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Minimum degree triangulation for rectangular domains

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Abstract

This paper describes an optimal triangulation algorithm for rectangles. We derive lower bounds on the maximum degree of triangulation, and show that our triangulation algorithm matches the lower bounds. Several important observations are also made, including a zig-zag condition that can verify whether a triangulation can minimizes the maximum degree to 4 or not. In addition, this paper identifies the necessary and sufficient condition that there exists a maximum degree 4 triangulation for convex polygons, and gives a linear time checking algorithm.

Keywords: Design of algorithms; Mesh generation; Rectangle and convex polygons; Min-max degree triangulation

1. Introduction

Mesh generation is one of the most import aspects in many computational disciplines, including computation fluid dynamics, image rendering, electromagnetic simulations, and resonance of piezoelectric crystal [2,8,9]. The quality of simulation results depend heavily on the quality of the mesh used in the simulation. As a result mesh generation is a very important step in all unstructured mesh related computations.

During mesh generation we often do not know where we should put the vertices, therefore when given a rectangular parametric domain, and the number of interval points on the four sides of this domain, we have to come up with an *initial triangulation*, which connects only vertices on the domain boundary. After the initial mesh is constructed, we add vertices to where they are needed as indicated by local point spacing requirement, and adjust the triangulation accordingly [3–5,7].

There has been a rich body of literature on the problem of triangulating two- and three-dimensional computation domains [1]. Triangulation techniques on various two-dimensional domains, including simple polygons, polygons with holes, point set, and planar straightline graph, were studied in [1]. For example, Delaunay triangulation produces a good triangulation in $O(n \log n)$ time for point sets, whee n is the number

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of the points. Also various quality measurements of triangulation were proposed, including the minimum triangulation angle, the maximum triangulation angle, and the minimum height of triangles.

The focus of this paper is to find a triangulation that minimizes the maximum degree for triangulation on rectangles and convex polygons. Note that we use the maximum degree of all vertices as a rough estimate on the mesh quality, since when a vertex has a large degree in a triangulation, skewed triangles with sharp angles will often occur. These sharp angles cause numerical instability in mesh computation, and should be avoided [1]. Since the quality measurement is the maximum degree of the triangulation, we do not cope with the geometry information, and the input consists of only the number of points along the rectangles or convex polygons. This simplification transforms our problem into a combinatorial optimization problem.

The minimization of maximum degree is difficult on general graphs. For example, Jansen showed that the problem to triangulate a plane geometric graph with degree at most seven is NP-complete [6]. This paper shows that for rectangles, there exists an algorithm that when given the number of vertices along the rectangle, computes an triangulation that minimizes the maximum degree of all vertices. In addition, we also generalize the results to convex polygons, and identify the necessary and sufficient conditions that there exists a maximum degree 4 triangulation.

The rest of the paper is organized as follows. Section 2 formally defines the triangulation problem and gives several lower bounds. Section 3 describes our optimal algorithm that matches the lower bounds. Section 4 generalizes the lower bounds and zig-zag conditions from rectangle to convex polygons. Finally, Section 5 concludes with possible future work and extension.

2. Rectangle triangulation

The problem of rectangle triangulation can be stated as follows. Given four numbers, a, b, c, and d, which represent the number of intervals along four sides of a rectangle, we want to triangulate the rectangle so that the maximum number of edges adjacent to a single vertex is minimized. Fig. 1 gives an example in which there are 3, 3, 2, 2 intervals along the sides of a rectangle, with the maximum degree being 6.

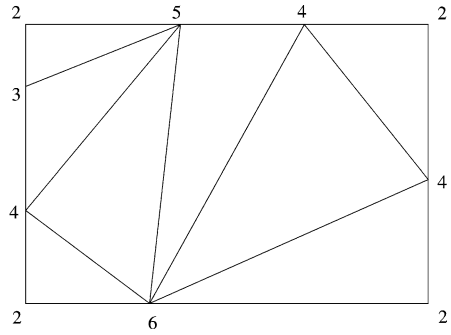


Fig. 1. An instance of rectangle triangulation. The degree of each vertex is indicated next to the vertex.

2.1. General lower bounds

Before we describe our algorithm we establish several lower bounds on the maximum degree for the rectangle triangulation problem. We first need to count the number of edges and triangles in a rectangle triangulation. In addition, we denote those edges along the boundary as "boundary" edges, and those that are added during the triangulation as "internal" edges.

Lemma 1. *Let n be the total number of intervals along the rectangle R. Every triangulation on R has $n-3$ internal edges inside R, and $n-2$ triangles.*

Proof. Consider an n-sided polygon R, which has a sum of $(n-2)\pi$ of its inner angles. That means the number of triangles from any triangulation is $n-2$. In addition, each edge adds a new triangle into the triangulation so the number of internal edges within R is $n-3$. □

We now count the number of corner vertices that have degree 2, i.e., those corners that are not connected to any other vertices. We will refer to such corner as "isolated", and show that there are at least 2 isolated corners.

Lemma 2. *The number of isolated corners is at least* 2.

Proof. Consider a corner c. If c is not isolated, i.e., c is connected to a vertex b, then there exists another corner a such that a, b and c form a triangle (see Fig. 2 for an illustration). Apparently a will not connect to any vertex, so it is isolated. As a result we will have at least two isolated corner in every triangulation. □

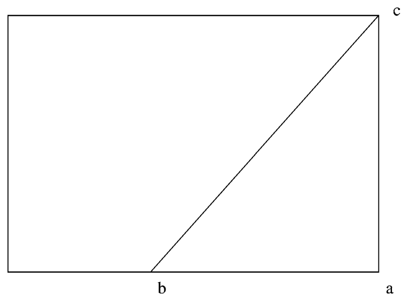


Fig. 2. An illustration that a non-isolated corner will introduce an isolated one.

Theorem 3. *For every rectangle triangulation with at least* 5 *intervals along the boundary, there exists a vertex with degree at least* 4.

Proof. From Lemma 1 there are exactly $n-3$ internal edges and n boundary edges. Each of them will be counted twice from both endpoints so the total number of degrees of all vertices is $4n-6$. Also from Lemma 2 we have at least two isolated corner with degree 2, therefore the total number of degrees besides these two isolated corners is $4n-10$. As a result at least one vertex among those remaining $n-2$ vertices will have at least $\lceil (4n-10)/(n-2) \rceil$ adjacent edges. This number is at least 4 when n is at least 5. □

We establish another general lower bound on the maximum degree for rectangle triangulation. We need this lower bound in order to prove that our triangulation algorithm, which will be introduced in the next section, is optimal.

Theorem 4. *Consider a rectangle R that has a, b, c, and d intervals on four boundaries. The maximum degree for any triangulation is at least* $\lceil (a-2)/(b+c+d-1) \rceil + 3$.

Proof. Let A be the boundary with a intervals. We use N to denote the set of the $b+c+d-1$ vertices that are not on A. From Lemma 1 we know that there are $a+b+c+d-3$ internal edges. Since every one of these edges must connect to a vertex in N, there exists a vertex in N that is adjacent to

$$\left\lceil \frac{a+b+c+d-3}{b+c+d-1} \right\rceil = \left\lceil \frac{a-2}{b+c+d-1} \right\rceil + 1$$

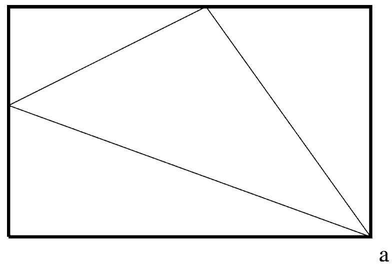


Fig. 3. The vertex a is of degree 4, but is not connected to two adjacent vertices along the boundary.

edges. Adding the two boundary edges and the lower bound follows. □

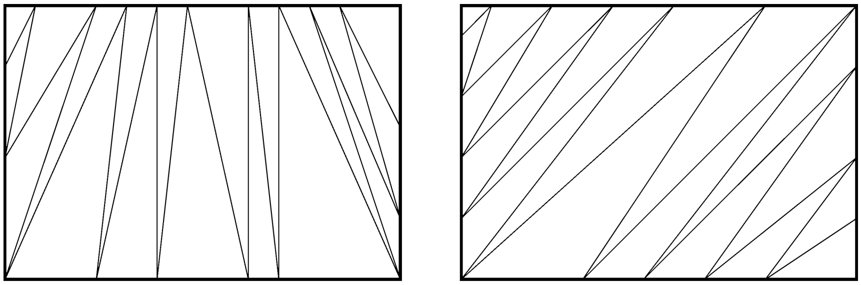
2.2. Specific lower bounds

Theorem 3 establishes a general lower bound on the maximum degree, and we will further improve this bound when the numbers of intervals along the boundary are in a particular condition. We will establish the necessary and sufficient condition for being able to find a triangulation with maximum degree 4—any configuration not satisfying this condition will have a maximum degree at least 5.

Theorem 5. *A rectangle R with at least* 7 *intervals on its boundary has a maximum degree* 4 *triangulation if and only if one of two following conditions holds. This will be referred to as the "zig-zag" condition.*

- *The number of intervals on one side of R and the number of intervals on the other three sides differ by at most* 1.
- *The number of intervals on two adjacent sides of R and the number of intervals on the other two sides differ by at most* 1.

Proof. A rectangle R with at least 7 intervals on its boundary has a maximum degree 4 triangulation only when every vertex of degree 4 is connected to adjacent vertices along the boundary. The reason is that no more than 2 internal edges are adjacent to a vertex, and the internal edges induce a connected graph. Note that we need at least 7 intervals to avoid the configuration in Fig. 3, in which the vertex a is *not* connected to two adjacent vertices along the boundary, and it does have only degree of 4.



(a) (b)

Fig. 4. Two cases of zig-zag.

Now the vertices with degree 4 must be connected in zig-zag fashion, i.e., they will go to different sides of the rectangle back and forth since they have to connect to adjacent vertices along the boundary. In addition, those vertices on the same side cannot connect to each other, therefore the zig-zag is either going from one side to the other three, or between two groups—each has two adjacent sides of R, as indicated in Fig. 4. Notice that the number of intervals on the two sides of zig-zagging could differ by 1, as in Fig. 4(a), or be the same, as in Fig. 4(b). □

The implication of Theorem 5 is that if we are given a rectangle R that does not satisfy the zig-zag condition, we know immediately that the maximum degree is at least 5.

3. The optimal algorithm

We now describe our optimal algorithm $\mathcal{Z}$. Let A be the side that has the maximum number of intervals in R, and a be the number of intervals in A. We will consider only two cases—when a is greater than $b + c + d + 1$, or less than $b + c + d - 1$. If neither case is true the zig-zag condition applies and $\mathcal{Z}$ simply zig-zags the rectangle with optimal degree 4.

3.1. *The case when $a > b + c + d + 1$*

Since A has a very large number of intervals, the idea is to evenly distribute them to the vertices on the other three sides. The algorithm $\mathcal{Z}$ first identifies the $a - 2$ *intervals* in the middle of A, then evenly distributes them to the $b + c + d - 1$ vertices not in A.

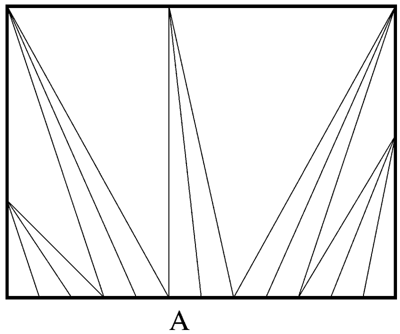


Fig. 5. Distributing the intervals in A to vertices not in A.

Fig. 5 illustrates an example of distributing 10 intervals in the middle of A to 5 vertices not in A, so that each vertex not in A has degree 3.

The algorithm $\mathcal{Z}$ produces an optimal maximum degree triangulation. The algorithm guarantees that a vertices not in A will be connected to at most $\lceil (a-2)/(b+c+d-1) \rceil + 1$ vertices on A, since there are $a - 2$ intervals and $b+c+d-1$ vertices. The maximum degree is then bounded by $\lceil (a-2)/(b+c+d-1) \rceil + 3$ after adding the two boundary edges. This number matches the lower bound in Theorem 4.

3.2. *The case when $a < b + c + d - 1$*

Now we consider the second case when $a < b + c + d - 1$. Let A be the side with a intervals, C be the opposite side of A (with c intervals), and B and D be the two adjacent sides of A, with b and d intervals each. Without loss of generality, we assume that $b \geqslant d$, therefore $a + b \geqslant c + d$ since a has the maximum number of intervals. We further assume that $a + b > c + d + 1$, otherwise the zig-zag condition

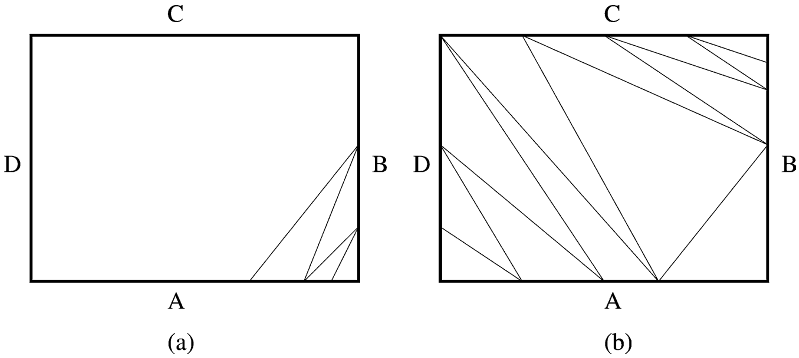


Fig. 6. We zig-zag on $\lceil (e+1)/2 \rceil$ intervals on A with $\lfloor (e+1)/2 \rfloor$ intervals in B in the corner (a), then zig-zag on the remaining part (b).

holds and we simply use the zig-zag to obtain the optimal maximum degree 4 triangulation.

3.2.1. Corner removal

The key idea in the algorithm $\mathcal{Z}$ is to remove some of the intervals from A and B, so that the number of intervals in C and D is the same as those remaining in A and B. In that case $\mathcal{Z}$ can apply the zig-zag method and obtain a low degree triangulation. Let $e = (a+b) - (c+d)$. $\mathcal{Z}$ first zig-zags on $\lceil (e+1)/2 \rceil$ intervals on A with $\lfloor (e+1)/2 \rfloor$ intervals in B, as shown in Fig. 6(a). We denote this as *corner zig-zagging*. Notice that $\lceil (e+1)/2 \rceil$ and $\lfloor (e+1)/2 \rfloor$ differ by at most one so the zig-zagging is possible. After removing $\lceil (e+1)/2 \rceil$ and $\lfloor (e+1)/2 \rfloor$ from A and B, we have introduced a new edge on the corner, so the number of intervals on A and B becomes $a+b-\lceil (e+1)/2 \rceil - \lfloor (e+1)/2 \rfloor + 1 = a+b-e = c+d$, and $\mathcal{Z}$ can zig-zag the rest of the rectangle (Fig. 6(b)). We denote this as *interior zig-zagging*.

Now we need to show that A and B indeed have at least $\lceil (e+1)/2 \rceil$ and $\lfloor (e+1)/2 \rfloor$ intervals so that the "corner-removal" method will work.

Lemma 6. *$a \geqslant \lceil (e+1)/2 \rceil$ and $b \geqslant \lfloor (e+1)/2 \rfloor$, where $e = (a+b) - (c+d)$.*

Proof. We prove the inequalities by contradiction. Suppose that $a < \lceil (e+1)/2 \rceil$, then

$$a < \left\lceil \frac{a+b-c-d+1}{2} \right\rceil \leqslant \frac{a+b-c-d+1}{2} + \frac{1}{2}.$$

This implies $a < b-c-d+2 \leqslant b$, which contradicts to the assumption that a is the maximum among four sides. Again suppose that $b < \lfloor (e+1)/2 \rfloor \leqslant (a+b-c-d)/2$, which implies $b < a-c-d+1$, and $a \geqslant b+c+d$. This is also a contradiction to the assumption that $a < b+c+d$–1. □

3.2.2. Final adjustment

The construction above gives a maximum degree 6 triangulation—the endpoints of the new interval introduced in the corner removal may have two edges from interior zig-zagging, one edge from the new interval, one from the corner zig-zagging, and two from the boundary. In order to reduce the maximum degree to 5 we need a final adjustment. We have two cases to consider—one when the corner removal removes all the intervals from either A or B, and the case that it leaves at least one interval on both A and B. Note that the corner removal cannot remove all the intervals from both A and B since that will leave only one interval on the A and B, but we have at least two intervals on C and D.

In the first case the corner removal removes all intervals from either A or B. Let the edge (f, g) be the internal edge introduced by the corner removal, and g be the one that is not at a corner (Fig. 7(a)). When the section between g and h has one more intervals than the section between h and f, the corner zig-zagging may start from g in order to control the maximum degree within 4 (Fig. 7(a)). In that case g may have degree 6 when the interior zig-zagging introduces two additional edges to g. $\mathcal{Z}$ can reduce the degree of g by changing the *orientation* of the corner zig-zagging, i.e., starting the zig-zagging from g instead of from f, as shown in Fig. 7(b). This change add an edge to

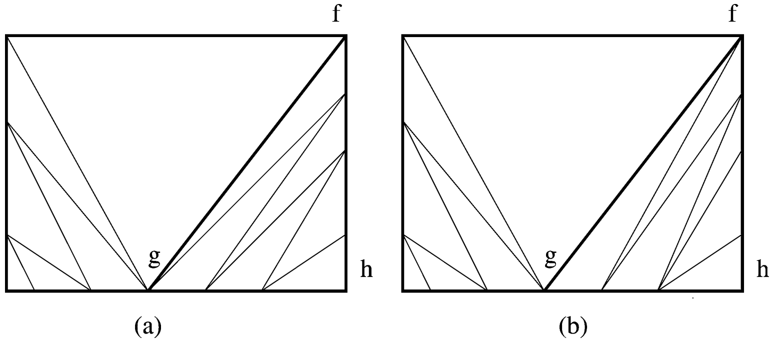


Fig. 7. To change the orientation of the corner zig-zagging.

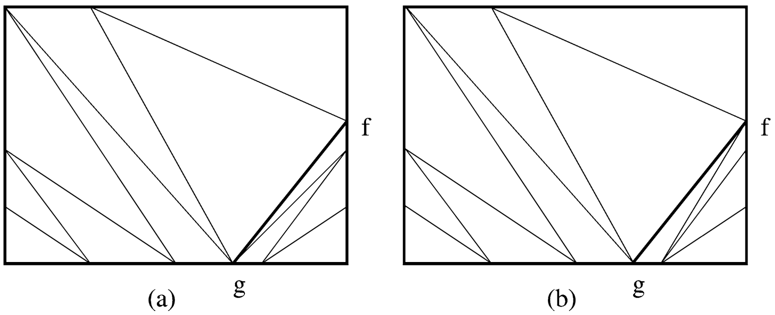


Fig. 8. Final adjustment to reduce the maximum degree to 5.

both f and the last vertex of the corner zig-zagging, but the maximum degree overall is now 5.

In the second case the corner removal will not remove all the intervals from either A or B. Again consider the two end points f and g of the added edge during corner removal. If the interior zig-zagging starts from either of the endpoints, say f, f will have only one interior zig-zagging edge. In the worst case g could have degree 6 if the corner zig-zagging also starts from g (Fig. 8(a))—two boundary edges, two interior zig-zagging edges, one edge connecting to f, and an edge that starts the corner zig-zagging. In that case $\mathcal{Z}$ simply changes the orientation of the corner zig-zagging and let f starts. Similar to the construction in the first case we can still control the maximum degree to be 5 (Fig. 8(b)).

Now we consider the situation that the interior zig-zagging does not start at either f or g. That is, both f and g have two interior zig-zagging edges, and $\mathcal{Z}$ cannot avoid the maximum degree 6 simply by switching the corner zig-zagging orientation. Let us locate the two vertices that appear two steps before, or after f and g during the interior zig-zagging. If either of these two nodes exist, we replace the edge as indicated in Fig. 9(a), i.e., $\mathcal{Z}$ replaces the edge from i to f with the one from h to j. This again controls the maximum degree within 5. On the other hand if $\mathcal{Z}$ cannot locate the two nodes that appear two steps before, or after f and g, then since f and g both have two interior zig-zagging edges by assumption, the interior zig-zagging is like in Fig. 9(b). This is impossible since after the corner removal the interior zig-zagging always has equal number of intervals on both sides.

We now summarize the upper bounds from the algorithm $\mathcal{Z}$.

Theorem 7. *The algorithm $\mathcal{Z}$ produces an optimal triangulation of a rectangles. The execution time of algorithm $\mathcal{Z}$ is a linear function of the total number of intervals. Let a, b, c, d be the number of intervals of the four sides of a rectangle, and a be the maximum among them.*

- *If a is between $b + c + d - 1$ and $b + c + d + 1$ inclusively, the zig-zag condition holds and $\mathcal{Z}$ produces an optimal triangulation by zig-zagging.*

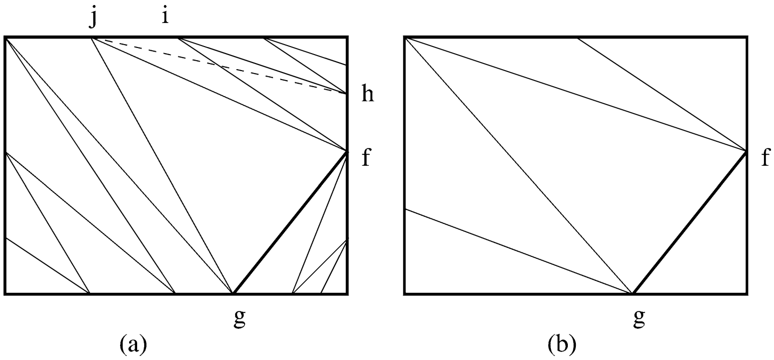


Fig. 9. Final adjustment to reduce the maximum degree to 5.

- *When $a > b + c + d + 1$, $\mathcal{Z}$ produces an optimal triangulation with maximum degree $\lceil (a-2)/(b+c+d-1) \rceil + 3$ according to Theorem 4.*
- *When $a < b + c + d - 1$ we have two cases. If the zig-zag condition holds then $\mathcal{Z}$ produces an optimal triangulation with maximum degree 4 by zig-zagging, else $\mathcal{Z}$ produces an optimal triangulation with maximum degree 5, according to the discussion in Theorem 5.*

4. Convex polygons

We generalize the previous results on rectangular domains to convex polygons. We first show that the general lower bound of 4 in Theorem 3 is still valid.

Theorem 8. *For every triangulation on a convex polygon with more than four sides, there exists a vertex with degree at least 4.*

Proof. It is easy to see that Lemmas 1 and 2 are still valid for polygons. Lemma 1 makes no difference between a rectangle with n intervals or a polygon with the same number of intervals. If a corner is not isolated, then it must connect to another vertex via an edge, which will divide a polygon into two halves. In each of these two smaller polygons there exists at least one isolated corner and Lemma 2 follows. With a similar argument as in Theorem 3 we conclude that the lower bound on the maximum degree is 4 for all polygons. □

Lemma 9. *An n-sided polygon P with at least $n + 3$ intervals on its boundary has a maximum degree 4 triangulation only if every node of degree 4 is connected to adjacent nodes along the boundary.*

Proof. The proof is as same as in Theorem 5. We only need to make sure that there are at least $n + 3$ intervals. □

With Lemma 9 in place we now identify the necessary and sufficient conditions that a polygon can have a maximum degree 4 triangulation.

Theorem 10. *An n-sided polygon P with at least $n+3$ intervals on its boundary has a maximum degree 4 triangulation if and only if the n sides of P can be partitioned into two sets A and B, each consisting of consecutive sides of P, and the numbers of intervals in A and B differ by at most 1. This "zig-zag condition" can be verified in $O(n)$ time.*

Proof. We only need to give an algorithm that can verify the zig-zag condition in linear time. Consider a ring with n numbers on it. We would like to verify that if there is a way to cut the ring into two parts so that the sums of the numbers on the two parts differ by at most 1.

First we compute the sum of all the numbers on the ring (denoted by S), which takes $O(n)$ time. We then use two pointers, *head* and *tail*, to indicate the two positions where the ring should be cut. Initially we set both head and tail to the same position in the ring. If the sum of the numbers between head and tail (inclusively) is less than $\lfloor S/2 \rfloor$. We advance the head pointer. If the sum of the numbers between head and tail (inclusive) is greater than $\lceil S/2 \rceil$, we advance the tail pointer. If neither of the above two conditions is true, we have a zig-zag condition, and the region between head and tail (inclusive) is the answer. If the tail pointer passes through the head pointer, or the starting position where the head pointer started, the zig-zag condition is not possible. It is easy to see that neither pointer will advance for more than $O(n)$ steps, and the time to update the sum of numbers between the head and the tail is $O(1)$, therefore the entire verification process takes $O(n)$ time. □

5. Conclusion

This paper describes an optimal triangulation algorithm for rectangles. We derive lower bounds on the maximum degree of rectangle triangulation, and develop optimal algorithms that produce triangulation with maximum degrees matching the lower bounds. We also observed a zig-zag condition that can verify whether a triangulation has a maximum degree 4 triangulation.

This paper does not cope with the actual geometric location of the points along the boundary, therefore a small degree triangulation can still result in sharp angles. However, a small degree triangulation is a good starting point for further optimization, since a large degree triangulation will produce sharp angles most of the time.

The lower bound results and the zig-zag condition theorem can be generalized from rectangles to convex polygons. We identify the zig-zag condition that defines the necessary and sufficient condition to control the maximum degree not exceeding 4. This is useful when a rectangle 2D domain has to be reshaped into a convex polygon to fit the computation domain better. This zig-zag condition can be verified in linear time.

Acknowledgements

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