Disjunctive attribute dependencies in formal concept analysis under the epistemic view of formal contexts

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Abstract

This paper considers an epistemic interpretation of formal contexts, interpreting blank entries in the context matrix as absence of information, which is in agreement with the usual focus on the extraction of implications between attributes. After recalling non-classical connections induced by rough sets and possibility theory in formal concept analysis (FCA), and the standard theory of attribute implications in FCA, this paper presents the notion of disjunctive attribute implications, which reflect additional information that can be extracted from an epistemic context. We show that they can be computed like standard attribute implications from the complementary context. The paper also recalls the logic of classical attribute implications, relying on works pertaining to functional dependencies in database theory, and proposes a dual logic for disjunctive attribute implications. A method for extracting the latter kind of rules from a formal context is proposed, using a counterpart of pseudo-intents. Lastly, the paper outlines a generalization of both conjunctive and disjunctive attribute implications under the form of rules, with a conjunction of conditions in the body and a disjunction of conditions in the head, that hold in a formal context under the epistemic view.

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1. Introduction

Formal context analysis (FCA) was developed in the eighties [37] as a tool for extracting knowledge from data describing objects in terms of Boolean attributes, a topic that has a great impact nowadays [3, 4, 5, 6, 9, 10, 26, 32, 39, 40, 45, 46, 48]. The data take the form of a relation between objects and attributes called a formal context. Knowledge is extracted from formal contexts in the form of if-then rules. Such data mining methods have been widely studied from a theoretical and applicational point of view [6, 14, 16, 26, 28, 36, 39, 40]. Traditionally, in FCA, only rules taking the form of Horn clauses are extracted (aka attribute implications, relating positive atoms representing attributes). They express information of the form "if objects of a context possess some attributes, then they also possess some other ones". Efficient techniques have been devised to extract these rules, using notions such as Galois connections, formal concepts, pseudo-intents [26], and stem bases [29].

←?? Why so many references?? Are they closely related to our topic?

However, there is an ambiguity as to the actual meaning of a formal context, described by a matrix containing crosses and blanks, whose lines are objects and columns are attributes or properties. A cross means that some object possesses a property. Nevertheless, the meaning of blank entries is more problematic.

Many authors and well-cited introductions to FCA (see [7]) interpret a blank in terms of one object that does not possess a property. However, it is noticeable that in their book [26], page 17, Ganter and Wille only give the meaning of crosses in the context matrix but do not say anything about the blank entries. Likewise Guigues and Duquenne [29] explicitly focus on extracting implications involving positive properties, and do not exploit the possibility of implications involving explicit negation of properties, which would be natural when interpreting blank entries as negative information. In our paper, we stick to interpreting blank entries as lack of knowledge, which we call the *epistemic interpretation* of formal contexts (as opposed to the standard complete information view of contexts, which could be named *ontic*. This is the natural assumption if we insist that extracted implications should only involve the presence of (positive) attributes.

Unknown or incomplete information has been studied in the framework of formal concept analysis, using so-called incomplete formal contexts [29, 47, 48] or partially-known formal concepts [33, 38]. The epistemic context considered in this paper, which follows the original view of Ganter and Wille, is in fact a more general framework in which the negative information is not known.

This paper then revisits the logic of attribute implications, in the scope of epistemic contexts. Moreover, we try to extract more general kinds of rules relating positive atoms, especially so-called disjunctive attribute implications. They express information of the form "if objects of a context possess a certain attribute, then they also possess one or more among some other ones". We show that these rules are naturally obtained from the complementary dual context, swapping blanks and crosses. Hence, the aim of this paper is different from the developments given in other papers with negative attributes, such as [34, 35, 36, 40]. By increasing the expressivity of attribute implivations, we try to extract more information from formal concepts, without the consideration of explicitly negative information (blank entries are associated with lack of information).

←The phrase "complementary dual" is unclear, dual is enough

Based on preliminary results in [2], we show that the theory of disjunctive attribute implications is a mirror image of the theory of standard attribute implications, replacing a formal context by its complement. We also introduce counterparts of formal concepts, pseudo-intent and minimal rule bases for disjunctive attribute implications, using non-classical connections in formal concept analysis. Finally, we outline a framework for extracting more general rules involving conjunctions of positive atoms in the body and disjunctions of positive atoms in the head.

The paper is structured as follows: Section 2 recalls basic notions useful in the sequel, especially non-classical connections induced by rough sets and possibility theory in FCA, stressing the epistemic view of contexts. Section 3 recalls the theory of attribute implications in FCA. We presents a logic for them, relying on works pertaining to functional dependencies in database theory, and discussing the meaning of the rules in the epistemic context. Section 4 presents disjunctive attribute implications and their logic. Section 5 proposes a method for extracting such rules from a formal context, using nonstandard connections a counterpart of pseudo-intents. Section 6 generalises conjunctive and disjunctive attribute implications to rules of the form "A and B imply C or D".

2. Preliminaries

This section recalls basic operators in FCA, namely the original ones leading to a Galois connection (whose definition is recalled in the Appendix), and newer ones related to modal logic, rough sets and possibility theory. Relationships between one another are highlighted.

2.1. Formal concept analysis

Relational datasets are interpreted in Formal Concept Analysis (FCA) as a set of **p**roperties or attributes $\mathcal{P}$, a set of **o**bjects $\mathcal{O}$ and a crisp relation between them $\mathcal{R} \subseteq \mathcal{O} \times \mathcal{P}$. The triple $(\mathcal{O}, \mathcal{P}, \mathcal{R})$ is called *formal context*, or simply *context*, and mappings ${}^{\uparrow}\colon 2^{\mathcal{O}} \to 2^{\mathcal{P}}$, ${}^{\downarrow}\colon 2^{\mathcal{P}} \to 2^{\mathcal{O}}$ between subsets of objects $X \subseteq \mathcal{O}$ and subsets of attributes $Y \subseteq \mathcal{P}$ are defined as follows:[2]

$$\begin{aligned} X^{\uparrow} &= \{a \in \mathcal{P} \mid \text{for all } x \in X, (x,a) \in \mathcal{R}\} \\ &= \{a \in \mathcal{P} \mid \text{if } x \in X, \text{ then } (x,a) \in \mathcal{R}\} \qquad (1) \\ Y^{\downarrow} &= \{x \in \mathcal{O} \mid \text{for all } a \in Y, (x,a) \in \mathcal{R}\} \\ &= \{x \in \mathcal{O} \mid \text{if } a \in Y, \text{ then } (x,a) \in \mathcal{R}\} \qquad (2) \end{aligned}$$

A *concept* in the context $\mathcal{C} = (\mathcal{O}, \mathcal{P}, \mathcal{R})$ is defined to be a pair (X, Y), where $X \subseteq \mathcal{O}$, $Y \subseteq \mathcal{P}$, which satisfies $X^{\uparrow} = Y$ and $Y^{\downarrow} = X$. The element X of the concept (X, Y) is the *extent* and Y the *intent*. Alternatively, one may define a concept (X, Y) as a maximal subset of the form $X \times Y \subseteq \mathcal{R}$. If we represent relations $\mathcal{R}$ by Boolean matrices R where $R(x, y) = 1$ if and only if $(x, y) \in \mathcal{R}$, a formal concept in $\mathcal{R}$ is a maximal rectangle of 1's in matrix R.

Nonetheless, in formal concept analysis, the statement $R(x, y) = 0$ [3] could be ambiguous: it may mean either that x does not possess property y, or that it is unknown whether this is so or not as first highlighted in [8]. As mentioned in the introduction, the first view, which could be called ontic (x does not possess property y) is often adopted. However, interpreting the entries of matrix R with the epistemic view, $R(x, y) = 1$ means that it is known that x possesses property y, and then its negation $R(x, y) = 0$ means

[2] Ganter and Wille used originally the notation $'$ for this operator, hence they were called derivation operators. We have changed the notation in order to differentiate between the mapping on the set of objects and the mapping on the set of attributes.

[3] Despite the fact that this notation may suggest falsity.

that *it is not known* whether x possesses property y. Then, the original definition of formal context can also represent incomplete information.

The epistemic view has been advocated by Holzer [8, 30], Obiedkov [33], noticeably. These authors highlight the fact that you need three kinds of entries in a context, to fully account for incomplete information, namely whether an object is known to possess a property, known not to possess it or not known to possess it. It yields a three-valued context. They point out that usual contexts can be obtained by assuming complete information (without the "don't know" entries). However we would also get a usual context assuming there is no negative information (thus yielding a positive epistemic context). We believe that this assumption is less committing and more realistic than assuming complete information.[4]

Let $\mathcal{C}(\mathcal{O}, \mathcal{P}, \mathcal{R})$ be the set of concepts in a context $(\mathcal{O}, \mathcal{P}, \mathcal{R})$, which is a complete lattice [12, 26], with the inclusion order on the left argument or the opposite of the inclusion order on the right argument, that is, for each $(X_1, Y_1), (X_2, Y_2) \in \mathcal{C}(\mathcal{O}, \mathcal{P}, \mathcal{R})$, we have $(X_1, Y_1) \leq (X_2, Y_2)$ if $X_1 \subseteq X_2$ (or, equivalently, $Y_2 \subseteq Y_1$). The meet $\wedge$ and join $\vee$ operators are defined by:

$$\begin{aligned}(X_1, Y_1) \wedge (X_2, Y_2) &= (X_1 \cap X_2, (Y_1 \cup Y_2)^{\downarrow\uparrow}) \\ (X_1, Y_1) \vee (X_2, Y_2) &= ((X_1 \cup X_2)^{\uparrow\downarrow}, Y_1 \cap Y_2)\end{aligned}$$

for all $(X_1, Y_1), (X_2, Y_2) \in \mathcal{C}(\mathcal{O}, \mathcal{P}, \mathcal{R})$.

Example 1. *We consider an example of formal context $(\mathcal{O}, \mathcal{P}, \mathcal{R})$ given in Table 1 where $\mathcal{O} = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $\mathcal{P} = \{a_1, a_2, a_3, a_4, a_5\}$. The cross mark "$\times$" indicates that the related object satisfies the corresponding attribute, otherwise the entry is blank and indicates lack of information.*

2.2. FCA operators, possibility theory and modal operators

The standard FCA operator $^\uparrow$ is one among four modal operators that can be defined on a formal context, viewing it as an accessibility relation. Given a context $(\mathcal{O}, \mathcal{P}, \mathcal{R})$, these modal operators respectively denoted by

[4] Positive epistemic contexts can be turned into complete contexts applying the closed world assumption as often done in the database setting. However, it makes sense for properties which are usually not satisfied, so that we can assume that the property does not hold unless it is explicitly known that it holds. This assumption is hardly ever made in FCA, and we shall not make it in the sequel.

Table 1: Relation $\mathcal{R}$ of the formal context $\mathcal{K}$.

$\mathcal{R}$	a_1	a_2	a_3	a_4	a_5
x_1			$\times$	$\times$	$\times$
x_2	$\times$				$\times$
x_3			$\times$		
x_4	$\times$	$\times$	$\times$	$\times$	
x_5		$\times$	$\times$	$\times$	$\times$
x_6			$\times$		

${}^{\uparrow_\Pi} : 2^{\mathcal{O}} \to 2^{\mathcal{P}}$, ${}^{\uparrow_N} : 2^{\mathcal{O}} \to 2^{\mathcal{P}}$, ${}^{\uparrow_\Delta} : 2^{\mathcal{O}} \to 2^{\mathcal{P}}$, ${}^{\uparrow_\nabla} : 2^{\mathcal{O}} \to 2^{\mathcal{P}}$ can be defined for each subset of objects $X \subseteq \mathcal{O}$ as follows [15, 28, 39]:[5]

$$\begin{aligned}
X^{\uparrow_\Pi} &= \{a \in \mathcal{P} \mid \text{ there exists } x \in X, \text{ such that } (x,a) \in \mathcal{R}\} \\
X^{\uparrow_N} &= \{a \in \mathcal{P} \mid \text{ for all } x \in \mathcal{O}, \text{ if } (x,a) \in \mathcal{R}, \text{ then } x \in X\} \\
X^{\uparrow_\Delta} &= \{a \in \mathcal{P} \mid \text{ for all } x \in \mathcal{O}, \text{if } x \in X, \text{ then } (x,a) \in \mathcal{R}\} \\
X^{\uparrow_\nabla} &= \{a \in \mathcal{P} \mid \text{ there exists } x \in \overline{X}, \text{ such that } (x,a) \in \overline{\mathcal{R}}\}
\end{aligned}$$

where $\overline{X}$ and $\overline{\mathcal{R}}$ are the complement of X and the complement relation of $\mathcal{R}$, respectively. The indices Π, N, Δ, ∇ refer to the four set functions in possibility theory, to which the above connections are similar [15]. Clearly the operator ${}^{\uparrow_\Delta}$ is nothing but the usual operator ${}^{\uparrow}$ of formal concept analysis defined in (1), and we drop the index Δ in the following. The operators ${}^{\uparrow_\Pi}$ and ${}^{\uparrow_N}$ were inspired by classical modal operators [28], but also rough set upper and lower approximations [39] and possibility theory [15].

These mappings can analogously be defined on subsets of attributes, namely: ${}^{\downarrow^\Pi} : 2^{\mathcal{P}} \to 2^{\mathcal{O}}$, ${}^{\downarrow^N} : 2^{\mathcal{P}} \to 2^{\mathcal{O}}$, ${}^{\downarrow} : 2^{\mathcal{P}} \to 2^{\mathcal{O}}$ and ${}^{\downarrow^\nabla} : 2^{\mathcal{P}} \to 2^{\mathcal{O}}$ are defined, for each $Y \subseteq \mathcal{P}$, as follows:

$$\begin{aligned}
Y^{\downarrow^\Pi} &= \{x \in \mathcal{O} \mid \text{ there exists } a \in Y, \text{ such that } (x,a) \in \mathcal{R}\} \\
Y^{\downarrow^N} &= \{x \in \mathcal{O} \mid \text{ for all } a \in \mathcal{P}, \text{ if } (x,a) \in \mathcal{R}, \text{ then } a \in Y\} \\
Y^{\downarrow} &= \{x \in \mathcal{O} \mid \text{ for all } a \in \mathcal{P}, \text{ if } a \in Y, \text{ then } (x,a) \in \mathcal{R}\} \\
Y^{\downarrow^\nabla} &= \{x \in \mathcal{O} \mid \text{ there exists } a \in \overline{Y}, \text{ such that } (x,a) \in \overline{\mathcal{R}}\}.
\end{aligned}$$

In [21], the above definitions were expressed using rows or columns of

[5]These notations are borrowed from possibility theory as in [15].

the relation table associated with the formal context. Specifically, given a context $\mathcal{C} = (\mathcal{O}, \mathcal{P}, \mathcal{R})$, the information in the rows of the relation $\mathcal{R}$ is contained in $R(x) = \{a \in \mathcal{P} \mid (x, a) \in \mathcal{R}\}$, for every object $x \in \mathcal{O}$, and the one associated with the columns of $\mathcal{R}$ is in $R^{-1}(a) = \{x \in \mathcal{O} \mid (x, a) \in \mathcal{R}\}$, for every property $a \in \mathcal{P}$, and the four above operators can be rewritten as follows:

$$\begin{aligned} X^{\uparrow_\Pi} &= \{a \in \mathcal{P} \mid R^{-1}(a) \cap X \neq \varnothing\} \\ X^{\uparrow_N} &= \{a \in \mathcal{P} \mid R^{-1}(a) \subseteq X\} \\ X^{\uparrow} &= \{a \in \mathcal{P} \mid X \subseteq R^{-1}(a)\} \\ X^{\uparrow_\triangledown} &= \{a \in \mathcal{P} \mid R^{-1}(a) \cup X \neq \mathcal{O}\} \end{aligned}$$

Clearly, these definitions are equivalent to the ones given above. Notice that the equivalence between the two definitions of the dual sufficiency operator holds because any attribute $a \in \mathcal{P}$ satisfies $R^{-1}(a) \cup X \neq \mathcal{O}$ if and only if there exists $x' \in \mathcal{O}$ such that $x' \notin R^{-1}(a) \cup X$, which is equivalent to $x' \notin R^{-1}(a)$ and $x \notin X$, i.e., $x \in \overline{X}$ and $(x', a) \in \overline{\mathcal{R}}$.

The operators ${}^{\uparrow_\Pi}$, ${}^{\downarrow_\Pi}$ are called *possibility*, ${}^{\uparrow_N}$, ${}^{\downarrow_N}$ *necessity* and ${}^{\uparrow_\triangledown}$, ${}^{\downarrow_\triangledown}$ *dual sufficiency operators*; the classical ones ${}^{\uparrow}$, ${}^{\downarrow}$ are called *sufficiency operators*.[6] The four above operators of possibility, necessity, sufficiency and dual sufficiency can be composed in order to form various types of connections or closure operators [10, 21, 23, 26, 28, 32, 37, 39]. As a consequence, several variants of lattices can be built, in the style of the *classical formal concept lattice* $\mathcal{C}(\mathcal{O}, \mathcal{P}, \mathcal{R})$ [10]. In this paper, we shall mainly be interested in so-called "object-oriented concept lattices", based on the isotone Galois connection $({}^{\uparrow_N}, {}^{\downarrow^\Pi})$.

The following results recall different well-known properties of the modal operators, which will be used in this paper. The first one presents the behavior with the whole set and the empty set.

Lemma 1. *Given two sets $\mathcal{O}$ and $\mathcal{P}$, and the subsets $X \subseteq \mathcal{O}$, $Y \subseteq \mathcal{P}$, we have*

[6]Counterparts of these notions in possibility theory [19] are respectively called potential possibility, actual necessity, actual possibility and potential necessity.

$$
\begin{aligned}
\mathcal{O}^{\uparrow_N} &= \mathcal{P} & \qquad \mathcal{P}^{\downarrow^N} &= \mathcal{O} \\
\varnothing^{\uparrow} &= \mathcal{P} & \varnothing^{\downarrow} &= \mathcal{O} \\
\varnothing^{\uparrow_\Pi} &= \varnothing & \varnothing^{\downarrow^\Pi} &= \varnothing \\
X^{\uparrow} &\subseteq X^{\uparrow_\Pi} & Y^{\downarrow} &\subseteq Y^{\downarrow^\Pi}
\end{aligned}
$$

The following proposition expresses counterparts, in FCA, of maxitivity and minitivity properties of set-functions in possibility theory [19, 20, 22] (see also Proposition 5.1 in [15]), but also modalities in the KD logic, and of approximations in rough set theory as well.

Proposition 1. *Given two sets $\mathcal{O}$ and $\mathcal{P}$, two index sets I and J and the families of subsets $\{X_i \subseteq \mathcal{P} \mid i \in I\}$, $\{Y_j \subseteq \mathcal{P} \mid j \in J\}$.*

$$
\begin{aligned}
\left(\bigcup_{i\in I} X_i\right)^{\uparrow} &= \bigcap_{i\in I} X_i{}^{\uparrow} & \qquad \left(\bigcup_{j\in J} Y_j\right)^{\downarrow} &= \bigcap_{j\in J} Y_j{}^{\downarrow} \\
\left(\bigcup_{i\in I} X_i\right)^{\uparrow_\Pi} &= \bigcup_{i\in I} X_i{}^{\uparrow_\Pi} & \left(\bigcup_{j\in J} Y_j\right)^{\downarrow^\Pi} &= \bigcup_{j\in J} Y_j{}^{\downarrow^\Pi} \\
\left(\bigcap_{i\in I} X_i\right)^{\uparrow_N} &= \bigcap_{i\in I} X_i{}^{\uparrow_N} & \left(\bigcap_{j\in J} Y_j\right)^{\downarrow^N} &= \bigcap_{j\in J} Y_j{}^{\downarrow^N} \\
\left(\bigcap_{i\in I} X_i\right)^{\uparrow_\nabla} &= \bigcup_{i\in I} X_i{}^{\uparrow_\nabla} & \left(\bigcap_{j\in J} Y_j\right)^{\downarrow^\nabla} &= \bigcup_{j\in J} Y_j{}^{\downarrow^\nabla}
\end{aligned}
$$

However, we only have inequalities of the following kind:

$$
\begin{aligned}
\left(\bigcap_{i\in I} X_i\right)^{\uparrow} &\supseteq \bigcup_{i\in I} X_i{}^{\uparrow} & \left(\bigcap_{j\in J} Y_j\right)^{\downarrow} &\supseteq \bigcup_{j\in J} Y_j{}^{\downarrow} \\
\left(\bigcap_{i\in I} X_i\right)^{\uparrow_\Pi} &\subseteq \bigcap_{i\in I} X_i{}^{\uparrow_\Pi} & \left(\bigcap_{j\in J} Y_j\right)^{\downarrow^\Pi} &\subseteq \bigcap_{j\in J} Y_j{}^{\downarrow^\Pi} \\
\left(\bigcup_{i\in I} X_i\right)^{\uparrow_N} &\supseteq \bigcup_{i\in I} X_i{}^{\uparrow_N} & \left(\bigcup_{j\in J} Y_j\right)^{\downarrow^N} &\supseteq \bigcup_{j\in J} Y_j{}^{\downarrow^N} \\
\left(\bigcup_{i\in I} X_i\right)^{\uparrow_\triangledown} &\subseteq \bigcap_{i\in I} X_i{}^{\uparrow_\triangledown} & \left(\bigcup_{j\in J} Y_j\right)^{\downarrow^\triangledown} &\subseteq \bigcap_{j\in J} Y_j{}^{\downarrow^\triangledown}
\end{aligned}
$$

The properties of the operators $^{\uparrow_N}$ and $^{\downarrow^N}$ clearly remind of the characteristic property of necessity measures $N(A \cap B) = \min(N(A), N(B))$ and the inequality $N(A \cup B) \geq \max(N(A), N(B))$ for subsets A and B of a set U. They also have counterparts in modal logic, i.e., the theorems of KD modal logic $\Box(p \wedge q) \iff \Box p \wedge \Box q$ and $\Box p \vee \Box q \to \Box(p \vee q)$, and counterparts in rough set theory (e.g., the lower approximation of the intersection of two sets is the intersection of their lower approximations). Similar remarks can be made for the operators $^{\uparrow_\Pi}$ and $^{\downarrow^\Pi}$, regarding the possibility measure Π and the modal possibility $\Diamond$, as well as rough set upper approximations. The properties of the operators $^{\uparrow}$ and $^{\downarrow}$ clearly remind of the characteristic property of the guaranteed possibility in possibility theory $\Delta(A \cup B) = \min(\Delta(A), \Delta(B))$ and the inequality $\Delta(A \cap B) \geq \max(\Delta(A), \Delta(B))$ for subsets A and B of a set U [19, 22]. For modal logic counterparts of $^{\uparrow}$ and $^{\uparrow_\triangledown}$, see the pioneering work of Gargov et al. [27], and more recently in [16, 24].

The meaning of the four operators was also analyzed in [21] in the FCA perspective and the following equalities were obtained, based on the rows of the relation $\mathcal{R}$.

$$
\begin{aligned}
X^{\uparrow_\Pi} &= \bigcup_{x\in X} R(x) & X^{\uparrow} &= \bigcap_{x\in X} R(x) \\
X^{\uparrow_N} &= \bigcap_{x\notin X} \overline{R(x)} & X^{\uparrow_\triangledown} &= \bigcup_{x\notin X} \overline{R(x)}
\end{aligned}
$$

where $\overline{R(x)} = \overline{R}(x)$. The above equalities are easy consequences of Proposition 1. They also are counterparts of expressions of set-functions in possibility theory in terms of possibility distributions $\pi(x) = \Pi(\{x\}) \in [0,1]$, namely, $\Pi(A) = \max_{x\in A} \pi(x), N(A) = \min_{x\notin A} 1-\pi(x), \Delta(A) = \min_{x\in A} \pi(x), \nabla(A) = \max_{x\notin A} 1 - \pi(x)$ [20].

Now, we can express these equalities in terms of sets of properties possessed by single objects instead of rows of R. Given $x \in \mathcal{O}$, we straightforwardly obtain that $R(x) = \{x\}^{\uparrow} = \{x\}^{\uparrow_\Pi}$ and so, we directly can ensure that:

$$X^{\uparrow_\Pi} = \bigcup_{x\in X} \{x\}^{\uparrow} \qquad\qquad X^{\uparrow} = \bigcap_{x\in X} \{x\}^{\uparrow} \tag{3}$$

For necessity and dual sufficiency operators we get:

$$X^{\uparrow_N} = \bigcap_{x\notin X} \overline{\{x\}^{\uparrow}} \qquad\qquad X^{\uparrow_\nabla} = \bigcup_{x\notin X} \overline{\{x\}^{\uparrow}} \tag{4}$$

In possibility theory, we have that $N(X) \leq \Pi(X)$ as soon as the possibility distribution π on $\mathcal{O}$ is normalized, that is $\pi(x) = 1$ for some x in $\mathcal{O}$ [20, 22]. In the case of operator $^{\uparrow_\Pi}$, we can see its restriction to singleton sets as a set-valued possibility distribution, denoted by $^{\uparrow_\pi} : \mathcal{O} \to 2^{\mathcal{P}}$ and defined as $x^{\uparrow_\pi} = \{x\}^{\uparrow_\Pi}$, which provides values in $2^{\mathcal{P}}$ instead of [0, 1], equipped with the inclusion ordering [15]. Hence, $^{\uparrow_\Pi}$ is normalized if $X^{\uparrow_\Pi} = \mathcal{P}$ for some $X \in 2^{\mathcal{O}}$,[7] which implies by the monotonicity of $^{\uparrow_\Pi}$ that $\mathcal{P}^{\uparrow_\Pi} = \mathcal{P}$. It means, under the epistemic view, that there is no property that is unknown for all objects, i.e., the relation $\mathcal{R}$ does not contain a blank column (for all $a \in \mathcal{P}, R^{-1}(a) \neq \varnothing$ holds), which also implies that [15]:

$$X^{\uparrow_N} \subseteq X^{\uparrow_\Pi},$$

which is the counterpart of the basic inequality $N(X) \leq \Pi(X)$ in possibility theory. We may require this condition of a non-empty column as an attribute whose value is unknown for all objects is useless.

In possibility theory, we also have that $\Delta(A) \leq \nabla(A)$ as soon as the possibility distribution π is bottom normalized, that is $\pi(x) = 0$ for some x in $\mathcal{O}$ [19, 22]. Hence, we intuitively expect that

$$X^{\uparrow_\Delta} \subseteq X^{\uparrow_\nabla}.$$

[7] And not $x^{\uparrow_\pi} = \mathcal{P}$, which would require a row full of crosses in the relation of the context. This is different from possibility measures defined on a chain, for which $\Pi(\mathcal{O}) = 1$ is equivalent to the existence of $x \in \mathcal{O}$ such that $\pi(x) = 1$.

as it means that if a property applies to all objects in X, there is at least one object outside X to which it is unknown whether this property applies. Clearly this presupposes that there is no property that applies to all objects, i.e., the relation $\mathcal{R}$ does not contain a column full of crosses (for all $a \in \mathcal{P}, R^{-1}(a) \neq \mathcal{O}$ holds).

Dually, considering the above operators applied to a set of properties Y, we should also require that there is no blank row and no row full of crosses in $\mathcal{R}$. In any case objects obeying all properties or for which all properties in $\mathcal{P}$ are unknown can be harmlessly omitted from $\mathcal{O}$, and likewise properties that apply to all objects or not known to apply to any object can be deleted from $\mathcal{P}$.

Definition 1. *A formal context without blank rows and columns and without full rows and columns will be called* normalized.

Properties dual to the ones provided in Lemma 1 hold for normalized contexts. More specifically:

- $\varnothing^{\uparrow_{\mathrm{N}}} = \{a \in \mathcal{P} \mid R^{-1}(a) = \varnothing\} = \varnothing$ except if there are blank columns in R (i.e., properties unknown for all objects).
- $\mathcal{O}^{\uparrow} = \{a \in \mathcal{P} \mid R^{-1}(a) = \mathcal{O}\} = \varnothing$ except if there are full columns in R (i.e., properties known to be satisfied by all objects).
- $\mathcal{O}^{\uparrow_{\Pi}} = \{a \in \mathcal{P} \mid R^{-1}(a) \neq \varnothing\} = \mathcal{P}$ except if there are blank columns in R (i.e., properties unknown for all objects).
- $\varnothing^{\downarrow_{\mathrm{N}}} = \{x \in \mathcal{O} \mid R(x) = \varnothing\} = \varnothing$ except if there are blank rows in R (i.e., objects with no known property).
- $\mathcal{P}^{\downarrow} = \{x \in \mathcal{O} \mid R(x) = \mathcal{P}\} = \varnothing$ except if there are full rows in R (i.e., objects satisfying all properties).
- $\mathcal{P}^{\uparrow_{\Pi}} = \{x \in \mathcal{O} \mid R(x) \neq \varnothing\} = \mathcal{O}$ except if there are blank rows in R (i.e., objects with no known property).

Moreover, using a normalized context the existing inclusions between the operators can be synthetized in a single one [15]:

$$X^{\uparrow_{\Delta}} \cup X^{\uparrow_{N}} \subseteq X^{\uparrow_{\nabla}} \cap X^{\uparrow_{\Pi}}$$

3. Attribute implications: from FCA to logic

Attribute implications were originally introduced by Guigues and Duquenne [29] (see also Ganter and Wille [26]) and they have been one of the most important parts of formal concept analysis. Attribute implications are if-then rules involving conjunctions of attributes only, that are universally valid in the considered formal context. Obtaining attribute implications allows us to induce a rule-based system that can explain the content of the formal context.

In this section, we will recall and analyze the meaning of the validity of an attribute implication in a concept lattice, and try to cast the inference between attribute implications in a classical logic setting.

3.1. Attribute implications in formal concept analysis

First of all, we recall that an *implication between attributes* (or attribute implication, in short) is simply a pair of arbitrary subsets of attributes, denoted by $A \Rightarrow B$, where A and B are interpreted as conjunctions of attributes. A set of attribute implications is called *implicational system*. The satisfaction or validity of an attribute implication requires a basic definition.

Definition 2. *Given a set $\mathcal{P}$ of attributes or properties, and $A, B, M \subseteq \mathcal{P}$, we say that M* satisfies *the attribute implication $A \Rightarrow B$, or equivalently, $A \Rightarrow B$ is* valid in *M, if either $A \not\subseteq M$ or $B \subseteq M$, and it is denoted by $M \models A \Rightarrow B$.*

This notion was introduced by Ganter and Wille writing that M *respects* the attribute implication $A \Rightarrow B$. Hence, the three possible words (satisfy, valid and respect) can be used. In the rest of definitions we will use only one of them.

In Definition 2, the property $A \not\subseteq M$ or $B \subseteq M$ is equivalent to the logical implication, in classical logic, saying that if M contains the attributes of A, then it also contains the attributes in B, that is, if $A \subseteq M$, then $B \subseteq M$. Moreover, the contrapositive law can be applied and another equivalent property is obtained, that is, if $B \not\subseteq M$, then $A \not\subseteq M$. Therefore, the satisfaction property given in Definition 2 can equivalently be rewritten in at least three different ways.

This definition is extended to a family of subsets of attributes.

Definition 3. *Given $A, B \subseteq \mathcal{P}$, a family of subsets $\mathcal{M} \subseteq 2^{\mathcal{P}}$* satisfies the attribute implication $A \Rightarrow B$*, if every subset $M \in \mathcal{M}$ satisfies $A \Rightarrow B$. It is denoted by $\mathcal{M} \models A \Rightarrow B$.*

This last definition intends to formally define the real notion of a valid attribute implication in formal concept analysis.

Definition 4. *Given a formal context $\mathcal{C} = (\mathcal{O}, \mathcal{P}, \mathcal{R})$, and $A, B \subseteq \mathcal{P}$, we say that $\mathcal{C}$* satisfies the attribute implication $A \Rightarrow B$ *if the family $\mathcal{R}_{\mathcal{O}} = \{R(x) \mid x \in \mathcal{O}\}$ satisfies the attribute implication, where $R(x) = \{a \in \mathcal{P} \mid (x, a) \in \mathcal{R}\}$. In this case, we also say that $A \Rightarrow B$ is* an implication of the context*, and we denote it by $\mathcal{C} \models A \Rightarrow B$.*

Notice that the original definition given by Ganter and Wille considers the system of object intents $\{x^{\uparrow} \mid x \in \mathcal{O}\}$, but clearly this is the same set as $\mathcal{R}_{\mathcal{O}}$. If we consider the formal context $\mathcal{C}$ introduced in Example 4, for instance, we have that $\mathcal{C}$ satisfies the attribute implication $A \Rightarrow B = \{a_4, a_5\} \rightarrow \{a_3\}$.

Given an object $x \in \mathcal{O}$, the set $R(x)$ and $A \subseteq \mathcal{P}$, clearly we have that $x \in A^{\downarrow}$ if and only if $A \subseteq R(x)$. Therefore, as a consequence of the definition above, $R(x)$ satisfies the attribute implication $A \Rightarrow B$, if $A \subseteq R(x)$, then $B \subseteq R(x)$, or equivalently, if for every $x \in \mathcal{O}$, such that $x \in A^{\downarrow}$, then $x \in B^{\downarrow}$ also holds, that is, $A^{\downarrow} \subseteq B^{\downarrow}$. This is a well-known property in FCA. It reads: if an object satisfies all properties in A then it satisfies all properties in B. Because the pair $({}^{\uparrow}, {}^{\downarrow})$ is a Galois connection, the previous inclusion is equivalent to $B \subseteq A^{\downarrow\uparrow}$ [29]. This last property is fundamental for computing the attribute implications valid in a formal context [26]. A set of attributes A such that $A = A^{\downarrow\uparrow}$ is said to be closed.

An important point is that attribute implications of this kind only involve positive attributes, and not their negation. This is in agreement with our assumption that blank entries correspond to a lack of knowledge of whether one object satisfies a property or not. Indeed if blank entries represent the fact that some objects satisfy the negation of properties, there is no reason to assume that attribute implications do not involve such negated properties.

Some attribute implications are informative other ones are not. An attribute implication $A \Rightarrow B$ is said to be informative if $B \not\subseteq A$. It implies that an informative attribute implication $A \Rightarrow B$ is such that A is not closed. Moreover, it can be easily checked that $A \Rightarrow B$ is valid in a context $\mathcal{C}$ if and only if $A \Rightarrow B \setminus A$ is valid as well in $\mathcal{C}$. Among non-informative (tautological) attribute implications are $A \Rightarrow A$ and, thus equivalently $A \Rightarrow \varnothing$,[8] but also $\mathcal{P} \Rightarrow B$. They are trivially valid in all contexts. Other noticeable extreme attribute implications are as follows:

- That $A \Rightarrow \mathcal{P}$ is valid in a context $\mathcal{C}$ is equivalent to $A^{\downarrow} \subseteq \mathcal{P}^{\downarrow} = \{x \in \mathcal{O} \mid R(x) = \mathcal{P}\}$. It is satisfied only if all properties in A are satisfied only by objects that satisfy all properties (a situation not met in normal contexts). In that case, $A^{\downarrow} \neq \varnothing$.

  If the context is normalized, it satisfies the rule $A \Rightarrow \mathcal{P}$ only in the form $A^{\downarrow} = \mathcal{P}^{\downarrow} = \varnothing$. Two remarks are worth making:

  - Given the epistemic interpretation of blank entries, $A^{\downarrow} = \varnothing$ does not mean that properties in A are conflicting in the given context, because blanks mean ignorance. It means that there is no object whose *known* properties include A. Therefore, $A^{\downarrow} = \varnothing$ does not exclude the existence of objects having all properties in A, but we have not enough information about them.
  - If $A^{\downarrow} = \varnothing$, the rule $A \Rightarrow \mathcal{P}$ should not be derived from the context when $A \neq \mathcal{P}$, because this rule is useless for data-mining purposes. For instance consider the relation in Table 2. Choosing $A = \{a_1, a_2, a_3\}$, $A \Rightarrow \mathcal{P}$ clearly holds because $A^{\downarrow} = \mathcal{P}^{\downarrow} = \varnothing$. But extracting the attribute implication $\{a_1, a_2, a_3\} \Rightarrow \{a_4\}$ from this context would be weird.[9]

Table 2: Normalized context

$\mathcal{R}$	a_1	a_2	a_3	a_4
x_1	×	×		
x_2			×	×

[8] Note that it does not mean that the set A is conflicting since here $\varnothing$ expresses a tautology.

[9] This problem is due to the fact that an attribute implication behaves like a material implication, and the latter is true when its antecedent is false. Rules extracted from a context are extreme cases of association rules in data-mining, i.e., closer to a conditional object, that is undefined when its antecedent is a contradiction [17, 18]. An attribute implication $A \Rightarrow B$ where $B \neq A$ and $A^{\downarrow} = \varnothing$ is an association rule with confidence 1 but with zero support.

- That $\varnothing \Rightarrow B$ be valid in a context $\mathcal{C}$ means $\varnothing^{\downarrow} = \mathcal{O} = B^{\downarrow}$ and is satisfied only if all objects possess all properties in B. In normalized contexts, this attribute implication never holds.

Now, we must emphasize again the conjunctive nature of sets of attributes involved in attribute implications. Namely, given $A = \{a_1, \ldots, a_m\}, B = \{1, \ldots, m\}$, we say that $A \Rightarrow B$ is a valid implication of the context $\mathcal{C}$ when, for any object $x \in \mathcal{O}$, if a_1 *and* a_2 *and* $\ldots$ *and* a_m is satisfied by x then b_1 *and* b_2 *and* $\ldots$ *and* b_r are also satisfied by x. Therefore, an alternative, logic-oriented, writing of the attribute implication in a formal context $\mathcal{C} = (\mathcal{O}, \mathcal{P}, \mathcal{R})$ can be

$$a_1(x) \wedge a_2(x) \wedge \cdots \wedge a_m(x) \rightarrow b_1(x) \wedge b_2(x) \wedge \cdots \wedge b_r(x)$$

for all $x \in \mathcal{O}$, where the attributes a_i's and b_i's are understood as unary predicates, objects x are constants, and $a_i(x)$, $b_i(x)$ are grounded atoms, $\wedge$ is a conjunction, and $\rightarrow$ stands for an implication.

It is interesting to notice that considering the formal context $\mathcal{C}$ as a database, such an attribute implication rule is exactly a functional dependency in database theory, from a syntactic point of view.

It is well-known that inference from this kind of rules can be carried out via Armstrong axioms [3] for functional dependencies.[10] Fagin has shown [25] that such rules can be viewed as Horn clauses in classical logic, that reasoning with such rules with Armstrong axioms comes down to classical logic inference albeit on a more restricted language. Therefore, the connective $\wedge$ behaves as a classical conjunction, and $\rightarrow$ as a material implication.

3.2. The logic of attribute implications

From the above considerations, it is natural to define a classical logic version of reasoning with attribute implications in FCA. Specifically, we will see that the logic behind these implications is a restricted first order logic without functions and with less connectives. Notice that, although propositional logic has usually been considered for interpreting attribute implications in a logic framework, the required framework for defining a semantics more related to the semantics of attribute implications is a first order logic. Due

[10](a variant of these axioms appear in the seminal paper of Guigues and Duquenne and the book by Ganter and Wille).

to the conjunctive character of the attribute implications only a conjunction and an implication are needed and the logic will be called *conjunctive logic of attribute implications* (CLAI).

The syntax of CLAI will be based on an alphabet, denoted by $\mathfrak{A}_{\mathcal{O},\mathcal{P}}$, associated with the formal context $(\mathcal{O},\mathcal{P},\mathcal{R})$. It is composed of:

- The connective symbols $\wedge$ and $\rightarrow$, called conjunction and implication, respectively.
- The logic symbol $\top$.
- The auxiliary symbols "(" and ")".
- A variable symbol α.
- A set of constants symbols $\mathcal{O}$.
- A set of predicate symbols $\mathcal{P}$.

Notice that we obtain different alphabets depending on the sets $\mathcal{O}$ and $\mathcal{P}$. Indeed, for each formal context $(\mathcal{O},\mathcal{P},\mathcal{R})$, we have an alphabet where the set of constants is interpreted by the set of objects $\mathcal{O}$, and the set of predicates is associated with the set of attributes $\mathcal{P}$.

Let us define the strings of symbols allowed in the CLAI logic.

Definition 5 (Well-formed conjunctions and rules). *Given an alphabet $\mathfrak{A}_{\mathcal{O},\mathcal{P}}$, the symbol $\top$ and the strings $a(x)$, where $a \in \mathcal{P}$ and $x \in \mathcal{O}$, are called* grounded atoms*, the set of grounded atoms is denoted by $\mathcal{G}_{\mathcal{O},\mathcal{P}}$. An* atom*, in general, is an element in $\mathcal{G}_{\mathcal{O},\mathcal{P}}$ or a string $a(\alpha)$, where $a \in \mathcal{P}$.*

Well-formed conjunctions *(or simply* conjunctions*) are conjunctions of atoms, that is, given $A = \{a_1,\dots,a_m\} \subseteq \mathcal{P}$, $\mathcal{A} = a_1(\alpha) \wedge a_2(\alpha) \wedge \cdots \wedge a_m(\alpha)$ is a (well-formed) conjunction.*

Well-formed rules *take the form $\mathcal{A} \rightarrow \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are (well-formed) conjunctions or $\top$.*

The whole set of grounded conjunctions (resp. rules) is denoted by $\mathcal{C}_{\mathcal{G}_{\mathcal{O},\mathcal{P}}}$ (resp. $\mathcal{R}_{\mathcal{G}_{\mathcal{O},\mathcal{P}}}$), and the set of well-formed conjunctions (resp. rules), grounded or not, built from alphabet $\mathfrak{A}_{\mathcal{O},\mathcal{P}}$, is denoted by $\mathcal{C}_{\mathfrak{A}_{\mathcal{O},\mathcal{P}}}$ (resp. $\mathcal{R}_{\mathfrak{A}_{\mathcal{O},\mathcal{P}}}$).

The definition of a CLAI program is given as usual.

Definition 6. *A* CLAI program *is a set* $\mathbb{P}\mathbf{rog}$ *of CLAI well-formed rules of the form* $\mathcal{A} \to \mathcal{B}$*. The antecedent is called* head *and the consequent* body *of the rule. As usual,* facts *are rules with body* $\top$*. Free occurrences of variables in the program are assumed to be universally quantified.*

Note that the contradiction $\bot$ is not in the language, because there is no way to express a contradiction by means of attribute implications extracted from an epistemic context. In particular:

- $A \Rightarrow \varnothing$ is expressed by $\mathcal{A} \to \top$, a tautology.
- $A \Rightarrow \mathcal{P}$ is expressed by $\wedge_{a \in A} a(x) \to \wedge_{b \in \mathcal{P}} b(x)$
- $\varnothing \Rightarrow B$ is expressed as $\top \to \mathcal{B}$.

where $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_k\}$ are subsets of $\mathcal{P}$, and $\mathcal{A} = a_1(\alpha) \wedge a_2(\alpha) \wedge \cdots \wedge a_m(\alpha)$ and $\mathcal{B} = b_1(\alpha) \wedge b_2(\alpha) \wedge \cdots \wedge b_r(\alpha)$.

This is due to the epistemic interpretation of blank entries as ignorance: if one can only express knowledge of positive literals or ignorance about them, there is no way of expressing contradiction between statements.

Clearly, an implicational system of a formal context $(\mathcal{O}, \mathcal{P}, \mathcal{R})$ is syntactically equivalent to a CLAI program associated with the alphabet $\mathfrak{A}_{\mathcal{O},\mathcal{P}}$. All the previous notions define the syntax of the logic of attribute implications. Inference rules can be based on Armstrong axioms, here in the form proposed by Fagin [25], i.e., one logical axiom and two inference rules:

- Reflexivity $\vdash \bigwedge_{i=1}^{m} a_i(\alpha) \to a_i(\alpha)$, for all $i \in \{1, \ldots, m\}$
- Closure under right conjunction
  - $\bigwedge_{i=1}^{m} a_i(\alpha) \to \bigwedge_{j=1}^{r} b_j(\alpha) \vdash \bigwedge_{i=1}^{m} a_i(\alpha) \to b_j(\alpha)$, for all $j \in \{1, \ldots, r\}$
  - $\{\bigwedge_{i=1}^{m} a_i(\alpha) \to b_j(\alpha) \colon j \in \{1, \ldots, r\}\} \vdash \bigwedge_{i=1}^{m} a_i(\alpha) \to \bigwedge_{j=1}^{r} b_j(\alpha)$
- Transitivity: $\bigwedge_{i=1}^{m} a_i(\alpha) \to \bigwedge_{j=1}^{r} b_i(\alpha), \bigwedge_{j=1}^{r} b_i(\alpha) \to \bigwedge_{k=1}^{p} c_k(\alpha)\} \vdash \bigwedge_{i=1}^{m} a_i(\alpha) \to \bigwedge_{k=1}^{p} c_k(\alpha)$

Fagin [25] has shown that inference based on this syntax and this axiom and these inference rules is in agreement with classical logic, namely an attribute implication $A \to B$ is a consequence of a set of attribute implications $\mathcal{K}$, if and only if, viewed as a material implication, $A \to B$ is a consequence of material implication counterparts of the set of attribute implications $\mathcal{K}$.

Next, the Boolean semantics of CLAI can be introduced, based on the first-order model semantics.

Definition 7. *An* interpretation I *is a mapping from the set of grounded atoms* $\mathcal{G}_{\mathcal{O},\mathcal{P}}$ *to the values set* $\{0,1\}$, $I\colon \mathcal{G}_{\mathcal{O},\mathcal{P}} \to \{0,1\}$, *defined as follows:*

- $I(\top) = 1$.
- $I(a(x))$ *is an element of* $\{0,1\}$, *for all* $x \in \mathcal{O}$ *and* $a \in \mathcal{P}$.

and I *is inductively extended to the whole set of well-formed conjunctions and rules, then denoted by* $\hat{I}\colon \mathcal{C}_{\mathfrak{A}_{\mathcal{O},\mathcal{P}}} \cup \mathcal{R}_{\mathfrak{A}_{\mathcal{O},\mathcal{P}}} \to \{0,1\}$, *as follows. Given two grounded conjunctions* $\mathcal{A} = a_1(x) \wedge a_2(x) \wedge \cdots \wedge a_m(x)$ *and* $\mathcal{B} = b_1(x) \wedge b_2(x) \wedge \cdots \wedge b_r(x)$, *we have*

- $\hat{I}(a_1(x) \wedge \cdots \wedge a_m(x)) = \min_{i=1}^{m} I(a_i(x))$
- $\hat{I}(\mathcal{A} \to \mathcal{B}) = \begin{cases} 1 & \text{if } \hat{I}(\mathcal{A}) \leq \hat{I}(\mathcal{B}) \\ 0 & \text{otherwise} \end{cases}$

Therefore, the symbols $\wedge$ *and* $\to$ *are interpreted as in classical logic, that is, as the "and" connective and the material implication, respectively.*

For a non-grounded conjunction $\mathcal{A} \in \mathcal{C}_{\mathfrak{A}_{\mathcal{O},\mathcal{P}}}$ *and rule* $\mathcal{A} \to \mathcal{B} \in \mathcal{R}_{\mathfrak{A}_{\mathcal{O},\mathcal{P}}}$ *(a formula with the variable) the interpretation* $\hat{I}$ *is defined as follows:*

$$\begin{aligned} \hat{I}(\mathcal{A}) &= \min\{\hat{I}(\mathcal{A}[\alpha/x]) \mid x \in \mathcal{O}\} \\ \hat{I}(\mathcal{A} \to \mathcal{B}) &= \min\{\hat{I}(\mathcal{A} \to \mathcal{B}[\alpha/x]) \mid x \in \mathcal{O}\} \end{aligned}$$

where $\mathcal{A}[\alpha/x]$ *and* $\mathcal{A} \to \mathcal{B}[\alpha/x]$ *respectively denote the grounded conjunction and rule obtained after substituting the variable* α *by the constant* x.

The set of all interpretations on $\mathcal{G}_{\mathcal{O},\mathcal{P}}$ *will be denoted by* $\mathcal{I}$.

From the previous definitions the notion of satisfiability and model can be defined.

Definition 8. *An interpretation* $I \in \mathcal{I}$ satisfies *a rule* $\mathcal{A} \to \mathcal{B}$, *if* $\hat{I}(\mathcal{A} \to \mathcal{B}) = 1$. *An interpretation* I *is a* model *of a CLAI program* $\mathbb{P}\mathbf{rog}$, *if all rules in* $\mathbb{P}\mathbf{rog}$ *are satisfied by* I.

In the setting of a context $\mathcal{C} = (\mathcal{O}, \mathcal{P}, \mathcal{R})$, we define the associated interpretation $I_\mathcal{C}$ as: $I_\mathcal{C}(a(x)) = 1$ if and only if $(x, a) \in \mathcal{R}$, for all $x \in \mathcal{O}$ and $a \in \mathcal{P}$. The following result semantically relates the attribute implications of a context to the rules in the corresponding CLAI.

Proposition 2. *Given a formal context $\mathcal{C} = (\mathcal{O}, \mathcal{P}, \mathcal{R})$, the subsets $A = \{a_1, \dots, a_m\}, B = \{b_1, \dots, b_r\} \subseteq \mathcal{P}$, and the interpretation $I_\mathcal{C}$ defined on $\mathcal{G}_{\mathcal{O},\mathcal{P}}$ as above, it holds that $\mathcal{C}$ satisfies the attribute implication $A \Rightarrow B$, if and only if $I_\mathcal{C}$ satisfies the rule*

$$a_1(\alpha) \wedge a_2(\alpha) \wedge \cdots \wedge a_m(\alpha) \to b_1(\alpha) \wedge b_2(\alpha) \wedge \cdots \wedge b_r(\alpha).$$

Proof. Let us denote $\mathcal{A} = a_1(\alpha) \wedge a_2(\alpha) \wedge \cdots \wedge a_m(\alpha)$ and $\mathcal{B} = b_1(\alpha) \wedge b_2(\alpha) \wedge \cdots \wedge b_r(\alpha)$. By Definition 4, if $\mathcal{C}$ satisfies the attribute implication $A \Rightarrow B$, then for every $x \in \mathcal{O}$ such that $A \subseteq R(x)$, we have that $B \subseteq R(x)$. Hence, given $x \in \mathcal{O}$ satisfying that $(x, a) \in \mathcal{R}$, for all $a \in A$, we have that $(x, b) \in \mathcal{R}$, for all $b \in B$. Therefore, since

$$\begin{aligned} \hat{I}_\mathcal{C}(\mathcal{A}[\alpha/x]) = \hat{I}_\mathcal{C}(a_1(x) \wedge \cdots \wedge a_m(x)) &= \min\{I_\mathcal{C}(a_1(x)), \dots, I_\mathcal{C}(a_m(x))\} \\ \hat{I}_\mathcal{C}(\mathcal{B}[\alpha/x]) = \hat{I}_\mathcal{C}(b_1(x) \wedge \cdots \wedge b_r(x)) &= \min\{I_\mathcal{C}(b_1(x)), \dots, I_\mathcal{C}(b_r(x))\} \end{aligned}$$

given $x \in \mathcal{O}$, if $\hat{I}(\mathcal{A}[\alpha/x]) = 1$, then we also obtain that $\hat{I}_\mathcal{C}(\mathcal{B}[\alpha/x]) = 1$, that is, $\hat{I}_\mathcal{C}(\mathcal{A}[\alpha/x]) \le \hat{I}_\mathcal{C}(\mathcal{B}[\alpha/x])$, which implies by Definition 7 that $\hat{I}_\mathcal{C}(\mathcal{A}[\alpha/x] \to \mathcal{B}[\alpha/x]) = 1$ and so,

$$\begin{aligned} 1 &= \min\{\hat{I}_\mathcal{C}(\mathcal{A}[\alpha/x]) \to \hat{I}_\mathcal{C}(\mathcal{B}[\alpha/x]) \mid x \in \mathcal{O}\} \\ &= \min\{\hat{I}_\mathcal{C}((\mathcal{A} \to \mathcal{B})[\alpha/x]) \mid x \in \mathcal{O}\} \\ &= \hat{I}_\mathcal{C}(\mathcal{A} \to \mathcal{B}) \end{aligned}$$

which proves that I satisfies the rule $\mathcal{A} \to \mathcal{B}$.

The converse follows analogously. □

Example 2. *Given the context $\mathcal{C}$ introduced in Example 4, we have that $\mathcal{C}$ satisfies the attribute implication:*

$$a_1(\alpha) \wedge a_3(\alpha) \quad \to \quad a_2(\alpha) \wedge a_4(\alpha)$$

since the interpretation $I_\mathcal{C}$ satisfies the rule as seen below:

$$\begin{aligned} &\hat{I}_\mathcal{C}(a_1(\alpha) \wedge a_3(\alpha) \to a_2(\alpha) \wedge a_4(\alpha)) = \\ &\qquad = \min\{\hat{I}_\mathcal{C}(a_1(x) \wedge a_3(x)) \to \hat{I}_\mathcal{C}(a_2(x) \wedge a_4(x)) \mid x \in \mathcal{O}\} \\ &\qquad = \min\{\min\{a_1(x), a_3(x)\} \to \min\{a_2(x), a_4(x)\} \mid x \in \mathcal{O}\} \\ &\qquad = \min\{0 \to 0, 0 \to 0, 0 \to 0, 1 \to 1, 0 \to 0, 0 \to 0\} = 1 \end{aligned}$$

$\square$

The soundness and completeness of CLAI with respect to this semantics is clear, given the results obtained by Fagin [25] on the translation of functional dependencies into Horn clauses in classical logic.

Remark. Note that the attribute implication $\{a_3\} \Rightarrow \{a_4\}$ is valid in the context whose relation appears in Table 2. Armstrong axioms enable (using reflexivity and transitivity) to derive the attribute implication $\{a_2, a_3\} \Rightarrow \{a_4\}$ which makes sense in classical logic, but may be viewed as questionable in the scope of knowledge extraction from data, since no object in the context satisfies both a_2 and a_3, so that such rules hold trivially. Finding a proof theory that prevents the deduction of rules whose antecedents have empty extents, from a set of attribute implications that hold non-trivally in a context, is a topic for further research.

4. Disjunctive attribute implications in formal concept analysis

This section presents disjunctive counterparts of attribute implications. Especially, one can try to devise the counterpart of CLAI for this new type of attribute implications.

4.1. From complementary contexts to disjunctive attribute implications

Different authors [2, 5, 6, 34, 35, 36] have highlighted the importance of considering information hidden in a formal context in the form of zeros or empty entries. Indeed such entries are not directly exploited by the usual derivation operators $^{\uparrow}$, $^{\downarrow}$. In this section we will consider a dual ~~positive~~ epistemic context in order to apply the previous results for obtaining extra information in the form of disjunctive implications, where we interpret $(x, a) \notin \mathcal{R}$ as the knowledge that x does not possess property a. Then one possibility to take into account this kind of data is to consider the dual, i.e., formally, the complement of the original context. ←dual positive = negative ??? It would be clearer

Definition 9. *Given a formal context $\mathcal{C} = (\mathcal{O}, \mathcal{P}, \mathcal{R})$, its dual context is the epistemic context $\overline{\mathcal{C}} = (\mathcal{O}, \mathcal{P}^{\neg}, \overline{\mathcal{R}})$, where $\mathcal{P}^{\neg}$ contains the negated versions of attributes in $\mathcal{P}$, that is, $\mathcal{P}^{\neg} = \{\bar{a} \mid a \in \mathcal{P}\}$, and $\overline{\mathcal{R}} \subseteq \mathcal{O} \times \mathcal{P}^{\neg}$ is the complementary relation, which is defined as $(x, \bar{a}) \in \overline{\mathcal{R}}$, if $(x, a) \notin \mathcal{R}$.*

Therefore, the object x is related to the attribute $\bar{a}$ via $\overline{\mathcal{R}}$, when it is known that x does not have the attribute a,[11] while a blank entry indicates again ignorance. For instance, the dual context of the one in Table 1 is in Table 3.

Table 3: Relation $\overline{\mathcal{R}}$ of the dual formal context $\overline{\mathcal{C}}$.

$\overline{\mathcal{R}}$	$\bar{a}_1$	$\bar{a}_2$	$\bar{a}_3$	$\bar{a}_4$	$\bar{a}_5$
x_1	×	×			
x_2		×	×	×	
x_3	×	×		×	×
x_4					×
x_5	×				
x_6	×	×		×	×

Notice that, when a cross means that some object possesses a property and a blank means that it does not possess such property, the notion of 'dual context' coincides with the notion of 'complementary context' [40] or yet 'opposite context' [34]. The FCA operators defined by Equations (1) and (2) will be denoted by $\bar{\uparrow}$ and $\bar{\downarrow}$ on the ~~complementary~~ dual context. In the following, for any subset of attributes A we denote by $A^{\neg}$ the set $\{\bar{a}_i \mid a_i \in A\}$. In particular, operator $\bar{\downarrow}$ applied to a subset of attributes $A^{\neg}$ in $\overline{\mathcal{R}}$, yields the set of objects which, in $\mathcal{R}$, have no property in A. The following lemma recalls well-known equalities between the modal operators introduced in Subsection 2.2 [2, 23].

Lemma 2. *Given a formal context $\mathcal{C} = (\mathcal{O}, \mathcal{P}, \mathcal{R})$, its dual $\overline{\mathcal{C}} = (\mathcal{O}, \mathcal{P}^{\neg}, \overline{\mathcal{R}})$, the mappings ${}^{\uparrow_N}\colon 2^{\mathcal{O}} \to 2^{\mathcal{P}}$, ${}^{\downarrow_\Pi}\colon 2^{\mathcal{P}} \to 2^{\mathcal{O}}$, ${}^{\bar{\uparrow}}\colon 2^{\mathcal{O}} \to 2^{\mathcal{P}^{\neg}}$, ${}^{\bar{\downarrow}}\colon 2^{\mathcal{P}^{\neg}} \to 2^{\mathcal{O}}$, and the subsets $X \subseteq \mathcal{O}$, $Y \subseteq \mathcal{P}$, the following equalities hold.*

$$X^{\uparrow_N} = (\overline{X})^{\bar{\uparrow}\neg} \qquad Y^{\downarrow_\Pi} = \overline{(Y^{\neg})^{\bar{\downarrow}}}$$

For example, from the dual context of Example 4 (Table 3), the implication $\{\bar{a}_1, \bar{a}_4\} \to \{\bar{a}_2, \bar{a}_5\}$ is valid in $\overline{\mathcal{C}} = (\mathcal{O}, \mathcal{P}^{\neg}, \overline{\mathcal{R}})$. It corresponds to the conjunctive rule $\bar{a}_1(\alpha) \wedge \bar{a}_4(\alpha) \to \bar{a}_2(\alpha) \wedge \bar{a}_5(\alpha)$ in the associated logic CLAI. Notice that this implication is interpreted in the dual context as "if an object

[11] Note that the overbar is not a logical connective, strictly speaking. $\bar{a}$ is just another attribute.

does not possess a_1 and a_4, then it does not possess a_2 and a_5 either", however, this meaning completely changes in the original context. In terms of original attributes, it is clear that such a rule can equivalently be expressed as "if any of a_2, a_5 holds for an object, then one of a_1, a_4 holds too" in the original context.

We are thus in a position to introduce another kind of attribute implications we name disjunctive. It is again a pair of arbitrary subsets of attributes A and B, each one interpreted as the disjunction of its attributes.

Definition 10. *Given a set of attributes $\mathcal{P}$ and two subsets $A = \{a_1, \ldots, a_m\}$, $B = \{b_1, \ldots, b_r\} \subseteq \mathcal{P}$, a* disjunctive attribute implication between A and B *is denoted by $\bigvee A \mapsto \bigvee B$, or simply $A \mapsto B$. We say that a subset $M \subseteq \mathcal{P}$ of attributes* satisfies the disjunctive attribute implication $A \mapsto B$ *if, when there exists $a_i \in A \cap M$, then there exists $b_j \in B \cap M$.*

The definition of satisfiability of a disjunctive attribute implication for a subset of attributes and more generally in a formal context is defined in a natural way as for conjunctive attribute implications (Definitions 3 and 4).

Definition 11. *Given a formal context $\mathcal{C} = (\mathcal{O}, \mathcal{P}, \mathcal{R})$, and two subsets $A, B \subseteq \mathcal{P}$ of attributes, a disjunctive attribute implication $A \mapsto B$ is valid in $\mathcal{C}$, when for each object $x \in \mathcal{O}$, if there exists a_i such that $(x, a_i) \in \mathcal{R}$, then there exists $b_j \in B$ such that $(x, b_j) \in \mathcal{R}$.*

Recall again that, in the above definition, disjunctive attribute implications are interpreted directly on the original epistemic context $\mathcal{C}$, and that the use of the dual context is only instrumental to compute disjunctive implications by means of the standard technique, since we cannot use negation in the logical setting for a positive epistemic context.

The reader may have noticed that $x^{\uparrow} = x^{\uparrow_\Pi}$. Hence, the validity of a disjunctive attribute implication can mathematically be also rewritten noting the equivalence between the statement "there exists $a_i \in A$ such that $a_i \in \mathcal{R}(x)$", in other words "$A \cap x^{\uparrow_\Pi} \neq \varnothing$", and the statement $x \in A^{\downarrow^\Pi}$. As a consequence, we can define the validity of a disjunctive attribute implication using a non-classical operator on sets of properties.

Proposition 3. *Given a context $\mathcal{C} = (\mathcal{O}, \mathcal{P}, \mathcal{R})$, the following statements are equivalent.*

- $A \mapsto B$ *is valid in* $\mathcal{C}$
- $A^{\downarrow^{\Pi}} \subseteq B^{\downarrow^{\Pi}}$

Proof. $A \mapsto B$ valid in $\mathcal{C}$ means "for all $x \in \mathcal{O}$, if there exists $a_i \in A$ s.t. $a_i \in x^{\uparrow_{\Pi}}$ then there exists $b_j \in B$ s.t. $b_j \in x^{\uparrow_{\Pi}}$". This also means "If $x \in A^{\downarrow^{\Pi}}$ then $x \in B^{\downarrow^{\Pi}}$", since $A \cap x^{\uparrow_{\Pi}} \neq \varnothing$ is equivalent to $x \in A^{\downarrow^{\Pi}}$, as we noticed above. $\square$

From this result we can formally relate conjunctive and disjunctive attribute implications, as follows.

Corollary 1. *Let* $\mathcal{C} = (\mathcal{O}, \mathcal{P}, \mathcal{R})$ *be a context, and two subsets of attributes* A, $B \subseteq \mathcal{P}$. *The following statements are equivalent.*

- $(A^{\neg})^{\downarrow} \subseteq (B^{\neg})^{\downarrow}$
- $A^{\neg} \Rightarrow B^{\neg}$ *is a valid conjunctive attribute implication rule in* $\overline{\mathcal{C}}$
- $B \mapsto A$ *is a valid disjunctive attribute implication rule in* $\mathcal{C}$
- $B^{\downarrow^{\Pi}} \subseteq A^{\downarrow^{\Pi}}$

Proof. All that needs to be proven is that $B^{\downarrow^{\Pi}} \subseteq A^{\downarrow^{\Pi}}$ is equivalent to $A^{\neg} \Rightarrow B^{\neg}$. The latter means

$$\{x \mid A^{\neg} \subseteq \overline{R}(x)\} \subseteq \{x \mid B^{\neg} \subseteq \overline{R}(x)\}$$

Now, $A^{\neg} \subseteq \overline{R}(x)$ means that for all $\bar{a} \in A^{\neg}$, we have $\bar{a} \in \overline{R}(x)$, and since $(x, \bar{a}) \in \overline{\mathcal{R}}$ means $(x, a) \notin \mathcal{R}$, the inclusion $A^{\neg} \subseteq \overline{R}(x)$ is equivalent to the sentence "if $a \in A$, then $(x, a) \notin \mathcal{R}$". Therefore, $\{x \mid A^{\neg} \subseteq \overline{R}(x)\} = \{x \mid A \cap R(x) = \varnothing\}$. Since this reasoning is independent of the subset of attributes, we obtain that: $A^{\neg} \Rightarrow B^{\neg}$ is equivalent to

$$\{x \mid A \cap R(x) = \varnothing\} \subseteq \{x \mid B \cap R(x) = \varnothing\}$$

By contraposition, this inclusion is equivalent to $\{x \mid B \cap R(x) \neq \varnothing\} \subseteq \{x \mid A \cap R(x) \neq \varnothing\}$, that is $B^{\downarrow^{\Pi}} \subseteq A^{\downarrow^{\Pi}}$, which finishes the proof. $\square$

Example 3. *In the formal context of Table 3, we can check that the following conjunctive implication is valid:* $\{\bar{a}_4, \bar{a}_5\} \Rightarrow \{\bar{a}_1\}$ *and thus its "contrapositive" implication* $\{a_1\} \mapsto \{a_4, a_5\}$ *is a disjunctive attribute implication valid in the formal context of Table 1.* $\square$

Interesting special cases of disjunctive attribute implications are as follows, for every subset $A, B \subseteq \mathcal{P}$, where $A \neq \varnothing, B \neq \mathcal{P}$.

- That $A \mapsto \mathcal{P}$ be valid in $\mathcal{C}$ means $A^{\downarrow^{\Pi}} \subseteq \mathcal{P}^{\downarrow^{\Pi}}$, which holds trivially since $A \subseteq \mathcal{P}$, and is then non-informative. More generally all disjunctive attribute implications of the form $A \mapsto B$ where $A \subseteq B$ are not informative.

- $A \mapsto \varnothing$ never holds in normalized contexts, because otherwise $A^{\downarrow^{\Pi}} = \varnothing$, which implies that no object possesses any property in A. As a consequence, since A contains at least one attribute, we obtain at least one blank column, which is not possible in a normalized context.

- $\varnothing \mapsto B$ is ever true, due to the monotonicity of the operator ${}^{\downarrow^{\Pi}}$.

- $\mathcal{P} \mapsto B$ is not valid except if no attribute outside B is known to be satisfied by at least one object.

4.2. The logic DLAI of disjunctive attribute implications

From Proposition 3 and Corollary 1 the relation between the conjunctive and disjunctive attribute implications is obtained via the ~~complementary~~ dual context, and the following statements are equivalent:

- $A^{\neg} \Rightarrow B^{\neg}$ is a valid conjunctive attribute implication rule in $\overline{\mathcal{C}}$

- $B \mapsto A$ is a valid disjunctive attribute implication rule in $\mathcal{C}$

for all $A, B \subseteq \mathcal{P}$. By Proposition 2, $A^{\neg} \Rightarrow B^{\neg}$, where $A = \{a_1, \ldots, a_m\}, B = \{b_1, \ldots, b_r\} \subseteq \mathcal{P}$, is expressed in classical logic as

$$\bar{a}_1(\alpha) \wedge \cdots \wedge \bar{a}_m(\alpha) \to \bar{b}_1(\alpha) \wedge \cdots \wedge \bar{b}_r(\alpha)$$

As a consequence, noticing that $\bar{a}(x)$ is true if and only if $a(x)$ is false, and because the following formulae are equivalent in classical logic:

$$\begin{array}{rcl} \neg a_1(\alpha) \wedge \cdots \wedge \neg a_m(\alpha) & \to & \neg b_1(\alpha) \wedge \cdots \wedge \neg b_r(\alpha) \\ b_1(\alpha) \vee \cdots \vee b_r(\alpha) & \to & a_1(\alpha) \vee \cdots \vee a_m(\alpha), \end{array}$$

we can define a counterpart, called disjunctive logic of attribute implications (DLAI), of the language CLAI with a disjunction $\vee$ in place of the conjunction, and contradiction symbol $\perp$ instead of a tautology symbol $\top$, in order to encode, and reason with, disjunctive attribute implications. The new obtained alphabet for defining DLAI is denoted by $\mathfrak{D}_{\mathcal{O},\mathcal{P}}$

Definition 12 (Well-formed disjunctive rules). *Given an alphabet* $\mathfrak{D}_{\mathcal{O},\mathcal{P}}$*, the symbol* $\perp$ *and the strings* $a(x)$*, where* $a \in \mathcal{P}$ *and* $x \in \mathcal{O}$*, are called* grounded atoms *of DLAI. An* atom in DLAI *is either a grounded atom or a string* $a(\alpha)$*, where* $a \in \mathcal{P}$*.*

The set of grounded atoms in DLAI is denoted by $\mathcal{D}_{\mathcal{O},\mathcal{P}}$ *and the set of atoms is denoted by* $\mathfrak{D}_{\mathcal{O},\mathcal{P}}$*.* Well-formed disjunctive rules in DLAI *take the form* $\mathcal{A} \to \mathcal{B}$ *where* $\mathcal{A}$ *and* $\mathcal{B}$ *are disjunctions of atoms or* $\perp$*.*

The set of grounded disjunctive rules is denoted by $\mathcal{R}_{\mathcal{D}_{\mathcal{O},\mathcal{P}}}$, and the set of well-formed disjunctive rules (grounded or not) from alphabet $\mathfrak{D}_{\mathcal{O},\mathcal{P}}$, is denoted by $\mathcal{R}_{\mathfrak{D}_{\mathcal{O},\mathcal{P}}}$.

Disjunctive implications are naturally encoded in DLAI:

Definition 13. *Given a set of attributes* $\mathcal{P}$ *and two subsets* $A = \{a_1, \dots, a_m\}$*,* $B = \{b_1, \dots, b_r\} \subseteq \mathcal{P}$*, a* disjunctive attribute implication $A \mapsto B$ between A and B *is encoded by the following rule in the associated DLAI:*

$$a_1(\alpha) \vee \cdots \vee a_m(\alpha) \to b_1(\alpha) \vee \cdots \vee b_r(\alpha)$$

Given a context $\mathcal{C} = (\mathcal{O}, \mathcal{P}, \mathcal{R})$, the *associated DLAI* is the logic obtained from the alphabet $\mathfrak{D}_{\mathcal{O},\mathcal{P}}$.

Special kinds of rules involving $\perp$ can be laid bare (they are contrapositive versions of conjunctive rules $\top \to \mathcal{A}$ and $\mathcal{B} \to \top$ in CLAI):

- $a_1(\alpha) \vee \cdots \vee a_m(\alpha) \to \perp$ encodes disjunctive attribute implications with empty head. Using the classical logic embedding it comes down to claiming that for all $x \in \mathcal{O}, \wedge_{i=1}^{n} \neg a_i(x)$ holds, which can be called a negative fact, since it says that all properties a_i, with $i \in \{1, \dots, n\}$, are false for all objects.

- $\perp \to \mathcal{B}$, which is a tautology in DLAI.

The definition of a DLAI program is given as usual.

Definition 14. *A* DLAI program *is a set of DLAI well-formed disjunctive rules of the form* $\mathcal{A} \to \mathcal{B}$*.* Negative facts *are disjunctive rules with head* $\perp$*.*

The semantics of DLAI is similar to the one of CLAI. An interpretation I is again a mapping from the set of grounded atoms to $\{0, 1\}$, however here it satisfies that $I(\perp) = 0$. Moreover, it is extended to the whole set of well-formed disjunctive rules, $\hat{I}\colon \mathcal{R}_{\mathfrak{D}_{\mathcal{O},\mathcal{P}}} \to \{0,1\}$. If $\mathcal{A} = a_1(x) \vee a_2(x) \vee \cdots \vee a_m(x)$ and $\mathcal{B} = b_1(x) \vee b_2(x) \vee \cdots \vee b_r(x)$ are two grounded disjunctions, the well-formed disjunctive rule $\mathcal{A} \to \mathcal{B} \in \mathcal{R}_{\mathcal{D}_{\mathcal{O},\mathcal{P}}}$ is interpreted by $\hat{I}$ as follows:

$$\hat{I}(\mathcal{A} \to \mathcal{B}) = \begin{cases} 1 & \text{if } \max_{i=1}^{m} I(a_i(x)) \leq \max_{i=1}^{k} I(b_i(x)) \\ 0 & \text{otherwise.} \end{cases}$$

The interpretation of a non-grounded disjunctive rule $\mathcal{A} \to \mathcal{B} \in \mathcal{R}_{\mathfrak{D}_{\mathcal{O},\mathcal{P}}}$ is similarly defined to the conjunctive case, that is,

$$\hat{I}(\mathcal{A} \to \mathcal{B}) = \min\{\hat{I}(\mathcal{A} \to \mathcal{B}[\alpha/x]) \mid x \in \mathcal{O}\}$$

where $\mathcal{A} \to \mathcal{B}[\alpha/x]$ denotes the grounded rule obtained after substituting the variable α by the constant x. The following result shows that the notion of validity is compatible with the notion of satisfiability in DLAI.

Proposition 4. *Consider a formal context $\mathcal{C} = (\mathcal{O}, \mathcal{P}, \mathcal{R})$, the subsets of attributes $A = \{a_1, \ldots, a_m\}, B = \{b_1, \ldots, b_r\} \subseteq \mathcal{P}$, and the interpretation $I_{\mathcal{C}}$ defined on $\mathcal{D}_{\mathcal{O},\mathcal{P}}$ as $I_{\mathcal{C}}(a(x)) = 1$, if $(x, a) \in \mathcal{R}$, for all $x \in \mathcal{O}$ and $a \in \mathcal{P}$. Then $\mathcal{C}$ satisfies the disjunctive attribute implication $A \mapsto B$ if and only if $I_{\mathcal{C}}$ satisfies the rule*

$$a_1(\alpha) \vee a_2(\alpha) \vee \cdots \vee a_m(\alpha) \to b_1(\alpha) \vee b_2(\alpha) \vee \cdots \vee b_r(\alpha)$$

Proof. The proof is similar to the one given to Proposition 2.

Moreover, using such a duality, the following axiom and rules of inference can be derived from Armstrong's for deduction in DLAI:

- Reflexivity $\bigvee_{i=1}^{m} a_i(\alpha)$, for all $i \in \{1, \ldots, m\}$
- Closure under left disjunction
  - $\bigvee_{i=1}^{m} a_i(\alpha) \to \bigvee_{j=1}^{r} b_j(\alpha) \vdash a_i(\alpha) \to \bigvee_{j=1}^{r} b_j(\alpha)$, for all $i \in \{1, \ldots, m\}$
  - $\{a_i(\alpha) \to \bigvee_{j=1}^{r} b_j(\alpha) | \; i \in \{1, \ldots, m\}\} \vdash \bigvee_{i=1}^{m} a_i(\alpha) \to \bigvee_{j=1}^{r} b_j(\alpha)$.
- Transitivity:
  $\bigvee_{i=1}^{m} a_i(\alpha) \to \bigvee_{j=1}^{r}, \bigvee_{j=1}^{r} b_j(\alpha) \to \bigvee_{k=1}^{p} c_k(\alpha)\} \vdash \bigvee_{i=1}^{m} a_i(\alpha) \to \bigvee_{k=1}^{p} c_k(\alpha)$.

The soundness and completeness of this rule-based system with respect to the semantics of disjunctive attribute implications is easy to show, again via duality, based on Fagin's results [25].

5. Object-oriented concept lattices and disjunctive implications

In this section, we define the counterpart of concept lattices for the possibility and necessity modalities applied to a formal context, and relate them to usual concept lattices in the dual context.

5.1. Object-Oriented Concept Lattices

Object-oriented concept lattices (OOCL), originally introduced by Yao [38], and property-oriented concept lattices (POCL), given by Duntsch and Gediga [23, 28], were introduced as extensions, to formal contexts, of upper and lower approximations in rough set theory, replacing the indiscernibility relation between objects by the relation linking objects and properties. Both notions are in mirror image to each other (exchanging objects and properties) and they have been studied in many papers [9, 14, 15, 40].

An *object-oriented concept* is a pair (X, Y), with $X \subseteq \mathcal{O}$, $Y \subseteq \mathcal{P}$, such that $X = Y^{\downarrow^{\Pi}}$ and $Y = X^{\uparrow_N}$. Yao [39] justified this name through the sentence: "if an object has a property in Y, then the object belongs to X". Moreover, X is the set of objects possessing at least one attribute in Y.

Formally, $X = Y^{\downarrow^{\Pi}}$ means that if $x \in X$ then there exists $a \in Y$ such that $(x, a) \in \mathcal{R}$. And $Y = X^{\uparrow_N}$ means that $Y = \{a \mid R^{-1}(a) \subseteq X\}$.

Actually, $Y \subseteq X^{\uparrow_N}$ reads in terms of characteristic functions:

$$\mu_Y(a) \leq \mu_R(x, a) \rightarrow \mu_X(x), \text{ for all } x \in \mathcal{O}$$

and $Y^{\downarrow_\Pi} \subseteq X$ reads

$$\mu_X(x) \geq \mu_R(x, a) \wedge \mu_Y(a), \text{ for all } a \in \mathcal{P}$$

Hence the equalities $Y = X^{\uparrow_N}$ and $Y^{\downarrow_\Pi} = X$ are reached by maximizing Y and minimizing X.

Therefore, it is clear that if $X = Y^{\downarrow^{\Pi}}$ and $Y = X^{\uparrow_N}$ together, then Y is the maximal subset of $\mathcal{P}$, such that for all $x \in X$, there exists $a \in Y$, with $(x, a) \in \mathcal{R}$, and $(x', a) \notin \mathcal{R}$, for all $x' \notin X$. Alternatively, X is the minimal subset of $\mathcal{O}$, such that for all $a \in Y$ there exists $x \in X$, with $(x, a) \in \mathcal{R}$, and for all $x' \notin X, (x', a') \in \mathcal{R}$ implies $a' \notin Y$.

Example 4. *Consider the normalized formal context $(\mathcal{O}, \mathcal{P}, \mathcal{R})$ given in Table 4 where $\mathcal{O} = \{x_1, x_2, x_3\}$ and $\mathcal{P} = \{a_1, a_2, a_3, a_4\}$. Given $Y = \{a_1, a_2\}$*

Table 4: Relation $\mathcal{R}$ of $\mathcal{C}$.

$\mathcal{R}$	a_1	a_2	a_3	a_4
x_1	$\times$	$\times$	$\times$	
x_2			$\times$	
x_3				$\times$

and $X = \{x_1\}$*, we have that*

$$\begin{aligned} Y^{\downarrow^\Pi} = \{a_1, a_2\}^{\downarrow^\Pi} = \{x_1\} &= X \\ X^{\uparrow_N} = \{x_1\}^{\uparrow_N} = \{a_1, a_2\} &= Y \end{aligned}$$

$Y = \{a_1, a_2\}$ *is the largest subset of properties satisfied by* x_1 *and only by* x_1*. Likewise,* $\{x_1\}$ *is the smallest subset of objects satisfying at least one property in* $Y = \{a_1, a_2\}$*.* □

If we consider the above recalled verbal definition of an object-oriented concept (X, Y) by Yao [39], we should complement it as follows: "if an object has a property in Y, then the object belongs to X, and Y is the largest subset satisfying this claim". Dually we also have that "if a property in Y is satisfied by an object, then the object belongs to X, and X is the smallest set of objects satisfying this claim." ←These remarks are not highlighted in the introduction. Should we do it ?

The set of object-oriented concepts is denoted by $C_o(\mathcal{O}, \mathcal{P}, \mathcal{R})$. It is a complete lattice [38], with the meet $\wedge$ and join $\vee$ operators defined, for each $(X_1, Y_1), (X_2, Y_2) \in L_o(\mathcal{O}, \mathcal{P}, \mathcal{R})$, as follows:

$$(X_1, Y_1) \wedge_o (X_2, Y_2) = ((Y_1 \cap Y_2)^{\downarrow^\Pi}, Y_1 \cap Y_2) \tag{5}$$

$$(X_1, Y_1) \vee_o (X_2, Y_2) = (X_1 \cup X_2, (X_1 \cup X_2)^{\uparrow_N}) \tag{6}$$

The following result relating object-oriented concepts and concepts in the usual sense is recalled with the notation considered in this paper as follows [28, 32].

Proposition 5. *A pair* (X, Y) *is an object-oriented concept in the context* $\mathcal{C} = (\mathcal{O}, \mathcal{P}, \mathcal{R})$ *if and only if the pair* $(\overline{X}, Y^\neg)$ *is a concept in the usual sense in the dual context* $\overline{\mathcal{C}} = (\mathcal{O}, \mathcal{P}^\neg, \overline{\mathcal{R}})$*, i.e., a maximal subset* $\overline{X} \times Y \subseteq \overline{\mathcal{R}}$*.*

As a consequence, a pair (X, Y) is an object-oriented concept in the context $\mathcal{R}$ also means that $\overline{X} \times Y$ is a maximal subset of $\overline{\mathcal{R}}$, or equivalently $\overline{\overline{X} \times Y}$

is a minimal subset of $\mathcal{O} \times \mathcal{P}$ that contains $\mathcal{R}$, where $\overline{\overline{X} \times Y}$ expresses the material implication $\overline{Y} + X$ where $X + Y = \overline{\overline{X} \times \overline{Y}}$ (alternatively, objects outside X do not have properties in Y). It means that properties in Y may be possessed only by objects in X. It is consistent with the terminology "object-oriented concepts". Clearly this $N - \Pi$ connection comes down to checking for maximal rectangles $\overline{X} \times Y$ of 0's in matrix R, which ensures minimal relations of the form $\overline{Y} + X$ containing $\mathcal{R}$.

5.2. Minimal base of disjunctive attribute implications

The notions of minimal base [1, 11, 26, 29] induced by a formal context can be easily translated into the object-oriented concept lattice framework. This subsection introduces such notions and various related results.

The first definition introduces a notion of soundness for a set of disjunctive implications.

Definition 15. *Given a context $\mathcal{C} = (\mathcal{O}, \mathcal{P}, \mathcal{R})$, an arbitrary set of disjunctive implications is called* disjunctive implicational system. *When the context $\mathcal{C}$ satisfies every disjunctive implication in a disjunctive implicational system $\mathcal{D}$, i.e., $\mathcal{D}$ is a set of disjunctive implications of the context $\mathcal{C}$, then we say that $\mathcal{D}$ is* sound *for context $\mathcal{C}$ (or $\mathcal{C}$-sound for short).*

Next, the notion of (semantic) deduction for conjunctive rules, after the corresponding definition in [26], is adapted to the disjunctive framework.

Definition 16. *A disjunctive implication $A \mapsto B$* follows *(*semantically*) from a disjunctive implicational system $\mathcal{D}$ if each subset of $\mathcal{P}$ respecting $\mathcal{D}$ also respects $A \mapsto B$.*

More notions can be defined for disjunctive implicational systems.

Definition 17. *Given a context $\mathcal{C} = (\mathcal{O}, \mathcal{P}, \mathcal{R})$, we say that a disjunctive implicational system $\mathcal{D}$ is:*

- closed *if every implication following from $\mathcal{D}$ is already contained in $\mathcal{D}$.*
- complete *for context $\mathcal{C}$ ($\mathcal{C}$-complete, for short) if it is $\mathcal{C}$-sound and every implication of $\mathcal{C}$ follows from $\mathcal{D}$.*[12]

[12] What we call $\mathcal{C}$-soundness and $\mathcal{C}$-completeness should not be confused with soundness and completeness in the logical sense of equivalence between syntactic and semantic deduction, e.g., in CLAI or DLAI.

- minimal *or* non-redundant*, if none of the disjunctive implications* $A \mapsto B$*, follows from* $\mathcal{B} \setminus \{A \mapsto B\}$*; i.e., there is a subset of* $\mathcal{P}$ *that satisfies* $\mathcal{B} \setminus \{A \mapsto B\}$ *but does not satisfy* $A \mapsto B$*.*

The use of minimal bases is fundamental in order to provide a compact description of the information contained in a context.

Definition 18. *An implicational system* $\mathcal{D}$ *is a* minimal disjunctive base *for a formal context* $\mathcal{C} = (\mathcal{O}, \mathcal{P}, \mathcal{R})$*, if it is a* $\mathcal{C}$*-sound,* $\mathcal{C}$*-complete and minimal disjunctive implicational system for* $\mathcal{C}$*.*

As in FCA, the first question is to ensure the existence of this kind of bases for every context. The next subsection follows the ideas of Guigues and Duquenne [29] for defining a standard disjunctive base.

5.3. Disjunctive pseudo-intents

Pseudo-intents play an important role in FCA for computing conjunctive implications of any context $(\mathcal{O}, \mathcal{P}, \mathcal{R})$. They were defined recursively in [26] as follows.

Definition 19. *A subset of attributes* $A \subseteq \mathcal{P}$ *is called* pseudo-intent *of the context* $(\mathcal{O}, \mathcal{P}, \mathcal{R})$ *if* $A^{\downarrow\uparrow} \neq A$ *and* $B^{\downarrow\uparrow} \subset A$ *holds for every pseudo-intent* $B \subseteq \mathcal{P}$*, such that* $B \subset A$*.*

One of the first attribute bases is the well-known *Duquenne-Guigues base* [29], often simply called the *stem base*, consisting of the set of attribute implications

$$\mathcal{L} = \{A \Rightarrow (A^{\downarrow\uparrow} \setminus A) \mid A \text{ is a pseudo-intent}\}.$$

Theorem 1 ([29]). *The Duquenne-Guigues base is non-redundant and* $\mathcal{C}$*-complete.*

Example 5. *The Duquenne-Guigues base of the context* $\mathcal{C}$ *in Example 1 (Table 1), removing the redundant information in the consequent, is the following:*

$$\begin{array}{rcl} a_2(\alpha) & \rightarrow & a_3(\alpha) \wedge a_4(\alpha) \\ a_4(\alpha) & \rightarrow & a_3(\alpha) \\ a_1(\alpha) \wedge a_3(\alpha) & \rightarrow & a_2(\alpha) \wedge a_4(\alpha) \\ a_3(\alpha) \wedge a_5(\alpha) & \rightarrow & a_4(\alpha) \end{array}$$

Explain how to get the rules??? □

From the relationship between FCA and object-oriented concept lattices [31, 32], a suitable notion of disjunctive pseudo-intent is presented next.

Definition 20. *Let $(\mathcal{O}, \mathcal{P}, \mathcal{R})$ be a context, a subset of attributes $A \subseteq \mathcal{P}$ is an* object-oriented pseudo-intent, *if $A^{\downarrow^{\Pi}\uparrow_N} \neq A$ and $B^{\downarrow^{\Pi}\uparrow_N} \subset A$, for all object-oriented pseudo-intents $B \subseteq \mathcal{P}$, with $B \subset A$.*

Next, we will show that the previous definition is really suitable, namely, it is a dual definition of the original one. Before that, the following technical lemma is introduced.

Lemma 3. *Given a context $(\mathcal{O}, \mathcal{P}, \mathcal{R})$ and $A \subseteq \mathcal{P}$, we have that*

$$A^{\downarrow^{\Pi}\uparrow_N} \;=\; (A^{\neg})^{\bar{\downarrow}\bar{\uparrow}\neg}. \tag{7}$$

Proof. From Lemma 2 we obtain that $A^{\downarrow^{\Pi}\uparrow_N} = \left(\overline{\overline{(A^{\neg})^{\bar{\downarrow}}}} \right)^{\bar{\uparrow}\neg} = (A^{\neg})^{\bar{\downarrow}\bar{\uparrow}\neg}$ □

Proposition 6. *Given a context $\mathcal{C} = (\mathcal{O}, \mathcal{P}, \mathcal{R})$ and its dual $\overline{\mathcal{C}} = (\mathcal{O}, \mathcal{P}^{\neg}, \overline{\mathcal{R}})$, we have that $A \subseteq \mathcal{P}$ is an object-oriented pseudo-intent of $\mathcal{C}$ if and only if $A^{\neg} \subseteq \mathcal{P}^{\neg}$ is a pseudo-intent of $\overline{\mathcal{C}}$.*

Proof. The computation of both pseudo-intents and object-oriented pseudo-intents is similar. It begins from the empty set, from which new (object-oriented) pseudo-intents are computed. The proof will be done by induction in the computation of the (object-oriented) pseudo-intents and Lemma 3.

The basic case is when only one (object-oriented) pseudo-intent is computed. In this case, the result straightforwardly arises from Lemma 3. For example, if this set is the empty set (which is the first set considered by the algorithm in page 85 of [26]), we clearly have that $\varnothing \subseteq \mathcal{P}$ is an object-oriented pseudo-intent of $\mathcal{C}$ and $\varnothing^{\neg} \subseteq \mathcal{P}^{\neg}$ is a pseudo-intent of $\overline{\mathcal{C}}$. Hence, in this elementary case, the theorem holds.

Now, assume the property is satisfied for a subset of n (object-oriented) pseudo-intents. Therefore, we have a set of object-oriented pseudo-intents $\{B_1, \ldots, B_n\}$ of $\mathcal{C}$ and the set of pseudo-intents $\{B_1^{\neg}, \ldots, B_n^{\neg}\}$ of $\overline{\mathcal{C}}$.

Let $A \subseteq \mathcal{P}$ be the new computed object-oriented pseudo-intent of $\mathcal{C}$. Hence, we have that $A^{\bar{\downarrow}^{\Pi}\bar{\uparrow}_N} \neq A$ and so, $A \neq \left((A^{\neg})^{\bar{\downarrow}\bar{\uparrow}}\right)^{\neg}$ holds by Equation 7. Applying the properties of $\neg$, we obtain that $A^{\neg} \neq (A^{\neg})^{\bar{\downarrow}\bar{\uparrow}}$.

Now, we need to prove that $(B_i^{\neg})^{\bar{\downarrow}\bar{\uparrow}} \subset A^{\neg}$, for every pseudo-intent $B_i^{\neg} \subset A^{\neg}$, with $i \in \{1, \ldots, n\}$. If $B_i^{\neg} \subset A^{\neg}$, by the induction hypothesis, we have that B_i is an object-oriented pseudo-intent in the set $\{B_1, \ldots, B_n\}$ and $B_i \subset A$. Therefore, as A is an object-oriented pseudo-intent, then we have, by Definition 20, that $B_i^{\bar{\downarrow}^{\Pi}\bar{\uparrow}_N} \subset A$ and, by Equation (7), we have

$$\left((B_i^{\neg})^{\bar{\downarrow}\bar{\uparrow}}\right)^{\neg} = B_i^{\bar{\downarrow}^{\Pi}\bar{\uparrow}_N} \subset A$$

which is equivalent to $(B_i^{\neg})^{\bar{\downarrow}\bar{\uparrow}} \subset A^{\neg}$.

The converse implication is proved analogously. □

As a consequence, the usual algorithms developed to compute pseudo-intents, such as Next Closure [26] can be used to compute object-oriented pseudo-intents.

Example 6. *Using* Next Closure *on the context $\overline{\mathcal{C}}$ associated with Table 3, we have that the set of pseudo-intents are:*

$$\{\{\bar{a}_3\}, \{\bar{a}_4\}, \{\bar{a}_1, \bar{a}_5\}, \{\bar{a}_2, \bar{a}_5\}, \{\bar{a}_1, \bar{a}_2, \bar{a}_4\}\}$$

Thus, by Proposition 6, the set of object-oriented pseudo-intents of the original context $\mathcal{C}$ given in Example 1 is:

$$\{\{a_3\}, \{a_4\}, \{a_1, a_5\}, \{a_2, a_5\}, \{a_1, a_2, a_4\}\}$$

□

As a consequence of the definition of object-oriented pseudo-intent, Proposition 6, and Theorem 1, the following result is obtained.

Theorem 2. *The set of implications*

$$\mathcal{L}_o = \left\{(A^{\downarrow^{\Pi}\uparrow_N} \setminus A) \mapsto A \mid A \textit{ is an object-oriented pseudo-intent}\right\}$$

is non-redundant and $\mathcal{C}$-complete, and it is called disjunctive pseudo-intent base *of disjunctive implications.*

Proof. Given an object-oriented pseudo-intent $A \subseteq \mathcal{P}$ and the implication $(A^{\downarrow^{\Pi}\uparrow_N} \setminus A) \mapsto A$, by Corollary 1 and Lemma 2, it is valid in $\mathcal{C}$ if and only if $A^{\neg} \Rightarrow (A^{\neg})^{\downarrow\uparrow} \setminus A^{\neg}$ is valid in $\overline{\mathcal{C}}$. Moreover, by Proposition 6, $A^{\neg}$ is a pseudo-intent in $\overline{\mathcal{C}}$. As a consequence,

$$(A^{\downarrow^{\Pi}\uparrow_N} \setminus A) \mapsto A \in \mathcal{L}_o \quad \text{if and only if} \quad A^{\neg} \Rightarrow ((A^{\neg})^{\downarrow\uparrow} \setminus A^{\neg}) \in \overline{\mathcal{L}}$$

where $\overline{\mathcal{L}} = \{A^{\neg} \Rightarrow ((A^{\neg})^{\downarrow\uparrow} \setminus A^{\neg}) \mid A \text{ is a pseudo-intent of } \overline{\mathcal{C}}\}$, which clearly is non-redundant and $\mathcal{C}$-complete by Theorem 1. Thus, $\mathcal{L}_o$ is also non-redundant and $\mathcal{C}$-complete. □

Notice that this result extends Proposition 7 and the appendix given in [2], and straightforwardly proves the minimality of the set $\mathcal{L}_o$.

Example 7. *From Theorem 2 and Proposition 6, the following set of disjunctive implications is the disjunctive pseudo-intent base of the context $\mathcal{C}$ given in Example 1.*

$$\begin{aligned} a_2(\alpha) \vee a_4(\alpha) &\rightarrow a_3(\alpha) \\ a_2(\alpha) &\rightarrow a_4(\alpha) \\ a_2(\alpha) \vee a_4(\alpha) &\rightarrow a_1(\alpha) \vee a_5(\alpha) \\ a_1(\alpha) \vee a_4(\alpha) &\rightarrow a_2(\alpha) \vee a_5(\alpha) \\ a_5(\alpha) &\rightarrow a_1(\alpha) \vee a_2(\alpha) \vee a_4(\alpha) \end{aligned}$$

Note that the second rule could be also considered as a conjunctive rules, which does not contradict the information given in Example 7. □

6. Toward hybrid conjunctive-disjunctive implications

In this section, we outline a natural extension of conjunctive and disjunctive attribute implications, and show how to define their satisfiability with respect to a context in terms of operators from FCA.

It is tempting to generalize the syntaxes of CLAI and DLAI so as to use both disjunctions and conjunctions in the language. We can consider a generalized language LAI containing rules of the form $\varphi \rightarrow \psi$ where φ be a propositional formula without negation, in disjunctive normal form (e.g., $\vee_{i=1}^{n}(\wedge_{k=1}^{n_i} a_{ik}(x)))$, and ψ be a propositional formula without negation, in conjunctive normal form (e.g., $\wedge_{j=1}^{m}(\vee_{\ell=1}^{m_j} b_{j\ell}(x)))$.

Such rules are thus of the form $\mu_1 \vee \cdots \vee \mu_n \rightarrow \kappa_1 \wedge \ldots \wedge \kappa_m$ where $\mu_i = \wedge_{k=1}^{n_i} a_{ik}(x)$, and $\kappa_j = \vee_{\ell=1}^{m_j} b_{j\ell}(x)$. They can be interpreted with respect to a formal context $\mathcal{C}$ as follows. Let A_i be the set of attributes appearing in μ_i, and B_j the set of attributes appearing in κ_j. Let $\mathbb{A} = \{A_1, \ldots, A_m\}$ and $\mathbb{B} = \{B_1, \ldots, B_m\}$.

Definition 21. *A* hybrid attribute implication *associated with a LAI rule $\varphi \to \psi$ defined as above is of the form $\mathbb{A} \twoheadrightarrow \mathbb{B}$.*

The validity of a hybrid attribute implication $\mathbb{A} \twoheadrightarrow \mathbb{B}$ with respect to a context $\mathcal{C}$ can then be defined:

Definition 22. *$\mathcal{C}$ satisfies $\mathbb{A} \twoheadrightarrow \mathbb{B}$ if and only if for all objects x in $\mathcal{O}$, there exists a subset of attributes $A_i \in \mathbb{A}$ such that if x satisfies all attributes in A_i, then x satisfies at least one attribute in each $B_j \in \mathcal{B}$.*

Under this definition, and as in classical logic, every hybrid attribute implication can be divided into simpler implications. For example, the validity of implication $\mathbb{A} \twoheadrightarrow \mathbb{B}^1 \cup \mathbb{B}^2$ is equivalent to the implications $\mathbb{A} \twoheadrightarrow \mathbb{B}^1$ and $\mathbb{A} \twoheadrightarrow \mathbb{B}^2$. Moreover, the validity of $\mathbb{A}^1 \cup \mathbb{A}^2 \twoheadrightarrow \mathbb{B}$ is equivalent to the implications $\mathbb{A}^1 \twoheadrightarrow \mathbb{B}$ and $\mathbb{A}^2 \twoheadrightarrow \mathbb{B}$.

Given the above semantics with respect to a context, these hybrid rules are equivalent to a set of more elementary rules we call conjunctive-disjunctive implications (cd-implications for short). Namely:

Proposition 7. *$\mathcal{C}$ satisfies $\mathbb{A} \twoheadrightarrow \mathbb{B}$ if and only if $\mathcal{C}$ satisfies $\{A_i\} \twoheadrightarrow \{B_j\}$ for all $i \in \{1, \ldots n\}$, $j \in \{1, \ldots m\}$.*

A cd-rule $\{A\} \twoheadrightarrow \{B\}$ can be written as $A_c \twoheadrightarrow B_d$ for short, where the body set is understood as a conjunction of attributes, and the head as a disjunction of attributes. It corresponds in the language LAI to a rule of the form

$$a_1(\alpha) \wedge \cdots \wedge a_m(\alpha) \to b_1(\alpha) \vee \cdots \vee b_r(\alpha)$$

where $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_r\}$. Extending the semantics of CLAI and DLAI, the satisfaction of the cd-implication $A_c \twoheadrightarrow B_d$ by a context $\mathcal{C}$ can be defined by:

$$\begin{aligned} 1 &= \hat{I}_{\mathcal{C}}(a_1(\alpha) \wedge \cdots \wedge a_m(\alpha) \to b_1(\alpha) \vee \cdots \vee b_r(\alpha)) \\ &= \min\{\hat{I}_{\mathcal{C}}(a_1(x) \wedge \cdots \wedge a_m(x) \to b_1(x) \vee \cdots \vee b_r(x)) \mid x \in \mathcal{O}\} \end{aligned}$$

which is equivalent to the following inequality for all $x \in \mathcal{O}$

$$\min\{I_{\mathcal{C}}(a_1(x)), \ldots, I_{\mathcal{C}}(a_m(x))\} \leq \max\{I_{\mathcal{C}}(b_1(x)), \ldots, I_{\mathcal{C}}(b_r(x))\} \quad (8)$$

The next result describes in terms of FCA operators when a given context satisfies a cd-implication.

Theorem 3. *A context $\mathcal{C}$ satisfies a cd-implication $A_c \twoheadrightarrow B_d$ if and only if $A^{\downarrow} \subseteq B^{\downarrow^{\Pi}}$.*

PROOF. Let $A = \{a_1, \ldots, a_m\}$, $B = \{b_1, \ldots, b_r\}$. Assume $x \in A^{\downarrow}$, we have that $(x, a) \in \mathcal{R}$, for all $a \in A$. Consequently, $\min\{I_{\mathcal{C}}(a_1(x)), \ldots, I_{\mathcal{C}}(a_m(x))\} = 1$, and by Equation (8) we obtain that there exists $b \in B$ such that $I_{\mathcal{C}}(b(x)) = 1$, that is, $(x, b) \in \mathcal{R}$, which shows that $x \in B^{\downarrow^{\Pi}}$.

The other implication follows similarly. $\square$

Note that the cd-implication $A_c \twoheadrightarrow B_d$ is non-informative if and only if $A \cap B \neq \varnothing$. Special cases of cd-implications are worth discussing, when A or B are the empty set or the whole set:[13]

- That $A_c \twoheadrightarrow \varnothing$ is valid in $\mathcal{C}$ means $A^{\downarrow} \subseteq \varnothing^{\downarrow^{\Pi}} = \varnothing$. Therefore, we get $A^{\downarrow} = \varnothing$ which means, if blank entries refer to the negation of properties, that properties in A are then mutually exclusive in the context.[14] In the epistemic view, $A^{\downarrow} = \varnothing$ only means that there is no object *known* to satisfy all properties in A, so that the contradiction inside A is only potential.

- $A_c \twoheadrightarrow \mathcal{P}_d$ is valid when $A^{\downarrow} \subseteq \mathcal{P}^{\downarrow^{\Pi}}$, which always holds, since this rule is not informative.

- If $\varnothing \twoheadrightarrow B_d$ is valid in $\mathcal{C}$, then $\varnothing^{\downarrow} = \mathcal{O} \subseteq B^{\downarrow^{\Pi}}$, i.e., each object has at least one property in B, which is equivalent to write $\top \to \vee_{b \in B} b(\alpha)$ in LAI.

[13]We will avoid writing $\varnothing$ with the subscripts c or d.

[14]This is very different from what the attribute implication $A \Rightarrow \varnothing$ means (i.e., it always holds and is encoded as a tautology $\wedge_{a \in A} a(\alpha) \to \top$ in CLAI; see subsection 3.1). On the contrary, in LAI, $A_c \twoheadrightarrow \varnothing$ is encoded as $\wedge_{a \in A} a(\alpha) \to \bot$.

- $\mathcal{P}_c \twoheadrightarrow B_d$ is valid in normalized contexts, since $\mathcal{P}^{\downarrow} = \varnothing \subseteq B^{\downarrow^{\Pi}}$, which is a tautology, and can be written as $\perp \rightarrow \vee_{b \in B} b(\alpha)$ in LAI. It is then non-informative.

Next, a particular context will be considered from which different cd-implications will be derived.

Example 8. *Let $\mathcal{C} = (\mathcal{O}, \mathcal{P}, \mathcal{R})$ be a formal context where the set $\mathcal{O} = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $\mathcal{P} = \{a_1, a_2, a_3, a_4, a_5\}$ and the relation $\mathcal{R}$ is represented in Table 5.*

Table 5: Relation $\mathcal{R}$ of the formal context $\mathcal{K}$.

$\mathcal{R}$	a_1	a_2	a_3	a_4	a_5
x_1	×		×		×
x_2	×			×	×
x_3			×		
x_4	×	×	×	×	
x_5		×	×		×
x_6		×	×		×

We have that $\mathcal{C}$ does not satisfy the implication

$$a_3(\alpha) \wedge a_5(\alpha) \rightarrow a_1(\alpha) \vee a_4(\alpha)$$

because: $\{a_3, a_5\}^{\downarrow} = \{x_1, x_5, x_6\} \not\subseteq \{x_1, x_2, x_4\} = \{a_1, a_2\}^{\downarrow^{\Pi}}$. If we consider now the cd-implication:

$$a_3(\alpha) \wedge a_5(\alpha) \rightarrow a_1(\alpha) \vee a_2(\alpha)$$

we have that $\mathcal{C}$ satisfies this implication since $\{a_3, a_5\}^{\downarrow} = \{x_1, x_5, x_6\} \subseteq \{x_1, x_2, x_4, x_5, x_6\} = \{a_1, a_2\}^{\downarrow^{\Pi}}$. Moreover, this implication cannot be obtained from either Duquenne-Guigues base or disjunctive pseudo-intent base, as we show next. The Duquenne-Guigues base of the context $\mathcal{C}$, removing the redundant information in the consequent, is the following:

$$\begin{aligned} a_2(\alpha) &\rightarrow a_3(\alpha) \\ a_4(\alpha) &\rightarrow a_1(\alpha) \\ a_1(\alpha) \wedge a_2(\alpha) \wedge a_3(\alpha) &\rightarrow a_4(\alpha) \\ a_1(\alpha) \wedge a_3(\alpha) \wedge a_4(\alpha) &\rightarrow a_2(\alpha) \end{aligned}$$

On the other hand, the disjunctive pseudo-intent base of the context $\mathcal{C}$ is:

←The 4th rule is strange as a_2 appears on both sides

$$\begin{array}{rcl} a_4(\alpha) & \to & a_1(\alpha) \\ a_2(\alpha) & \to & a_3(\alpha) \\ a_1(\alpha) \vee a_4(\alpha) & \to & a_2(\alpha) \vee a_5(\alpha) \\ a_1(\alpha) \vee a_2(\alpha) & \to & a_4(\alpha) \vee a_5(\alpha) \\ a_1(\alpha) \vee a_5(\alpha) & \to & a_2(\alpha) \vee a_3(\alpha) \vee a_4(\alpha) \\ a_5(\alpha) & \to & a_1(\alpha) \vee a_2(\alpha) \vee a_4(\alpha) \end{array}$$

THE 4th RULE SHOULD BE $a_1(\alpha) \to a_2(\alpha) \vee a_3(\alpha) \vee a_4(\alpha)$???

Notice that, these bases have two common implications. Furthermore, there is no direct inference for obtaining the cd-implication:

$$a_3(\alpha) \wedge a_5(\alpha) \to a_1(\alpha) \vee a_2(\alpha)$$

from the union of these bases. □

Thus, these results indicate that one may extract more information from a formal context than the one provided by conjunctive and disjunctive rules independently. There is room for further research, namely

- finding a methodology to efficiently extract cd-implications, and especially a minimal base of them that is $\mathcal{C}$-sound and $\mathcal{C}$-complete.

- finding a suitable proof system for reasoning with cd-implications.

Note that the language logic of cd-implications LAI is again a fragment of the one of classical logic. In particular, as seen above, for complete Boolean contexts, LAI includes connectives $\wedge, \vee, \to$ and both constants $\perp$ and $\top$. Note that because LAI can encode the rule $\wedge_{a \in A} a(\alpha) \to \perp$, it can encode classical logic statements such as $a(\alpha) \to \neg b(\alpha)$ in the form $a(\alpha) \wedge b(\alpha) \to \perp$, even if LAI does not include the negation symbol (negation statements can be expressed in the form $a(\alpha) \to \perp$).

However, while the method of extraction of disjunctive attribute implications can take advantage of standard methods applied to the dual view of a formal context (entries are then interpreted in terms of "known-to-be-false/possibly true" rather than "known-to-be-true/unknown"), it is clear that disjunctive rules as well as cd-implications also make sense in the original understanding of a formal context in terms of "known-to-be-true/unknown", as they are arguably weaker than original attribute implications encoded as Horn clauses. However, extreme cd-implications of the form $a(\alpha) \wedge b(\alpha) \to \bot$ are questionable under the epistemic view of a formal context (it is not a Horn clause anyway) and only make sense if crosses mean true and blanks mean false. Indeed a formal context, in its epistemic view in terms of known/unknown entries can never express conflicting information between pieces of positive knowledge, since in that case $A^{\downarrow}$ is only a lower approximation of the set of objects that satisfy all properties in A. We may also delete cd-rules of the form $A_c \twoheadrightarrow \varnothing$ as non supported by any example.

The full LAI should arguably be equivalent to classical logic, with the $\bot$ symbol included (for context entries interpreted in terms of true/false). The above discussion suggests that another variant of the LAI formalism, without the $\bot$ symbol (for epistemic contexts where sure falsity is not expressed), would also be worth studying further.

7. Conclusions and future work

In this paper, we have proposed a variant and a generalisation of standard attribute implications in formal context analysis, when the Boolean entries have an epistemic flavor in terms of known/unknown to be true. We have shown that using the dual context, we can define disjunctive attribute implications, which enable us to model the fact that one of several properties hold for all objects satisfying another given property. We have also developed the counterpart of known approaches for extracting such rules, based on the disjunctive counterparts of formal concepts and pseudo-intents. We have defined two logics of attribute implications for conjunctive and disjunctive ones respectively, and outlined a logic where both conjunctions and disjunctions appear, and where contradiction appears in the language or not depending on the interpretation of contexts (ontic or epistemic). We provided evidence to show that such logics are in agreement with classical first order logic, albeit with a restricted language.

This work differs from frameworks considered in [5, 34, 36] which also use negative attributes. In such papers, the importance of simultaneously using positive and negative attributes in the *same* context has been highlighted, which departs from our approach. For example, in [36], the authors include an example in which the relation $a \to \bar{b}$ between the attributes a and b is obtained, when a mixed context is used. It is the result of putting together a classical ~~standard~~ context where blanks stand for negative information, and its complement, in order to obtain standard attribute implications involving negative attributes.

Extracting rules from such mixed contexts under complete information may require the explicit use of logical negation in the attribute implication language, which may lead to a framework with the same expressiveness as classical logic and so, obtaining all relationships among attributes of a formal context with fully informed objects. All developments and algorithms given in classical logic could then be used to design computational procedures for making deductions from datasets interpreted as formal contexts. However, the question of knowing whether standard attribute implications obtained from a mixed context are more expressive than sets of cd-implications obtained from a classical context is open.

In the incomplete setting, where contexts are interpreted as positive knowledge, we may consider using, instead of CLAI and its variants, a fragment of a modal language (with necessity modalities prefixing positive atoms only) that could be a special case of the frameworks of Holzer [30] and Obiedkov [33] for more general epistemic contexts involving both positive and negative information. Its connection to the modal-like logic of incomplete information of Banerjee and Dubois [4] with possibilistic semantics would also be worth investigating.

Appendix

There are two dual versions of Galois connection [12, 13], the antitone and the isotone ones. The original notion is usually associated with the antitone version.

Definition 23. *Let* $(P_1, \leq_1)$ *and* $(P_2, \leq_2)$ *be posets, and* ${}^\downarrow\colon P_1 \to P_2$, ${}^\uparrow\colon P_2 \to P_1$ *be two mappings, the pair* $({}^\uparrow, {}^\downarrow)$ *forms an* (antitone) Galois connection *between* P_1 *and* P_2 *if and only if:*

1. ${}^\uparrow$ *and* ${}^\downarrow$ *are order-reversing.*
2. $x \leq_1 x^{\downarrow\uparrow}$ *for all* $x \in P_1$.
3. $y \leq_2 y^{\uparrow\downarrow}$ *for all* $y \in P_2$.

The isotone version is also called adjunction [11, 13].

Definition 24. *Let* $(P_1, \leq_1)$ *and* $(P_2, \leq_2)$ *be posets, and* ${}^{\downarrow}\colon P_1 \to P_2$*,* ${}^{\uparrow}\colon P_2 \to P_1$ *be two mappings, the pair* $({}^{\uparrow}, {}^{\downarrow})$ *forms an* isotone Galois connection *between* P_1 *and* P_2 *if and only if:*

1. ${}^{\uparrow}$ *and* ${}^{\downarrow}$ *are order-preserving.*
2. $x \leq_1 x^{\downarrow\uparrow}$ *for all* $x \in P_1$*.*
3. $y^{\uparrow\downarrow} \leq_2 y$ *for all* $y \in P_2$*.*

There are arguments for both versions, although the difference is not significant at the theoretical level, since we can pass from one to the other one swapping a poset by its dual, for example, going from $(P_2, \leq_2)$ to $(P_2, \leq_2^{\partial})$, where $\leq_2^{\partial}$ is the opposite of the ordering $\leq_2$, that is, given $y, z \in P_2$, we say that $y \leq_2^{\partial} z$, if $z \leq_2 y$, which is also written as $y \geq_2 x$.

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