Optimisation Problems as Decision Problems: The case of fuzzy Optimisation Problems

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To Janusz, Professor Janusz Kacprzyk, an outstanding researcher, an example to follow but, especially, a friend with whom we have shared much of our lives.

Abstract

The importance that decision-making problems and optimisation problems have today in all aspects of life is beyond all doubt. Despite that importance, both problems tend to be thought of as following different routes, when they have, in fact, a "symbiotic" relation. Here, we consider the different decision problems that arise when different kinds of information and framework of behaviour are considered, and we explore the corresponding optimisation problems that can be derived for searching the best possible decision. We explore the case where Fuzzy Mathematical Programming problems are obtained as well as other new ones in the fuzzy context.

Keywords: Decision Making, Fuzzy Sets, Fuzzy Mathematical Programming

1. Introduction.

Decision making is an activity that is inherent in the behaviour of people. When a decision only affects the person who decides, it may have a low impact, but when it may affect third parties (other people, a company, a market, etc.) it is essential to know in depth the mechanisms and methodologies that govern such a decision process.

In the hypothetical case that a decision-maker (a) knows all the possible consequences of their decisions, (b) knows all the alternatives, and (c) can choose rationally and knowingly, then the optimal choice will be the action whose consequence is the preferred by the decision-maker from among all the set of possible consequences that their action may have.

However, this apparently simple path is windy, complicated and sometimes almost impossible to follow due to the amount of information that needs to be collected. In simple terms, a decision maker would need to know all the alternatives, to be able to have access to all their consequences, to know how to order the results of the actions or ensure that a decision has only one consequence and so on.

Another problem arise when we assume that the context in which decision making is developed can change, the information available could be partial, etc. All of these circumstances, casuistic and options are studied in the Decision Theory area, whose first antecedent is found in [Von].

One of the most studied problems in Decision Theory considers a single decision-maker, that the information available is of a probabilistic nature, and the results of the different options that the decision-maker has (the rewards), can be ordered according to a criterion that the decision-maker establishes and which allows him/her to take the best decision among the possible ones.

As stated before, these ideal conditions are hard to meet. In the framework that defines the real world in which the problem is posed we may not know all the available alternatives, or we may not apply the criterion that we want to order the consequences or, more frequently, the information on the problem's parameters is not probabilistic, but of whatever other nature. Thus, depending on the characteristics of the problem, a wide range of models appears: single person decision problem with a finite number of numerically valued results, single person with incomplete information, group decision making problems, and so on.

Going further, what we want also to highlight here is that it is not just a matter of making decisions, but also a matter of making the best decision according to the framework in which we are acting, simply because we generally seek to optimise our results.

In this context, our aim here is to explore the connections between decision making problems and optimisation problems, starting with those of Mathematical Programming, formulating the later on the basis of the characteristics of the data and the information available in the former.

As a result, we expect contributing rationality to the field of optimisation and giving greater amplitude to decision making, facilitating the comparisons and allowing new optimisation and decision models to appear, which up to date have not been considered. This would be of particular relevance when new frameworks of behaviour are introduced for the decision maker.

The paper is organized as follows. In Section 2 the essential elements of a General Decision Problem are introduced. Then in Section 3, and departing from the Taxonomy of Ignorance by [Smi], the different natures that the information can have are described, and as a consequence the different frameworks in which decision and optimisation problems can be posed are presented. With these elements, Section 4 explores the case where the available information is fuzzy. We consider the potential decision making problems and the corresponding Optimisation Problems, together with their formulation. Section 5 is devoted to conclusions and further discussion.

2. Description of the General Decision Problem

In very general terms, a decision problem consists in choosing the best alternative in a set of actions, where a certain measure of "quality" of each of them is available. By "quality" of an alternative or action one understands the degree of agreement between the result that is obtained by applying it and the decision-maker's criteria. As expected, the quality must be evaluated from the information available. In the majority of real situations, upon knowing the context in which the decision problem arises, external factors exist (which are usually named as nature's states) that, together with the characteristics of the available information, condition the result of an action. Thus, if the decision-maker chooses a certain alternative x, he receives a reward , that we can note as $f(x, e_t)$, where e_t is the state of nature (i.e. the current situation of the external factors). That gain or reward may have different interpretations depending, for instance, on whether the information that we have on the data that define the problem is linguistic, visual, numerical, etc.

Thus, from a formal point of view, the simplest decision problem contains the following seven basic elements:

1) A **decision-maker**, individual person or team of people that act as one, interested in solving the problem posed.
2) A space of **alternatives**, actions or possible decisions, X, which contains at least two elements (if only one single possible action exists then there is no problem of choice).

3) A set of states of nature, E, that describes the **Environment** in which the problem is posed, i.e. that the decision-maker does not control and therefore they condition the results of their actions.
4) A set of **results**, rewards or consequences, each of which is associated to a pair constituted by an alternative and a state of nature.
5) A relationship that establishes the **preferences** of the decision-maker on the possible results that can be obtained.
6) A characterisation of the **available information** that indicates, in terms of knowledge, what we know about the elements that participate in the problem.
7) A **framework** of behaviour K that establishes the rules to guide the decision-maker in making the selection.

Therefore, if $E = \{e_1, e_2 \dots e_m\}$ is the set of possible states of nature, that is to say the Environment of the problem, and $X = \{x_1, x_2 \dots x_n\}$ is the set of alternatives, the results will be represented as an application

$$f: X \times E \rightarrow U$$

where $U = \{f(x_1, e_1), f(x_1, e_2) \dots f(x_n, e_m)\}$ is a set of utility values (the payment, benefit, reward or, ultimately, consequence that is associated to each alternative-state).

When some (simple) formal conditions arise, the problem can be represented by a matrix such as the following,

	e_1	e_2	…	e_m
x_1	r_{11}	r_{12}	…	r_{1m}
x_2	r_{21}	r_{21}	…	r_{2m}
…	…	…	…	…
x_n	r_{n1}	r_{n2}	…	r_{nm}

where $r_{ij} = f(x_i, e_j)$, $i = 1, \dots, n$; $j = 1, \dots, m$.

Comparing $f(x_i, e_j)$ by means of the order relation (the criterion) that the decision-maker has established, a measure of the suitability of x_i with respect to such a criterion can be obtained. So that the aggregation of these measures will provide the overall "quality" of the action that, finally, will indicate the course of action to follow (the decision that must be taken in terms of the available information).

Summarizing, a unipersonal decision problem can be represented by a sextet $(X, E, f, \leq, I, K)$ that includes the set of actions to be taken, the environment, the results, the relation that orders said results, which naturally will have to be consistent with the characteristics that the set U (numerical, linguistic, etc.) has, the available information and the framework of behaviour in which the decision-maker makes decisions.

It should be noted that other more specific aspects can be superimposed to this scheme, which, as a whole, end up characterising the situation, i.e., the decision problem that is being considered:

a) The decision-maker, as the entity that takes decisions, can be a single individual or also a set of people. In this latter case, we would have a group decision making (GDM) problem in which the different individuals interact to take a decision. Below, we shall consider the unipersonal case and we shall refer to this henceforth as the General Decision Problem.

b) The problem can take into account repeated decision making over a certain time period, that is to say, it can be stated as a multi-stage problem. Here, we consider only single-stage problems.

c) In general, the consequences are measured numerically. However other alternatives are possible. In other words the set U where the function f takes values, does not have to be a sub-set of the real line, making sense to imagine linguistically established consequences, which express a quality, or that are even affected by other characteristics, such as can be the case of random, symbolic, uncertain etc., values.

d) Likewise, the decision-maker must establish a proper order on possible results, which will ultimately be what enables them to take the decision which is most interesting in the Environment where the problem is being developed, and which can be conditioned by the Framework of Behaviour K, i.e. by the rules of behaviour to be followed. As well as this, when the consequences are not valued numerically, it is also possible to resort to building a utility function of the results. When the consequences can be valued numerically, and also when a utility function is available, what remains to be solved may be seen as an optimisation problem. But the order used by the decision-maker does not necessarily have to be unique, i.e. that the decision-maker can have different criteria simultaneously (cost, distance, accessibility, security, …) to hierarchize their preferences. If this were the case, this would give rise to the denominated Multi-Criteria Decision-Making (MCDM) Problems which we will not be dealing with herein. In any case, it is interesting to point out that it is perfectly reasonable to have group decision problems (GDM) with multiple criteria (MCDM). On this point we will make no initial supposition.

e) The environment of the problem is described by the states of nature. The type of information available will exercise an influence in the environment, so that given a specific environment, the decision could be different depending if the information we have on the states of nature is linguistic or probabilistic since that, depending on the case, the theoretical tools that we can use will be different (potentially leading to different solutions). In any case the number of states of nature, just like the number of actions, can be finite, which is not synonymous of easy, infinite or infinite numerable.

f) The information available can come from different sources. Also, it may have different properties (which generates aggregation, comprehension, and information fusion problems), it may not be precisely defined or it may be conditioned by some special characteristic (provisionality, incoherences, etc.). In short, for the available information we have one of the following two possibilities:

   - We do not know or we ignore, partially or totally, some of the elements that take part into the problem. In this case, we say that the information available is incomplete and we have a General Decision Problem with Incomplete Information (GDPII).
   - All the elements that take part into the problem are precisely known, so that there is no ignorance regarding this. In this case we shall talk about a General Decision Problem with Complete Information (GDPCI).

In the case of problems with complete information, it is clear that they can be understood as an optimisation problems, which depending on each particular case, could be solved either exactly (with methods from the Mathematical Programming area) or approximately (for example using Metaheuristics). However, when the information is not complete, the possible solving methods will vary depending on the nature of the information. A proper classification of the different types of information is needed, to properly understand, in turn, which are the most appropriated methods to use.

Besides considering the cases of complete or incomplete information, the framework of behaviour in which the problem is posed should also be considered. The concept of “best decision” varies according to the framework being considered. For example, our decisions are not the same when we have enough time for analysing the alternatives as to when we have to decide fast due to some emergency.

The following section delves deeper into these aspects. On the one hand, a classification of the different types of information that we may take into account is presented, and on the other hand the different (behavioural), frameworks that we can consider for the solution to the problem at hand are presented.

3. Types of information available and frames of behaviour.

Decision making situations with complete information are not usual. In order to obtain effective operative and applicable solutions for real world situations, there will be aspects we are not aware of, i.e., that we do not know for whatever reason. Among which are the lack of knowledge of our own lack of knowledge or the ignorance of what we do not know because we despise it and do not give importance to the data that we cannot understand [Bam].

In reality we do not consciously exercise ignorance, but that we do not know or despise what is unknown. As pointed out in [Rav], more often than we would like: “The real world outside the laboratory, where things are messy and unpredictable, is to be ignored”.

Several forms of ignorance can arise in a decision-making situation, and reviewing all of them is completely out of the scope of this paper. The field of ignorance that we shall consider is based on the Smithson’s Taxonomy of Ignorance [Smi] which we shortly describe in Fig. 1.

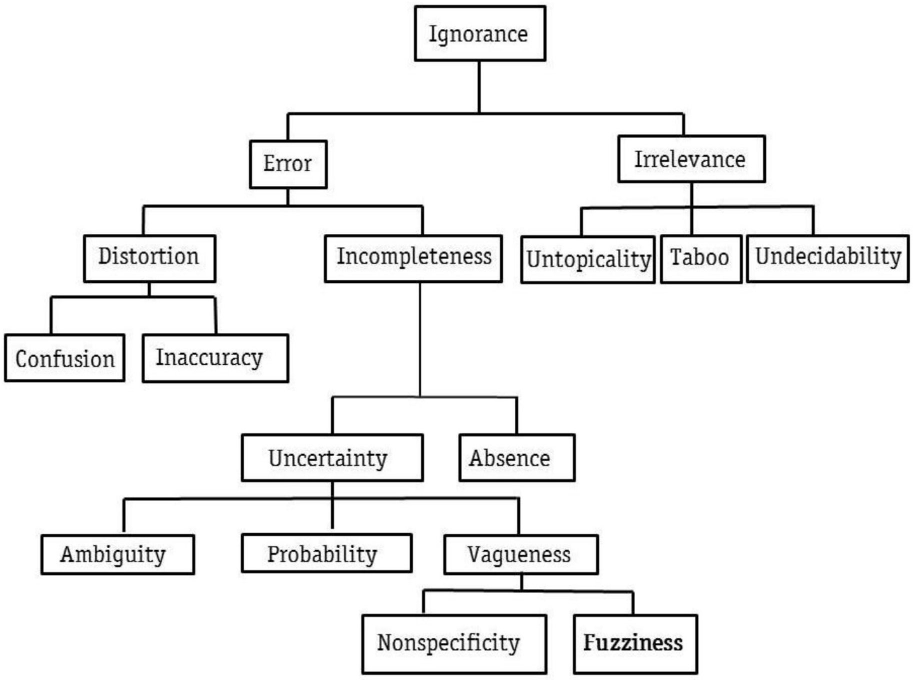

Fig. 1. Taxonomy of ignorance

According to this taxonomy, a General Decision Problem should initially distinguish between two possible types of ignorance. We refer to the first as error-type Ignorance, abbreviated to Error, and the second is known as Ignorance associated with Irrelevant, Irrelevance for short. Whilst the former covers ignorance due to incomplete knowledge, the latter contemplates ignorance derived from deliberately neglecting data due to considering them irrelevant. This first division refers to admitting the possibility of later reviewing the available information, replacing it eventually in the event of errors, or of declaring possible anomalies as being irrelevant, despising them and not including them in the model.

Ignorance by irrelevance is divided into three types: a) information discarded by Untopicality, b) that which is deliberately omitted, i.e. Taboo, and c) that characterised by Undecidability (we do not know, for example, if certain data are true or false and that is not relevant either, as often occurs in judicial contexts). Whatever the nature of the irrelevance-type information, it does not usually arise in decision-making problems and therefore we shall not be concerned with it.

Ignorance caused by errors may be due to:

a) A distortion, originated in confusions or inexactness. A clear example is rounding numerical data to facilitate handling in our problem. In such situation, the erroneous or distorted data can be reviewed and eventually replaced. These distortions can, in turn, be of two types: human or automatic confusions, and inaccuracies. For example, when the unit of measurement is confused (centimetres instead of meters), in the first case, or when approximate quantities are expressed, in the second one.

b) Certain incompleteness of the information. It can be provoked by two other possibilities, depending on whether there is Uncertainty over the data (understood as that the information is gradual and therefore not complete), or there is a real Absence of information (related with lack of knowledge on what is ignored). It would be the case for example of not knowing all the data of a time series, but knowing that they exist, or not knowing that those data exist (absence of knowledge). The main difference between information uncertainty and absence of information is that we can associate a measure to the former. At some point, when a measure of information of uncertainty reaches its maximum, we can say that the situation is the same as the absence of information. Therefore, we will left out the latter case in the rest of this paper.

So going deeper into the Uncertainty case, then we can classify it by three different forms, according to whether the information available is vague, probabilistic or ambiguous. Vagueness refers to the fact that the data are given in ranges of values (it is between 6 and 8, between spring and summer …). The probability arises when randomness take part into the information available (you will win a prize with a probability of 0.01) and ambiguity is present when we have a finite number of possibilities for our data (we can go to France, to China, to the Caribbean …).

Up to this point, the taxonomy is obviously general. When we concentrate on the most particular area of decision making problems under uncertainty, and with the aim of later focusing in the corresponding problem solving methods, we can even further analyze the type of the information available. In this way, we can naturally reach what we called the Environment of the problem.

When we deal with ambiguous or vague information, we may consider:

a) If the information is vague, we may found Fuzziness [Zad] in the data or they are not completely specified (there is Nonspecificity). In the first case, the concepts associated to our data do not have clearly defined frontiers (young, tall, etc.). In this case fuzzy set theory, or more generally Soft Computing, are the right tools to manage such kind of information. In the second, the nonspecificity supposes that handling these data demands greater precision, a better definition, to be able to operate with efficacy (it is not enough to say "in the 20th century", "in Europe", etc.).

b) Ambiguous information always appears in situations in which the definition of a state can have different interpretations. For example, stating that someone is "a brilliant person" does not clarify the sense of the characteristic (we could be referring to the fact that the person is elegant, intelligent, smart …) so a series of possibilities appear to confer ambiguity to the information being handled. That ambiguity can be reduced by refining the discourse universe on the parameters we are talking about.

The essential difference between Vagueness and Ambiguity is graduality, in the sense that some information can be more precise than another one. For the purposes of decision-making, Vagueness and Ambiguity will define identical Environments E, and therefore the same states of nature, since adjusting the granularity or degree of belonging of the different parameters to consider, in one case or another the corresponding information can be made to be of one or another type.

As relevant examples for Environments with vagueness (identical for the ambiguous-type information) we can cite:

- Environment of Certainty: it is characterised because there is sureness regarding the values that the data will take in the future; an essential nuance with regard to certitude. As an example we can know that it will rain tomorrow (certainty), which does not coincide with having certitude over something: 7.45 l/m^2 of water collected.
- Environment of Possibility: arises when a possibility distribution is established on the data. This is somehow similar to a probability distribution, but more associated to the concept of feasibility than to randomness and therefore without any formal axiomatic.
- Environment of Undefinition: arises when the data are not well specified, although they are known to a certain extent. This would be the case, for instance, with data like cool, agreeable and hot.

c) When the information available is affected by randomness, surely the most studied situation in Decision Theory, the following three Environments can be considered:

- Environment of Certitude: in this case, the current state of the nature is perfectly known. In general terms, it is in this environment where Optimisation Problems arise (find the best value in a set, conforming to a certain pre-established criterion).
- Environment of Risk: this appears when the decision-maker only knows the probability distribution of the different states of nature. The essential tool to solve the problems associated to this environment is the Criterion of Expected Utility. It should be noted that when a given state of nature has a probability of 1 of occurrence, then the situation fits an environment of Certitude.
- Environment of Random Uncertainty (uncertainty for short in the sequel): just the list of possible states of nature is available with no additional information regarding their occurrences. In this environment, such well-known decision rules as Laplace, Maximin, etc. among many others, can be applied.

Despite the type of information available, it is clear that the order of the alternatives provided by the decision-maker is a direct function of the Framework of Behaviour, in which the problem is defined. In other words, given a decision problem, the chosen alternative will change depending on the Context.

Although the number of possible Frameworks of Behaviour (Frames for short) for decision making is practically unlimited, we highlight some Frames as being the most influential from the decision maker point of view.

Specifically, we refer to the following:

- Neutral Frame, which arises when the context of the problem is exempt of peculiarities that can influence the decisions to be taken. In general, this Frame, in which the decision-maker chooses the courses of action with rational criteria, without external incidences and for variables that take real positive values, is usually assumed for the theoretical study of solutions to the problem.
- Frame of Competition, which arises when two or more decision-makers with partially or totally opposed interests take part into in the problem; each of whom act in their own benefit. In these circumstances a decision may by chosen that, not being the best, is that which most benefits the decision-maker for the prejudice that it may cause to his/her opponent.
- Ethical Frame, in which the decisions are taken prioritising the common good of the social surroundings in which the problem is developed. Often, the decisions that are chosen are the optimum in terms that cannot be measured quantitatively, although they can qualitatively: the best alternative for the survival of a company may be the suppression of x job posts. However, in a Frame of Labour Ethics that optimum decision can be substituted by another much less damaging for the persons involved.
- Adversarial Decision Making Frame, in this frame, decision maker may resort to sub-optimal solutions because there is an external observer (adversary) trying to diminish decision maker rewards. For example, he may not choose the optimal route because he is being observed and tries to confuse the adversary.
- Emergency Frame, appears when exceptional circumstances are given (e.g. catastrophes, accidents, …) and in which the best possible decision among those available must be taken, which are usually not all of those possible. Hopefully, that best decision in the emergency frame may match the optimal solution (the one corresponding to the neutral frame). Nevertheless, in the majority of cases it will not happen due to different factors such as the lack of resources to explore all the range of alternatives, the possible disappearance of alternatives, the sudden unfeasibility of some others, etc. In this frame, a good solution strategy can be to protocolise the problem, in such a way that when the emergency arises then an action protocol that minimises as far as possible the risks of a bad action can be consulted, being able to increase the possibility that the solution to the problem in an Emergency Frame and the optimal solution to the problem coincide.
- Sustainability Frame, associated to the concept of "sustainable decisions". In parallel to how "sustainable development" [Bru], a decision is sustainable if it satisfies the current decision maker expectations (i.e. be optimal in some sense), but without compromising choices on the problem that may be made in the future. It therefore makes perfect sense that we consider the Sustainability Frames to take decisions which, fitting the needs of the problem facing us, allow to solve the problem again when it reappears, without being conditioned by the preceding decisions.
- Dynamic Frame, in which the concept of best solution changes with time as the conditions are subject to dynamic changes also. A simple example illustrate this. We wish to decide the route that we will follow to reach a specific place. The information available advises choosing route R1 over R2. But as we will not be the only ones who solve that problem, at that same moment, there may be a large number of persons who isolate and independently also take route R1. Thus R1, which appeared the fastest, safest, shortest or whatever, will become a "non-recommendable" route, and R2 may become the optimal route, without having even put into practice the choice of R1. This Dynamic Frame is typical of Problems of Transport, Investment, Management and many other areas [Yan].

Although the description we have given has been done describing each Frame separately, it is clear that a confluence of frames could occur. The following figure describes this situation, both for a problem with complete information as well as with incomplete information, which in each case would add a third dimension to the figure: that of the source of available information.

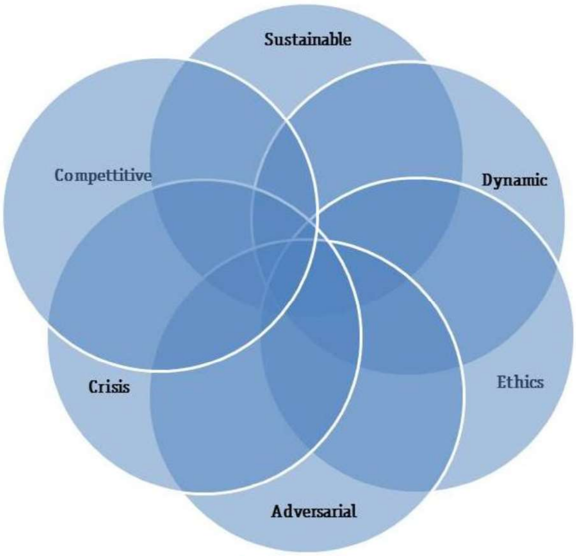

Figure 2: Concurrence of Frames of Behaviour and types of information

4. Particular Decision Problems: Optimization Problems

Given a decision problem, we consider two different perspectives. The first is called Model Formulation and the second is Model Solving. From the point of view of formulation, we need to define the elements (X, E, f, $\leq$, I, K). From the point of view of model solving and independently of the environment and the frame of behaviour, we always need to take the best decision, i.e. find the alternative that maximises our results. It is at this point, where the decision problem becomes an optimisation problem.

The optimisation problem can be defined once the features of the decision problems are fully determined, particularly the type of information available and the frame in which it will be developed. Thus, the optimisation problem can be represented as (X^I_K, E^I_K, f^I_K, $\leq I_K$), where I_k stands for a given type of information (I) and a frame of behaviour (K).

Then the problem we have is to find that alternative $x^* \in X^I_K$ such that

$$f^I_K(x^*) = \mathrm{Max}^I_K \{f^I_K(x): x \in X^I_K; f^I_K: X^I_K \times E^I_K \rightarrow U^I_K\}$$

where f^I_K is a function that gives the reward associated to each alternative, for each state of nature that is considered. Said reward can be numerical, linguistic, etc.

As we described above, let us now distinguish two cases: when the information available is complete or incomplete.

Problems with Complete Information

If the current situation has the following features: the available information is complete, the results are valued in the real line, the set of alternatives $X \subset R^n$ and the environment of certainty, i.e. we know exactly what the state of nature is; the problem that we have is to find that alternative $x^* \in X$ such that

$$f_K(x^*) = \mathrm{Max}_K \{f_K(x): x \in X_K; f_K: X_K \times E_K \rightarrow U_K\}$$

This situation can be stated as a General Optimisation problem (GOP), formulated in the following terms:

$$\begin{array}{ll} \text{Maximise} & f(x) \\ \text{Subject to} & g_i(x) \leq 0, \text{ for } i \in M = \{1, \ldots, k\} \\ & h_j(x) = 0, \text{ for } j \in L = \{1, \ldots, l\} \\ & \quad x \in K \end{array}$$

where f, $g_i(x)$ and $h_j(x)$, $i \in M$ and $j \in L$, are functions that take values in the real line and K is a set of rules, eventually constraints, that define the Frame of Behaviour in which the problem is to be developed.

In the context of the GOP, the frame of behaviour K appears as constraints on the values of the variables, as functions $g_i(x)$ and $h_j(x)$, $i \in M$ and $j \in L$ or as both.

The function f is called objective function. Constraints $g_i(x) \leq 0$, $i \in M$ are called inequality constraints, while the $h_j(x) = 0$, $j \in L$, are called equality constraints. A vector $x \in X$ that satisfies all the constraints is said to be a feasible solution to the problem, with the feasible set of the problem being constituted by all the feasible solutions. As a patent result, resolving the previous GOP consists in finding a feasible solution x* such that

$$f(x^*) \geq f(x), \text{ for any feasible solution } x$$

A solution x* is called the optimal solution to the problem.

Given that any constraint in equality can be decomposed into two constraints in inequality (one in $\leq$ and the other in $\geq$) and that, multiplying by -1, any constraint in $\geq$ becomes represented in $\leq$; henceforth we shall represent a GOP considering only inequality constraints.

If we now assume the ideal situation of a Neutral Frame, and we add the constraint $x \geq 0$ (i.e. K is defined by the positive orthant), the problem to be solved is formulated as follows,

$$\begin{array}{ll} \text{Maximise} & f(x) \\ \text{Subject to} & g_i(x) \leq 0, \text{ for } i \in I = \{1, \ldots, m\} \\ & \quad x \geq 0 \end{array} \qquad (1)$$

The reader can observe here that this is a Mathematical Programming problem. We can resort to a wide set of methods to solve it.

Mathematical Programming problems can be divided into two major groups: those of Convex Programming, characterised because the functions that take part into them are convex, i.e. satisfy that

$$h(\alpha x + \beta y) \leq \alpha h(x) + \beta h(y), \ \forall x,y \in X \text{ and } \forall \alpha, \beta \in R \ (\alpha+\beta = 1, \alpha \geq 0, \beta \geq 0)$$

and those of non-Linear Programming, which describe optimisation problems in which the functions that define them are neither linear nor convex. These latter do not usually have standard methods for solving them, but more "ad hoc" methodologies.

Problems with Incomplete Information

If the current situation has the following features: the available information is incomplete, with fuzziness that can be properly modelled with fuzzy techniques, the set of alternatives and the results are also fuzzy, then the corresponding optimization problem consists in finding the alternative x* such that

$$f^f{}_K(x^*) = \mathrm{Max}^f{}_K \{f^f{}_K(x): x \in X^f{}_K; f^f{}_K: X^f{}_K \times E^f{}_K \rightarrow U^f{}_K\}$$

where the superscript ^{f} stands for "fuzzy". This problem is a Fuzzy Mathematical Programming problem that can adopt different versions and need specific solution methods.

Therefore, from now on we shall consider problems that can be formulated as,

$$\begin{array}{ll} \text{Maximise} & f(x)^f \\ \text{Subject to:} & g_i(x)^f \leq^f 0, \text{ for } i \in I \\ & \quad x \in K \end{array} \tag{2}$$

where the objective function, the set of constraints or the coefficients that take part into the problem, one by one, partially, or all together are of a fuzzy nature.

We consider in the following that the functions f and g_i, $i \in I$, are convex and that the coefficients that take part into the problem are given by LR type fuzzy numbers.

We recall that a fuzzy set A, with membership function μ_A is said to be convex if

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_A(x_1), \mu_A(x_2)\}, x_1, x_2 \in X, \lambda \in [0,1]$$

(alternatively, a fuzzy set is convex if all its α-cuts are convex). Likewise, [Tak], a fuzzy set K^f in R^n, with membership function μ_K, is a (fuzzy) cone if all its α-cuts are cones in R, $\forall\alpha\in[0,1]$.

Therefore, and as before, if K constrains the solutions to the positive orthant, we obtain the model that was first studied in [Bel]:

$$\begin{array}{ll} \text{Maximise} & z = f(x)^f \\ \text{Subject to:} & g_i(x)^f \leq^f 0, \text{ para } i \in I \\ & \quad x \geq 0 \end{array}$$

When all the elements that take part in the problem are convex, we have a Fuzzy Convex Programming problem. Depending on the specific characteristics of the different fuzzy elements that define it, we will have different types of Fuzzy Convex Programming problems.

Among these, the most important due to their application to the real world are those of Fuzzy Linear Programming (FLP), those of Fuzzy Quadratic Programming (FQP), those of Fuzzy Geometrical Programming (FGP) and, because of their theoretical interest and novelty, those of Fuzzy Conic Programming (FCP). These variants are described next.

4.1 Fuzzy Linear Programming

The most general FLP problem can be stated as follows:

$$\begin{array}{ll} \text{Maximise} & z = c^f x \\ \text{Subject to:} & A^f x \leq^f b^f \\ & \quad x \geq 0 \end{array} \tag{3}$$

where c^f and b^f are vectors of n and m fuzzy numbers, respectively, A^f is an n×m matrix of fuzzy numbers and $\leq^f$ is a comparison relation among fuzzy numbers.

The description of the origin of FLP, its different taxonomies, models, methods, extensions and applications, is beyond the scope of this paper. The interested reader is referred to [Luh2], [Ver1], [Ver2] and [Lod].

We recall that (3) can be resolved from the Representation Theorem of Fuzzy Sets, taking into account its α-cuts, $\forall\alpha\in[0,1]$. For illustrative purposes, we will show next how such solution approach works, as it may help to solve other similar problems.

Let's suppose that in (3) all the elements that take part into the problem are well known, with the exception of the set of constraints which we consider as fuzzy. In other words, the decision-maker allows violations up to a certain value that he/she predetermines. In the following model, the fuzziness in the constraints is represented by the symbol $\leq_f$:

$$\begin{array}{ll} \text{Maximise} & z = cx \\ \text{Subject to:} & Ax \leq_f b \\ & x \geq 0 \end{array} \qquad (4)$$

Formally, this means that for each constraint $a_i x \leq_f b_i$, $i \in I$, there is a continuous and not decreasing membership function $\mu_i(\cdot)$ defined by the following expression,

$$\mu_i(a_i x, b_i) = \begin{cases} 1 & \text{if} \quad a_i x \leq b_i \\ 1 - (a_i x - b_i)/d_i & \text{if } d_i \leq a_i x \leq b_i + d_i \\ 0 & \text{if} \quad a_i x > b_i + d_i \end{cases}$$

In such a way that whenever the constraint is completely satisfied, the degree of its fulfilment is 1, but that violations of the constraint are admitted up to a maximum value b_i+d_i, from which the degree of fulfilment is zero.

Considering the α-cuts of the constraints, [Ver1] we can find a fuzzy solution for our problem [Ver1] from the optimal parametric solution $x^*(\alpha)$, $\alpha\in[0,1]$, from the following parametric PL model,

$$\begin{array}{ll} \text{Maximise} & z = cx \\ \text{Subject to} & Ax \leq b + d\,(1-\alpha) \\ & x \geq 0,\ \alpha\in[0,1] \end{array} \qquad (5)$$

This model, is a conventional (non-fuzzy) model that acts as an auxiliary model in the search for a fuzzy solution for (4) and illustrates how, in general, if we have a fuzzy optimisation problem P^f, calculating its α-cuts, $\forall\alpha\in[0,1]$, we build a collection of α-problems that we can solve (for each α) with standard methods. We thus obtain a set of solutions $x^*(\alpha)$, $\alpha\in[0,1]$, from which, by application of the Theorem of Representation, we build the final fuzzy solution x^f of P^f.

A special case of this type of problems are those of Fuzzy Fractional Programming, initially introduced in [Luh1]. In general, a Fractional Programming problem is established in the following terms,

$$\begin{array}{ll} \text{Maximise} & z = f_0(x) \\ \text{Subject to:} & Px \leq g \\ & Ax = b \end{array}$$

Where the objective function f_0 is defined as

$$f_0(x) = (cx + d) / (ex + f)$$

and $ex + f > 0$, and c,d,e and f are real coefficients.

These problems, by simple transformations, can be easily linearised, and changed to reach the following structure:

$$\begin{array}{ll} \text{Maximise} & cy + dz \\ \text{Subject to:} & Py - gz \le 0 \\ & Ay - bz = 0 \\ & ey + fz = 1 \\ & \quad z \ge 0 \end{array}$$

when some of the elements that take part into the problem are of a fuzzy nature we can obtain Fractional Fuzzy Programming Problems which in each case will have different structures: fuzzy constraints, fuzzy objective, etc. and obviously can be solved by means of the above parametric method.

4.2 Fuzzy Quadratic Programming

Within Convex Programming, we can find the field of Quadratic Programming, which can be understood a generalisation of Linear Programming. Quadratic Programming problems are applied in a wide variety of areas [Sil]: ranging from Least Squares approximation and estimation problems through to those of planning insurance and re-insurance, passing through those which are perhaps the most referred to: those of selection and optimisation of economic portfolios.

In general, a quadratic programming problem is posed as follows,

$$\begin{array}{ll} \text{Maximise} & z = cx + \tfrac{1}{2}\, x\, H\, x \\ \text{Subject to:} & Ax \le b \\ & \quad x \ge 0 \end{array} \tag{6}$$

where c, b and A have known meanings and H is a symmetrical matrix, of appropriate dimensions, which is positive semi-definite, so that the quadratic shape xHx is convex and the objective function is also convex.

From here, and in the same ways as with FLP problems, we can consider different models of Fuzzy Quadratic Programming according to where fuzziness may appear. For example, in the coefficients of the objective function,

$$\begin{array}{ll} \text{Maximise} & z = c^f x + \tfrac{1}{2}\, xH^f x \\ \text{Subject to:} & Ax \le b \\ & \quad x \ge 0 \end{array} \tag{7}$$

in the constraints,

$$\begin{array}{ll} \text{Maximise} & z = cx + \tfrac{1}{2}\, xHx \\ \text{Subject to:} & Ax \le_f b \\ & \quad x \ge 0 \end{array} \tag{8}$$

or in the coefficients that define the constraints,

$$\begin{array}{ll} \text{Maximise} & z = cx + \tfrac{1}{2}\, xHx \\ \text{Subject to:} & A^f x \le^f b^f \\ & \quad x \ge 0 \end{array} \tag{9}$$

It is obvious that combinations among these three models also make sense, i.e., Quadratic Programming problems with fuzzy costs and constraints.

A first approach to these problems can be found in [Bec] and a survey of models and solution methods in [Sil]. However, we note that we can apply the parametric method explained in the previous sub-section to any of these three problems (7)-(9) to obtain auxiliary models that, in each case, are no more than conventional Quadratic Programming parametric problems, solvable with standard algorithms.

4.3 Geometric Programming

A Geometric Programming problem in standard form is an optimisation problem that has a very special structure. To specify its definition, [Duf], we remember that we use the term monomial function or simply monomial, to refer to a function of the shape

$$f(x) = c\, x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$$

where c > 0 and the values are real numbers. The constants c and a_i are usually named as coefficient and exponent of the monomial, respectively.

A sum of monomials, i.e. a function with the form

$$f(x) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}}$$

where $c_k > 0$ is called posynomic or posynomial. Any monomial is a posynomial.

Then, a Geometric Programming problem is formulated as,

$$\begin{array}{ll} \text{Maximise} & f(x) \\ \text{Subject to} & g_i(x) \le 1, \text{ para } i \in K = \{1, \ldots, k\} \\ & h_j(x) = 1, \text{ para } j \in L = \{1, \ldots, l\} \end{array} \qquad (10)$$

where the $h_j(x)$ are given by monomial functions, the objective f and the constraints g_i are posynomial functions and the variables x, by definition, are strictly positive, i.e., $x_i > 0$, i = 1, …, n.

From a model like (10), some generalizations are possible, like Fuzzy Geometric Programming models, i.e. problems in which some or all of the elements that define it are of a fuzzy nature:

- Geometric Programming problems with fuzzy coefficients and/or exponents, derived from situations in which fuzzy numbers gives the corresponding parameters. We shall denominate these as Fuzzy Numbers based Geometric Programming problems.

- Geometric Programming problems with fuzzy constraints, i.e., Geometric Programming problems in which some, or all, of the constraints can be slightly violated. We shall denominate these problems as Fuzzy Constrained Geometric Programming problems.

- Fuzzy Geometric Programming problems, i.e. Geometric Programming problems in which coefficients, exponents and constraints of a fuzzy nature are simultaneously considered.

With regard to solution methods for each of these three types of problems, the most immediate, intuitive and effective approach is to employ a parametric one [Ver1], i.e., based on the determination of the α-cuts, $\alpha \in [0,1]$, of the fuzzy elements that take part into the problem.

Geometric Programming problems are in almost all cases associated to Engineering applications. Among these, energy control, concentration of impurities (doping profile), the design of digital circuits, logistics, the composition of floors (floor planning), the wiring of circuits or the calculation of reticular steel structures (truss design), to cite but a few, stand out [Boy]. But, although they are real-world applications, it is more that reasonable to suppose that the data available have a fuzzy nature; yet in reality there are not many known models of Geometric Programming in the specialised literature which consider fuzzy elements, thus making it an important area and with growing interest [Cao], [Zho], [Liu].

4.4 Fuzzy Conic Programming

A set K is a convex cone if it is a convex set and additionally it is a cone, i.e. for any $x_1, x_2 \in K$ and $\theta_1, \theta_2 \geq 0$, it is verified

$$\theta_1 x_1 + \theta_2 x_2 \in K$$

A Conic Programming Problem is posed in the following terms,

$$\begin{array}{ll} \text{Minimise} & cx \\ \text{Subject to} & Ax = b,\ x \in K \end{array} \qquad (11)$$

where $c \in R^n$, $b \in R^m$, A is a matrix $(n \times m)$ of real numbers and K is a convex cone.

The dual problem of (11) is,

$$\begin{array}{ll} \text{Maximise} & by \\ \text{Subject to} & A'y + s = c,\ s \in K^* \end{array} \qquad (12)$$

where A' is the transposed matrix of A and K* is the dual cone [Tak] of K, defined by

$$K^* = \{s \in R^n : sx \geq 0,\ \forall x \in K\}$$

The applications of Conic Programming are very varied and amongst which we can highlight [Lob] antenna array weight design, portfolio optimisation with loss risk constraints, finite impulse response filter design, equilibrium of systems with piecewise-linear springs or grasping forte optimisation. However, and despite the novelty and interest of its applications, there is little work found in the area of fuzzy sets related to Fuzzy Conic Programming (FCP).

An important concept is that of fuzzy convex cone. A fuzzy set K in R^n is a fuzzy convex cone if all its α-cuts K_α, $\forall \alpha \in [0,1]$, are convex cones. From that, taking into account some of the already known results on duality in Fuzzy Mathematical Programming [Ver3], different possibilities arise to study dual problems of the possible Fuzzy Conic Programming problems that may appear, depending on whether the fuzziness is in the coefficients of the objective function or in some element of the constraints, whether it be the cone itself, the constraints, the coefficients in the constraints or a model that involves two or more of these elements.

But in addition to these lines of research, there are other options that depend on the characteristics of the cone K that takes part into the problem. The most frequently used cones are the Positive Octant, the Lorentz Cone or Second Order Cone and the Positive Semi-defined Cone. Thus, for instance, we could consider certain fuzziness in the case of the Lorentz Cone,

$$K = \left\{ x \in \mathbb{R}^n : x_n^2 \geq \sum_{i=1}^{n-1} x_i^2,\ x_n \geq 0 \right\}$$

To work with a Fuzzy Lorentz Cone K_f, i.e., defined as

$$K_f = \left\{ x \in \mathbb{R}^n : x_n^2 \ \geq_f \ \sum_{i=1}^{n-1} x_i^2, \ x_n \geq 0 \right\}$$

where, as we have been representing throughout this work, the symbol $\geq_f$ makes reference to the fact that the constraint that defines K_f, is fuzzy.

In this case would have the following new Fuzzy Conic Programming problem,

Minimise cx
Subject to $Ax = b$, $x \in K_f$

and its possible variants.

It seems clear that the definition of fuzzy dual cone and, from that, the approach and resolution of new Fuzzy Conic Programming models, is a task of undoubted interest which will have to be undertaken immediately, as well as analysing the possibility of integrating the different Frames of Behaviour that we defined above in Section 3, for decision making in terms of sustainability, of crisis, etc., but now defining those frames from the concept of cone, whether it be fuzzy or not.

Concluding Remarks

In decision making contexts, making the best decision is equivalent to find the optimal alternative, thus the connection between a decision problems and an optimization problems is clear. One key aspect of these problems is that one of the kind of available information. Considering different types of information available and frames of behaviour, which maps to a set of constraints restricting the alternative availables, may lead to a myriad of scenarios/situations/models. Taking into account all these aspects, we have explored here the different models that may arise when we consider incomplete information and vagueness, and particularly when the vagueness is associated with fuzziness. As a consequence we have contributed some rationality to the field of optimisation by facilitating the comparisons and allowing new optimisation and decision models to appear, which up to date have not been considered in the literature. The proposal and analysis of solution methods for these new optimization models found, mainly those of Fuzzy Conic Programming, but also those derived from the new defined frameworks of behaviour, will be approached in future works.

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References

[Bam] G. Bammer, M. Smithson and Goolabri Group: The Nature of Uncertainty. In "Uncertainty and Risk: Multidisciplinary Perspectives" (G. Bammer and M. Smithson Eds.). Earthscan (2009), 289-303.

[Bec] C.R. Bector and S. Chandra: Linear and quadratic programming under fuzzy environment. In "Fuzzy Mathematical Programming and Fuzzy Matrix Games" (C.R. Bector and S. Chandra Eds.). Studies in Fuzziness and Soft Computing 169 (2005) 57-94

[Bel] R.E. Belman and L.A. Zadeh: Decision-Making in a Fuzzy Environment. Man. Sci. 1 (1970), B-141-B-164.

[Boy] Boyd S, Kim SJ, Vandenberghe L and Hassibi A: A tutorial on geometric programming. Optimization and Engineering 8 (2007) 67–127.

[Bru] Our Common Future: Brundtland Report. 20 March 1987. United Nations.

[Cao] B.Y. Cao, Fuzzy geometric programming. Series: Applied Optimization Vol.76: Kluwer Academic Publishers, Dordrecht, 2002.

[Duf] Duffin R, Peterson E, Zener C (1967) Geometric programming—theory and application.Wiley, New York

[Far] Farhadinia B and Kamyad AV: Weak and strong duality theorems for Fuzzy Conic optimization problems. Iranian Journal of Fuzzy Systems 10,1 (2013) 143-152

[Liu] Shiang-Tai Liu: Geometric programming with fuzzy parameters in engineering optimization. International Journal of Approximate Reasoning 46 (2007) 484–498

[Lob] Lobo MS, Vandenberghe L, Boyd S and Lebret H: Applications of second-Order cone programming. Linear Algebra and its Applications 284 (1998) 193-228.

[Lod] W. Lodwick and P. Thipwiwatpotjana: Flexible and Generalized Uncertainty Optimization. Theory and Methods. Studies in Computational Intelligence. Springer (2017)

[Luh1] Luhandjula, MK: Fuzzy approaches for multiple objective linear fractional optimization. Fuzzy Sets and Systems 13, 1 (1984) 11-23

[Luh2] Luhandjula, MK: Fuzzy optimization: Milestones and perspectives. Fuzzy Sets and Systems 274 (2015) 4–11.

[Rav] Jerome Ravetz: Preface. In “Uncertainty and Risk: Multidisciplinary Perspectives” (G. Bammer and M. Smithson Eds.). Earthscan (2009), xiii-xvi

[Sil] R. C. Silva, C. Cruz and J.L. Verdegay: Fuzzy costs in quadratic programming problems. Fuzzy Optimization and Decision Making 12, 3 (2013), 231-248.

[Smi] M. Smithson: Ignorance and Uncertainty:Emerging Paradigms. Springer Verlag (1989), New York.

[Tak] Takeda E and Nishida T: Multiple criteria decision problems with fuzzy domination structures. Fuzzy Sets and Systems 3 (1980) 123-136.

[Ver1] J.L. Verdegay: Fuzzy mathematical programming. In: Fuzzy Information and Decision Processes (M.M. Gupta and E. Sánchez (eds) (1982) 231-237.

[Ver2] J.L. Verdegay: Progress on Fuzzy Mathematical Programming: A personal perspective. Fuzzy Sets and Systems 281 (2015) 219-226.

[Ver3] J.L. Verdegay: A Dual Approach to Solve the Fuzzy Linear Programming Problem. Fuzzy Sets and Systems, 14,2 (1984), 131-141.

[Von] J. von Neumann and O. Morgenstern. Theory of Games and Economic Behavior. John Wiley and Sons, New York, 1944.

[Yan] D. Yankelevich: https://www.linkedin.com/pulse/prediciendo-un-futuro-predicho-daniel-yankelevich?trk=hp-feed-article-title-like

[Zad] L.A. Zadeh, Fuzzy sets. Inform. and Control (1965) 338–353

[Zho] Xue-Gang Zhou, Xiao-Peng Yang and Bing-Yuan Cao: Posynomial geometric programming problem subject to max–min fuzzy relation equations. Information Sciences 328 (2016) 15–25