Sharp bounds of Jensen type for the generalized Sugeno integral

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Abstract

In this paper we provide two-sided attainable bounds of Jensen type for the generalized Sugeno integral of *any* measurable function. The results extend the previous results of Román-Flores et al. for increasing functions and Abbaszadeh et al. for convex and concave functions. We also give corrections of some results of Abbaszadeh et al. As a by-product, we obtain sharp inequalities for symmetric integral of Grabisch. To the best of our knowledge, the results in the real-valued functions context are presented for the first time here.

Keywords: Jensen inequality; Sugeno integral; Shilkret integral; q-integral; seminormed fuzzy integral; monotone measure.

1 Introduction

Let (X, \mathcal{A}) be a measurable space, where \mathcal{A} is a σ -algebra of subsets of a nonempty set X. A monotone measure on \mathcal{A} is a nondecreasing set function $\mu \colon \mathcal{A} \to \overline{\mathbb{R}}_+$, i.e. $\mu(A) \leq \mu(B)$ whenever $A \subset B$ with $\mu(\emptyset) = 0$, where $\overline{\mathbb{R}}_+ = [0, \infty]$. We denote the range of μ by $\mu(\mathcal{A})$ and the class of all monotone measures on (X, \mathcal{A}) by $\mathcal{M}_{(X, \mathcal{A})}$. The class of all \mathcal{A} -measurable functions $f \colon X \to Y$ is denoted by $\mathcal{F}_{(X,Y)}$, where $Y \subset \overline{\mathbb{R}}_+$. A binary map $\circ \colon \overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ is said to be nondecreasing if $a \circ b \leq c \circ d$ for all $a \leq c$ and $b \leq d$. The generalized Sugeno integral of $f \in \mathcal{F}_{(X,\overline{\mathbb{R}}_+)}$ on $A \in \mathcal{A}$ is defined as

$$\int_{\mathcal{O},A} f \,\mathrm{d}\mu := \sup_{t \ge 0} \left\{ t \circ \mu(A \cap \{f \ge t\}) \right\},\tag{1}$$

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where $\{f \ge t\} = \{x \in X : f(x) \ge t\}, \mu$ is a monotone measure on \mathcal{A} and \circ is a nondecreasing binary map. Commonly encountered examples of the generalized Sugeno integral include the Sugeno integral [34]

$$\oint_{A} f \,\mathrm{d}\mu = \sup_{t \ge 0} \left\{ t \land \mu(A \cap \{f \ge t\}) \right\},\tag{2}$$

the Shilkret integral [32]

$$\int_{Y,A} f \, \mathrm{d}\mu = \sup_{t \ge 0} \left\{ t \cdot \mu(A \cap \{f \ge t\}) \right\},$$

the q-integral [11, 12] and the seminormed fuzzy integral [5, 33]. Here and subsequently, $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$.

One of the most important inequalities in mathematics, economics and information theory is the Jensen inequality. The classical integral form of Jensen inequality states that

$$\int_{X} H(f(x)) \mathsf{P}(\mathrm{d}x) \ge H\Big(\int_{X} f(x) \mathsf{P}(\mathrm{d}x)\Big),\tag{3}$$

where $(X, \mathcal{A}, \mathsf{P})$ is a probability measure space, H is a real-valued convex function on an interval I of the real line, $f(x) \in I$ for all x and $f \in L^1(\mathsf{P})$. Numerous applications of the Jensen inequality are presented in [25, 30].

Given a function H and nondecreasing binary operations \circ, \star , we say that a *lower Jensen* type bound holds for the generalized Sugeno integral if there exists a function \widehat{H} such that for any $f \in \mathcal{F}_{(X,Y)}$

$$\int_{\circ,A} H(f) \,\mathrm{d}\mu \geqslant \widehat{H}\Big(\int_{\star,A} f \,\mathrm{d}\mu\Big). \tag{4}$$

Replacing " \geq " with " \leq " in (4) gives an *upper Jensen type bound*. The study of Jensen type inequalities for the Sugeno integral was initiated by Román-Flores and Chalco-Cano [31]. They provided bounds for strictly monotone nonnegative real functions and continuous monotone measure. Since then, the fuzzy integral counterparts of the Jensen inequality have been studied by Caballero and Sadarangani [7], Daraby and Rahimi [9], as well as Jaddi et al. [19]. Kaluszka et al. [21] presented necessary and sufficient conditions for the validity of Jensen type inequalities for the generalized Sugeno integrals under monotonicity condition.

As one of the referees pointed out, the result of Theorem 3.1 in [19] is a special case of Theorem 2.3 in [21]. Indeed, if $H: Y \to Y$ is a differentiable convex function with $H'(y) \ge 1$ for each $y \in Y$, then it is nondecreasing and left-continuous on Y, thus the result (necessary and sufficient condition for the Jensen integral inequality) follows from [21, Theorem 2.3]. Corollaries 3.2, 3.3 and 3.4 from [19] are in fact the results from [21] after Theorem 2.3

therein. Moreover, the assumption on continuity of monotone measure μ is also a superfluous constraint in [19]. Also, the result for the discrete case (Theorem 4.1 in [19]) is immediate.

Abbaszadeh et al. 1 obtained new Jensen type inequalities using concavity/convexity of H, but some of these results are not valid (see counterexamples below). Generalizations of Jensen integral inequality for the pseudo-integral are proven by Pap and Strboja [29]. Agahi et al. |2| extended the Jensen type inequality on *g*-expectation with general kernels. Costa [8] provided fuzzy versions of Jensen inequalities type integral for convex and concave fuzzy-interval-valued functions.

In this article, we use a new method of proof to establish some Jensen type inequalities for the generalized Sugeno integral of any measurable function H. We also improve and correct the Jensen type inequalities for the Sugeno integral previously proposed in the literature. Moreover, we give the Jensen type bounds for the symmetric Sugeno integral introduced by Grabisch [13], which have not been considered in the literature so far.

The paper is organized as follows. In Section 2, we derive sharp lower and upper bounds for the generalized Sugeno integral and nonnegative function H without the assumptions of convexity, concavity or monotonicity of H. In Section 3, we deduce some Jensen type bounds from a Liapunov type inequality for nonnegative concave functions. Our final section provides a Jensen type inequality for the *-symmetric Sugeno integral having both upper and lower estimates.

$\mathbf{2}$ Jensen type bounds for nonnegative functions

We say that a monotone measure μ is weakly subadditive on $A \in \mathcal{A}$, if $\mu(A) \leq \mu(A \cap B) +$ $\mu(A \cap B^c)$ for all B, where $B^c = X \setminus B$. A measure μ is weakly superadditive on A, if " \leq " is replaced by " \geq " in the definition of weak subadditivity on A. Clearly, any subadditive measure is weakly subadditive on any measurable set A, but a weakly subadditive measure need not be subadditive. For example, the monotone measure μ on $X = \{1, 2, 3\}$ defined by $\mu(\{1,2,3\}) = 2$, $\mu(\{k\}) = 0.5$ and $\mu(\{k,l\}) = 1$ for all k, l, is not subadditive while it is weakly subadditive on $A = \{1, 2\}$.

Throughout the paper, $\inf H(A) = \inf_{y \in A} H(y)$, $\sup H(A) = \sup_{y \in A} H(y)$, $\inf H = \inf H(\overline{\mathbb{R}}_+)$, and $\sup H = \sup H(\overline{\mathbb{R}}_+)$ for any function $H \colon \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ and $A \subset \overline{\mathbb{R}}_+$. Denote by $H(p_-)$ and $H(p_{+})$ the lower left-hand limit and the lower right-hand limit of H at p, respectively, that is, $H(p_{-}) = \liminf_{\varepsilon \to 0} H((p - \varepsilon, p))$ and $H(p_{+}) = \liminf_{\varepsilon \to 0} H((p, p + \varepsilon))$. Hereafter, $H(0_{-}) = 0$. First, we give lower bounds of Jensen type.

Theorem 2.1. Suppose that $\circ: \overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ is a nondecreasing map such that $a \circ 0 = 0$ for all a and $x \mapsto x \circ y$ is a left-continuous function for any fixed y. Suppose also that $f, H(f) \in \mathcal{F}_{(X,\overline{\mathbb{R}}_+)}$ and $p = \int_A f \, \mathrm{d}\mu < \infty$.

(i) The following inequality holds

$$\int_{\circ,A} H(f) \,\mathrm{d}\mu \ge \left[\left(H(p_{-}) \wedge \inf H([p,\infty]) \right) \circ p \right] \lor \left[\inf H \circ \mu(A) \right].$$
(5)

There is equality in (5) for $f = \mu(A)\mathbb{1}_A$ if H is left-continuous at p and $H(p) = \inf H([p,\infty])$.

(ii) If μ is weakly subadditive on A, then

$$\int_{\circ,A} H(f) \,\mathrm{d}\mu \ge \left[\left(H(p_+) \wedge \inf H([0,p]) \right) \circ \left(\mu(A) - p \right) \right] \lor \left[\inf H \circ \mu(A) \right]. \tag{6}$$

The equality holds in (6) if $f = y_0 \mathbb{1}_A$, H is right-continuous at p, $H(p) = \inf H([0, p])$ and $H(y_0) = \inf H$ for some y_0 .

Proof. (i) Assume that p > 0, as the bound (5) is trivial for p = 0. Let $h(\varepsilon) = \inf H([p-\varepsilon,\infty])$ for $\varepsilon \in (0,p)$. Define $H_0(s) = \inf H$ for $s and <math>H_0(s) = h(\varepsilon)$ for $s \ge p - \varepsilon$. Clearly, $H(s) \ge H_0(s)$ for all $s \in \mathbb{R}_+$. Thus, we have from the monotonicity of the generalized Sugeno integral that

$$\int_{\circ,A} H(f) \, \mathrm{d}\mu \ge \int_{\circ,A} H_0(f) \, \mathrm{d}\mu = \sup_{0 \le t \le \inf H} \{t \circ \mu(A)\} \lor \sup_{t > \inf H} \{t \circ \mu(A \cap \{H_0(f) \ge t\})\}$$
$$= \left[\inf H \circ \mu(A)\right] \lor \left[h(\varepsilon) \circ \mu(A \cap \{H_0(f) \ge h(\varepsilon)\})\right]$$
$$= \left[\inf H \circ \mu(A)\right] \lor \left[h(\varepsilon) \circ \mu(A \cap \{f \ge p - \varepsilon\})\right].$$

It is well known that $\mu(A \cap \{f \ge y\}) \ge p$ for all y < p, where $p = \int_A f \, d\mu$ (see [37, Lemma 9.7]). Therefore,

$$\int_{\mathcal{O},A} H(f) \, \mathrm{d}\mu \ge \big[\inf H \circ \mu(A)\big] \lor \big[h(\varepsilon) \circ p\big].$$

By left-continuity of $x \mapsto x \circ p$ and monotonicity of $h(\varepsilon)$, we obtain

$$\int_{\circ,A} H(f) d\mu \ge \left[\inf H \circ \mu(A)\right] \vee \lim_{\varepsilon \to 0} \left[h(\varepsilon) \circ p\right]$$

$$= \left[\inf H \circ \mu(A)\right] \vee \left[\lim_{\varepsilon \to 0} h(\varepsilon) \circ p\right]$$

$$= \left[\inf H \circ \mu(A)\right] \vee \left[\left(H(p_{-}) \wedge \inf H([p,\infty])\right) \circ p\right].$$
(7)

Equality holds in (7) for $f = \mu(A)\mathbb{1}_A$ provided that H is left-continuous at p and $H(p) = \inf H([p,\infty])$.

(ii) Let $h(\varepsilon) = \inf H([0, p + \varepsilon])$ for all $\varepsilon > 0$. Put $H_0(s) = \inf H$ if $s > p + \varepsilon$ and $H_0(s) = h(\varepsilon)$ if $s \leq p + \varepsilon$. Weak subadditivity of μ implies that

$$\int_{\circ,A} H(f) d\mu \ge \int_{\circ,A} H_0(f) d\mu = \left[\inf H \circ \mu(A)\right] \lor \sup_{t>\inf H} \left\{t \circ \mu(A \cap \{H_0(f) \ge t\})\right\}$$
$$= \left[\inf H \circ \mu(A)\right] \lor \left[h(\varepsilon) \circ \mu(A \cap \{f \le p + \varepsilon\})\right]$$
$$\ge \left[\inf H \circ \mu(A)\right] \lor \left[h(\varepsilon) \circ (\mu(A) - \mu(A \cap \{f > p + \varepsilon\}))\right].$$

It follows from [37, Lemma 9.7] that $\mu(A \cap \{f > y\}) \leq p < \infty$ for all y > p. By the monotonicity of $h(\varepsilon)$ and left-continuity of map $y \mapsto y \circ (\mu(A) - p)$, we get

$$\int_{\circ,A} H(f) d\mu \ge \left[\inf H \circ \mu(A)\right] \vee \lim_{\varepsilon \to 0} \left[h(\varepsilon) \circ (\mu(A) - p)\right]$$
$$= \left[\inf H \circ \mu(A)\right] \vee \left[(H(p_{+}) \wedge \inf H([0, p])) \circ (\mu(A) - p)\right]. \tag{8}$$

There is equality in (8) for $f = y_0 \mathbb{1}_A$, if H is right-continuous at p, $H(y_0) = \inf H$ and $H(p) = \inf H([0,p])$. Here and subsequently, $\infty \cdot 0 = 0$.

Remark 2.1. The bound (5) (resp. (6)) is sharp for each p, if function H is nondecreasing and left-continuous (resp. nonincreasing and right-continuous). Moreover, if μ is a subadditive monotone measure and H is a continuous function, then $H(p_{-}) = H(p_{+}) = H(p)$ and we have from (5) and (6) that

$$\int_{\circ,A} H(f) \,\mathrm{d}\mu \ge \left[\inf H([p,\infty]) \circ p\right] \lor \left[\inf H([0,p]) \circ (\mu(A) - p)\right] \lor \left[\inf H \circ \mu(A)\right]. \tag{9}$$

Assume additionally that H is quasiconvex, that is, H is nonincreasing on [0, a] and nondecreasing on $[a, \infty]$ for some $a \in (0, \infty)$ [6, p. 99]. Then the bound (9) is attainable for every p as $H(p) = \inf H([p, \infty])$ or $H(p) = \inf H([0, p])$.

Now we provide some consequences of Theorem 2.1 for the Sugeno integral.

Corollary 2.1. Assume that $H: \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ is nondecreasing and left-continuous at p, where $p = \int_A f \, d\mu < \infty$ and $f \in \mathcal{F}_{(X,\overline{\mathbb{R}}_+)}$. Then the following sharp bound holds

$$\oint_{A} H(f) \,\mathrm{d}\mu \ge H(p) \wedge p. \tag{10}$$

Proof. Apply Theorem 2.1 (i) with $\circ = \wedge$. Inequality (10) is attainable if $f = \mu(A) \mathbb{1}_A$.

Corollary 2.1 generalizes Corollary 3.3 of [3], Lemma 1 of [7] and [21, Theorem 2.1] for $\circ = \star = \wedge$ and $Y = \overline{\mathbb{R}}_+$.

The following example shows that the equality in (10) may be achieved by a nonconstant function f.

Example 2.1. Let $X = \mathbb{R}_+$, $A = \{1, 2, 3, 4, 5\}$ and μ be the counting measure on \mathbb{R}_+ , which means that $\mu(B) = \infty$, if B is an infinite subset of \mathbb{R}_+ and $\mu(B) = \operatorname{card}(B)$, if B is a finite subset of \mathbb{R}_+ . Take $H(x) = x^2/3$ and f(x) = x. Then

$$\int_{A} f \, \mathrm{d}\mu = \sup_{t \ge 0} \left\{ t \land \mu(A \cap \{f \ge t\}) \right\} = \max_{i \in A} \{ i \land (6-i) \} = 3$$
$$\int_{A} H(f) \, \mathrm{d}\mu = \max_{i \in A} \left\{ (i^2/3) \land (6-i) \right\} = 3,$$

so the bound (10) is reached if f(x) = x.

The next Corollary is a corrected version of Theorem 5.1 in [1].

Corollary 2.2. Suppose that $H: \mathbb{R}_+ \to \mathbb{R}_+$ is a convex function which attains its infimum at point a. The sharp inequality

$$\int_{A} H(f) \,\mathrm{d}\mu \ge H(p) \wedge p \tag{11}$$

holds for any $f \in \mathcal{F}_{(X,\mathbb{R}_+)}$ such that $p = \int_A f \, \mathrm{d}\mu \in [a,\infty)$.

It is easy to check that Theorem 5.1 in [1] is not true without the additional assumption that $\varphi'(p) = 1$, but this assumption implies that $\varphi(p) \leq \varphi'(p)p = p$, where we follow the notation in [1].

Counterexample 2.1. Let us consider the space $([0, 5], \mathcal{A}, \mu)$ with the Lebesgue measure μ . Take $\varphi(x) = (x - 0.5)^2$ and f(x) = x. Clearly, φ is a differentiable convex function and $\varphi(x) \leq x\varphi'(x)$ for $x \in [0, 5]$. All assumptions of Theorem 5.1 from [1] are satisfied. It is easy to check that $f_{[0,5]} f d\mu = \sup_{t \geq 0} \{t \land (5-t)\} = 2.5$ and

$$\oint_{[0,5]} \varphi(f) \, \mathrm{d}\mu = \sup_{t \in [0, 0.25)} \left\{ t \wedge (5 - 2\sqrt{t}) \right\} \vee \sup_{t \in [0.25, 4.5^2]} \left\{ t \wedge (4.5 - \sqrt{t}) \right\}$$
$$= \frac{10 - \sqrt{19}}{2} \in (2.82, 2.821).$$

Thus, $\varphi(f_{[0,5]} f d\mu) = 4 > f_{[0,5]} \varphi(f) d\mu$. Note that $\varphi'(f_{[0,5]} f d\mu) = 4 \neq 1$, but (11) holds, i.e.

$$\int_{A} \varphi(f) \,\mathrm{d}\mu \geqslant \varphi(p) \wedge p = 2.5$$

Next we provide a lower bound of Jensen type for the Shilkret integral.

Corollary 2.3. If $H: \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing left-continuous at $p = \int_A f \, d\mu$, where $f \in \mathcal{F}_{(X,\mathbb{R}_+)}$ and $\mu(A) < \infty$, then the following attainable bound is valid

$$\int_{\cdot,A} H(f) \,\mathrm{d}\mu \ge H(p)p. \tag{12}$$

Proof. Setting $\circ = \cdot$ in Theorem 2.1 (i), we get

$$\int_{\cdot,A} H(f) \,\mathrm{d}\mu \ge \left[(H(p) \wedge H(p)) \cdot p \right] \vee \left[H(0) \cdot \mu(A) \right] \ge H(p)p.$$

Example 2.2. Let X = [0, 1] and $\mu = \lambda^q$, where λ is the Lebesgue measure and q > 0. Take $H(x) = x^{1/q}$ and $f(x) = x^q$. Then

$$\int_{X} f \, \mathrm{d}\mu = \sup_{0 \le t \le 1} \{ t \land (1 - t^{1/q})^q \} = 0.5^q,$$
$$\int_{Y,X} H(f) \, \mathrm{d}\mu = \sup_{0 \le t \le 1} \{ t \cdot (1 - t)^q \} = \frac{1}{q} \left(\frac{q}{1 + q} \right)^{q+1}$$

We get from (12) that $\int_{X} H(f) d\mu \ge 0.5^{q+1}$.

Example 2.3. Using Corollary 2.3 for H(x) = ax, a > 0, we obtain the following inequality for the Shilkret integral

$$\int_{\cdot,A} f \,\mathrm{d}\mu \geqslant \left(\oint_A f \,\mathrm{d}\mu \right)^2,$$

where $f \in \mathcal{F}_{(X,\mathbb{R}_+)}$. This bound is obvious (see the geometric interpretation of the Sugeno integral and the Shilkret integral), but it shows that the equality in (5) may hold not only for piecewise constant functions f when $\circ = \cdot$.

Recall that a nondecreasing map $\otimes : [0,1]^2 \to [0,1]$ is said to be a *fuzzy conjunction* if $1 \otimes 1 = 1$ and $0 \otimes 1 = 1 \otimes 0 = 0 \otimes 0 = 0$ (see [12, Definition 2]). The special case of the fuzzy conjunction is a *semicopula*, which has extra limit conditions $a \otimes 1 = 1 \otimes a = a$ (cf. [16]). Dubois et al. [11] introduced and studied the *q*-integral defined as

$$\int_{\mu}^{\otimes} f = \sup_{t \in [0,1]} \left\{ \mu(\{f \ge t\}) \otimes t \right\},\tag{13}$$

where \otimes denotes a fuzzy conjunction, $f \in \mathcal{F}_{(X,[0,1])}$ and X is a finite set (see also [12]). This definition is motivated by alternative ways of using weights of qualitative criteria in min- and max-based aggregations, that make intuitive sense as tolerance thresholds. In the literature, we can find Jensen type bounds for q-integral if H is a nondecreasing function (see [21, Theorems 2.1-2.3 and Theorem 3.3]). Now we give their counterparts for a quasiconvex function H. **Corollary 2.4.** Assume that a fuzzy conjunction \otimes is left-continuous in the second coordinate, $\mu(\mathcal{A}) \subset [0,1]$ and $H: [0,1] \to [0,1]$ is a quasiconvex function which attains its infimum at point a_0 . Then, for all $f \in \mathcal{F}_{(X,[0,1])}$ such that $p = \int_{\mu}^{\otimes} f \in [a_0,1]$, we have

$$\int_{\mu}^{\infty} H(f) \ge p \otimes H(p_{-}).$$

Proof. Put $a \circ b = (b \wedge 1) \otimes (a \wedge 1)$ in (1). Note that $a \circ 0 = 0$ as $0 \leq 0 \otimes a \leq 0 \otimes 1 = 0$ for all a. Moreover,

$$\int_{\mu}^{\infty} H(f) = \sup_{t \in [0,1]} \left\{ t \circ \mu(\{H(f) \ge t\}) \right\} = \sup_{t \ge 0} \left\{ t \circ \mu(\{H(f) \ge t\}) \right\} = \int_{\circ, X} H(f) \, \mathrm{d}\mu.$$

The assertion follows from Theorem 2.1 (i).

Applying Theorem 2.1 one can also obtain lower bounds by means of $\int_{\circ,A} f \, d\mu$ instead of the Sugeno integral.

Corollary 2.5. Assume that a semicopula $S: [0,1]^2 \to [0,1]$ is left-continuous in the first coordinate, $\mu(\mathcal{A}) \subset [0,1]$ and $H: [0,1] \to [0,1]$ is a left-continuous and nondecreasing function on $[a_0,1]$ for some $a_0 \in [0,1]$. Then the following sharp inequality for the seminormed fuzzy integral holds for all $f \in \mathcal{F}_{(X,[0,1])}$

$$\int_{\mathbf{S},A} H(f) \,\mathrm{d}\mu \geqslant \mathbf{S}(H(p_S), p_S),$$

where $p_S := \int_{S,A} f \, d\mu \in [a_0, 1].$

Proof. Take $a \circ b = S(a \wedge 1, b \wedge 1)$ in (1). It is clear that

$$\int_{\circ,A} f \,\mathrm{d}\mu = \int_{\mathrm{S},A} f \,\mathrm{d}\mu = \sup_{t \in [0,1]} \mathrm{S}\big(t, \mu(A \cap \{f \ge t\})\big).$$

As $S(a, b) \leq a \wedge b$ for all a, b, we have $a_0 \leq p_S \leq f_A f d\mu \leq 1$. Moreover, from Theorem 2.1 (i) and monotonicity of H on $[a_0, 1]$ we get

$$\int_{\mathbf{S},A} H(f) \,\mathrm{d}\mu \ge \mathrm{S}\Big(H\Big(\oint_A f \,\mathrm{d}\mu\Big), \oint_A f \,\mathrm{d}\mu\Big) \ge \mathrm{S}(H(p_S), p_S)$$

This bound is reached for $f = \mu(A) \mathbb{1}_A$ if $S(\mu(A), \mu(A)) = \mu(A)$.

Next we find some upper bounds of Jensen type. Let $H: \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ be a Borel measurable function. Denote by $H(p^-)$ and $H(p^+)$ the upper left-hand limit and the upper right-hand limit of H at p, respectively, that is, $H(p^-) = \lim_{\varepsilon \to 0} \sup H((p - \varepsilon, p))$ with $H(0^-) = 0$, and $H(p^+) = \lim_{\varepsilon \to 0} \sup H((p, p + \varepsilon))$.

Theorem 2.2. Let $\circ: \overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ be a nondecreasing map such that $x \mapsto x \circ y$ is right-continuous for any fixed y and $a \circ 0 = 0$ for all a. Assume that $f, H(f) \in \mathcal{F}_{(X,\overline{\mathbb{R}}_+)}$ and $p = \int_A f \, d\mu < \infty$.

(i) The following bound is valid

$$\int_{\circ,A} H(f) \,\mathrm{d}\mu \leqslant \left[\left(H(p^+) \lor \sup H([0,p]) \right) \circ \mu(A) \right] \lor \left[\sup H \circ p \right].$$
(14)

The equality holds in (14) for $f = y_0 \mathbb{1}_A$ if H is right-continuous at p, $H(p) = \sup H([0, p])$ and $H(y_0) = \sup H$ for some y_0 .

(ii) If μ is weakly superadditive on A, then

$$\int_{\circ,A} H(f) \,\mathrm{d}\mu \leqslant \left[\left(H(p^{-}) \lor \sup H([p,\infty]) \right) \circ \mu(A) \right] \lor \left[\sup H \circ \left(\mu(A) - p \right) \right].$$
(15)

The equality in (15) is attained for $f = \mu(A) \mathbb{1}_A$ if H is left-continuous at p and $H(p) = \sup H([p, \infty])$.

Proof. (i) Let $h(\varepsilon) = \sup H([0, p + \varepsilon])$ for all $\varepsilon > 0$. Put $H_0(s) = \sup H$ for $s > p + \varepsilon$ and $H_0(s) = h(\varepsilon)$ for $s \leq p + \varepsilon$. Evidently, $H(s) \leq H_0(s)$ for all s. Therefore

$$\int_{\circ,A} H(f) d\mu \leq \int_{\circ,A} H_0(f) d\mu = \sup_{0 \leq t \leq h(\varepsilon)} \{t \circ \mu(A)\} \lor \sup_{t > h(\varepsilon)} \{t \circ \mu(A \cap \{H_0(f) \geq t\})\}$$
$$= [h(\varepsilon) \circ \mu(A)] \lor [\sup H \circ \mu(A \cap \{f > p + \varepsilon\})].$$

As $\mu(A \cap \{f > y\}) \leq p$ for y > p, we get, from the right-continuity of $x \mapsto x \circ p$, that

$$\int_{\circ,A} H(f) \,\mathrm{d}\mu \leqslant \left[\left(H(p^+) \lor \sup H([0,p]) \right) \circ \mu(A) \right] \lor \left[\sup H \circ p \right].$$
(16)

The equality holds in (16), if H is right-continuous at p, $H(p) \ge \sup H([0, p])$, and $f = y_0 \mathbb{1}_A$, where y_0 is such that $H(y_0) = \sup H$.

(ii) As the bound (15) is obvious for p = 0, we assume that p > 0. Put $h(\varepsilon) = \sup H([p - \varepsilon, \infty])$ for $\varepsilon \in (0, p)$. Set $H_0(s) = \sup H$ for $s and <math>H_0(s) = h(\varepsilon)$ for $s \ge p - \varepsilon$. Since $H(s) \le H_0(s)$ for all s, we get

$$\int_{\circ,A} H(f) \, \mathrm{d}\mu \leqslant \int_{\circ,A} H_0(f) \, \mathrm{d}\mu = \left[h(\varepsilon) \circ \mu(A)\right] \vee \left[\sup H \circ \mu(A \cap \{f$$

Clearly, $\mu(A) \ge \mu(A \cap \{f and <math>\mu(A \cap \{f \ge p - \varepsilon\}) \ge p$, so $\int_{\circ, A} H(f) \, \mathrm{d}\mu \le \left[h(\varepsilon) \circ \mu(A)\right] \lor \left[\sup H \circ (\mu(A) - p)\right].$

Taking the limit as $\varepsilon \to 0$, we obtain (15) with equality if H is left-continuous at $p, H(p) \ge \sup H([p, \infty])$ and $f = \mu(A) \mathbb{1}_A$.

Remark 2.2. The bound (14) (resp. (15)) is sharp for each p, if H is a nondecreasing rightcontinuous function (resp. nonincreasing left-continuous function). Given a superadditive monotone measure μ and a continuous quasiconcave function H, Theorem 2.2 implies that

$$\int_{\circ,A} H(f) d\mu \leq \left[\sup H([0,p]) \circ \mu(A) \right] \vee \left[\sup H([p,\infty]) \circ \mu(A) \right]$$
$$\vee \left[\sup H \circ p \right] \vee \left[\sup H \circ (\mu(A) - p) \right]$$
(17)

and the bound (17) is sharp for every p.

The following result is an immediate consequence of Theorem 2.2.

Corollary 2.6. Assume that a continuous function H is increasing on [0, c] and decreasing on $[c, \infty]$, where $c \in [a, b] \subset \mathbb{R}_+$. If $f \in \mathcal{F}_{(X, [a, b])}$ and $p = \oint_A f \, d\mu \leq c$, then

$$\oint_{A} H(f) \, \mathrm{d}\mu \leqslant (H(p) \lor p) \land H(c) \land \mu(A).$$

Moreover, if $c and <math>\mu$ is a weakly superadditive monotone measure on A, then

$$\int_{A} H(f) \, \mathrm{d}\mu \leqslant \left(H(p) \lor (\mu(A) - p) \right) \land H(c) \land \mu(A)$$

Proof. Recall that $\oint_A f \, d\mu \leq \mu(A)$. Apply Theorem 2.2 with $\circ = \land$ and observe that $(H(p) \land \mu(A)) \lor (H(c) \land p) = (H(p) \lor p) \land H(c) \land \mu(A)$ for $p \leq c$ and $(H(p) \land \mu(A)) \lor (H(c) \land (\mu(A) - p)) = (H(p) \lor (\mu(A) - p)) \land H(c) \land \mu(A)$ for p > c.

As some nondecreasing binary maps \circ are not left-continuous (see e.g. [23, Example 1.24]), we provide modifications of Theorems 2.1 and 2.2, which hold true without any continuity assumption on \circ . Let us recall that a monotone measure μ is *continuous from below* (resp. *from above*) if $\lim_{n\to\infty} \mu(A_n) = \mu(\lim_{n\to\infty} A_n)$ for all $A_n \in \mathcal{A}$ such that $A_n \subset A_{n+1}$ (resp. $A_{n+1} \subset A_n$) for $n \in \mathbb{N}$. We say that μ is *continuous*, if it is both continuous from below and from above. The following result generalizes Theorem 1 in [31].

Theorem 2.3. Let $H: \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ and $\circ: \overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ be a nondecreasing map such that $a \circ 0 = 0$ for all $a, f \in \mathcal{F}_{(X,\overline{\mathbb{R}}_+)}$ and $p = \int_A f \, d\mu < \infty$. Let μ be a continuous monotone measure on X.

(i) The following inequalities hold true

$$\int_{\circ,A} H(f) \,\mathrm{d}\mu \ge \left[\inf H([p,\infty]) \circ p\right] \lor \left[\inf H \circ \mu(A)\right],\tag{18}$$

$$\int_{\circ,A} H(f) \,\mathrm{d}\mu \leqslant \big[\sup H([0,p]) \circ \mu(A) \big] \lor \big[\sup H \circ p \big].$$
(19)

There is equality in (18) for $f = \mu(A)\mathbb{1}_A$ if $H(p) = \inf H([p,\infty])$. Equality holds in (19) if $f = y_0\mathbb{1}_A$, $H(p) = \sup H([0,p])$ and $H(y_0) = \sup H$ for some y_0 .

(ii) If μ is weakly subadditive on A, then

$$\int_{\circ,A} H(f) \,\mathrm{d}\mu \ge \left[\inf H([0,p]) \circ (\mu(A) - p)\right] \lor \left[\inf H \circ \mu(A)\right].$$
(20)

The bound (20) is reached by $f = y_0 \mathbb{1}_A$ if $H(p) = \inf H([0,p])$ and $H(y_0) = \inf H$ for some y_0 .

(iii) If μ is weakly superadditive on A, then

$$\int_{\circ,A} H(f) \,\mathrm{d}\mu \leqslant \big[\sup H([p,\infty]) \circ \mu(A) \big] \lor \big[\sup H \circ (\mu(A) - p) \big].$$
(21)

The equality is attained in (21) for $f = \mu(A)\mathbb{1}_A$ if $H(p) = \sup H([p,\infty])$.

Proof. The proof of Theorem 2.3 is similar to those of Theorem 2.1 and 2.2; just put $\varepsilon = 0$ and use the fact that if μ is a continuous monotone measure, then $\mu(A \cap \{f \ge p\}) \ge p$ and $\mu(A \cap \{f > p\}) \le p$. The last statement follows easily from the bounds $\mu(A \cap \{f \ge y\}) \ge p$ for y < p and $\mu(A \cap \{f \ge y\}) \le p$ for y > p (see also [37, Lemma 9.5]).

Note that the bounds in Theorem 2.3 may be better than their counterparts in Theorems 2.1 and 2.2.

Example 2.4. Let μ be the Lebesgue measure on X = [0, 2]. If f(x) = x and $H(x) = 0.5\mathbb{1}_{\{1\}}(x) + x^2\mathbb{1}_{(1,2]}(x)$, then $p = \int_X f \, d\mu = 1$ and $\int_X H(f) \, d\mu = 1$. Inequality (5) gives us the trivial bound $\int_X H(f) \, d\mu \ge 0$, as $H(1_-) = 0$, while from (18) we get $\int_X H(f) \, d\mu \ge 0.5$.

Remark 2.3. Corollary 3.6 of [1] gives the upper bound for the Sugeno integral of a concave function, but the following counterexample shows that the result is false if m > 0, where $m \in \partial \varphi(p)$. We follow the notation of [1]. Let X = A = [0, 1] and μ be the Lebesgue measure. Take $\varphi(x) = \sqrt{x}$ and f(x) = 0.5x. Then f(X) = [0, 0.5], p = 1/3 and $m = \varphi'(p) = 0.5\sqrt{3} > 0$. By Corollary 3.6 in [1] we get

$$\int_{A} \varphi(f) \,\mathrm{d}\mu \leqslant \frac{m}{m+1} (0.5-p) + \frac{1}{m+1} \varphi(p). \tag{22}$$

An easy computation shows that $\int_A \varphi(f) d\mu = 0.5$ and the right-hand side of (22) is approximately equal to 0.39, so inequality (22) is invalid.

3 Jensen inequalities for nonnegative concave functions

In this section we give some Liapunov type inequalities, that is, we evaluate the integral $\int_{\circ,A} H(f) d\mu$ by means of integrals $\int_{\circ,A} G(f) d\mu$ and $\int_{\bullet,A} f d\mu$. As a consequence, we obtain some new Jensen type inequalities for nonnegative concave functions.

Theorem 3.1. Let $\circ, \bullet: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be nondecreasing maps such that $a \circ 0 = a \bullet 0 = 0$ for all a. Let $H: \mathbb{R}_+ \to \mathbb{R}_+$, $\mu \in \mathcal{M}_{(X,\mathcal{A})}$, $A \in \mathcal{A}$, $f \in \mathcal{F}_{(X,\mathbb{R}_+)}$ and $p = \int_{\bullet,A} f \, d\mu < \infty$. Assume that $(a + b) \circ c \leq (a \circ c) + (b \circ c)$ for all a, b, c. If there exists $m_p \in \mathbb{R}$ such that $H(y) \leq H(p) + m_p(y - p)$ for $y \geq 0$, then the following attainable bound holds true

$$\int_{\mathcal{O},A} H(f) \,\mathrm{d}\mu \leqslant \inf_{c \in \mathbb{R}} \left\{ \left[\left(H(p) + m_p(c-p) \right)^+ \circ \mu(A) \right] + \int_{\circ,A} (m_p(f-c))^+ \,\mathrm{d}\mu \right\},\tag{23}$$

where $a^+ = a \lor 0$.

Proof. By the assumption on H, we obtain

$$\int_{\circ,A} H(f) d\mu \leq \int_{\circ,A} \left(H(p) + m_p(c-p) + m_p(f-c) \right) d\mu$$

$$\leq \int_{\circ,A} \left[\left(H(p) + m_p(c-p) \right)^+ + \left(m_p(f-c) \right)^+ \right] d\mu,$$
(24)

where $c \in \mathbb{R}$. It is easy to check that the generalized Sugeno integral has the scale translation property, i.e.,

$$\int_{\circ,A} (a+f) \,\mathrm{d}\mu \leqslant (a \circ \mu(A)) + \int_{\circ,A} f \,\mathrm{d}\mu \tag{25}$$

for all $a \ge 0$ under the condition $(x + y) \circ z \le (x \circ z) + (y \circ z)$ for all $x, y, z \ge 0$ (see [4]). Inequality (23) follows from (24) and (25). Bound (23) is reached by the function $f = (a \bullet \mu(A))\mathbb{1}_A$, where $a \ge 0$, if $\mu(A) \bullet \mu(A) = \mu(A)$ and the map \bullet is associative. This follows from (23) applied to $c = a \bullet \mu(A)$.

Denote by $\partial H(x)$ the subdifferential of a concave function H at point x (see [1]).

Corollary 3.1. Let $\mu \in \mathcal{M}_{(X,\mathcal{A})}, H: \mathbb{R}_+ \to \mathbb{R}_+$ be a concave function and $m_p \in \partial H(p)$, where $p = \int_A f \, d\mu < \infty$ and $f \in \mathcal{F}_{(X,\mathbb{R}_+)}$. Then

$$\int_{A} H(f) \,\mathrm{d}\mu \leqslant \left[H(p) \wedge \mu(A) \right] + \int_{A} \left(m_p(f-p) \right)^+ \mathrm{d}\mu. \tag{26}$$

Proof. Put c = p and $\circ = \bullet = \land$ in (23).

Corollary 3.1 shows that Theorem 3.1 is a generalization of Theorem 4.3 in [1]. Indeed, bound (26) was given in [1, Theorem 4.3] under the assumption that H is an increasing concave function, $A = \{x_1, x_2, \ldots, x_n\}$ and $f(x_1) \ge f(x_2) \ge \ldots \ge f(x_n)$. Note that for $m_p \ge 0$ we get

$$\begin{aligned} \oint_A \left(m_p(f-p) \right)^+ \mathrm{d}\mu &= \sup_{t \ge 0} \left\{ t \land \mu(A \cap \{ m_p(f-p)^+ \ge t \}) \right\} \\ &= \max_i \left\{ \left(m_p(f(x_i) - p)^+ \right) \land \mu_i \right\} = \max_i \left\{ \left(m_p(f(x_i) - p) \right) \land \mu_i \right\}, \end{aligned}$$

where $\mu_i = \mu(\{x_1, ..., x_i\}).$

By (23) we also obtain the following inequality for the Sugeno integral

$$\int_{A} H(f) \,\mathrm{d}\mu \leqslant \left[(H(p) - pm_p)^+ \wedge \mu(A) \right] + \int_{A} (m_p)^+ f \,\mathrm{d}\mu,\tag{27}$$

where $p = \int_A f \, d\mu$ and $f \in \mathcal{F}_{(X,\mathbb{R}_+)}$. Further, if $0 < m_p \leq 1$ and $f \in \mathcal{F}_{(X,[0,1])}$, then combining the fact that $m_p y \leq m_p \wedge y$ for $m_p, y \in [0,1]$ with comonotone minitivity of the Sugeno integral, we obtain the Jensen type bound of the form

$$\oint_{A} H(f) \,\mathrm{d}\mu \leqslant \left[(H(p) - pm_p)^+ \wedge \mu(A) \right] + (m_p \wedge p).$$

The following example shows that the infimum in (23) can be achieved at $c \notin \{0, p\}$.

Example 3.1. Let $X = \mathbb{R}$, A = [0, 5] and μ be the Lebesgue measure. Take $H(x) = \sqrt{x}$ and f(x) = x. Then $p = \int_A f \, d\mu = 2.5$ and

$$\int_{A} H(f) \,\mathrm{d}\mu = \frac{-1 + \sqrt{21}}{2} \approx 1.7913$$

Write $g(c) = [(H(p) + m_p(c-p))^+ \wedge \mu(A)] + f_A (m_p(f-c))^+ d\mu$. Clearly, $g(c) = [(\sqrt{2.5} + m_p(c-2.5))^+ \wedge 5] + [((-m_pc)^+ \wedge 5) \vee (\frac{m_p}{m_p+1}(5-c)^+)],$

where $m_p = H'(p) = 1/\sqrt{10}$. After an easy calculation we get $\inf_{c \in \mathbb{R}} g(c) = g(-2.5) \approx 1.8019$, so the difference between the upper bound (23) and the exact value of integral $f_A H(f) d\mu$ is about 0.0106.

We also give a Jensen type inequality for the Shilkret integral.

Corollary 3.2. Let $H : \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable and concave function. Then for all $f \in \mathcal{F}_{(X,\mathbb{R}_+)}$ and $\mu \in \mathcal{M}_{(X,\mathcal{A})}$ we get

$$\int_{\cdot,A} H(f) \, \mathrm{d}\mu \leqslant H(p)\mu(A) + \left[(H'(p))^+ - H'(p)\mu(A) \right] p, \tag{28}$$

where $p = \int_{A} f d\mu < \infty$. In particular, if $\mu(A) = 1$ and $H'(p) \ge 0$, then

$$\int_{A} H(f) \, \mathrm{d}\mu \leqslant H\left(\int_{A} f \, \mathrm{d}\mu\right).$$

Proof. Take c = 0, $m_p = H'(p)$ and $\circ = \bullet = \cdot$ in Theorem 3.1. Observe that $0 \leq H(0) \leq H(p) - H'(p)p$.

4 Jensen type bounds for real-valued functions

Let $\star: \mathbb{R}_+ \times \mathbb{R}_- \to \mathbb{R}$ be a nondecreasing map, where $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}_- = (-\infty, 0]$. Suppose that $f \in \mathcal{F}_{(X,\mathbb{R})}$ and write $f^+ = f \vee 0$ and $f^- = (-f) \vee 0$. We define the \star -symmetric Sugeno integral of f on $A \in \mathcal{A}$ by the formula

$$\operatorname{Su}_{\mu,A}^{\star}(f) := \left(\oint_{A} f^{+} d\mu \right) \star \left(- \oint_{A} f^{-} d\mu \right),$$
(29)

provided that $\int_A f^+ d\mu < \infty$ and $\int_A f^- d\mu < \infty$. Kawabe [22] examined properties of the +-symmetric Sugeno integral while Grabisch proposed to use the symmetric Sugeno integral defined by (29) with the operator $a \otimes b = \operatorname{sign}(a+b)(|a| \vee |b|)$, where $a, b \in \mathbb{R}$ (see [13, 16]).

We derive both lower and upper bound on the *-symmetric Sugeno integral of H(f)by means of the Sugeno integrals $p_1 := f_A f^+ d\mu$ and $p_2 := f_A f^- d\mu$, where $H : \mathbb{R} \to \mathbb{R}$ is a nondecreasing function such that H(0) = 0. By the assumption on H, we have $H(f(x)) \lor 0 =$ $H_1(f^+(x))$ and $(-H(f(x))) \lor 0 = H_2(f^-(x))$ for all $x \in X$, where $H_1(x) = H(x)$ and $H_2(x) = -H(-x)$ for $x \ge 0$. Of course, functions $H_1, H_2 : \mathbb{R}_+ \to \mathbb{R}_+$ are nondecreasing, $H_1(0) = H_2(0) = 0$ and

$$\operatorname{Su}_{\mu,A}^{\star}(H(f)) = \left(\oint_{A} H_{1}(f^{+}) \,\mathrm{d}\mu \right) \star \left(- \oint_{A} H_{2}(f^{-}) \,\mathrm{d}\mu \right).$$
(30)

Since \star is a nondecreasing binary map, we can apply Theorems 2.1-2.3 to obtain two-sided bounds on $\operatorname{Su}_{\mu,A}^{\star}(H(f))$. Below, we provide the upper bound. Assume, for simplicity of exposition, that $\mu \in \mathcal{M}_{(X,\mathcal{A})}$ is continuous. Further, assume that $p_1, p_2 < \infty$ and $p_1 \leq \sup H$. Putting $\circ = \wedge$ in (18) and (19) we get

$$\int_{A} H_2(f^-) d\mu \ge H_2(p_2) \wedge p_2 = \left(-H(-p_2)\right) \wedge p_2,$$

$$\int_{A} H_1(f^+) d\mu \le \left(H(p_1) \wedge \mu(A)\right) \vee \left(\sup H \wedge p_1\right) = \left(H(p_1) \vee p_1\right) \wedge \mu(A),$$
(31)
(32)

because $p_1 \leq \mu(A)$. As a consequence of (30)-(32), we obtain the following bound

$$\operatorname{Su}_{\mu,A}^{\star}(H(f)) \leqslant \left[\left(H(p_1) \lor p_1 \right) \land \mu(A) \right] \star \left[H(-p_2) \lor (-p_2) \right].$$
(33)

The equality is reached in (33) for $f(x) = \mu(B)\mathbb{1}_B(x) - \mu(A \setminus B)\mathbb{1}_{A \setminus B}(x)$, where $B \subset A$ is such that $\mu(B), \mu(A \setminus B) < \infty$ and $H(\mu(B)) = \mu(B)$. Summing up, we arrive at the following result.

Theorem 4.1. Let $H: \mathbb{R} \to \mathbb{R}$ be a nondecreasing function such that H(0) = 0. Then the sharp inequality (33) holds true for any continuous monotone measure μ and for all $f \in \mathcal{F}_{(X,\mathbb{R}_+)}$ such that the integrals $f_A f^+ d\mu$, $f_A f^- d\mu$ are finite and $f_A f^+ d\mu \leq \sup H$. **Example 4.1.** Let $X = A = \{1, 2, 3\}$. Suppose that

$$\begin{split} \mu(\{1\}) &= 0.1, \qquad \mu(\{1,2\}) = 0.4, \qquad f(1) = -1, \\ \mu(\{2\}) &= 0.25, \qquad \mu(\{1,3\}) = 0.3, \qquad f(2) = 0.3, \\ \mu(\{3\}) &= 0.2, \qquad \mu(\{2,3\}) = 0.6, \qquad f(3) = 1 \end{split}$$

and $\mu(\{1, 2, 3\}) = 1$. If $H(x) = x^3$, then

$$p_1 = 0.3, \quad p_2 = 0.1, \quad \oint_A H_1(f^+) \,\mathrm{d}\mu = 0.2, \quad \oint_A H_2(f^-) \,\mathrm{d}\mu = 0.1.$$

Hence, we get from (33) that

$$0.1 = 0.2 + (-0.1) = \operatorname{Su}_{\mu,A}^+(H(f)) \leq p_1 + (-p_2) = 0.2,$$

$$0.2 = 0.2 \oslash (-0.1) = \operatorname{Su}_{\mu,A}^{\oslash}(H(f)) \leq p_1 \oslash (-p_2) = 0.3.$$

Example 4.2. Assume that $X = \mathbb{R}$, A = [-3, 1] and $\mu = \sqrt{\lambda}$, where λ is the Lebesgue measure. Put f(x) = x and $H(x) = x \mathbb{1}_{\mathbb{R}_+}(x) + 2x \mathbb{1}_{\mathbb{R}_-}(x)$. Then

$$p_1 = \frac{\sqrt{5} - 1}{2}, \quad p_2 = \frac{\sqrt{13} - 1}{2}, \quad \oint_A H_1(f^+) \,\mathrm{d}\mu = \frac{\sqrt{5} - 1}{2}, \quad \oint_A H_2(f^-) \,\mathrm{d}\mu = 1.5.$$

It follows from (33) that

$$\frac{\sqrt{5}-4}{2} = \operatorname{Su}_{\mu,A}^{+}(H(f)) \leqslant p_1 - p_2 = \frac{\sqrt{5}-\sqrt{13}}{2},$$

$$-1.5 = \operatorname{Su}_{\mu,A}^{\otimes}(H(f)) \leqslant p_1 \otimes (-p_2) = \frac{1-\sqrt{13}}{2} \approx -1.3.$$

Similar result as in (33) can be obtained provided that H is nonincreasing and H(0) = 0. If μ is subadditive and $H: \mathbb{R} \to \mathbb{R}_+$ is nonincreasing for $x \leq 0$, nondecreasing for $x \geq 0$ and H(0) = 0, then

$$\begin{split} & \oint_A H(f) \, \mathrm{d}\mu \leqslant \sup_{t \ge 0} \left\{ t \land \mu \left(A \cap \{f \ge 0\} \cap \{H(f) \ge t\} \right) \right\} \\ & \quad + \sup_{t \ge 0} \left\{ t \land \mu \left(A \cap \{-f \ge 0\} \cap \{\widetilde{H}(-f) \ge t\} \right) \right\} \\ & \quad = \oint_A H(f^+) \, \mathrm{d}\mu + \oint_A \widetilde{H}(f^-) \, \mathrm{d}\mu, \end{split}$$

where $\widetilde{H}(x) = H(-x)$ for $x \ge 0$. Thus, the upper bound can be derived from (14) or (19). Clearly, for any $\mu \in \mathcal{M}_{(X,\mathcal{A})}$,

$$\oint_{A} H(f) \,\mathrm{d}\mu \ge \oint_{A} H(f^{+}) \,\mathrm{d}\mu \lor \oint_{A} \widetilde{H}(f^{-}) \,\mathrm{d}\mu,$$

so we can also give a lower bound on $f_A H(f) d\mu$. Further, in a similar way as above, we can also estimate the (\star, \circ) -asymmetric integral defined by

$$\operatorname{Su}_{\mu,\nu,A}^{\star,\circ}(H(f)) := \left(\int_{\circ,A} H(f) \vee 0 \,\mathrm{d}\mu\right) \star \left(-\int_{\circ,A} (-H(f)) \vee 0 \,\mathrm{d}\nu\right),$$

where $\mu, \nu \in \mathcal{M}_{(X,\mathcal{A})}$. See e.g. [28] for the motivation of this definition with $\circ = \land$ and $\star = \oslash$.

5 Conclusions

In this paper, we have provided optimal lower/upper bounds of the Jensen type for the generalized Sugeno integral of measurable real-valued functions. As a consequence, we have obtained the Jensen type inequalities for the Sugeno integral, Shilkret integral and q-integral. Our results generalize and improve a number of known results.

The Jensen type inequalities for fuzzy integrals can be a useful tool to solve both theoretical and practical problems in many areas of research as the concept of the Sugeno integral has numerous applications. The Sugeno integral plays important role in decision-making problems under uncertainty and multi-criteria decision problems [10]. The famous Hirsch index [17], which is closely related to the Sugeno integral [36], is widely used in evaluation of research performance of individual scientists, research groups and universities. Nurukawa and Torra [27] described the use of the Sugeno integral in decision making when modeling auctions. An application of risk theory can be found in [20]. The Sugeno integral was applied to describe a face recognition using modular neural networks with a fuzzy logic method [24]. Hu [18] proposed a fuzzy data mining method with the Sugeno fuzzy integral that can effectively find a compact set of fuzzy if-then classification rules. Some applications of the Sugeno integral, we refer to [1, 14, 15, 16, 37, 38].

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