Discrete feedback control for highly nonlinear neutral stochastic delay differential equations with Markovian switching

Chunhui Mei[a,b], Chen Fei[c], Mingxuan Shen[a], Weiyin Fei[a,*], Xuerong Mao[d]

[a] *The Key Laboratory of Advanced Perception and Intelligent Control of High-end Equipment, Ministry of Education, and School of Mathematics–Physics and Finance, Anhui Polytechnic University, Wuhu, 241000, China*
[b] *School of Science, Nanjing University of Science and Technology, Nanjing, Jiangsu 210094, China*
[c] *Business School, University of Shanghai for Science and Technology, Shanghai, 200093, China*
[d] *Department of Mathematics and Statistics, University of Strathclyde, Glasgow G1 1XH, UK*

Abstract

In this article, it is proved that feedback controllers can be designed to stabilize nonlinear neutral stochastic systems with Markovian switching (NSDDEwMS in short) only by using discrete observed state sequences. Due to the superlinear coefficients, the neutral term and the discrete observation data, many routine methods and techniques for the study of stochastic systems are not applicable. A new Lyapunov functional is constructed by using multiple M-matrices to prove that a given unstable NSDDEwMS can be stabilized if the control function can be designed to meet a couple of easy-to-be-verified rules. Finally, an example is given to illustrate the feasibility of the theoretical results.

Keywords: Neutral systems, Discrete observation, Highly nonlinear, Feedback control, Multiple M-matrices

1. Introduction

By considering the influence of stochastic factors and sudden change of systems parameters and structure, stochas-tic differential equations with Markovian switching (SDEwMS in short) as special hybrid systems have been widely developed in the past decades [1, 2]. As fundamental properties, the stability and stabilization play an irreplaceable role in the theory and applications of SDEwMS. For the unstable stochastic systems, the regular stabilization methods are based on the continuous-time observational data to design the feedback controller (continuous controller in short)[3, 4, 5]. However, due to consideration of the cost and practical operation, in 2013, Mao [6] designed a controller depending only on discrete state observations (discrete controller in short) to stabilize SDEwMS. The main method in [6] is to compare the discrete controller with the continuous one, and obtain exponential stability of the controlled system in the mean square sense by using the properties of Markov skeleton process. Base on Mao's work, under the weaker constraints on the coefficients of the system, You et al. [7] did not only establish the stabilization criteria in various senses, but also proposed a better state observation time interval by the method of Lyapunov functionals. Considering both the feedback control of discrete observations and the periodic intermittent control, Wu et al. [8] studied the synchronization of stochastic oscillators with time-varying coupling structure and Liu and Wu [9] gave a controller to stabilize the unstable system of ordinary differential equation. It is noted that the above criteria only work for the systems whose coefficients are linear or subject to the linear growth condition.

However, in the real world, many practical systems are not linear, such as finance model of constant elasticity of variance (CEV), the predator-prey model in ecology and so on [10, 11]. Therefore, the study of the stability and stabilization for the above highly nonlinear systems has attracted extensive attention. In 2013, Hu et al. [12, 13] used different methods to study the asymptotic behavior of highly nonlinear SDEwMS with delay. Fei et al. [14] further discussed the influence of different structures on the robust stability and boundedness of SDEwMS with delay and

*Corresponding author
Email addresses: mch413@163.com (Chunhui Mei), chenfei@usst.edu.cn (Chen Fei), smx1011@ahpu.edu.cn (Mingxuan Shen), wyfei@ahpu.edu.cn (Weiyin Fei), x.mao@strath.ac.uk (Xuerong Mao)

Preprint submitted to Elsevier *January 13, 2022*

superlinear coefficients. Recently, based only on discrete-time state observations, the Lyapunov functional method can be used to design the controller to stabilize the unstable highly nonlinear SDEwMS [15, 16]. Meanwhile, event-triggered stabilization and sliding-mode control of stochastic systems have also received further attention [17, 18, 19]. In particular, Zong et al. [20] explored the problem of delay tolerance for SDEs with global and nonglobal Lipschitz coefficients in general sense, and gave an upper bound of delay, in which the delay caused by discrete state observation is analyzed as a special case.

On the other hand, motivated by the dynamics of oscillators as well as the viscous aftereffect problems, neutral systems have attracted widespread and constant attention, and appeared frequently in population ecology, automatic control, chemical reactors, distributed networks and so on [21, 22, 23, 24]. Considering the influence of random factors, by the LaSalle theorem of stochastic version, Mao et al. [25] further investigated the almost sure stability of linear and nonlinear hybrid stochastic neutral systems, and obtained the rate of convergence. The properties of highly nonlinear NSDDEwMS are analyzed from different perspectives such as numerical method, decay rate in general sense and delay dependent stability [26, 27, 28]. Although continuous controllers are designed in [29, 30] to stabilize unstable neutral systems, there is little known on how to stabilize highly nonlinear NSDDEwMS by a discrete feedback control. Comparing with the existing results, we highlight the main features of this article as follows.

- We give a set of rules so that the feedback control based on the discrete state observations can be designed more easily by following the rules step by step.
- Several new mathematical techniques have been developed to cope with the difficulties due to the neutral term, super-linear coefficients, discrete-time state observations. For example, in the proof of Theorem 3.2, the conventional stopping technique can not deal with the existence of solutions of neutral systems in this paper. Therefore, we use the piecewise technique to prove the existence of the solution, and then construct the integral Lyapunov functional to obtain the asymptotic boundedness of the solution.
- Using multiple M-Matrices, we overcome the difficulties caused by superlinear coefficients in neutral systems and realize the exponential stabilization of controlled systems.

The rest of this paper is organized as follows. In Section 2, necessary notations, basic assumptions and a lemma are given. In Section 3, the existence, uniqueness and moment boundedness of the global solution of the controlled NSDDEwMS are addressed, and the corresponding conditions are designed for the controller to obtain the stability of the controlled NSDDEwMS. A numerical example is provided in Section 4, and the conclusion is drawn in Section 5.

2. Standing hypotheses

Notation: Let $\mathbb{R}_+ = [0,\infty)$. If D is a vector or matrix, its transpose is denoted by D^T. For $y \in \mathbb{R}^d$, $|y|$ is the Euclidean norm. If $D \in \mathbb{R}^{d\times m}$, $|D| = \sqrt{\text{trace}(D^T D)}$ and $\|D\|_2 = \sqrt{\sigma_{\max}(D^T D)}$ denote its Frobenius norm and spectral norm, respectively. By $D < 0$ and $D \leq 0$, we denote D is negative definite and non-positive real-valued square matrix. For the positive number h, $C_{h,d} = C([-h,0];R^d)$ means all continuous functions ϕ from $[-h,0] \to R^d$ and the norm is $\|\phi\| = \sup_{-h\leq\theta\leq 0} |\phi(\theta)|$. Let $B(t)$ be an m-dimensional Brownian motion and $(\Omega, \mathfrak{G}, \mathbb{P})$ be a complete probability space. $\{\mathfrak{G}_t\}_{t\geq 0}$ is a natural filtration satisfying the usual conditions. Let $q(t)$, $t \geq 0$, be a right-continuous Markov chain taking values in a finite state space $\Theta = \{1, 2, ..., M\}$ on the probability space with generator $Q = (q_{jk})_{M\times M}$ given by

$$\mathbb{P}\{q(t+\Delta) = k|q(t) = j\} = \begin{cases} 1 + q_{jj}\Delta + o(\Delta) & \text{if } j = k, \\ q_{jk}\Delta + o(\Delta) & \text{if } j \neq k, \end{cases}$$

where $\Delta \downarrow 0$. Here $q_{jk} \geq 0$ is the transition rate from j to k if $j \neq k$ while $q_{jj} = -\sum_{j\neq k} q_{jk}$. We always assume that the Brownian motion $B(\cdot)$ and the Markov chain $q(\cdot)$ are independent of each other.

Let

$$f : \mathbb{R}^d \times \mathbb{R}^d \times \Theta \times \mathbb{R}_+ \to \mathbb{R}^d,\ g : \mathbb{R}^d \times \mathbb{R}^d \times \Theta \times \mathbb{R}_+ \to \mathbb{R}^{d\times m} \text{ and } G : \mathbb{R}^d \to \mathbb{R}^d$$

are Borel measurable functions. Consider a nonlinear NSDDEwMS

$$d(y(t) - G(y(t-h))) = f(y(t), y(t-h), q(t), t)dt + g(y(t), y(t-h), q(t), t)dB(t) \tag{2.1}$$

on $t \geq 0$ with the initial data

$$\phi_0 = \{y(\theta) : -h \leq \theta \leq 0\} \in C_{h,d}, \; q(0) = j_0 \in \Theta, \tag{2.2}$$

where $h > 0$ is the system delay and $y(t) \in \mathbb{R}^d$ is the state vector (to make our mathematical analysis clear and concise, we only deal with the system, which the neutral term is independent of time and switching). Next, we give some fundamental assumptions, which are some restrictions on the coefficients and neutral term of equation (2.1).

Assumption 2.1. *Assume that for any $b > 0$, there exists a $L_b > 0$ such that*

$$|f(y_1, z_1, j, t) - f(y_2, z_2, j, t)|^2 \vee |g(y_1, z_1, j, t) - g(y_2, z_2, j, t)|^2 \leq L_b(|y_1 - y_2|^2 + |z_1 - z_2|^2) \tag{2.3}$$

for all $y_1, y_2, z_1, z_2 \in \mathbb{R}^d$ with $|y_1| \vee |y_2| \vee |z_1| \vee |z_2| \leq b$ and all $(j, t) \in \Theta \times \mathbb{R}_+$. In addition, for all $(y, z, j, t) \in \mathbb{R}^d \times \mathbb{R}^d \times \Theta \times \mathbb{R}_+$, there exist some positive numbers $L > 0$, $l_1 \geq 1$ and $l_2 \geq 1$ such that

$$|f(y, z, j, t)| \leq L(|y|^{l_1} + |z|^{l_1} + |y| + |z|) \quad \text{and} \quad |g(y, z, j, t)| \leq L(|y|^{l_2} + |z|^{l_2} + |y| + |z|). \tag{2.4}$$

Assume moreover that there is a constant $\varrho \in (0, 1)$ such that for all $y, z \in \mathbb{R}^d$

$$|G(y) - G(z)| \leq \varrho |y - z| \tag{2.5}$$

while $G(0) = 0$.

It is obvious that the condition (2.4) may be specialised as the linear growth condition by letting $l_1 = l_2 = 1$. This shows that the results of this paper are more general than the previous ones [31, 32]. In order to avoid the solution of the equation exploding in finite time, we also need to give a class of Khasminskii-type conditions to make the neutral system (2.1) has a global continuous solution.

Assumption 2.2. *For l_1, l_2 and ϱ in Assumption 2.1, there exist some positive numbers $\zeta_1, \zeta_2, \zeta_3, \zeta_4, p, l$ such that*

$$p \geq (2l_1) \vee (l_1 + 2l_2 - 1), \quad l \geq (l_1 + 1) \vee (2l_2 - l_1 + 1), \quad \zeta_1 > \zeta_2 \Big(\frac{1+\varrho}{1-\varrho}\Big)^{p-2} \tag{2.6}$$

while

$$(y - G(z))^T f(y, z, j, t) + \frac{p-1}{2}|g(y, z, j, t)|^2 \leq -\zeta_1 |y|^l + \zeta_2 |z|^l + \zeta_3 |y|^2 + \zeta_4 |z|^2 \tag{2.7}$$

for all $(y, z, j, t) \in \mathbb{R}^d \times \mathbb{R}^d \times \Theta \times \mathbb{R}_+$.

The following lemma is commonly used in dealing with neutral systems.

Lemma 2.3. *[33] For vectors $y, z \in \mathbb{R}^d$ and $b \geq 1$, under codition (2.5), we have*

$$|y - G(z)|^b \leq (1+\varrho)^{b-1}|y|^b + \varrho(1+\varrho)^{b-1}|z|^b,$$
$$|y - G(z)|^b \geq (1-\varrho)^{b-1}|y|^b - \varrho(1-\varrho)^{b-1}|z|^b.$$

Theorem 2.4. *For any given initial data (2.2), under Assumptions 2.1 and 2.2, the NSDDEwMS (2.1) has a unique global continuous solution such that $\sup_{-h \leq t < \infty} \mathbb{E}|y(t)|^p < \infty$.*

Proof. Let $U(y) = |y|^p$. By the Itô formula,

$$\begin{aligned} dU(y(t) - G(y(t-h))) = &L_1 U(y(t), y(t-h), q(t), t)dt \\ &+ p|y(t) - G(y(t-h))|^{p-2}(y(t) - G(y(t-h)))^T g(y(t), y(t-h), q(t), t)dB(t), \end{aligned}$$

where the functional $L_1 U$ is defined by

$$\begin{aligned} L_1 U(y, z, j, t) = &p|y - G(z)|^{p-2}(y - G(z))^T f(y, z, j, t) + \frac{p}{2}|y - G(z)|^{p-2}|g(y, z, j, t)|^2 \\ &+ \frac{p(p-2)}{2}|y - G(z)|^{p-4}|(y - G(z))^T g(y, z, j, t)|^2 \\ &\le p|y - G(z)|^{p-2}\Big[(y - G(z))^T f(y, z, j, t) + \frac{p-1}{2}|g(y, z, j, t)|^2\Big]. \end{aligned}$$

Using Lemma 2.3 and Assumption 2.2, we get

$$\begin{aligned} L_1 U(y, z, j, t) \le &p|y - G(z)|^{p-2}\Big[- \zeta_1|y|^l + \zeta_2|y|^l + \zeta_3|z|^2 + \zeta_4|z|^2\Big] \\ \le &- p\zeta_1(1-\varrho)^{p-3}(|y|^{p+l-2} - \varrho|y|^l|z|^{p-2}) + p\zeta_2(1+\varrho)^{p-3}(|y|^{p-2}|z|^l + \varrho|z|^{p-2+l}) \\ &+ p(1+\varrho)^{p-3}(|y|^{p-2} + \varrho|z|^{p-2})(\zeta_3|y|^2 + \zeta_4|z|^2). \end{aligned}$$

Using the Young inequality,

$$\begin{aligned} &|y|^l|z|^{p-2} \le \frac{l}{p+l-2}|y|^{p+l-2} + \frac{p-2}{p+l-2}|z|^{p+l-2}, \quad |y|^{p-2}|z|^2 \le \frac{p-2}{p}|y|^p + \frac{2}{p}|z|^p, \\ &|y|^{p-2}|z|^l \le \frac{p-2}{p+l-2}|y|^{p+l-2} + \frac{l}{p+l-2}|z|^{p+l-2}, \quad |y|^2|z|^{p-2} \le \frac{2}{p}|y|^p + \frac{p-2}{p}|z|^p. \end{aligned}$$

Hence

$$L_1 U(y, z, j, t) \le -\beta_1|y|^{p+l-2} + \beta_2|z|^{p+l-2} + \beta_3|y|^p + \beta_4|z|^p, \tag{2.8}$$

where

$$\begin{aligned} \beta_1 = &p\zeta_1(1-\varrho)^{p-3} - p\zeta_1\varrho(1-\varrho)^{p-3}\frac{l}{p+l-2} - p\zeta_2(1+\varrho)^{p-3}\frac{p-2}{p+l-2}, \\ \beta_2 = &p\zeta_1\varrho(1-\varrho)^{p-3}\frac{p-2}{p+l-2} + p\zeta_2(1+\varrho)^{p-3}\frac{l}{p+l-2} + p\zeta_2\varrho(1+\varrho)^{p-3}, \\ \beta_3 = &\zeta_3(p+2\varrho)(1+\varrho)^{p-3} + \zeta_4(p-2)(1+\varrho)^{p-3}, \\ \beta_4 = &\zeta_3\varrho(p-2)(1+\varrho)^{p-3} + \zeta_4(p\varrho+2)(1+\varrho)^{p-3}. \end{aligned}$$

Recalling (2.6), it is easy to show that $\beta_1 > \beta_2 > 0$ and $\beta_4 > 0$, we can then rewrite (2.8) as

$$\begin{aligned} L_1 U(y, z, j, t) \le &C_1 - \beta_2|y|^{p+l-2} + \beta_2|z|^{p+l-2} - \beta_4|y|^p + \beta_4|z|^p \\ \le &C_1 - \beta_4\Big(|y|^p + \frac{\beta_2}{\beta_4}|y|^{p+l-2}\Big) + \beta_4\Big(|z|^p + \frac{\beta_2}{\beta_4}|z|^{p+l-2}\Big), \end{aligned}$$

where $C_1 = \max_{s\ge0}\Big(- (\beta_1 - \beta_2)s^{p+l-2} + (\beta_3 + \beta_4)s^p\Big)$. Let $U(y) = |y|^p$ and $U_2(y) = |y|^p + \frac{\beta_2}{\beta_4}|y|^{p+l-2}$, we easy verify Assumption 3 in [34]. Thus, we can get our result by using Theorem 1 and Corollary 1 in [34]. □

3. Boundedness and stabilization

Obviously, the system satisfying Assumptions 2.1 and 2.2 is not necessarily stable. As mentioned earlier, we will design a discrete controller $u(y([t/\tau]\tau), q(t), t)$ in the drift term to stabilize the unstable NSDDEwMS (2.1), where τ is the gap between two discrete observations. That is, we will discuss the asymptotic behavior of the new NSDDEwMS

$$\begin{aligned} d(y(t) - G(y(t-h))) = &[f(y(t), y(t-h), q(t), t) + u(y([t/\tau]\tau), q(t), t)]dt \\ &+ g(y(t), y(t-h), q(t), t)dB(t), \quad t \ge 0, \end{aligned} \tag{3.1}$$

where $u : \mathbb{R}^d \times \Theta \times \mathbb{R}_+ \to \mathbb{R}^d$ is a Borel measurable function. In this article, we will make the following rule about the controller u.

Assumption 3.1. *For each $j \in \Theta$ and $y, z \in \mathbb{R}^d$, there exists a constant $\vartheta > 0$ satisfies*

$$|u(y, j, t) - u(z, j, t)| \le \vartheta |y - z|, \quad \forall t \ge 0. \tag{3.2}$$

In addition, for each $j \in \Theta$, assume that $u(0, j, t) \equiv 0, \forall t \ge 0$.

For each $(y, j) \in \mathbb{R}^d \times \Theta$, Assumption 3.1 implies

$$|u(y, j, t)| \le \vartheta |y|, \quad \forall t \ge 0. \tag{3.3}$$

3.1. Boundedness

In the following part, we will show that the new NSDDEwMS (3.1) can inherit the properties of the original system (2.1), such as the existence and asymptotic boundedness of continuous solution $y(t)$ on $[-h, \infty)$.

Theorem 3.2. *For any given initial data (2.2), under Assumptions 2.1, 2.2 and 3.1, the NSDDEwMS (3.1) has a unique solution $y(t)$ on $[-h, \infty)$ a.s. Moreover, the solution $y(t)$ satisfies that*

$$\sup_{-h \le t < \infty} \mathbb{E}|y(t)|^p < \infty. \tag{3.4}$$

Proof. Set $\bar{y}(t) = y(t) - G(y(t-h))$. The controlled NSDDEwMS (3.1) may be rewritten as

$$d\bar{y}(t) = \big(f(y(t), y(t-h), q(t), t) + u(y(t - \varphi(t)), q(t), t)\big)dt + g(y(t), y(t-h), q(t), t)dB(t),$$

where $\varphi(t) = t - i\tau \quad$ for $i\tau \le t < (i+1)\tau, \ i = 0, 1, 2, ...$

Step 1. Let $U(y) = |y|^p$ again. By the Itô formula,

$$dU(\bar{y}(t)) = L_2U(y(t), y(t-h), y(t-\varphi(t)), q(t), t)dt + p|\bar{y}(t)|^{p-2}\bar{y}(t)^T g(y(t), y(t-h), q(t), t)dB(t),$$

where L_2U is defined by

$$L_2U(y, z, \hat{z}, j, t) = L_1U(y, z, j, t) + p|y - G(z)|^{p-2}(y - G(z))^T u(\hat{z}, j, t),$$

in which $\hat{z}$ refers to the state $y(t - \varphi(t))$ of the control function u. Recalling (2.8), then using Assumption 3.1 and Lemma 2.3, we give

$$\begin{aligned} L_2U(y, z, \hat{z}, j, t) \le & -\beta_1|y|^{p+l-2} + \beta_2|z|^{p+l-2} + \beta_3|y|^p + \beta_4|z|^p + p\vartheta|y - G(z)|^{p-1}|\hat{z}| \\ \le & -\beta_1|y|^{p+l-2} + \beta_2|z|^{p+l-2} + \beta_3|y|^p + \beta_4|z|^p + p\vartheta(1+\varrho)^{p-2}(|y|^{p-1} + \varrho|z|^{p-1})|\hat{z}|. \end{aligned}$$

By the Young inequality,

$$\begin{aligned} p\vartheta(1+\varrho)^{p-2}|y|^{p-1}|\hat{z}| =& \Big(p\frac{\varepsilon^2}{2}|\hat{z}|^p\Big)^{1/p}\Big(\frac{p2^{1/(p-1)}\vartheta^{p/(p-1)}(1+\varrho)^{(p-2)p/(p-1)}|y|^p}{\varepsilon^{2/(p-1)}}\Big)^{(p-1)/p} \\ \le& \frac{\varepsilon^2}{2}|\hat{z}|^p + \frac{(p-1)2^{1/(p-1)}\vartheta^{p/(p-1)}(1+\varrho)^{(p-2)p/(p-1)}}{\varepsilon^{2/(p-1)}}|y|^p, \\ p\vartheta\varrho(1+\varrho)^{p-2}|z|^{p-1}|\hat{z}| =& \Big(p\frac{\varepsilon^2}{2}|\hat{z}|^p\Big)^{1/p}\Big(\frac{p2^{1/(p-1)}(\vartheta\varrho)^{p/(p-1)}(1+\varrho)^{(p-2)p/(p-1)}|z|^p}{\varepsilon^{2/(p-1)}}\Big)^{(p-1)/p} \\ \le& \frac{\varepsilon^2}{2}|\hat{z}|^p + \frac{(p-1)2^{1/(p-1)}(\vartheta\varrho)^{p/(p-1)}(1+\varrho)^{(p-2)p/(p-1)}}{\varepsilon^{2/(p-1)}}|z|^p. \end{aligned}$$

Hence

$$L_2U(y, z, \hat{z}, j, t) \le -\beta_1|y|^{p+l-2} + \beta_2|z|^{p+l-2} + \hat{\beta}_3|y|^p + \hat{\beta}_4|z|^p + \varepsilon^2|\hat{z}|^p, \tag{3.5}$$

where $\hat{\beta}_3 = \frac{(p-1)2^{1/(p-1)}\vartheta^{p/(p-1)}(1+\varrho)^{(p-2)p/(p-1)}}{\varepsilon^{2/(p-1)}} + \beta_3$, $\hat{\beta}_4 = \frac{(p-1)2^{1/(p-1)}(\vartheta\varrho)^{p/(p-1)}(1+\varrho)^{(p-2)p/(p-1)}}{\varepsilon^{2/(p-1)}} + \beta_4$.

For $t \in [0, \tau]$, $y(t - \varphi(t)) = y(0)$, thus we have

$$\begin{aligned}
&L_2U(y(t), y(t-h), y(0), q(t), t)\\
&\le -\beta_1|y(t)|^{p+l-2} + \beta_2|y(t-h)|^{p+l-2} + \hat{\beta}_3|y(t)|^p + \hat{\beta}_4|y(t-h)|^p + \varepsilon^2|y(0)|^p\\
&\le C_2 - \hat{\beta}_4\Big(|y(t)|^p + \frac{\beta_2}{\hat{\beta}_4}|y(t)|^{p+l-2}\Big) + \hat{\beta}_4\Big(|y(t-h)|^p + \frac{\beta_2}{\hat{\beta}_4}|y(t-h)|^{p+l-2}\Big),
\end{aligned}$$

where $C_2 := \max_{s\ge 0}\Big(-(\beta_1 - \beta_2)s^{p+l-2} + (\hat{\beta}_3 + \hat{\beta}_4)s^p\Big) + \varepsilon^2|y(0)|^p$. From Theorem 1 in [34], NSDDEwMS (3.1) has a unique continuous solution $y(t)$ on $[0, \tau]$. Next, for $t \in [\tau, 2\tau]$, $y(t - \varphi(t)) = y(\tau)$, thus we obtain

$$\begin{aligned}
&L_2U(y(t), y(t-h), y(\tau), q(t), t)\\
&\le -\beta_1|y(t)|^{p+l-2} + \beta_2|y(t-h)|^{p+l-2} + \hat{\beta}_3|y(t)|^p + \hat{\beta}_4|y(t-h)|^p + \varepsilon^2|y(\tau)|^p\\
&\le C_3 - \hat{\beta}_4\Big(|y(t)|^p + \frac{\beta_2}{\hat{\beta}_4}|y(t)|^{p+l-2}\Big) + \hat{\beta}_4\Big(|y(t-h)|^p + \frac{\beta_2}{\hat{\beta}_4}|y(t-h)|^{p+l-2}\Big),
\end{aligned}$$

where $C_3 := \max_{s\ge 0}\Big(-(\beta_1 - \beta_2)s^{p+l-2} + (\hat{\beta}_3 + \hat{\beta}_4)s^p\Big) + \varepsilon^2|y(\tau)|^p$. Similarly, NSDDEwMS (3.1) still has a unique continuous solution $y(t)$ on $[\tau, 2\tau]$. Continuing this procedure inductively, we can see that NSDDEwMS (3.1) has the unique globe solution.

Step 2. Applying the Itô formula, together with (3.5), yields

$$\begin{aligned}
d(e^{\varepsilon t}U(\bar{y}(t))) =& e^{\varepsilon t}\Big(\varepsilon U(\bar{y}(t)) + L_2U(y(t), y(t-h), y(t-\varphi(t)), q(t), t)\Big)dt\\
&+ pe^{\varepsilon t}|\bar{y}(t)|^{p-2}\bar{y}^T(t)g(y(t), y(t-h), q(t), t)dB(t)\\
\le& e^{\varepsilon t}\Big(\varepsilon|\bar{y}(t)|^p - \beta_1|y(t)|^{p+l-2} + \beta_2|y(t-h)|^{p+l-2} + \hat{\beta}_3|y(t)|^p + \hat{\beta}_4|y(t-h)|^p\\
&+ \varepsilon^2|y(t-\varphi(t))|^p\Big)dt + pe^{\varepsilon t}|\bar{y}(t)|^{p-2}\bar{y}^T(t)g(y(t), y(t-h), q(t), t)dB(t)\\
\le& e^{\varepsilon t}\Big(-\beta_1|y(t)|^{p+l-2} + \beta_2|y(t-h)|^{p+l-2} + \bar{\beta}_3|y(t)|^p + \bar{\beta}_4|y(t-h)|^p\\
&+ \varepsilon^2|y(t-\varphi(t))|^p\Big)dt + pe^{\varepsilon t}|\bar{y}(t)|^{p-2}\bar{y}^T(t)g(y(t), y(t-h), q(t), t)dB(t),
\end{aligned} \tag{3.6}$$

where $\bar{\beta}_3 = \varepsilon(1+\varrho)^{p-1} + \hat{\beta}_3$, $\bar{\beta}_4 = \varepsilon\varrho(1+\varrho)^{p-1} + \hat{\beta}_4$. We now define

$$U_1(t) = \beta_2\int_{t-h}^{t} e^{\varepsilon(w+h)}|y(w)|^{p+l-2}dw + \bar{\beta}_4\int_{t-h}^{t} e^{\varepsilon(w+h)}|y(w)|^p dw.$$

By the simple calculation,

$$dU_1(t) = (\beta_2 e^{\varepsilon(t+h)}|y(t)|^{p+l-2} - \beta_2 e^{\varepsilon t}|y(t-h)|^{p+l-2} + \bar{\beta}_4 e^{\varepsilon(t+h)}|y(t)|^p - \bar{\beta}_4 e^{\varepsilon t}|y(t-h)|^p)dt.$$

This, together with (3.6), gives

$$d(e^{\varepsilon t}U(\bar{y}(t)) + U_1(t)) \le e^{\varepsilon t}\Big[\hat{H}(y(t)) + \varepsilon^2|y(t-\varphi(t))|^p\Big]dt + pe^{\varepsilon t}|\bar{y}(t)|^{p-2}\bar{y}^T(t)g(y(t), y(t-h), q(t), t)dB(t),$$

where $\hat{H}(s) = -(\beta_1 - \beta_2 e^{\varepsilon h})|s|^{p+l-2} + (\bar{\beta}_3 + \bar{\beta}_4 e^{\varepsilon h})|s|^p$. Recalling (2.6), we may choose $\varepsilon > 0$ sufficiently small for $\beta_1 - \beta_2 e^{\varepsilon h} > 0$, which implies

$$C_4 = \sup_{s\ge 0}\hat{H}(s) < \infty.$$

Hence, we have

$$\begin{aligned}
e^{\varepsilon t}\mathbb{E}U(\bar{y}(t)) \le \mathbb{E}(e^{\varepsilon t}U(\bar{y}(t)) + U_1(t)) \le& U(\bar{y}(0)) + U_1(0) + \mathbb{E}\int_0^t e^{\varepsilon w}\Big[C_4 + \varepsilon^2|y(w-\varphi(w))|^p\Big]dw\\
\le& U(\bar{y}(0)) + U_1(0) + \frac{C_4 e^{\varepsilon t}}{\varepsilon} + \varepsilon e^{\varepsilon t}\sup_{-h\le w\le t}\mathbb{E}|y(w)|^p.
\end{aligned}$$

We therefore get

$$\begin{aligned}\mathbb{E}|y(t)-G(y(t-h))|^p \le& U(\bar{y}(0))+U_1(0)+\frac{C_4}{\varepsilon}+\varepsilon\sup_{-h\le w\le t}\mathbb{E}|y(w)|^p\\ :=& C_5+\varepsilon\sup_{-h\le w\le t}\mathbb{E}|y(w)|^p.\end{aligned}$$

Again, using Lemma 2.3, we obtain

$$(1-\varrho)^{p-1}\mathbb{E}|y(t)|^p-\varrho(1-\varrho)^{p-1}\mathbb{E}|y(t-h)|^p\le\mathbb{E}|y(t)-G(y(t-h))|^p\le C_5+\varepsilon\sup_{-h\le w\le t}\mathbb{E}|y(w)|^p,$$

which implies

$$(1-\varrho)^{p-1}\mathbb{E}|y(t)|^p\le C_5+\left(\varrho(1-\varrho)^{p-1}+\varepsilon\right)\sup_{-h\le w\le t}\mathbb{E}|y(w)|^p.$$

Thus, we get

$$\begin{aligned}(1-\varrho)^{p-1}\sup_{-h\le w\le t}\mathbb{E}|y(w)|^p\le&(1-\varrho)^{p-1}(\|\phi_0\|^p+\sup_{0\le w\le t}\mathbb{E}|y(w)|^p)\\ \le&(1-\varrho)^{p-1}\|\phi_0\|^p+C_5+(\varrho(1-\varrho)^{p-1}+\varepsilon)\sup_{-h\le w\le t}\mathbb{E}|y(w)|^p.\end{aligned}$$

Choosing $\varepsilon>0$ sufficiently small such that $\varepsilon<\frac{\ln\beta_1-\ln\beta_2}{h}\wedge(1-\varrho)^p$, we have

$$\sup_{-h\le w\le t}\mathbb{E}|y(w)|^p\le\frac{(1-\varrho)^{p-1}\|\phi_0\|^p+C_5}{(1-\varrho)^p-\varepsilon}:=C_6,$$

Letting $t\to\infty$, we there therefore obtain the desired result (3.4). □

Remark 3.3. *(i) For $t=i\tau, i=1,2,...,$ the bounded time-varying delay $\varphi(t)$ caused by discrete observations is not differentiable, so we can not directly use the relevant results in [34] to explain the the existence, uniqueness and boundedness of the solution of NSDDEwMS (3.1).*

(ii) Combining this theorem with condition (2.6), we can get

$$\sup_{0\le t<\infty}\mathbb{E}|f(y(t),y(t-h),q(t),t)|^2<\infty\quad\textit{and}\quad\sup_{0\le t<\infty}\mathbb{E}|g(y(t),y(t-h),q(t),t)|^2<\infty.$$

3.2. *Exponential stabilization*

In this section, we will show that under certain conditions, the feedback controller which depends on discrete-time state observations can achieve exponential stabilization under the premise of ensuring the asymptotic boundedness of the solution of the controlled NSDDEwMS (3.1). Our main methods are multiple M-matrices and Lyapunov functional. Regarding the definition and fundamental properties on M-matrix, see [1, section 2.6]. Now we state the first condition in term of the M-matrix.

Assumption 3.4. *For any $j\in\Theta$, there are positive constants ζ_{ji}, $\hat{\zeta}_{ji}(i=1,2,4)$ and real constants $\zeta_{j3},\hat{\zeta}_{j3}$ such that*

$$(y-G(z))^T[f(y,z,j,t)+u(y,j,t)]+\frac{1}{2}|g(y,z,j,t)|^2\le-\zeta_{j1}|y|^l+\zeta_{j2}|z|^l+\zeta_{j3}|y|^2+\zeta_{j4}|z|^2 \tag{3.7}$$

and

$$(y-G(z))^T[f(y,z,j,t)+u(y,j,t)]+\frac{l_1}{2}|g(y,z,j,t)|^2\le-\hat{\zeta}_{j1}|y|^l+\hat{\zeta}_{j2}|z|^l+\hat{\zeta}_{j3}|y|^2+\hat{\zeta}_{j4}|z|^2 \tag{3.8}$$

for all $(y,z,t)\in\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}_+$. Moreover, both

$$\mathcal{M}_1:=-2\mathrm{diag}(\zeta_{13},...,\zeta_{M3})-Q\quad\textit{and}\quad\mathcal{M}_2:=-(l_1+1)\mathrm{diag}(\hat{\zeta}_{13},...,\hat{\zeta}_{M3})-Q \tag{3.9}$$

are nonsingular M-matrices.

Remark 3.5. *From the assumptions in Section 2, we can see that the above rules are easy to realize in practice. For example, let's take* $u(y, j, t) = Dy$*, where* $D < 0$ *is a symmetric matrix such that* $2\sigma_{\max}(D) - \varrho\sigma_{\min}(D) \leq -4\zeta_3$ *(where* $\sigma_{\max}(D)$ *and* $\sigma_{\min}(D)$ *are the largest and smallest eigenvalues of matrix D and* $\|D\|_2 = \sqrt{\sigma_{\max}(D^T D)} = -\sigma_{\min}(D)$*). Then, for all* $(y, z, j, t) \in \mathbb{R}^d \times \mathbb{R}^d \times \Theta \times \mathbb{R}_+$

$$\begin{aligned}(y - G(z))^T u(y, j, t) \leq&\sigma_{\max}(D)|y|^2 + |G(z)|\|D\|_2|y|\\ \leq&\sigma_{\max}(D)|y|^2 - \varrho\sigma_{\min}(D)|z||y|\\ \leq&(\sigma_{\max}(D) - \frac{\varrho\sigma_{\min}(D)}{2})|y|^2 - \frac{\varrho\sigma_{\min}(D)}{2}|z|^2\\ \leq&-2\zeta_3|y|^2 - \frac{\varrho\sigma_{\min}(D)}{2}|z|^2.\end{aligned}$$

By (2.7), we have

$$(y - G(z))^T[f(y, z, j, t) + u(y, j, t)] + \frac{1}{2}|g(y, z, j, t)|^2 \leq -\zeta_1|y|^l + \zeta_2|z|^l - \zeta_3|y|^2 + (\zeta_4 - \frac{\varrho\sigma_{\min}(D)}{2})|z|^2$$

as well as

$$(y - G(z))^T[f(y, z, j, t) + u(y, j, t)] + \frac{l_1}{2}|g(y, z, j, t)|^2 \leq -\zeta_1|y|^l + \zeta_2|z|^l - \zeta_3|y|^2 + (\zeta_4 - \frac{\varrho\sigma_{\min}(D)}{2})|z|^2$$

while

$$\mathcal{M}_1 = 2\mathrm{diag}(\zeta_3, ..., \zeta_3) - Q \ \textit{ and } \ \mathcal{M}_2 = (l_1 + 1)\mathrm{diag}(\zeta_3, ..., \zeta_3) - Q$$

which are nonsingular M-matrices.

Set

$$(\pi_1, ..., \pi_M)^T := \mathcal{M}_1^{-1}(1, ..., 1)^T \ \text{ and } (\hat{\pi}_1, ..., \hat{\pi}_M)^T := \mathcal{M}_2^{-1}(1, ..., 1)^T. \tag{3.10}$$

Obviously, from the properties of nonsingular M-matrices, for all $j \in \Theta$, we have $\pi_j > 0$ and $\hat{\pi}_j > 0$. Then define a function $V : \mathbb{R}^d \times \Theta \to \mathbb{R}_+$ by

$$V(y, j) = \pi_j|y|^2 + \hat{\pi}_j|y|^{l_1+1}, \tag{3.11}$$

while define a functional $\mathcal{L}V : \mathbb{R}^d \times \mathbb{R}^d \times \Theta \times \mathbb{R}_+ \to \mathbb{R}$ by

$$\begin{aligned}\mathcal{L}V(y, z, j, t) =&2\pi_j\Big[(y - G(z))^T[f(y, z, j, t) + u(y, j, t)] + \frac{1}{2}|g(y, z, j, t)|^2\Big]\\ &+ (l_1 + 1)\hat{\pi}_j|y - G(z)|^{l_1-1}\Big[(y - G(z))^T[f(y, z, j, t) + u(y, j, t)]\\ &+ \frac{l_1}{2}|g(y, z, j, t)|^2\Big] + \sum_{k=1}^{M} q_{jk}(\pi_k|y - G(z)|^2 + \hat{\pi}_k|y - G(z)|^{l_1+1}).\end{aligned}$$

Assumption 3.6. *For each* $(y, z, j, t) \in \mathbb{R}^d \times \mathbb{R}^d \times \Theta \times \mathbb{R}_+$*, there exist positive numbers* δ_i $(i = 1, ..., 9)$ *and a function* $\Phi(y) \in C(\mathbb{R}^d; \mathbb{R}_+)$*, such that*

$$\delta_5 < \delta_4, \quad \delta_6 < 1, \quad \delta_7|y|^{l_1+l-1} \leq \Phi(y) \leq \delta_8 + \delta_9|y|^{l_1+l-1}, \tag{3.12}$$

as well as

$$\begin{aligned}\mathcal{L}V(y, z, j, t) + \delta_1(2\pi_j|y - G(z)|+(l_1 + 1)\hat{\pi}_j|y - G(z)|^{l_1})^2 + \delta_2|f(y, z, j, t)|^2 + \delta_3|g(y, z, j, t)|^2\\ \leq -\delta_4|y|^2 + \delta_5|z|^2 - \Phi(y) + \delta_6\Phi(z).\end{aligned} \tag{3.13}$$

In order to obtain our main results, we give the form of Lyapunov functional with multiple M-matrices as follows

$$\hat{V}(\hat{y}_t, \hat{q}_t, t) = V(y(t), q(t)) + \frac{\vartheta^2}{\delta_1(1-\varrho)^2}\int_{-\tau}^{0}\int_{t+s}^{t}\Big[|g(y(w), y(w-h), q(w), w)|^2 + \tau|f(y(w), y(w-h), q(w), w) + u(y(w-\varphi(w)), q(w), w)|^2\Big]dwds, \tag{3.14}$$

for $t \geq 0$, where $\hat{y}_t := \{y(t+\theta) : -2h \leq \theta \leq 0\}$ and $\hat{q}_t := \{q(t+\theta) : -2h \leq \theta \leq 0\}$ and V has been defined by (3.11). Define $y(\theta) = y(-h)$ for $\theta \in [-2h, -h)$ and $q(\theta) = j_0$ for $\theta \in [-2h, 0)$, so that $\hat{y}_t$ and $\hat{q}_t$ have a clear definition on $0 \leq t < 2h$. Similarly, we set $f(y, z, j, w) = f(y, z, j, 0)$, $g(y, z, j, w) = g(y, z, j, 0)$, $u(\hat{z}, j, w) = u(\hat{z}, j, 0)$ for $(y, z, \hat{z}, j, w) \in \mathbb{R}^{3d} \times \Theta \times [-2h, 0]$.

Next, we give a lemma which can be proved from the generalized Itô formula in [1].

Lemma 3.7. *For $t \geq 0$, $\hat{V}(\hat{y}_t, \hat{q}_t, t)$ is an Itô stochastic process and its Itô differential is*

$$d\hat{V}(\hat{y}_t, \hat{q}_t, t) = [LV(y(t), y(t-h), y(t-\varphi(t)), q(t), t) + \frac{\vartheta^2\tau}{\delta_1(1-\varrho)^2}J(t) - \frac{\vartheta^2}{\delta_1(1-\varrho)^2}\int_{t-\tau}^{t}J(w)dw]dt + dM(t), \tag{3.15}$$

where LV and J are defined as

$$\begin{aligned}
&LV(y(t), y(t-h), y(t-\varphi(t)), q(t), t)\\
&= 2\pi_{q(t)}\Big[\bar{y}(t)^T[f(y(t), y(t-h), q(t), t) + u(y(t-\varphi(t)), q(t), t)] + \frac{1}{2}|g(y(t), y(t-h), q(t), t)|^2\Big]\\
&\quad + (l_1+1)\hat{\pi}_{q(t)}|\bar{y}(t)|^{l_1-1}\Big[\bar{y}(t)^T[f(y(t), y(t-h), q(t), t) + u(y(t-\varphi(t)), q(t), t)]\\
&\quad + \frac{1}{2}|g(y(t), y(t-h), q(t), t)|^2\Big] + \frac{(l_1^2-1)}{2}\hat{\pi}_{q(t)}|\bar{y}(t)|^{l_1-3}|\bar{y}(t)^T g(y(t), y(t-h), q(t), t)|^2\\
&\quad + \sum_{k=1}^{M} q_{q(t)k}(\pi_k|\bar{y}(t)|^2 + \hat{\pi}_k|\bar{y}(t)|^{l_1+1})
\end{aligned}$$

and

$$J(t) = \tau|f(y(t), y(t-h), q(t), t) + u(y(t-\varphi(t)), q(t), t)|^2 + |g(y(t), y(t-h), q(t), t)|^2 \tag{3.16}$$

respectively. In addition, $M(t)$ is a local continuous martingale with $M(0) = 0$.

We may give first result of stabilization.

Theorem 3.8. *Under Assumptions 2.1, 2.2, 3.1, 3.4 and 3.6, further assume τ is sufficiently small such that*

$$\tau < \frac{(1-\varrho)\sqrt{\delta_1(\delta_4-\delta_5)}}{2\vartheta^2} \quad \text{and} \quad \tau \leq \frac{\sqrt{\delta_1\delta_2}(1-\varrho)}{\sqrt{2}\vartheta} \wedge \frac{\delta_1\delta_3(1-\varrho)^2}{\vartheta^2} \wedge \frac{(1-\varrho)}{4\sqrt{2}\vartheta}. \tag{3.17}$$

Then the solution $y(t)$ of the controlled system (3.1) obeys

$$\limsup_{t\to\infty}\frac{1}{t}\log(\mathbb{E}|y(t)|^q) < 0 \tag{3.18}$$

for any $q \in [2, p)$ and given initial data (2.2).

Proof. We divide the proof into three steps.

Step 1. By condition (3.17), we get

$$\frac{2\vartheta^2\tau^2}{\delta_1(1-\varrho)^2} \leq \delta_2 \quad \text{and} \quad \frac{\vartheta^2\tau}{\delta_1(1-\varrho)^2} \leq \delta_3.$$

This, together with (3.16), yields

$$\frac{\vartheta^2\tau}{\delta_1(1-\varrho)^2}J(t) \leq \delta_2|f(y(t),y(t-h),q(t),t)|^2+\delta_3|g(y(t),y(t-h),q(t),t)|^2+\frac{2\tau^2\vartheta^2}{\delta_1(1-\varrho)^2}|u(y(t-\varphi(t)),q(t),t)|^2$$
$$\leq \delta_2|f(y(t),y(t-h),q(t),t)|^2+\delta_3|g(y(t),y(t-h),q(t),t)|^2+\frac{2\tau^2\vartheta^4}{\delta_1(1-\varrho)^2}|y(t-\varphi(t))|^2.$$

On the other hand, recalling the definitions LV and $\mathcal{L}V$, and combining (3.3), we get

$$\begin{aligned}&LV(y(t),y(t-h),y(t-\varphi(t)),q(t),t)\\&\leq \mathcal{L}V(y(t),y(t-h),q(t),t)+[2\pi_{q(t)}-(l_1+1)\hat{\pi}_{q(t)}|\bar{y}(t)|^{l_1-1}]\bar{y}(t)^T[u(y(t),q(t),t)-u(y(t-\varphi(t)),q(t),t)]\\&\leq \mathcal{L}V(y(t),y(t-h),q(t),t)+\delta_1[2\pi_{q(t)}|\bar{y}(t)|+(l_1+1)\hat{\pi}_{q(t)}|\bar{y}(t)|^{l_1}]^2+\frac{\vartheta^2}{4\delta_1}|y(t)-y(t-\varphi(t))|^2.\end{aligned}$$

Thus

$$d\hat{V}(\hat{y}_t,\hat{q}_t,t) \leq \mathbb{L}\hat{V}(\hat{y}_t,\hat{q}_t,t)dt+dM(t), \tag{3.19}$$

where

$$\begin{aligned}\mathbb{L}\hat{V}(\hat{y}_t,\hat{q}_t,t) =&\mathcal{L}V(y(t),y(t-h),q(t),t)+\delta_1[2\pi_{q(t)}|\bar{y}(t)|+(l_1+1)\hat{\pi}_{q(t)}|\bar{y}(t)|^{l_1}]^2\\&+\delta_2|f(y(t),y(t-h),q(t),t)|^2+\delta_3|g(y(t),y(t-h),q(t),t)|^2\\&+\frac{\vartheta^2}{4\delta_1}|y(t)-y(t-\varphi(t))|^2+\frac{2\tau^2\vartheta^4}{\delta_1(1-\varrho)^2}|y(t-\varphi(t))|^2-\frac{\vartheta^2}{\delta_1(1-\varrho)^2}\int_{t-\tau}^t J(s)ds.\end{aligned} \tag{3.20}$$

Substituting (3.13) into (3.20), we have

$$\begin{aligned}\mathbb{L}\hat{V}(\hat{y}_t,\hat{q}_t,t) \leq& -\delta_4|y(t)|^2+\delta_5|y(t-h)|^2-\Phi(y(t))+\delta_6\Phi(y(t-h))\\&+\frac{\vartheta^2}{4\delta_1}|y(t)-y(t-\varphi(t))|^2+\frac{2\tau^2\vartheta^4}{\delta_1(1-\varrho)^2}|y(t-\varphi(t))|^2-\frac{\vartheta^2}{\delta_1(1-\varrho)^2}\int_{t-\tau}^t J(s)ds.\end{aligned}$$

It follows from (3.17) immediately that $\vartheta\tau \leq (1-\varrho)/4\sqrt{2}$, which implies

$$\frac{2\tau^2\vartheta^4}{\delta_1(1-\varrho)^2}|y(t-\varphi(t))|^2 \leq \frac{4\tau^2\vartheta^4}{\delta_1(1-\varrho)^2}|y(t)|^2+\frac{\vartheta^2}{8\delta_1}|y(t)-y(t-\varphi(t))|^2.$$

Then,

$$\begin{aligned}\mathbb{L}\hat{V}(\hat{y}_t,\hat{q}_t,t) \leq& -\Big(\delta_4-\frac{4\tau^2\vartheta^4}{\delta_1(1-\varrho)^2}\Big)|y(t)|^2+\delta_5|y(t-h)|^2-\Phi(y(t))+\delta_6\Phi(y(t-h))\\&+\frac{3\vartheta^2}{8\delta_1}|y(t)-y(t-\varphi(t))|^2-\frac{\vartheta^2}{\delta_1(1-\varrho)^2}\int_{t-\tau}^t J(s)ds.\end{aligned} \tag{3.21}$$

Moreover, under Assumptions 2.1, 3.1 and Theorem 3.2, especially Remark 3.3, we get

$$\sup_{0\leq t<\infty}\mathbb{E}|\mathbb{L}\hat{V}(\hat{y}_t,\hat{q}_t,t)| < \infty.$$

Step 2. Integrating (3.19) from 0 to t, and taking expectation, we have

$$e^{\varepsilon t}\mathbb{E}\hat{V}(\hat{y}_t,\hat{q}_t,t) \leq \hat{V}(\hat{y}_0,\hat{q}_0,0)+\mathbb{E}\int_0^t e^{\varepsilon s}(\varepsilon\hat{V}(\hat{y}_s,\hat{q}_s,s)+\mathbb{L}\hat{V}(\hat{y}_s,\hat{q}_s,s))ds, \tag{3.22}$$

for any $t \geq 0$. Substituting (3.21) into (3.22) yields

$$e^{\varepsilon t}\mathbb{E}\hat{V}(\hat{y}_t, \hat{q}_t, t) \leq \hat{V}(\hat{y}_0, \hat{q}_0, 0) + \mathbb{E}\int_0^t \varepsilon e^{\varepsilon s}\hat{V}(\hat{y}_s, \hat{q}_s, s)ds + \Xi_1 + \Xi_2 + \Xi_3 - \Xi_4, \tag{3.23}$$

where

$$\begin{aligned}
\Xi_1 =& \mathbb{E}\int_0^t e^{\varepsilon s}\Big[-\Big(\delta_4 - \frac{4\tau^2\vartheta^4}{\delta_1(1-\varrho)^2}\Big)|y(s)|^2 + \delta_5|y(s-h)|^2\Big]ds,\\
\Xi_2 =& \mathbb{E}\int_0^t e^{\varepsilon s}\Big[-\Phi(y(s)) + \delta_6\Phi(y(s-h))\Big]ds,\\
\Xi_3 =& \frac{3\vartheta^2}{8\delta_1}\mathbb{E}\int_0^t e^{\varepsilon s}|y(s) - y(s-\varphi(s))|^2 ds,\\
\Xi_4 =& \frac{\vartheta^2}{\delta_1(1-\varrho)^2}\mathbb{E}\int_0^t e^{\varepsilon s}\Big(\int_{s-\tau}^s J(w)dw\Big)ds.
\end{aligned}$$

By the substitution technique, we may compute that

$$\Xi_1 \leq \delta_5 e^{\varepsilon h}\int_{-h}^0 |\phi_0(s)|^2 ds - \Big(\delta_4 - \delta_5 e^{\varepsilon h} - \frac{4\tau^2\vartheta^4}{\delta_1(1-\varrho)^2}\Big)\mathbb{E}\int_0^t e^{\varepsilon s}|y(s)|^2 ds \tag{3.24}$$

and

$$\Xi_2 \leq \delta_6 e^{\varepsilon h}\int_{-h}^0 \Phi(\phi_0(s))ds - \Big(1 - \delta_6 e^{\varepsilon h}\Big)\mathbb{E}\int_0^t e^{\varepsilon s}\Phi(y(s))ds. \tag{3.25}$$

From the Fubini theorem,

$$\Xi_3 = \frac{3\vartheta^2}{8\delta_1}\int_0^t e^{\varepsilon s}\mathbb{E}|y(s) - y(s-\varphi(s))|^2 ds.$$

For $t \in [-h, h]$, we get

$$\begin{aligned}
\int_0^t e^{\varepsilon s}\mathbb{E}|y(s) - y(s-\varphi(s))|^2 ds \leq& 2\int_0^h e^{\varepsilon s}(\mathbb{E}|y(s)|^2 + \mathbb{E}|y(s-\varphi(s))|^2)ds\\
\leq& 4he^{\varepsilon h}\sup_{-h\leq w\leq h}\mathbb{E}|y(w)|^2 := C_7.
\end{aligned}$$

For $t \geq h$,

$$\begin{aligned}
\mathbb{E}|y(s) - y(s-\varphi(s))| \leq& \mathbb{E}|\bar{y}(s) - \bar{y}(s-\varphi(s))| + \varrho\mathbb{E}|y(s-h) - y(s-\varphi(s)-h)|\\
\leq& \mathbb{E}\Big|\int_{s-\varphi(s)}^s (f(y(w), y(w-h), q(w), w) + u(y(w-\varphi(w)), q(w), w))dw\\
&+ \int_{s-\varphi(s)}^s g(y(w), y(w-h), q(w), w)dB(w)\Big| + \varrho\mathbb{E}|y(s-h)) - y(s-\varphi(s)-h)|,
\end{aligned}$$

which implies

$$\begin{aligned}
&\mathbb{E}|y(s) - y(s-\varphi(s))|^2\\
\leq& (1 + 1/\varsigma)\mathbb{E}\Big|\int_{s-\varphi(s)}^s (f(y(w), y(w-h), q(w), w) + u(y(w-\varphi(w)), q(w), w)dw\\
&+ \int_{s-\varphi(s)}^s g(y(w), y(w-h), q(w), w)dB(w)\Big|^2 + (1+\varsigma)\varrho^2\mathbb{E}|y(s-h) - y(s-\varphi(s)-h)|^2\\
\leq& 2(1 + 1/\varsigma)\mathbb{E}\int_{s-\varphi(s)}^s \Big(\tau|f(y(w), y(w-h), q(w), w) + u(y(w-\varphi(w)), q(w), w)|^2\\
&+ |g(y(w), y(w-h), q(w), w)|^2\Big)dw + (1+\varsigma)\varrho^2\mathbb{E}|y(s-h) - y(s-\varphi(s)-h)|^2.
\end{aligned}$$

Setting $\varsigma = 1/\varrho - 1$ gives

$$\begin{aligned}
&\int_h^t e^{\varepsilon s}\mathbb{E}|y(s) - y(s-\varphi(s))|^2 ds \\
&\le \frac{2}{1-\varrho}\mathbb{E}\int_h^t e^{\varepsilon s}\int_{s-\varphi(s)}^s J(w)dwds + \varrho\int_h^t e^{\varepsilon s}\mathbb{E}|y(s-h) - y(s-\varphi(s)-h)|^2 ds \\
&\le \frac{2}{1-\varrho}\mathbb{E}\int_h^t e^{\varepsilon s}\int_{s-\tau}^s J(w)dwds + \varrho e^{\varepsilon h}\int_0^t e^{\varepsilon s}\mathbb{E}|y(s) - y(s-\varphi(s))|^2 ds.
\end{aligned}$$

Choosing $\varepsilon > 0$ such that $1 - \varrho e^{\varepsilon h} > 15(1-\varrho)/16$, we therefore have

$$\begin{aligned}
&\int_h^t e^{\varepsilon s}\mathbb{E}|y(s) - y(s-\varphi(s))|^2 ds \\
&\le \frac{2}{(1-\varrho)(1-\varrho e^{\varepsilon h})}\mathbb{E}\int_h^t e^{\varepsilon s}\int_{s-\tau}^s J(w)dwds + \frac{\varrho}{1-\varrho e^{\varepsilon h}}\int_0^h e^{\varepsilon s}\mathbb{E}|y(s) - y(s-\varphi(s))|^2 ds \\
&\le \frac{32}{15(1-\varrho)^2}\mathbb{E}\int_h^t e^{\varepsilon s}\int_{s-\tau}^s J(w)dwds + \frac{16\varrho}{15(1-\varrho)}C_7.
\end{aligned}$$

This implies

$$\begin{aligned}
\Xi_3 \le& \frac{3\vartheta^2}{8\delta_1}C_7 + \frac{3\vartheta^2}{8\delta_1}\Big(\frac{32}{15(1-\varrho)^2}\mathbb{E}\int_h^t e^{\varepsilon s}\int_{s-\tau}^s J(w)dwds + \frac{16\varrho}{15(1-\varrho)}C_7\Big) \\
\le& \frac{4}{5}\Xi_4 + \frac{(15+\varrho)\vartheta^2}{40\delta_1(1-\varrho)}C_7.
\end{aligned} \tag{3.26}$$

Substituting (3.24), (3.25) and (3.26) into (3.23) gives

$$\begin{aligned}
e^{\varepsilon t}\mathbb{E}\hat{V}(\hat{y}_t, \hat{q}_t, t) \le& C_8 + \hat{V}(\hat{y}_0, \hat{q}_0, 0) - \Big(\delta_4 - \delta_5 e^{\varepsilon h} - \frac{4\tau^2\vartheta^4}{\delta_1(1-\varrho)^2}\Big)\mathbb{E}\int_0^t e^{\varepsilon s}|y(s)|^2 ds \\
&- \delta_7\Big(1 - \delta_6 e^{\varepsilon h}\Big)\mathbb{E}\int_0^t e^{\varepsilon s}|y(s)|^{l_1+l-2}ds - \frac{1}{5}\Xi_4,
\end{aligned}$$

where $C_8 = \hat{V}(\hat{y}_0, \hat{q}_0, 0) + \delta_5 e^{\varepsilon h}\int_{-h}^0 |\phi_0(s)|^2 ds + \delta_6 e^{\varepsilon h}\int_{-h}^0 \Phi(\phi_0(s))ds + \frac{(15+\varrho)\vartheta^2}{40\delta_1(1-\varrho)}C_7$.

Step 3. Recalling the structure of $\hat{V}$ and condition (2.6), by the inequality $|y|^{l_1} \le |y|^2 + |y|^{l_1+l-2}$, we obtain

$$\begin{aligned}
\kappa_1 e^{\varepsilon t}\mathbb{E}|y(t)|^2 \le& C_8 + \Xi_5 - \frac{1}{5}\Xi_4 - \Big(\delta_4 - \delta_5 e^{\varepsilon h} - \frac{4\tau^2\vartheta^4}{\delta_1(1-\varrho)^2} - \varepsilon\kappa_2 - \varepsilon\kappa_3\Big)\mathbb{E}\int_0^t e^{\varepsilon s}|y(s)|^2 ds \\
&- \Big(\delta_7 - \delta_6\delta_7 e^{\varepsilon h} - \varepsilon\kappa_3\Big)\mathbb{E}\int_0^t e^{\varepsilon s}|y(s)|^{l_1+l-2}ds,
\end{aligned} \tag{3.27}$$

where $\kappa_1 = \min_{j\in\Theta}\pi_j$, $\kappa_2 = \max_{j\in\Theta}\pi_j$, $\kappa_3 = \max_{j\in\Theta}\hat{\pi}_j$, and

$$\Xi_5 = \frac{\varepsilon\vartheta^2}{\delta_1(1-\varrho)^2}E\int_0^t e^{\varepsilon s}\Big(\int_{-\tau}^0\int_{s+v}^s J(w)dwdv\Big)ds.$$

On the other hand, it is obvious that

$$\begin{aligned}
\Xi_5 \le& \frac{\varepsilon\vartheta^2}{\delta_1(1-\varrho)^2}\mathbb{E}\int_0^t e^{\varepsilon s}\Big(\tau\int_{s-\tau}^s \Big[\tau|f(y(w), y(w-h), q(w), w) + u(y(w-\varphi(w)), q(w), w)|^2 \\
&+ |g(y(w), y(w-h), q(w), w)|^2\Big]dw\Big)ds = \varepsilon\tau\Xi_4.
\end{aligned}$$

We may choose $0 < \varepsilon < \frac{\ln(1+\frac{1-\varrho}{16})}{h}$ such that

$$\delta_5 e^{\varepsilon h} + \varepsilon\kappa_2 + \varepsilon\kappa_3 \leq \delta_4 - \frac{4\tau^2\vartheta^4}{\delta_1(1-\varrho)^2}, \quad \delta_6 e^{\varepsilon h} + \frac{\varepsilon\kappa_3}{\delta_7} \leq 1, \quad \varepsilon\tau \leq \frac{1}{5}.$$

Plugging these into (3.27) gives

$$\mathbb{E}|y(t)|^2 \leq \frac{C_8}{\kappa_1} e^{-\varepsilon t}, \quad \forall t \geq 0. \tag{3.28}$$

Applying the Hölder inequality, combining (3.4) and (3.28), for any $q \in [2, p)$, we have

$$\mathbb{E}|y(t)|^q \leq \; C_6^{(q-2)/(p-2)} (C_8/\kappa_1)^{(p-q)/(p-2)} e^{-\varepsilon t(p-q)/(p-2)}.$$

Thus, the assertion (3.18) follows immediately. □

Finally, we can use the similar methods in [4, Theorem 4.5] and [15, Theorem 5.4] to show that under the same conditions, the control function u can also make the neutral system achieve exponential stabilization with probability 1.

Theorem 3.9. *For any given initial data (2.2), under the same conditions as Theorem 3.8, we can obtain that the solution of equation (3.1) obeys*

$$\limsup_{t\to\infty} \frac{1}{t} \log(|y(t)|) < 0 \;\; a.s. \tag{3.29}$$

4. Example

To explain the effectiveness of our given theory clearly, let's consider the following two-dimensional NSDDEwM-S:

$$d(y(t) - G(y(t-h))) = f(y(t), y(t-h), q(t), t)dt + g(y(t), y(t-h), q(t), t)dB(t) \tag{4.1}$$

in which the coefficients are defined by

$$f(y,z,1,t) = \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix} \begin{pmatrix} 1 + z_2^2 - 6y_1^2 \\ 1 + z_1^2 - 6y_2^2 \end{pmatrix}, \quad g(y,z,1,t) = \begin{pmatrix} y_1 z_2 & 0 \\ 0 & y_2 z_1 \end{pmatrix},$$
$$f(y,z,2,t) = \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix} \begin{pmatrix} 1 + 0.5z_2^2 - 4y_1^2 \\ 1 + 0.5z_1^2 - 4y_2^2 \end{pmatrix}, \quad g(y,z,2,t) = \begin{pmatrix} 0.5z_2^2 & 0 \\ 0 & 0.5z_1^2 \end{pmatrix}.$$

the neural term $G(z) = (0.1z_1, 0.1z_2)^T$, and $q(t) \in \Theta = \{1, 2\}$ is a Markov chain with its generator

$$Q = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}.$$

Through simple calculation, it can be deduced that equation (4.1) satisfies both Assumptions 2.1 and 2.2. Thus, it can be seen that equation (4.1) has a unique global solution by using Theorem 2.4. Moreover, we take $h = 1, \phi_0 = (1 + \cos(t), 1 + \sin(t))^T$ on $t \in [-1, 0]$ and $q(0) = 2$, from the numerical simulation, we can see that NSDDEwMS (4.1) is unstable. We illustrate this conclusion by simulation shown in Fig. 4.1. Next, we will design a discrete controller to stabilize NSDDEwMS (4.1). Let's give the control functions

$$u(y, 1, t) = -3y, \quad u(y, 2, t) = -2y,$$

which implies Assumption 3.1 holds with $\vartheta = 3$. In summary, all conditions of Theorem 3.2 can be satisfied, so that the controlled NSDDEwMS

$$d(y(t) - G(y(t-h))) = (f(y(t), y(t-h), q(t), t) + u(y(t-\varphi(t)), q(t), t))dt + g(y(t), y(t-h), q(t), t)dB(t) \tag{4.2}$$

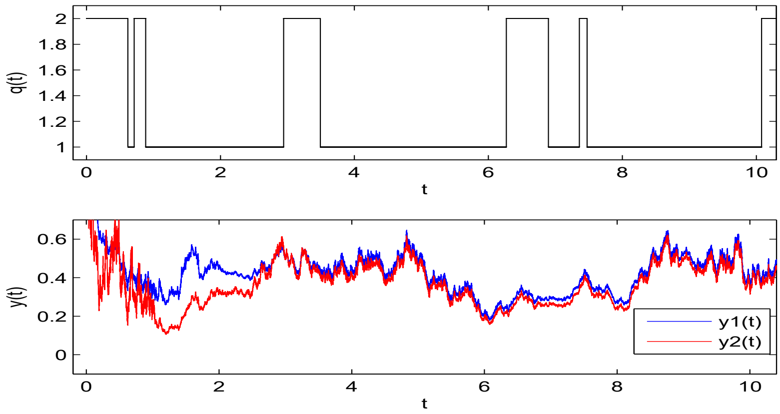

Fig. 4.1: Numerical simulation of the trajectories of the Markov chain and the solution $y(t)$ of the NSDDEwMS (4.1) using the tamed EM method [35] with step size 10^{-4}.

has a unique global continuous solution almost surely and the solution obeys that

$$\sup_{0\le t<\infty} \mathbb{E}|y(t)|^p < C_4, \quad \forall p \ge 6.$$

Let's verify the conditions in Assumption 3.4. For $(y, z, j, t) \in \mathbb{R} \times \mathbb{R} \times \Theta \times \mathbb{R}_+$, we have

$$\begin{aligned}
&(y - 0.1z)[f(y, z, j, t) + u(y, j, t)] + \frac{1}{2}|g(y, z, j, t)|^2 \\
&\le \begin{cases} -2.59625|y|^4 + |z|^4 - 1.9|y|^2 + 0.1|z|^2, & \text{if } j = 1, \\ -1.71285|y|^4 + 0.7625|z|^4 - 0.95|y|^2 + 0.05|z|^2, & \text{if } j = 2, \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
&(y - 0.1z)[f(y, z, j, t) + u(y, j, t)] + \frac{3}{2}|g(y, z, j, t)|^2 \\
&\le \begin{cases} -2.34525|y|^4 + 1.5|z|^4 - 1.9|y|^2 + 0.1|z|^2, & \text{if } j = 1, \\ -1.71825|y|^4 + 0.7625|z|^4 - 0.95|y|^2 + 0.05|z|^2, & \text{if } j = 2, \end{cases}
\end{aligned}$$

which implies $\delta_{13} = \hat{\delta}_{13} = -1.9$, $\delta_{23} = \hat{\delta}_{23} = -0.95$. Hence, both

$$\mathcal{M}_1 = \begin{pmatrix} 4.8 & -1 \\ -2 & 3.9 \end{pmatrix} \text{ and } \mathcal{M}_2 = \begin{pmatrix} 8.6 & -1 \\ -2 & 5.8 \end{pmatrix}$$

are nonsingular M-matrices. That is, Assumption 3.4 holds. Finally, let's test Assumption 3.6. Recalling (3.10) and (3.11), we have

$$\pi_1 = 0.293062, \; \pi_2 = 0.406699, \; \hat{\pi}_1 = 0.142022, \; \hat{\pi}_2 = 0.221387,$$

and

$$V(y, j) = \begin{cases} 0.293062|y|^2 + 0.142022|y|^4, & \text{if } j = 1, \\ 0.406699|y|^2 + 0.221387|y|^4, & \text{if } j = 2. \end{cases}$$

Choosing $\delta_1 = 0.6$, $\delta_2 = 0.01$ and $\delta_3 = 1.5$, we get

$$\begin{aligned}
&LU(y, z, j, t) + \delta_1(2\pi_j|y| + (l_1 + 1)\hat{\pi}_j|y|^{l_1})^2 + \delta_2|f(y, z, j, t)|^2 + \delta_3|g(y, z, j, t)|^2 \\
&\le -0.361137|y|^2 + 0.040629|z|^2 - \Phi(y) + 0.984233\Phi(z),
\end{aligned}$$

where $\Phi(y) = 2.414428y_1^4 + 2.423191y_2^4 + 0.744218y_1^6 + 0.9582075y_2^6$, which means Assumption 3.6 is also met.

By Theorems 3.8 and 3.9, when $\tau < 0.016431$, the controlled NSDDEwMS (4.2) is exponentially stable in L^q ($p > q \geq 2$) and almost surely as well. For numerical simulation, we take the same initial data as before and let $h = 1, \tau = 0.016$. The trajectories of the solution of equation (4.2) and the Markov chain are shown in Fig. 4.2.

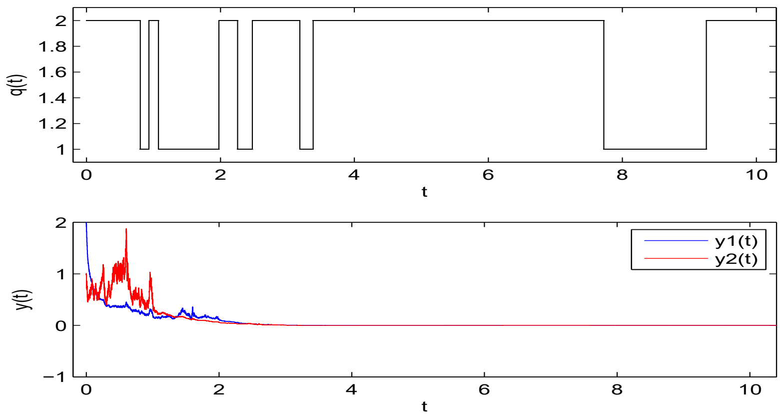

Fig. 4.2: Numerical simulation of the trajectories of the Markov chain and the solution $y(t)$ of the NSDDEwMS (4.2) with $\tau = 0.016$ using the tamed EM method [35] with step size 10^{-4}.

5. Conclusion

In this paper, it is shown that the feedback controller bases on the discrete observation state sequence can be used to stabilize highly nonlinear neutral stochastic systems, which is different from the traditional methods in the existing papers. Since the controlled NSDDEwMS (3.1) has the characteristics of superlinear coefficients and non differentiable variable delay, many existing stabilization techniques are not applicable here. We use a new method to obtain the moment boundedness of the controlled system. Under this premise, we use multiple M-matrices to describe a set of rules to ensure that the system (2.1) can be exponentially stabilized in the moment and almost surely sense as long as these rules are followed step by step.

The results of this paper can be applied to stochastic systems with G-Brownian motion, and hence the work of Yin et al [36] can be generalized. In addition, based on the results of our paper, we can also consider combining other control methods to obtain a better time interval for discrete-time state observations and further reduce the control costs.

Acknowledgements

The authors would like to thank the National Natural Science Foundation of China (71571001), the Natural Science Foundation of Universities in Anhui Province (KJ2020A0367), Startup Foundation for Introduction Talent of AHPU (2020YQQ066,2021YQQ058), Open Project of Anhui Province Center for International Research of Intelligent Control of High-end Equipment (IRICHE-01), the Royal Society (WM160014, Royal Society Wolfson Research Merit Award), the Royal Society and the Newton Fund (NA160317, Royal Society-Newton Advanced Fellowship), the Royal Society of Edinburgh (RSE1832), Shanghai Administration of Foreign Experts Affairs (21WZ2503700, the Foreign Expert Program) for their financial support.

Reference

[1] X. Mao, C. Yuan, Stochastic Differential Equations with Markovian Switching, Imperial college press, London, 2006.
[2] L. Fan, Q. Zhu, Mean square exponential stability of discrete-time Markov switched stochastic neural networks with partially unstable subsystems and mixed delays, Information Sciences 580 (2021) 243–259.
[3] L. Hu, X. Mao, Almost sure exponential stabilisation of stochastic systems by state-feedback control, Automatica 44 (2008) 465–471.
[4] X. Li, X. Mao, Stabilisation of highly nonlinear hybrid stochastic differential delay equations by delay feedback control, Automatica 112 (2020) 108657.
[5] J. Hu, W. Liu, F. Deng, X. Mao, Advances in stabilization of hybrid stochastic differential equations by delay feedback control, SIAM Journal on Control and Optimization 58 (2020) 735–754.
[6] X. Mao, Stabilization of continuous-time hybrid stochastic differential equations by discrete-time feedback control, Automatica 49 (2013) 3677–3681.
[7] S. You, W. Liu, J. Lu, X. Mao, Q. Qiu, Stabilization of hybrid systems by feedback control based on discrete–time state observations, SIAM Journal on Control and Optimization 53 (2015) 905–925.
[8] Y. Wu, S. Zhuang, W. Li, Periodically intermittent discrete observation control for synchronization of the general stochastic complex network, Automatica 110 (2019) 108591.
[9] L. Liu, Z. Wu, Intermittent stochastic stabilization based on discrete-time observation with time delay, Systems & Control Letters 137 (2020) 104626.
[10] Y. Zhu, K. Wang, Y. Ren, Dynamics of a mean-reverting stochastic volatility equation with regime switching, Communications in Nonlinear Science and Numerical Simulation 83 (2020) 105110.
[11] X. Mao, F. Wei, T. Wiriyakraikul, Positivity preserving truncated Euler-Maruyama method for stochastic Lotka-Volterra competition model, Journal of Computational and Applied Mathematics 394 (2021) 113566.
[12] L. Hu, X. Mao, Y. Shen, Stability and boundedness of nonlinear hybrid stochastic differential delay equations, Systems & Control Letters 62 (2013) 178–187.
[13] L. Hu, X. Mao, L. Zhang, Robust stability and boundedness of nonlinear hybrid stochastic differential delay equations, IEEE Transactions on Automatic Control 58 (2013) 2319–2332.
[14] W. Fei, L. Hu, X. Mao, M. Shen, Structured robust stability and boundedness of nonlinear hybrid delay systems, SIAM Journal on Control and Optimization 56 (2018) 2662–2689.
[15] C. Fei, W. Fei, X. Mao, D. Xia, L. Yan, Stabilisation of highly nonlinear hybrid systems by feedback control based on discrete-time state observations, IEEE Transactions on Automatic Control 65 (2020) 2899–2912.
[16] X. Li, X. Mao, D. S. Mukama, C. Yuan, Delay feedback control for switching diffusion systems based on discrete-time observations, SIAM Journal on Control and Optimization 58 (2020) 2900–2926.
[17] Y. Kao, J. Xie, C. Wang, H. R. Karimi, A sliding mode approach to H_∞ non-fragile observer-based control design for uncertain Markovian neutral-type stochastic systems, Automatica 52 (2015) 218–226.
[18] F. Li, Y. Liu, Event-triggered stabilization for continuous-time stochastic systems, IEEE Transactions on Automatic Control 65 (2020) 4031–4046.
[19] W. Ji, J. Qiu, L. Wu, H.-K. Lam, Fuzzy-affine-model-based output feedback dynamic sliding mode controller design of nonlinear systems, IEEE Transactions on Systems, Man, and Cybernetics: Systems 51 (2021) 1652–1661.
[20] X. Zong, T. Li, G. Yin, J. Zhang, Delay tolerance for stable stochastic systems and extensions, IEEE Transactions on Automatic Control 66 (2021) 2604–2619.
[21] E. Fridman, Introduction to Time-Delay Systems: Analysis and Control, Birkhäuser, Basel, 2014.
[22] Y. Kao, C. Wang, J. Xie, H. R. Karimi, W. Li, H_∞ sliding mode control for uncertain neutral-type stochastic systems with Markovian jumping parameters, Information Sciences 314 (2015) 200–211.
[23] H. Chen, P. Shi, C.-C. Lim, Stability analysis for neutral stochastic delay systems with Markovian switching, Systems & Control Letters 110 (2017) 38–48.
[24] H. Chen, C.-C. Lim, P. Shi, Stability analysis for stochastic neutral switched systems with time-varying delay, SIAM Journal on Control and Optimization 59 (2021) 24–49.
[25] X. Mao, Y. Shen, C. Yuan, Almost surely asymptotic stability of neutral stochastic differential delay equations with Markovian switching, Stochastic Processes and their Applications 118 (2008) 1385–1406.
[26] M. Obradović, M. Milošević, Stability of a class of neutral stochastic differential equations with unbounded delay and Markovian switching and the Euler–Maruyama method, Journal of Computational and Applied Mathematics 309 (2017) 244–266.
[27] M. Li, F. Deng, Almost sure stability with general decay rate of neutral stochastic delayed hybrid systems with lévy noise, Nonlinear Analysis: Hybrid Systems 24 (2017) 171–185.
[28] M. Shen, W. Fei, X. Mao, Y. Liang, Stability of highly nonlinear neutral stochastic differential delay equations, Systems & Control Letters 115 (2018) 1–8.
[29] W. Chen, S. Xu, Y. Zou, Stabilization of hybrid neutral stochastic differential delay equations by delay feedback control, Systems & Control Letters 88 (2016) 1–13.
[30] M. Shen, C. Fei, W. Fei, X. Mao, Stabilisation by delay feedback control for highly nonlinear neutral stochastic differential equations, Systems & Control Letters 137 (2020) 104645.
[31] L. Bai, Q. Li, H. Xu, Q. Zhang, Y. Zhang, Discrete-time feedback stabilization for hybrid neutral stochastic systems, Stochastics and Dynamics 18 (2018) 1850035.
[32] H. Chen, C. Yuan, On the asymptotic behavior for neutral stochastic differential delay equations, IEEE Transactions on Automatic Control 64 (2019) 1671–1678.
[33] X. Mao, Stochastic Differential Equations and Applications, 2nd ed., Horwood Publishing, Chichester, 2007.

[34] M. Shen, C. Fei, W. Fei, X. Mao, Boundedness and stability of highly nonlinear hybrid neutral stochastic systems with multiple delays, Science China Information Sciences 62 (2019) 202205.
[35] S. Deng, C. Fei, W. Fei, X. Mao, Tamed EM schemes for neutral stochastic differential delay equations with superlinear diffusion coefficients, Journal of Computational and Applied Mathematics 388 (2021) 113269.
[36] W. Yin, J. Cao, Y. Ren, G. Zheng, Improved results on stabilization of G-SDEs by feedback control based on discrete-time observations, SIAM Journal on Control and Optimization 59 (2021) 1927–1950.