# ON FORMAL CONCEPTS OF RANDOM FORMAL CONTEXTS 


#### Abstract

In formal concept analysis, it is well-known that the number of formal concepts can be exponential in the worst case. To analyze the average case, we introduce a probabilistic model for random formal contexts and prove that the average number of formal concepts has a superpolynomial asymptotic lower bound.


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## 1. Introduction

How many formal concepts does a formal context have? This is one of the fundamental problems in the theory of formal concept analysis - an application area of lattice theory which originates from Wille [6] to support data analysis and knowledge processing. In the graph-theoretic language, the problem asks the number of maximal bicliques of bipartite graphs. The problem of determining the number of formal concepts is proved to be \#P-complete by Kuznetsov [5, Theorem 1]. Even though the counting problem is hard in general, it is of interest to get a general idea of how large the number is.

It is well-known that the number of formal concepts can be exponential in the worst case, and it can be one in the best case. Such extremal formal contexts are obtained from contranomial scales and formal contexts defined by the empty relation. Since these examples appear to be highly atypical, it is natural to study the number of formal concepts in the average case. To this end, we introduce random formal contexts (Definition 3.2) and present an exact formula for the average number of formal concepts (Proposition 4.1). Lastly, we prove that the average

[^0]number of formal concepts has a superpolynomial asymptotic lower bound (Theorem 5.1), which is the main result of this article. Our theorem and its proof help to understand why a "typical" formal context has numerous formal concepts.

## 2. Preliminaries

2.1. Formal concept analysis. We recall basic notions in formal concept analysis which can be found in the textbook by Ganter and Wille [2, Chapter 1]. A (formal) context is defined to be a triple $K=(G, M, I)$ consists of two sets $G, M$, and a subset $I$ of $G \times M$. An element $g$ of $G$ is called an object, an element $m$ of $M$ is called an attribute, and $I$ is called the incidence relation of the context $K$. An object $g$ is said to have an attribute $m$ if a pair $(g, m)$ belongs to $I$. A context is often represented by a cross table whose rows and columns are indexed by objects and attributes, and the incidence relation is indicated by crosses as in Figure 1.


Figure 1. The cross table of a context.

Let $A$ be a set of objects and let $B$ be a set of attributes. The set of attributes that all objects in $A$ have in common is denoted by

$$
A^{\prime}=\bigcap_{g \in A}\{m \in M \mid(g, m) \in I\} .
$$

Similarly, the set of objects that have all attributes in $B$ is denoted by

$$
B^{\prime}=\bigcap_{m \in B}\{g \in G \mid(g, m) \in I\}
$$

A pair $(A, B)$ is defined to be a (formal) concept if $A^{\prime}=B$ and $B^{\prime}=A$; the first and second components are called the extent and intent of the concept. The set of concepts of a context $K$ is denoted by $\mathfrak{B}(K)$.
2.2. Asymptotic analysis. We recall two useful notations in asymptotic analysis: the little-oh notation and the Vinogradov notation. Let $\left(x_{n}\right)$ and $\left(y_{n}\right)$ be real sequences. For an arbitrary positive real number $\varepsilon$, if $\left|x_{n}\right|<\varepsilon\left|y_{n}\right|$ for sufficiently large $n$, then we write $x_{n}=o\left(y_{n}\right)$. If there is some positive real number $\gamma$ satisfying $\left|x_{n}\right| \leq \gamma\left|y_{n}\right|$ for sufficiently large $n$, then we write $y_{n} \gg x_{n}$.

## 3. Random contexts

In this section, we introduce a probabilistic model for random contexts. Although we provide its measure-theoretic formalization later for completeness, the randomness we consider might be best described by the following informal manner.

Let $n$ be a positive integer and take an $n$-set, say $U=\{1,2, \ldots, n\}$. For each element of $U$, we regard it as an object with probability $p$ and as an attribute with probability $1-p$, independently. Subsequently, for each pair $(g, m)$ of an object $g$ and an attribute $m$, we regard an object $g$ has an attribute $m$ with probability $q$, independently. We add that the probabilities $p$ and $q$ are not necessarily constants like $p=1 / 2$ and may be functions of $n$ like $q=1-1 / n$.

A similar probabilistic model with a fixed number of objects and attributes is used by Kovács in $[4, \S 2.1]$ to estimate the number of concepts. Those who familiar with random graph theory would instantly recognize that this is very much alike to the model for binomial random graphs [3, p. 2], which is also known as the Erdős-Rényi model. In this article, we content ourselves with this simplest model for random contexts. The readers may wish to skim through the next notation and definition if they are comfortable with this informal description of our probabilistic model.

Throughout this article, we use a convention to write random variables in bold. For basic concepts of probability theory, we refer the readers to a work by Bauer $[1$, Chapter I], for example.

Notation 3.1. Let $n$ be a positive integer and let $p$ and $q$ be real numbers belonging to the unit interval $[0,1]$. Set $U=\{1,2, \ldots, n\}$. Write $\Omega$ for the set of contexts $(G, M, I)$ with $G+M=U$ where + denotes the disjoint union. Define the probability measure $P=\kappa_{n, p, q}$ on the power set $2^{\Omega}$ by

$$
P\{(G, M, I)\}=p^{|G|}(1-p)^{|M|} q^{|I|}(1-q)^{|G \times M-I|} .
$$

The probability space $\left(\Omega, 2^{\Omega}, P\right)$ is our mathematical model for random contexts.
Definition 3.2. We call an $\Omega$-valued random variable $\boldsymbol{K}$ a random context and write $\boldsymbol{K} \sim \kappa_{n, p, q}$ if the distribution of $\boldsymbol{K}$ equals $\kappa_{n, p, q}$.

For a real-valued function $f$ on $\Omega$ and a random context $\boldsymbol{K}$, we write

$$
E(f \circ \boldsymbol{K})=\int f d P=\sum_{K \in \Omega} f(K) P\{K\}
$$

for the expectation.

## 4. Average number of concepts

Based on the notion of random contexts that is introduced in the previous section, we show an exact formula for the average number of concepts in this section.

Proposition 4.1. Let $\boldsymbol{K}$ be a random context with $\boldsymbol{K} \sim \kappa_{n, p, q}$. Then

$$
\begin{equation*}
E(|\mathfrak{B}(\boldsymbol{K})|)=\sum_{(a, b, c, d)}\binom{n}{a b c d} p^{a+c}(1-p)^{b+d} q^{a b}\left(1-q^{a}\right)^{d}\left(1-q^{b}\right)^{c} \tag{4.1}
\end{equation*}
$$

where the sum is taken over all non-negative integers with $a+b+c+d=n$.

Proof. Set $\boldsymbol{K}=(\boldsymbol{G}, \boldsymbol{M}, \boldsymbol{I})$. Let $A$ and $B$ be subsets of $U$. We write $\mathbf{1}_{\{(A, B) \in \mathfrak{B}(\boldsymbol{K})\}}$ for the indicator variable of an event that a pair $(A, B)$ is a concept of $\boldsymbol{K}$. By the linearity of expectation and the law of total probability, we may reduce the problem as

$$
\begin{aligned}
E(|\mathfrak{B}(\boldsymbol{K})|) & =\sum_{(A, B)} E\left(\mathbf{1}_{\{(A, B) \in \mathfrak{B}(\boldsymbol{K})\}}\right)=\sum_{(A, B)} P\{(A, B) \in \mathfrak{B}(\boldsymbol{K})\} \\
& =\sum_{(A, B, C, D)} P(\{(A, B) \in \mathfrak{B}(\boldsymbol{K})\} \cap\{\boldsymbol{G}=A+C\} \cap\{\boldsymbol{M}=B+D\})
\end{aligned}
$$

where the sums are taken over all tuples of subsets of $U$. Suppose that $(A, B, C, D)$ is an ordered partition of the set $U$. From the reduction, it is enough to show that

$$
\begin{aligned}
P(\{(A, B) \in \mathfrak{B}(\boldsymbol{K})\} & \cap\{\boldsymbol{G}=A+C\} \cap\{\boldsymbol{M}=B+D\}) \\
& =p^{|A+C|}(1-p)^{|B+D|} q^{|A \times B|}\left(1-q^{|A|}\right)^{|D|}\left(1-q^{|B|}\right)^{|C|}
\end{aligned}
$$

The cross table of a context in Figure 2 may help the readers to see why this claim holds. First, every element of $A+C$ must belong to $\boldsymbol{G}$ with probability $p^{|A+C|}$ (row


Figure 2. When $\{(A, B) \in \mathfrak{B}(\boldsymbol{K})\} \cap\{\boldsymbol{G}=A+C\} \cap\{\boldsymbol{M}=B+D\}$ occurs.
header), and every element of $B+D$ must belong to $\boldsymbol{M}$ with probability $(1-p)^{|B+D|}$ (column header). Second, every pair of $A \times B$ must belong to $I$ with probability $q^{|A \times B|}$ (upper-left corner). Next, every attribute in $D$ must not be shared by all objects in $A$ with probability $\left(1-q^{|A|}\right)^{|D|}$ (upper-right corner), and every object in $C$ must not have all attributes in $B$ with probability $\left(1-q^{|B|}\right)^{|C|}$ (lower-left corner). Last, the rest entries (lower-right corner) do not affect the occurrence of the event. The above argument establishes the claim and completes the proof.

## 5. Asymptotic lower bound

In this section, we study random contexts with constant probabilities $p=q=1 / 2$ in detail and prove that the average number of concepts has a superpolynomial asymptotic lower bound. The following is the main result of this article.

Theorem 5.1. Let $\left(\boldsymbol{K}_{n}\right)$ be a sequence of random contexts with $\boldsymbol{K}_{n} \sim \kappa_{n, \frac{1}{2}, \frac{1}{2}}$. Then

$$
E\left(\left|\mathfrak{B}\left(\boldsymbol{K}_{n}\right)\right|\right)>n^{\log n}
$$

for sufficiently large $n$. In particular, $E\left(\left|\mathfrak{B}\left(\boldsymbol{K}_{n}\right)\right|\right) \gg n^{\log n}$.

For a real number $x$, the integer part and fractional part of $x$ are denoted by $[x]$ and $\{x\}$. To obtain a lower bound for the average number of concepts of $\boldsymbol{K}_{n}$, we single out the specific term

$$
\begin{equation*}
t_{n}=\binom{n}{a_{n} b_{n} c_{n} d_{n}} p^{a_{n}+c_{n}}(1-p)^{b_{n}+d_{n}} q^{a_{n} b_{n}}\left(1-q^{a_{n}}\right)^{d_{n}}\left(1-q^{b_{n}}\right)^{c_{n}} \tag{5.1}
\end{equation*}
$$

in (4.1) for constant probabilities $p=q=1 / 2$ where

$$
\begin{align*}
& a_{n}=\left[\frac{\log n}{\log 2}\right], \quad b_{n}=\left[\frac{\log n}{\log 2}\right]+2\left\{\frac{n}{2}\right\}, \quad \text { and }  \tag{5.2}\\
& c_{n}=d_{n}=\left[\frac{n}{2}\right]-\left[\frac{\log n}{\log 2}\right]
\end{align*}
$$

Although this is just one term in the summation, it turns out to be large enough for our purpose. The asymptotic behavior of $t_{n}$ is described as follows.

Lemma 5.2. With notation in (5.1),

$$
\log t_{n}=\frac{\log ^{2} n}{\log 2}(1+o(1))
$$

To prove this asymptotic equivalence, we need some lemmas.
Lemma 5.3. With notation in (5.2),

$$
\log \binom{n}{a_{n} b_{n} c_{n} d_{n}}=n \log 2+2 \frac{\log ^{2} n}{\log 2}+o\left(\log ^{2} n\right)
$$

Proof. By the Stirling formula and the Taylor formula,

$$
\begin{aligned}
\log n!= & n \log n-n+o\left(\log ^{2} n\right), \\
\log a_{n}!= & \log \left(\frac{\log n}{\log 2}-\left\{\frac{\log n}{\log 2}\right\}\right)!=o\left(\log ^{2} n\right), \\
\log b_{n}!= & \log \left(\frac{\log n}{\log 2}-\left\{\frac{\log n}{\log 2}\right\}+2\left\{\frac{n}{2}\right\}\right)!=o\left(\log ^{2} n\right), \quad \text { and } \\
\log c_{n}!= & \log d_{n}!=\log \left(\frac{n}{2}-\left\{\frac{n}{2}\right\}-\frac{\log n}{\log 2}+\left\{\frac{\log n}{\log 2}\right\}\right)! \\
= & \log \left(\frac{n}{2}-\frac{\log n}{\log 2}+o(\log n)\right)! \\
= & \left(\frac{n}{2}-\frac{\log n}{\log 2}+o(\log n)\right) \log \left(\frac{n}{2}-\frac{\log n}{\log 2}+o(\log n)\right) \\
& -\left(\frac{n}{2}-\frac{\log n}{\log 2}+o(\log n)\right)+o(\log 2 n) \\
= & \left(\frac{n}{2}-\frac{\log n}{\log 2}\right) \log \left(\frac{n}{2}-\frac{\log n}{\log 2}+o(\log n)\right)-\frac{n}{2}+o\left(\log ^{2} n\right) \\
= & \left(\frac{n}{2}-\frac{\log n}{\log 2}\right)\left(\log n-\log 2-\frac{2}{n} \frac{\log n}{\log 2}+o\left(\frac{\log ^{2} n}{n}\right)\right)-\frac{n}{2}+o\left(\log ^{2} n\right) \\
= & \frac{1}{2} n \log n-\frac{1}{2}(1+\log 2) n-\frac{\log ^{2} n}{\log 2}+o\left(\log ^{2} n\right) .
\end{aligned}
$$

Therefore

$$
\begin{aligned}
& \log \binom{n}{a_{n} b_{n} c_{n} d_{n}} \\
& \quad=n \log n-n-2\left(\frac{1}{2} n \log n-\frac{1}{2}(1+\log 2) n-\frac{\log ^{2} n}{\log 2}\right)+o\left(\log ^{2} n\right) \\
& \quad=n \log 2+2 \frac{\log ^{2} n}{\log 2}+o\left(\log ^{2} n\right)
\end{aligned}
$$

Lemma 5.4. With notation in (5.2),

$$
\left|\log \left(1-2^{-a_{n}}\right)^{d_{n}}\left(1-2^{-b_{n}}\right)^{c_{n}}\right|<2
$$

Proof. We may assume that $n>2$. Note that $c_{n}=d_{n} \leq n / 2$ and

$$
1-2^{-b_{n}} \geq 1-2^{-a_{n}}=1-2^{-\frac{\log n}{\log 2}+\left\{\frac{\log n}{\log 2}\right\}}=1-\frac{2^{\left\{\frac{\log n}{\log 2}\right\}}}{n}>1-\frac{2}{n}
$$

Hence

$$
\begin{aligned}
& \left|\log \left(1-2^{-a_{n}}\right)^{d_{n}}\left(1-2^{-b_{n}}\right)^{c_{n}}\right| \\
& \quad=-d_{n} \log \left(1-2^{-a_{n}}\right)-c_{n} \log \left(1-2^{-b_{n}}\right)<-n \log \left(1-\frac{2}{n}\right) \leq 2
\end{aligned}
$$

Proof of Lemma 5.2. By Lemmas 5.3 and 5.4,

$$
\begin{aligned}
\log t_{n}= & \log \binom{n}{a_{n} b_{n} c_{n} d_{n}}-\left(\frac{n}{2}-\left\{\frac{n}{2}\right\}\right) \log 2-\left(\frac{n}{2}+\left\{\frac{n}{2}\right\}\right) \log 2 \\
& -\left(\frac{\log n}{\log 2}-\left\{\frac{\log n}{\log 2}\right\}\right)\left(\frac{\log n}{\log 2}-\left\{\frac{\log n}{\log 2}\right\}+2\left\{\frac{n}{2}\right\}\right) \log 2 \\
& +\log \left(1-2^{-a_{n}}\right)^{d_{n}}\left(1-2^{-b_{n}}\right)^{c_{n}} \\
= & n \log 2+2 \frac{\log ^{2} n}{\log 2}-\frac{n}{2} \log 2-\frac{n}{2} \log 2-\frac{\log ^{2} n}{\log 2}+o\left(\log ^{2} n\right) \\
= & \frac{\log ^{2} n}{\log 2}+o\left(\log ^{2} n\right)=\frac{\log ^{2} n}{\log 2}(1+o(1))
\end{aligned}
$$

Proof of Theorem 5.1. By Proposition 4.1, we have $E\left(\left|\mathfrak{B}\left(\boldsymbol{K}_{n}\right)\right|\right) \geq t_{n}$. Set $\varepsilon=$ $1-\log 2=0.306 \cdots$. It follows from Lemma 5.2 that

$$
\log E\left(\left|\mathfrak{B}\left(\boldsymbol{K}_{n}\right)\right|\right) \geq \log t_{n}>\frac{\log ^{2} n}{\log 2}(1-\varepsilon)=\log ^{2} n
$$

for sufficiently large $n$, which proves the theorem.

| $n$ | $10^{1}$ | $10^{2}$ | $10^{3}$ | $10^{4}$ | $10^{5}$ | $10^{6}$ | $10^{7}$ | $10^{8}$ | $10^{9}$ | $10^{10}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $\delta_{n}$ | 1.467 | 0.860 | 0.646 | 0.566 | 0.477 | 0.416 | 0.386 | 0.347 | 0.316 | 0.299 |

Table 1. How large $n$ should be for the theorem?

In the end, we make a short comment on how large $n$ should be for the theorem. Table 1 shows the rounded values of

$$
\delta_{n}=\left|\frac{\log t_{n}}{\log ^{2} n / \log 2}-1\right|
$$

for $n=10^{1}, \ldots, 10^{10}$. The proof indicates that $n>10^{10}$ would be sufficient for the theorem.

## 6. Conclusions

In this article, we addressed the problem of how large the average number of concepts is. To this end, we introduced the distribution $\kappa_{n, p, q}$ for random contexts and presented an exact formula for the average number $E(|\mathfrak{B}(\boldsymbol{K})|)$ of concepts of a random context $\boldsymbol{K} \sim \kappa_{n, p, q}$. To establish a superpolynomial asymptotic lower bound, random contexts with constant probabilities $p=q=1 / 2$ were studied in detail. For a sequence of random contexts $\left(\boldsymbol{K}_{n}\right)$ with $\boldsymbol{K}_{n} \sim \kappa_{n, \frac{1}{2}, \frac{1}{2}}$, we proved that $E\left(\left|\mathfrak{B}\left(\boldsymbol{K}_{n}\right)\right|\right) \gg n^{\log n}$.

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