14) and (19), $\vec{\varepsilon}_1=\mathrm{diag}(\varepsilon_1^{(1)},\varepsilon_2^{(1)},\ldots,\varepsilon_N^{(1)})$ is the solution of matrix inequality (19), and the third relationship in (18) has been utilized to derive (37).

Similarly, we have

$$\begin{aligned}&\varepsilon_0^{(2)}\mathcal{F}^T(k,h)\mathcal{T}_2^T\mathcal{T}_2\mathcal{F}(k,h)-\varepsilon_0^{(2)}\xi_2^T(k,h)(I_2\otimes\mathscr{L}_1^T)(F_1^{(2)}+F_2^{(2)})^T\mathcal{T}_2\mathcal{F}(k,h)\\&\qquad+\varepsilon_0^{(2)}\xi_2^T(k,h)(I_2\otimes\mathscr{L}_1^T)\mathrm{Sym}((F_1^{(2)})^TF_2^{(2)})(I_2\otimes\mathscr{L}_1)\xi_2(k,h)\le 0\end{aligned}\tag{38}$$

and

$$\begin{aligned}&\mathcal{F}^T(k,h)\mathcal{T}_4^T(\vec{\varepsilon}_2\otimes I_n)\mathcal{T}_4\mathcal{F}(k,h)-2\xi_2^T(k,h)(I_2\otimes\mathscr{L}_2^T)\mathscr{U}_1^{(2)}\mathcal{T}_4\mathcal{F}(k,h)\\&\qquad+\xi_2^T(k,h)(I_2\otimes\mathscr{L}_2^T)\mathscr{U}_2^{(2)}(I_2\otimes\mathscr{L}_2)\xi_2(k,h)\le 0,\end{aligned}\tag{39}$$

where $\vec{\varepsilon}_2$ is the solution of matrix inequality (19), and the second and the forth relationships in (18) have been utilized, respectively, to derive (38) and (39).

Now, letting $\xi(k,h)=\mathrm{col}(\xi_1(k,h),\xi_2(k,h),\mathcal{F}(k,h),\mathcal{H}(k,h))$, substituting (33) into (31) and combining with inequalities (20), (22), (34), (37), (38) and (39), we have

$$\mathscr{J}\le\mathbb{E}\Big\{\xi(k,h)\Xi\xi(k,h)|\aleph(k,h)\Big\}\tag{40}$$

where $\Xi=\vec{\Psi}+\Gamma(\mathcal{P}_1+\mathcal{P}_2)^{-1}\Gamma^T$ and matrix $\vec{\Psi}$ is almost the same as matrix Ψ in (19) with only Ψ_{11} being substituted by

$$\begin{aligned}\vec{\Psi}_{11}=&\mathrm{diag}((\overline{\sigma}-\underline{\sigma}+1)\mathcal{Q}_1-\mathcal{P}_1,(\overline{\tau}-\underline{\tau}+1)\mathcal{Q}_2-\mathcal{P}_2)-(I_2\otimes\mathscr{L}_2^T)\mathscr{U}_2^{(1)}(I_2\otimes\mathscr{L}_2)\\&-\varepsilon_0^{(1)}(I_2\otimes\mathscr{L}_1^T)\mathrm{Sym}((F_1^{(1)})^TF_2^{(1)})(I_2\otimes\mathscr{L}_1)+\sum_{i=0}^{N}\theta_i(I_2\otimes\mathscr{L}_1^T)H^TH(I_2\otimes\mathscr{L}_1).\end{aligned}$$

The well-known Schur Complement Lemma [4] guarantees the validity of $\Xi<0$ from the inequality condition (19), which further leads to $\mathscr{J}\le 0$. After taking mathematical operation again, one gets

$$\mathbb{E}\{V(k+1,h+1)\}\le\mathbb{E}\{V_1(k+1,h)+V_2(k,h+1)\}.\tag{41}$$

In the following, we show that the trivial solution of (14) with $\nu(k,h)\equiv 0$ is stable in the mean square (the method used here has been firstly introduced in [28]). For any given scalar $\epsilon>0$, by resorting to the boundary initial condition (11), there exists one scalar $\delta\in(0,\epsilon)$ which is small enough such that

$$\max_{r\in\lfloor 0,N\rfloor}\sum_{(k,h)\in\mathcal{N}(r)}\mathbb{E}\{V(k,h)\}\le\epsilon^2\tag{42}$$

whenever $\|\varphi(k,h)\|\le\delta$ for $(k,h)\in\lfloor-\overline{\tau},0\rfloor\times\lfloor 0,\kappa_1\rfloor$ and $\|\phi(k,h)\|\le\delta$ for $(k,h)\in\lfloor 0,\kappa_2\rfloor\times\lfloor-\overline{\sigma},0\rfloor$ in (11), where the constant positive integer $N>\max\{\kappa_1,\kappa_2\}+\max\{\overline{\tau},\overline{\sigma}\}$ and the index set $\mathcal{N}(r)=:\{(k,h)|\ k+h=$

$r;\ k,h \in \mathbb{Z}_+\}$. Moreover, for any $r \geq N$, from the inequality (41), it can be shown that the following inequality holds:

$$\begin{aligned}\sum_{(k,h)\in\mathcal{N}(r+1)} \mathbb{E}\{V(k,h)\} \leq & \mathbb{E}\Big\{V_1(r,0) + \big(V_1(r-1,1) + V_2(r-1,1)\big) + \dots \\ & + \big(V_1(1,r-1) + V_2(1,r-1)\big) + V_2(0,r)\Big\} \\ = & \mathbb{E}\Big\{\big(V_1(r,0) + V_2(r,0)\big) + \big(V_1(r-1,1) + V_2(r-1,1)\big) + \dots \\ & + \big(V_1(1,r-1) + V_2(1,r-1)\big) + \big(V_1(0,r) + V_2(0,r)\big)\Big\} \\ = & \sum_{(k,h)\in\mathcal{N}(r)} \mathbb{E}\{V(k,h)\}, \end{aligned} \tag{43}$$

which means $\sum_{(k,h)\in\mathcal{N}(r)} \mathbb{E}\{V(k,h)\}$ is non-increasing with respect to r when $r \geq N$. It should be noted that when deriving (43), the initial conditions $\varphi(k,h) = 0$ for $(k,h) \in \lfloor -\overline{\tau}, 0\rfloor \times \lfloor \kappa_1 + 1, \infty)$ and $\phi(k,h) = 0$ for $(k,h) \in \lfloor \kappa_2 + 1, \infty) \times \lfloor -\overline{\sigma}, 0\rfloor$ in (11) have been utilized. (42) together with (43) guarantee that

$$\lambda_{\min}(\mathcal{P}_1 + \mathcal{P}_2)\mathbb{E}\{\|\tilde{x}(k,h)\|^2\} \leq \lambda_{\min}(\mathcal{P}_1 + \mathcal{P}_2)\mathbb{E}\{\|\eta(k,h)\|^2\} \leq \mathbb{E}\{V(k,h)\} \leq \epsilon^2$$

holds for any $(k,h) \in \mathbb{Z}_+ \times \mathbb{Z}_+$, i.e, system (14) is stable in the mean square.

To draw the conclusion that system (14) with $\nu(k,h) \equiv 0$ is globally asymptotically stable in the mean square, we still need to show $\lim_{k+h\to\infty} \mathbb{E}\{\|\tilde{x}(k,h)\|\} = 0$. The conclusion $\Xi < 0$ in (40) infers that there exists a constant $\mu > 0$ such that

$$\mathbb{E}\{(V(k+1,h+1) - V_1(k+1,h) - V_2(k,h+1))|\aleph(k,h)\} \leq -\mu\mathbb{E}\{\|\eta(k,h+1)\|^2|\aleph(k,h)\}.$$

Taking mathematical expectation on both sides of the above inequality and summing up both sides of it with k, h varying from 0 to N, where integer N is large enough, it is not difficult to obtain

$$\begin{aligned}\sum_{k=0}^{N}\sum_{h=0}^{N} \mathbb{E}\{\|\eta(k,h+1)\|^2\} \leq & \frac{1}{\mu}\Big(\sum_{k=0}^{N} \mathbb{E}\{V_1(k+1,0) - V_1(k+1,N+1)\} \\ & + \sum_{h=0}^{N} \mathbb{E}\{V_2(0,h+1) - V_2(N+1,h+1)\}\Big) \\ \leq & \frac{1}{\mu}\Big(\sum_{k=0}^{N} \mathbb{E}\{V_1(k+1,0)\} + \sum_{h=0}^{N} \mathbb{E}\{V_2(0,h+1)\}\Big) < \infty \end{aligned} \tag{44}$$

where the last step holds because of the bounded initial condition (11). From the necessary condition for the convergent positive series, it can be concluded from (44) that

$$\lim_{k+h\to\infty} \mathbb{E}\{\|\eta(k,h)\|\} = 0.$$

Second, we investigate the H_∞ performance for the output estimation error system (16) by assuming the zero-initial boundary condition. To obtain the H_∞ estimation information, define the index as follows:

$$\vec{\mathscr{J}} =: \mathbb{E}\Big\{\big[\sum_{s=1}^{3}(\Delta V_{1s}(k,h) + \Delta V_{2s}(k,h)) + \vec{z}^T(k,h)\vec{z}(k,h) - \gamma^2\vec{v}^T(k,h)\vec{v}(k,h)\big]|\aleph(k,h)\Big\}, \tag{45}$$

where $\vec{z}(k,h) = (\tilde{z}^T(k+1,h),\ \tilde{z}^T(k,h+1))^T$ and $\vec{v}(k,h) = (\nu^T(k+1,h),\ \nu^T(k,h+1))^T$.

The augmented state estimation error system (17) can be rewritten as

$$\begin{aligned}\eta(k+1,h+1) = & \vec{\mathcal{A}}\xi_1(k,h) + \vec{\mathcal{D}}\xi_2(k,h) + \mathcal{B}_1\mathcal{F}(k,h) + \vec{\mathcal{E}}\vec{v}(k,h) \\ & + (\alpha(k,h) - \bar{\alpha})\mathcal{B}_2\mathcal{F}(k,h) + \mathcal{H}(k,h)\omega(k,h),\end{aligned}$$

where matrix $\vec{\mathcal{E}}$ is defined in (19), and hence it can be derived that

$$\begin{aligned}
&\mathbb{E}\Big\{\eta^T(k+1,h+1)(\mathcal{P}_1+\mathcal{P}_2)\eta(k+1,h+1)|\aleph(k,h)\Big\}\\
&\quad=\mathbb{E}\Big\{\Big[\xi_1^T(k,h)\vec{\mathcal{A}}^T(\mathcal{P}_1+\mathcal{P}_2)\vec{\mathcal{A}}\xi_1(k,h)+\xi_2^T(k,h)\vec{\mathcal{D}}^T(\mathcal{P}_1+\mathcal{P}_2)\vec{\mathcal{D}}\xi_2(k,h)+\mathcal{F}^T(k,h)\mathcal{B}_1^T(\mathcal{P}_1+\mathcal{P}_2)\mathcal{B}_1\mathcal{F}(k,h)\\
&\qquad+\vec{\nu}^T(k,h)\vec{\mathcal{E}}^T(\mathcal{P}_1+\mathcal{P}_2)\vec{\mathcal{E}}\vec{\nu}(k,h)+2\xi_1^T(k,h)\vec{\mathcal{A}}^T(\mathcal{P}_1+\mathcal{P}_2)\big(\vec{\mathcal{D}}\xi_2(k,h)+\mathcal{B}_1\mathcal{F}(k,h)+\vec{\mathcal{E}}\vec{\nu}(k,h)\big)\\
&\qquad+2\xi_2^T(k,h)\vec{\mathcal{D}}^T(\mathcal{P}_1+\mathcal{P}_2)\big(\mathcal{B}_1\mathcal{F}(k,h)+\vec{\mathcal{E}}\vec{\nu}(k,h)\big)+2\mathcal{F}^T(k,h)\mathcal{B}_1^T(\mathcal{P}_1+\mathcal{P}_2)\vec{\mathcal{E}}\vec{\nu}(k,h)\\
&\qquad+\bar{\alpha}(1-\bar{\alpha})\mathcal{F}^T(k,h)\mathcal{B}_2^T(\mathcal{P}_1+\mathcal{P}_2)\mathcal{B}_2\mathcal{F}(k,h)+\mathcal{H}^T(k,h)(\mathcal{P}_1+\mathcal{P}_2)\mathcal{H}(k,h)\Big]|\aleph(k,h)\Big\}.
\end{aligned}\tag{46}$$

Moreover, it follows from the output estimation error system (16) that

$$\vec{z}^T(k,h)\vec{z}(k,h)=\xi_1^T(k,h)\mathrm{diag}(\mathcal{M}^T\mathcal{M},\mathcal{M}^T\mathcal{M})\xi_1(k,h).\tag{47}$$

Substituting (46) into (31) and combining with inequalities (20), (22), (34), (37), (38) and (39), we have

$$\vec{\mathscr{J}}\le\mathbb{E}\Big\{\vec{\xi}(k,h)\tilde{\Xi}\vec{\xi}(k,h)|\aleph(k,h)\Big\}\tag{48}$$

where $\vec{\xi}(k,h)=\mathrm{col}(\xi(k,h),\vec{\nu}(k,h))$ and $\tilde{\Xi}=\mathrm{diag}(\Psi,-\gamma^2 I_{2p})+\mathrm{col}(\Gamma,\vec{\mathcal{E}}^T(\mathcal{P}_1+\mathcal{P}_2))(\mathcal{P}_1+\mathcal{P}_2)^{-1}(\mathrm{col}(\Gamma,\vec{\mathcal{E}}^T(\mathcal{P}_1+\mathcal{P}_2)))^T$. Again from the Schur Complement Lemma [4], it is known that matrix $\tilde{\Xi}<0$ if and only if the inequality condition (19) holds. That is, under the condition (19), it is assured that for all $\vec{\xi}(k,h)\neq 0$,

$$\begin{aligned}
\mathbb{E}\{V(k+1,h+1)|\aleph(k,h)\}<&\mathbb{E}\{[(V_1(k+1,h)+V_2(k,h+1))-(\|\tilde{z}(k+1,h)\|^2+\|\tilde{z}(k,h+1)\|^2)\\
&+\gamma^2(\|\nu(k+1,h)\|^2+\|\nu(k,h+1)\|^2)]|\aleph(k,h)\}.
\end{aligned}$$

Taking mathematical expectation on both sides of the above inequality, the following inequalities can be obtained:

$$\begin{aligned}
\mathbb{E}\{V(k+1,0)\}=&\mathbb{E}\{V_1(k+1,0)+V_2(k+1,0)\},\\
\mathbb{E}\{V(k,1)\}\le&\mathbb{E}\{(V_1(k,0)+V_2(k-1,1))-(\|\tilde{z}(k,0)\|^2+\|\tilde{z}(k-1,1)\|^2)+\gamma^2(\|\nu(k,0)\|^2+\|\nu(k-1,1)\|^2)\},\\
\mathbb{E}\{V(k-1,2)\}\le&\mathbb{E}\{(V_1(k-1,1)+V_2(k-2,2))-(\|\tilde{z}(k-1,1)\|^2+\|\tilde{z}(k-2,2)\|^2)\\
&+\gamma^2(\|\nu(k-1,1)\|^2+\|\nu(k-2,2)\|^2)\},\\
&\vdots\\
\mathbb{E}\{V(2,k-1)\}\le&\mathbb{E}\{(V_1(2,k-2)+V_2(1,k-1))-(\|\tilde{z}(2,k-2)\|^2+\|\tilde{z}(1,k-1)\|^2)\\
&+\gamma^2(\|\nu(2,k-2)\|^2+\|\nu(1,k-1)\|^2)\},\\
\mathbb{E}\{V(1,k)\}\le&\mathbb{E}\{(V_1(1,k-1)+V_2(0,k))-(\|\tilde{z}(1,k-1)\|^2+\|\tilde{z}(0,k)\|^2)+\gamma^2(\|\nu(1,k-1)\|^2+\|\nu(0,k)\|^2)\},\\
\mathbb{E}\{V(0,k+1)\}=&\mathbb{E}\{V_1(0,k+1)+V_2(0,k+1)\}.
\end{aligned}$$

Adding up both sides of the above $k+2$ inequalities with k varying from 0 to $N_1\in\mathbb{Z}_+$ and considering the zero-initial boundary condition, we get the inequality given below:

$$\begin{aligned}
&\sum_{k=0}^{N_1}\Big\{\sum_{j=0}^{k}\mathbb{E}\{\|\tilde{z}(k-j,j)\|^2\}-\frac{1}{2}\mathbb{E}\{\|\tilde{z}(k,0)\|^2\}-\frac{1}{2}\mathbb{E}\{\|\tilde{z}(0,k)\|^2\}\Big\}\\
&\quad\le\sum_{k=0}^{N_1}\Big\{\sum_{j=0}^{k}\mathbb{E}\{V(k-j,j)\}-\sum_{j=0}^{k+1}\mathbb{E}\{V(k+1-j,j)\}\Big\}\\
&\qquad+\gamma^2\sum_{k=0}^{N_1}\Big(\sum_{j=0}^{k}\mathbb{E}\{\|\nu(k-j,j)\|^2\}-\frac{1}{2}\mathbb{E}\{\|\nu(k,0)\|^2\}-\frac{1}{2}\mathbb{E}\{\|\nu(0,k)\|^2\}\Big)\\
&\quad=\mathbb{E}\{V(0,0)\}-\sum_{j=0}^{N_1+1}\mathbb{E}\{V(N_1+1-j,j)\}\\
&\qquad+\gamma^2\sum_{k=0}^{N_1}\Big(\sum_{j=0}^{k}\mathbb{E}\{\|\nu(k-j,j)\|^2\}-\frac{1}{2}\mathbb{E}\{\|\nu(k,0)\|^2\}-\frac{1}{2}\mathbb{E}\{\|\nu(0,k)\|^2\}\Big)\\
&\quad\le\gamma^2\sum_{k=0}^{N_1}\Big(\sum_{j=0}^{k}\mathbb{E}\{\|\nu(k-j,j)\|^2\}-\frac{1}{2}\mathbb{E}\{\|\nu(k,0)\|^2\}-\frac{1}{2}\mathbb{E}\{\|\nu(0,k)\|^2\}\Big).
\end{aligned}$$

By letting $N_1\to\infty$, we have

$$\begin{aligned}
&\sum_{h=0}^{\infty}\sum_{k=0}^{\infty}\mathbb{E}\{\|\tilde{z}(k,h)\|^2\}-\frac{1}{2}\sum_{k=0}^{\infty}\mathbb{E}\{\|\tilde{z}(k,0)\|^2)\}-\frac{1}{2}\sum_{h=0}^{\infty}\mathbb{E}\{\|\tilde{z}(0,h)\|^2)\}\\
&\qquad\le\gamma^2\Big\{\sum_{h=0}^{\infty}\sum_{k=0}^{\infty}\mathbb{E}\{\|\nu(k,h)\|^2\}-\frac{1}{2}\sum_{k=0}^{\infty}\mathbb{E}\{\|\nu(k,0)\|^2)\}-\frac{1}{2}\sum_{h=0}^{\infty}\mathbb{E}\{\|\nu(0,h)\|^2)\}\Big\},
\end{aligned}$$

i.e.,

$$\|\tilde{z}\|_{l_2}^2\le\gamma^2\|\nu\|_{l_2}^2,$$

which completes the proof of Theorem 1. ■

To derive the explicit design scheme for the distributed H_∞ state estimation problem, we still need to introduce the following lemma whose proof is straightforward and therefore omitted here.

Lemma 1: [33] Let $\mathbb{P}=\operatorname{diag}(P_{11},P_{22},\dots,P_{NN})$ with $P_{ii}\in\mathbb{R}^{n\times n}$ $(i=1,2,\dots,N)$ being invertible matrices. If $X=\mathbb{P}\bar{U}$ for $\bar{U}\in\mathbb{R}^{nN\times mN}$, then we have $\bar{U}\in\mathcal{W}_{n\times m}\Leftrightarrow X\in\mathcal{W}_{n\times m}$.

We are now ready to deal with the distributed H_∞ estimation design problem in the following theorem.

Theorem 2: Consider the target plant (1)-(2) with output measurements (8) and let $\gamma>0$ be a prescribed constant scalar. For all $i=1,2,\dots,N$, the system in (12)-(13) is a distributed H_∞ state estimator on sensor i if there exist matrices $P_{lj}>0$ and $\mathcal{Q}_l>0$ $(j=0,1,\dots,N)$, positive diagonal matrices $\vec{\delta}_l$, $\vec{\varepsilon}_l$ and $\vec{\theta}=\operatorname{diag}(\theta_0,\theta_1,\dots,\theta_N)$, matrices $M_i\in\mathbb{R}^{q\times n}$ $(i\in\lfloor 1,N\rfloor)$, $\mathcal{X}_l\in\mathcal{W}_{n\times m}$ and positive scalars $\varepsilon_0^{(l)}$ $(l=1,2)$ such that the following matrix inequality holds:

$$\Phi=\begin{bmatrix}\vec{\Psi} & 0 & \tilde{\Gamma} & \Phi_{14}\\ * & -\gamma^2 I_{2p} & \Phi_{23} & 0\\ * & * & -(\mathcal{P}_1+\mathcal{P}_2) & 0\\ * & * & * & -I\end{bmatrix}<0 \tag{49}$$

where $\mathcal{P}_l = \text{diag}(P_{l0}, \mathscr{P}_l)$ with $\mathscr{P}_l = \text{diag}(P_{l1}, P_{l2}, \dots, P_{lN})$ $(l = 1, 2)$, $\vec{\Psi}$ is almost the same as matrix Ψ in (19) with only Ψ_{11} being substituted by

$$\begin{aligned}\vec{\Psi}_{11} =& \text{diag}((\overline{\sigma} - \underline{\sigma} + 1)\mathcal{Q}_1 - \mathcal{P}_1, (\overline{\tau} - \underline{\tau} + 1)\mathcal{Q}_2 - \mathcal{P}_2) - (I_2 \otimes \mathscr{L}_2^T)\mathscr{U}_2^{(1)}(I_2 \otimes \mathscr{L}_2) \\ & - \varepsilon_0^{(1)}(I_2 \otimes \mathscr{L}_1^T)\text{Sym}((F_1^{(1)})^T F_2^{(1)})(I_2 \otimes \mathscr{L}_1) + \sum_{i=0}^{N} \theta_i (I_2 \otimes \mathscr{L}_1^T) H^T H (I_2 \otimes \mathscr{L}_1),\end{aligned}$$

$\Phi_{14} = \text{col}(\vec{\mathcal{M}}^T, 0, 0, 0)$ with $\vec{\mathcal{M}} = \text{diag}(\mathcal{M}, \mathcal{M})$, $\tilde{\Gamma} = \text{col}(\tilde{\Gamma}_1^T, \tilde{\Gamma}_2^T, \tilde{\Gamma}_3^T, 0)$ with

$$\begin{aligned}\tilde{\Gamma}_1 =& \begin{bmatrix} (P_{10} + P_{20})A_1 & 0 & (P_{10} + P_{20})A_2 & 0 \\ 0 & \tilde{\Gamma}_{122} & 0 & \tilde{\Gamma}_{124} \end{bmatrix}, \\ \tilde{\Gamma}_2 =& (\mathcal{P}_1 + \mathcal{P}_2) \begin{bmatrix} (1_{N+1} \otimes D_1)\mathscr{L}_1 & (1_{N+1} \otimes D_2)\mathscr{L}_1 \end{bmatrix}, \\ \tilde{\Gamma}_3 =& \begin{bmatrix} \bar{\alpha}(P_{10} + P_{20})B_1 & (1 - \bar{\alpha})(P_{10} + P_{20})B_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{\Gamma}_{323} & \tilde{\Gamma}_{324} & -\mathcal{X}_1 & -\mathcal{X}_2 \end{bmatrix}, \\ \Phi_{23} =& \begin{bmatrix} E_1^T(P_{10} + P_{20}) & (1_N \otimes E_1)^T(\mathscr{P}_1 + \mathscr{P}_2) - \widetilde{\mathscr{W}}^T \mathcal{X}_1^T \\ E_2^T(P_{10} + P_{20}) & (1_N \otimes E_2)^T(\mathscr{P}_1 + \mathscr{P}_2) - \widetilde{\mathscr{W}}^T \mathcal{X}_2^T \end{bmatrix},\end{aligned}$$

where

$$\begin{aligned}\tilde{\Gamma}_{122} =& (\mathscr{P}_1 + \mathscr{P}_2)(I_N \otimes A_1) - \mathcal{X}_1(I_N \otimes S_1)\overline{\mathscr{C}}, \quad \tilde{\Gamma}_{124} = (\mathscr{P}_1 + \mathscr{P}_2)(I_N \otimes A_2) - \mathcal{X}_2(I_N \otimes S_1)\overline{\mathscr{C}}; \\ \tilde{\Gamma}_{323} =& \bar{\alpha}(\mathscr{P}_1 + \mathscr{P}_2)(I_N \otimes B_1), \qquad \tilde{\Gamma}_{324} = (1 - \bar{\alpha})(\mathscr{P}_1 + \mathscr{P}_2)(I_N \otimes B_2).\end{aligned}$$

Moreover, the state estimation gain matrices can be designed as follows:

$$\mathscr{K}_l = (\mathscr{P}_1 + \mathscr{P}_2)^{-1}\mathcal{X}_l, \quad l = 1, 2 \tag{50}$$

and the output estimation gain matrices M_i $(i = 1, 2, \dots, N)$ can be obtained directly as the solution of (49).

Proof: By using the Schur Complement Lemma [4] to inequality (49) and noticing the equalities in (50), it will be concluded that condition (19) holds under the validity of inequality (49). Hence, it follows from Theorem 1 that the result presented in this theorem is also tenable. ■

Remark 3: In this paper, the distributed H_∞ state estimation problem is studied for a class of stochastic 2-D systems with RVNs and time-varying delays. The main novelty lies in that 1) the proposed 2-D system is general enough to model the phenomena of RVNs, sensor saturations and time-delays; 2) a new energy-like quadratic function is employed to analyze the system stability and performance; and 3) intensive stochastic analysis is conducted to enforce the H_∞ performance for the addressed state estimation problem. It should be pointed out that the main results established in Theorem 2 contain all the information about the system parameters, the occurring probabilities of RVNs, the sensor saturation level as well as the bounds of the time-varying delays.

Remark 4: Note that, for the standard LMI system, the algorithm has a polynomial-time complexity. That is, the number $\mathcal{N}(\varepsilon)$ of flops needed to compute an ε-accurate solution is bounded by $O(\mathcal{M}\mathcal{N}^3 \log(\mathcal{V}/\varepsilon))$, where $\mathcal{M}$ is the total row size of the LMI system, $\mathcal{N}$ is the total number of scalar decision variables, $\mathcal{V}$ is a data-dependent scaling factor, and ε is relative accuracy set for algorithm. Obviously, the computational complexity of the LMI-based algorithms depends polynomially on the network size and the variable dimensions. In order to reduce the computation burden, a possible way is to obtain the estimator gains in a node-by-node way. Fortunately, research on LMI optimization is a very active area in the applied mathematics, optimization and the operations research community, and substantial speed-ups can be expected in the future.

Remark 5: It can be seen from the main results that the feasibility of the developed algorithm for estimator design would decrease with the increase of the occurring probabilities of randomly varying nonlinearities, the increase of

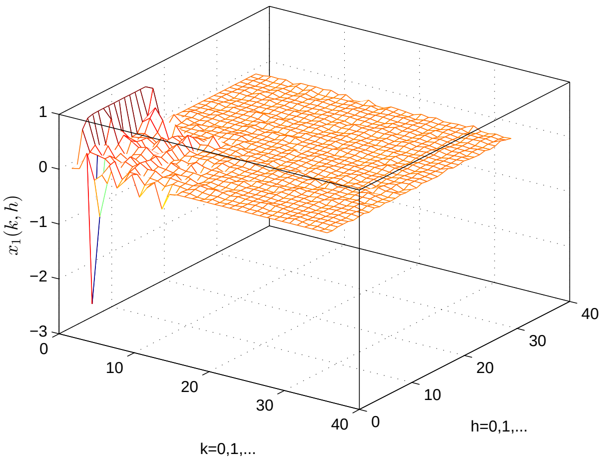

Fig. 1. Evolution of the first element of state $x(k,h)$.

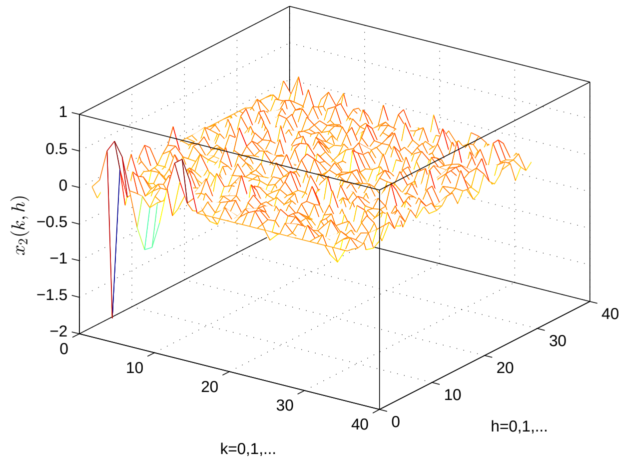

Fig. 2. Evolution of the second element of state $x(k,h)$.

the sensor saturation level, and the increase of the bounds of the interval-like time-varying delays. On the other hand, if the connectivity of the sensor network is improved, the sparseness issue will be eased and the feasibility of the proposed estimator design procedure will be enhanced.

IV. Illustrative Example

Consider a discrete 2-D delayed system with stochastic disturbances modeled by (1) with the following parameters:

$$A_1 = \begin{bmatrix} 0.12 & 0.08 \\ 0.1 & -0.12 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.05 & -0.06 \\ 0.04 & 0.09 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.04 & 0 \\ 0.08 & 0.05 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.2 & 0.5 \\ 0.1 & -0.2 \end{bmatrix},$$
$$D_1 = \begin{bmatrix} 0.16 & 0.02 \\ -0.14 & 0.04 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0.07 & -0.02 \\ 0.06 & 0.04 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0.1 & 0.6 \\ -0.6 & 0.8 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0.3 & 0.4 \\ 0.05 & -0.4 \end{bmatrix}.$$

The time-varying delays in both directions are $\tau(k) = 3 + 3|\sin(\frac{k}{2}\pi)|$ and $\sigma(h) = 2 + 5|\cos(\frac{h}{2}\pi)|$, respectively, with bounds as $\overline{\tau} = 6$, $\underline{\tau} = 3$, $\overline{\sigma} = 7$ and $\underline{\sigma} = 2$. For $u = (u_1, u_2)^T$, $v = (v_1, v_2)^T \in \mathbb{R}^2$, the nonlinearities $f_1(u,v) = (0.2u_1 + \tanh(0.04u_1) + 0.2v_1 - \tanh(0.1v_1), 0.2u_2 - \tanh(0.1u_2) + 0.2v_2 + \tanh(0.04v_2))^T$ and $f_2(u,v) = (0.2u_1 - \tanh(0.1u_1) + 0.2v_1 + \tanh(0.04v_1), 0.1u_2 + \tanh(0.05u_2) + 0.2v_2 + \tanh(0.04v_2))^T$ which obviously satisfy the conditions in (6) with

$$F_1^{(1)} = \begin{bmatrix} 0.2 & 0 & 0.1 & 0 \\ 0 & 0.1 & 0 & 0.2 \end{bmatrix}, \qquad F_2^{(1)} = \begin{bmatrix} 0.24 & 0 & 0.2 & 0 \\ 0 & 0.2 & 0 & 0.24 \end{bmatrix},$$
$$F_1^{(2)} = \begin{bmatrix} 0.1 & 0 & 0.2 & 0 \\ 0 & 0.1 & 0 & 0.2 \end{bmatrix}, \qquad F_2^{(2)} = \begin{bmatrix} 0.2 & 0 & 0.24 & 0 \\ 0 & 0.15 & 0 & 0.24 \end{bmatrix}.$$

It is assumed that the nonlinearities are randomly varying with the probability $\bar{\alpha} = 0.68$. The noise intensity function $\hbar(u,v) = (0.24\tanh u_1 + 0.2\tanh v_1, -0.15\sin u_2 + 0.1\cos v_2)^T$ which is subject to the constraint (5) with

$$H = \begin{bmatrix} 0.24 & 0 & 0.2 & 0 \\ 0 & 0.15 & 0 & 0.1 \end{bmatrix},$$

and the matrix M_0 for deriving the output signal $z(k,h)$ in equation (2) is taken to be $[0.105 \quad -0.068]$.

The initial boundary condition associated with system (1) is taken to be $x(k,h) = (0.1\tan(k+h), 0.7\sin(kh))^T$ for $(k,h) \in \lfloor -6, 0 \rfloor \times (0, 13\rfloor$, $x(k,h) = (0.8\tanh(k-h), 0.2\cos(k+h))^T$ for $(k,h) \in (0, 14\rfloor \times \lfloor -7, 0\rfloor$ and

$x(k,h) = (0,0)^T$ otherwise. Moreover, the exogenous disturbance input $\nu(k,h) = (6\sin((k+7)(h+8)), 2\cos(k+h))^T$ for $(k,h) \in \lfloor 0, 24 \rfloor \times \lfloor 0, 23 \rfloor$ and $\nu(k,h) = (0,0)^T$ otherwise. The corresponding dynamical evolutions of the state $x(k,h)$ are shown in Figs. 1-2.

The sensor network considered here is represented by a directed graph $\mathscr{G} = (\mathscr{V}, \mathscr{E}, \mathcal{L})$ formed by 6 sensors, where the set of edges $\mathscr{E} = \{(1,1),(1,2),(2,2),(2,3),(2,5),(3,3),(3,4),(3,6),(4,4),(4,5),(5,1)(5,5),(6,1),(6,6)\}$ and the adjacency elements associated with the edges of the graph are $l_{ij} = 1$. The matrices in the output measurement equation (8) are assumed to be

$$C_1 = \begin{bmatrix} -0.3 & 0.1 \\ -0.1 & 0.5 \end{bmatrix}, \quad W_1 = \begin{bmatrix} 0.15 & 0 \\ 0.3 & 0.2 \end{bmatrix}; \quad C_2 = \begin{bmatrix} 0.1 & 0.2 \\ 0.5 & 0.15 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0.15 & 0.65 \\ 0 & -0.2 \end{bmatrix};$$
$$C_3 = \begin{bmatrix} -0.1 & 0.17 \\ 0.04 & 0.5 \end{bmatrix}, \quad W_3 = \begin{bmatrix} 0.14 & 0.7 \\ 0.07 & -0.3 \end{bmatrix}; \quad C_4 = \begin{bmatrix} 0.3 & 0.1 \\ 0.5 & 0.09 \end{bmatrix}, \quad W_4 = \begin{bmatrix} 0.19 & -0.22 \\ 0 & -0.19 \end{bmatrix};$$
$$C_5 = \begin{bmatrix} -0.15 & 0.1 \\ 0 & 0.25 \end{bmatrix}, \quad W_5 = \begin{bmatrix} 0.2 & 0.03 \\ 0.14 & 0.8 \end{bmatrix}; \quad C_6 = \begin{bmatrix} 0.25 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad W_6 = \begin{bmatrix} 0.15 & 0 \\ 0.66 & 0.18 \end{bmatrix}.$$

The matrices S_1 and S_2 employed for dealing with the nonlinear saturated function $g(\cdot)$ with saturation level vector $u_{\max} = (6,8)^T$ are taken to be $S_1 = \mathrm{diag}(0.28, 0.32)$ and $S_2 = \mathrm{diag}(1.14, 1.09)$, which easily means that $S = \mathrm{diag}(0.86, 0.77)$.

With the parameters given above, it is aimed to design a distributed state estimator in the form of (12)-(13) for the stochastic 2-D target plant in (1)-(2) with 6 sensor measurement outputs (8). By utilizing the Matlab Toolbox, it is found that, for the given H_∞ performance index $\gamma \geq 2.684$, a solution can always be obtained for the matrix inequality (49) in Theorem 2, which means that the minimum of the index for characterizing the H_∞ performance is $\gamma^* = 2.684$. For example, the solution corresponding to the case of $\gamma = 2.684$ is obtained as follows (here only part of the solution is given for space consideration): $\varepsilon_0^{(1)} = 0.4240$, $\varepsilon_0^{(2)} = 1.2463$, $\theta_0 = 1.1367$ and

$$M_2 = \begin{bmatrix} 0.0746 & 0.0174 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 0.0725 & 0.0184 \end{bmatrix}, \quad M_6 = \begin{bmatrix} 0.0752 & -0.0088 \end{bmatrix};$$
$$P_{10} = \begin{bmatrix} 0.7265 & 0.1308 \\ 0.1308 & 0.3952 \end{bmatrix}, \qquad P_{20} = \begin{bmatrix} 0.3928 & -0.0394 \\ -0.0394 & 0.2607 \end{bmatrix}.$$

Moreover, the state estimation gain matrices can be explicitly designed as follows according to (50) (for the same reason of space consideration, only part of the block sub-matrices are given):

$$K_{111} = \begin{bmatrix} -0.2456 & -0.0282 \\ -0.4363 & -0.4501 \end{bmatrix}, \quad K_{125} = \begin{bmatrix} 0.1229 & 0.0583 \\ -0.2466 & -0.0256 \end{bmatrix}, \quad K_{161} = \begin{bmatrix} -0.0760 & 0.0626 \\ -0.3339 & -0.2271 \end{bmatrix},$$
$$K_{223} = \begin{bmatrix} -0.0518 & -0.2134 \\ -0.0331 & 0.3227 \end{bmatrix}, \quad K_{245} = \begin{bmatrix} 0.4249 & 0.2376 \\ -0.1825 & -0.0270 \end{bmatrix}, \quad K_{251} = \begin{bmatrix} -0.8504 & -0.0804 \\ -0.0619 & 0.2170 \end{bmatrix}.$$

It follows immediately from Theorem 2 that for all $i = 1, 2, \ldots, 6$, the system in (12)-(13) is a distributed H_∞ state estimator on sensor i for the target plant (1)-(2) with output measurements (8).

With the estimator gain matrices given above, to illustrate the effectiveness of the designed estimators with more visuality, Figs. 3-4 show the dynamical evolutions of the state estimation error $\tilde{x}_1(k,h)$ for sensor 1, Fig. 5 and Fig. 6 present the dynamical evolutions of the output estimation errors for sensor 3 and sensor 5 respectively, which further demonstrate the validity of the results obtained in Section III (for space saving purpose, we only list four figures here).

Furthermore, it can be shown that the occurring probability $\bar{\alpha}$ of the RVNs does affect the feasibility of the proposed results. In this example, the effective interval for the feasibility of the matrix inequality (49) is $[0.6799, 1]$. If we utilize the usual estimation method other than the distributed idea employed here, i.e., each sensor estimates the

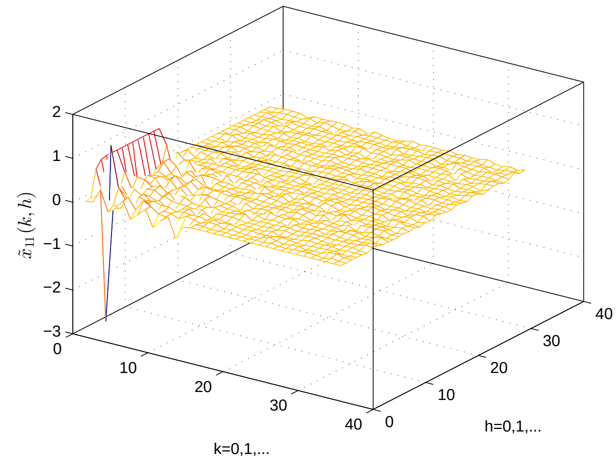

Fig. 3. Evolution of the estimation error $\tilde{x}_{11}$ from sensor 1.

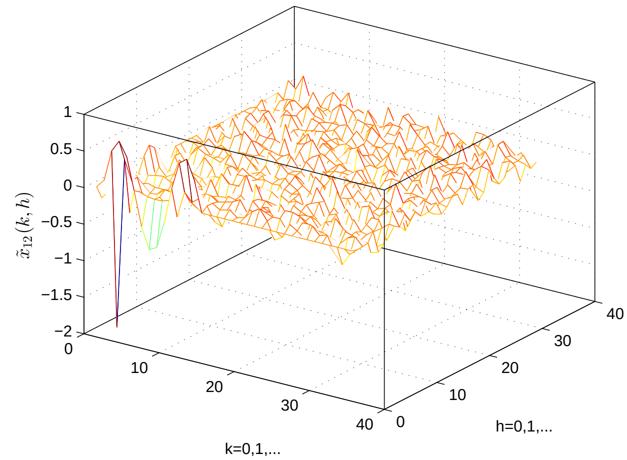

Fig. 4. Evolution of the estimation error $\tilde{x}_{12}$ from sensor 1.

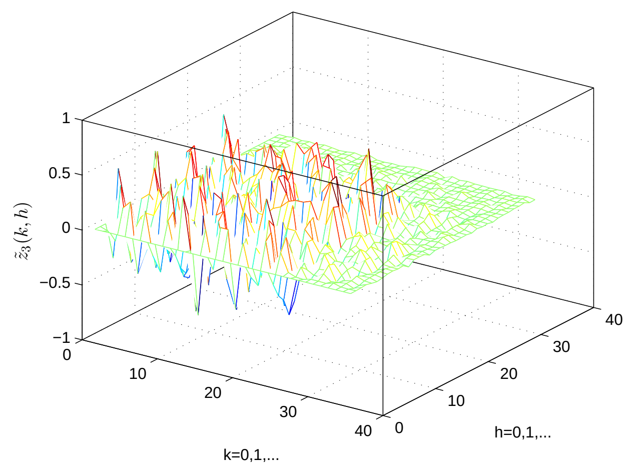

Fig. 5. Evolution of the output estimation error $\tilde{z}_3(k,h)$ from sensor 3.

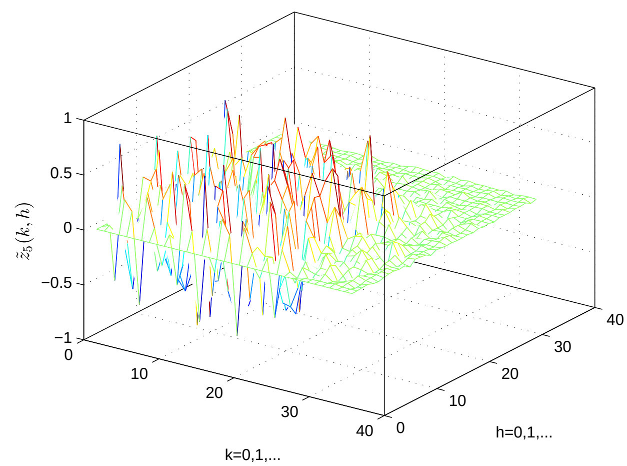

Fig. 6. Evolution of the output estimation error $\tilde{z}_5(k,h)$ from sensor 5.

states of the target plant by only its own measured outputs, it can be shown that the minimum index for characterizing the state estimation H_∞ performance is $\gamma^{**} = 2.763$, which further infers that the distributed estimation scheme makes the H_∞ attenuation level smaller.

V. Conclusions

In this paper, we have addressed the distributed H_∞ state estimation problem for the stochastic 2-D systems with time-varying delays. RVNs have been introduced in the target plant to reflect the nonlinear disturbances which appear in a probabilistic way and are changeable randomly in terms of their types and intensity. Due to the fact that there is no centralized processor which can capable of collecting all the measurements from the sensors, this paper has designed the distributed state estimators which estimate the states of the target plant in a distributed way. More specifically, each individual sensor estimates the states of the target plant based on not only its own but also its neighboring sensors' measurements according to certain topology. By using the Kronecker product and the inequality technique, an energy-like function has been introduced to derive some sufficient criteria under which the estimation error system is globally asymptotically stable in the mean square and the H_∞ performance constraint is also guaranteed. Explicit representation of the estimation gains has been given in terms of the solution of certain matrix inequality. Furthermore, the effectiveness of the proposed design scheme has been checked by a numerical example.

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