On the Cubicity of Bipartite Graphs

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Abstract

A unit cube in k-dimension (or a k-cube) is defined as the cartesian product $R_1 \times R_2 \times \cdots \times R_k$, where each R_i is a closed interval on the real line of the form $[a_i, a_i + 1]$. The cubicity of G, denoted as $cub(G)$, is the minimum k such that G is the intersection graph of a collection of k-cubes. Many NP-complete graph problems can be solved efficiently or have good approximation ratios in graphs of low cubicity. In most of these cases the first step is to get a low dimensional cube representation of the given graph.

It is known that for a graph G, $cub(G) \leq \left\lfloor \frac{2n}{3} \right\rfloor$. Recently it has been shown that for a graph G, $cub(G) \leq 4(\Delta+1)\ln n$, where n and Δ are the number of vertices and maximum degree of G, respectively. In this paper, we show that for a bipartite graph $G = (A \cup B, E)$ with $|A| = n_1$, $|B| = n_2$, $n_1 \leq n_2$, and $\Delta' = \min\{\Delta_A, \Delta_B\}$, where $\Delta_A = max_{a\in A} d(a)$ and $\Delta_B = max_{b\in B} d(b)$, $d(a)$ and $d(b)$ being the degree of a and b in G respectively, $cub(G) \leq 2(\Delta' + 2)\lceil \ln n_2 \rceil$. We also give an efficient randomized algorithm to construct the cube representation of G in $3(\Delta' + 2)\lceil \ln n_2 \rceil$ dimensions. The reader may note that in general Δ' can be much smaller than Δ.

Keywords: Cubicity, algorithms, intersection graphs.

1 Introduction

Let $\mathcal{F}$ be a family of non-empty sets. An undirected graph G is an intersection graph for $\mathcal{F}$ if there exists a one-one correspondence between the vertices of G and the sets in $\mathcal{F}$ such that two vertices in G are adjacent if and only if the corresponding sets have non-empty intersection. If $\mathcal{F}$ is a family of intervals on real line, then G is called an *interval graph*. If $\mathcal{F}$ is a family of intervals on real line such that all the intervals are of equal length, then G is called a *unit interval graph*.

A unit cube in k-dimensional space or a k-cube is defined as the cartesian product $R_1 \times R_2 \times \cdots \times R_k$, where each R_i is a closed interval on the real line of the form $[a_i, a_i + 1]$. A k-cube representation of a graph is a mapping of the vertices of G to k-cubes such that two vertices in G

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are adjacent if and only if their corresponding k-cubes have a non-empty intersection. The *cubicity* of G is the minimum k such that G has a k-cube representation. Note that a k-cube representation of G using cubes with unit side length is equivalent to a k-cube representation where the cubes have side length c for some fixed positive number c. The graphs of cubicity 1 are exactly the class of unit interval graphs.

The concept of cubicity was introduced by F. S. Roberts [9] in 1969. This concept generalizes the concept of unit interval graphs. If we require that each vertex of G correspond to a k-dimensional axis-parallel box $R_1 \times R_2 \times \cdots \times R_k$, where each R_i, $1 \leq i \leq k$, is a closed interval of the form $[a_i, b_i]$ on the real line, then the minimum dimension required to represent G is called its *boxicity* denoted as $box(G)$. Clearly $box(G) \leq cub(G)$, for a graph G. It has been shown that deciding whether the cubicity of a given graph is at least three is NP-complete [11]. Computing the boxicity of a graph was shown to be NP-hard by Cozzens in [5]. This was later strengthened by Yannakakis [11], and finally by Kratochvil [6] who showed that deciding whether boxicity of a graph is at most two itself is NP-complete.

Thus, it is interesting to design efficient algorithms to represent small cubicity graphs in low dimension. There have been many attempts to bound the cubicity of graph classes with special structure. The cube and box representations of special classes of graphs like hypercubes and complete multipartite graphs were investigated in [1, 2, 3, 7, 8, 9, 10].

1.1 Our results

Recently Chandran *et al.* [4] have shown that for a graph G, $cub(G) \leq 4(\Delta + 1)\ln n$, where n and Δ are the number of vertices and maximum degree of G, respectively. In this paper, we present an efficient randomized algorithm to construct a cube representation of *bipartite graphs* in low dimension. In particular, we show that for a bipartite graph $G = (A \cup B, E)$, $cub(G) \leq 2(\Delta' + 2)\lceil \ln n_2 \rceil$, where $|A| = n_1$, $|B| = n_2$, $n_1 \leq n_2$, and $\Delta' = \min\{\Delta_A, \Delta_B\}$, where $\Delta_A = \max_{a \in A} d(a)$ and $\Delta_B = \max_{b \in B} d(b)$, $d(a)$ and $d(b)$ being the degree of a and b in G, respectively. The algorithm presented in this paper is not very different from that of [4] but this has the advantage that it gives a better result in the case of bipartite graphs. Note that, Δ' can be much smaller than Δ in general, where Δ is the maximum degree of G. In particular, when $|A| \ll |B|$, then the bound for cubicity given in this paper can be much better than that given in [4]. Also, the complexity of our algorithm is comparable with the complexity of the algorithm proposed in [4].

2 Preliminaries

Let $G = (A \cup B, E)$ be a simple, finite bipartite graph. Let $|A| = n_1$, $|B| = n_2$, and $n_1 \leq n_2$. Let $N(v) = \{w \in V(G) | vw \in E(G)\}$ be the set of neighbors of v. Degree of a vertex v, denoted as $d(v)$, is defined as the number of edges incident on v. That is, $d(v) = |N(v)|$. Suppose Δ_A denote the maximum degree in A and Δ_B denote the maximum degree in B. That is, $\Delta_A = \max_{a \in A} d(a)$ and $\Delta_B = \max_{b \in B} d(b)$.

For a graph G, let G' be a graph such that $V(G') = V(G)$. Then, G' is a *super graph* of G if $E(G) \subseteq E(G')$. We define the *intersection* of two graphs as follows: if G_1 and G_2 are two graphs such that $V(G_1) = V(G_2)$, then the intersection of G_1 and G_2 denoted as $G = G_1 \cap G_2$ is a graph with $V(G) = V(G_1) = V(G_2)$ and $E(G) = E(G_1) \cap E(G_2)$.

Let $I_1, I_2, \ldots, I_k$ be k unit interval graphs such that $G = I_1 \cap I_2 \cap \cdots \cap I_k$, then $I_1, I_2, \ldots, I_k$ is called an *unit interval graph representation* of G. The following equivalence is well known.

Theorem 2.1 ([9])**.** *The minimum k such that there exists a unit interval graph representation of G using k unit interval graphs $I_1, I_2, \ldots, I_k$ is the same as $cub(G)$.*

3 Construction

Let $G = (A \cup B, E)$ be a bipartite graph. In this section we describe an algorithm to efficiently compute a cube representation of G in $2(\Delta' + 2)\lceil \ln n_2 \rceil$ dimensions, where $\Delta' = \min\{\Delta_A, \Delta_B\}$.

Definition 3.1. *Let π be a permutation of the set $\{1, 2, \ldots, n\}$ and $X \subseteq \{1, 2, \ldots, n\}$. The projection of π onto X denoted as π_X is defined as follows. Let $X = \{u_1, u_2, \ldots, u_r\}$ be such that $\pi(u_1) < \pi(u_2) < \ldots < \pi(u_r)$. Then $\pi_X(u_1) = 1, \pi_X(u_2) = 2, \ldots, \pi_X(u_r) = r$.*

Definition 3.2. *A graph $G = (V, E)$ is a unit interval graph if and only if there exists a function $f : V \longrightarrow R$ and a constant c such that $(u, v) \in E(G)$ if and only if $|f(u) - f(v)| \le c$.*

Remark: Note that the above definition is consistent with the definition of the unit interval graphs given at the beginning of the introduction.

Let $G = (A \cup B, E)$ be a bipartite graph. Given a permutation of the vertices of A, we construct a unit interval graph $U(\pi, A, B, G)$ as follows. Let $f : A \cup B \longrightarrow R$ be such that if $v \in A$, then $f(v) = \pi(v)$ and if $v \in B$, then $f(v) = n + \min_{x \in N(v)} \pi(x)$. Two vertices $u, v \in A \cup B$ are made adjacent if and only if $|f(u) - f(v)| \le n$, where $n = |A| + |B| = n_1 + n_2$.

Claim 1: Let $G' = U(\pi, A, B, G)$. Then G' is a supergraph of G.

Proof. Suppose $(a, b) \in E(G)$. Without loss of generality suppose $a \in A$ and $b \in B$. Let $s = \min_{x \in N(b)} \pi(x)$. So, $f(b) = n + s$. As $f(a) = \pi(a)$ and $a \in N(b)$, $\pi(a) \ge s$. Therefore we have, $|f(b) - f(a)| = n + s - \pi(a) \le n$. Thus $(a, b) \in E(G')$. Hence G' is a supergraph of G. □

Remark: Note that if we reverse the roles of A and B in the above construction, i.e., if we start with a permutation of the vertices of B rather than that of A, then the resulting unit interval graph will be denoted as $U(\pi, B, A, G)$. Clearly, $U(\pi, B, A, G)$ will also be a super graph of G.

RANDUNIT
Input: A bipartite graph $G = (A \cup B, E)$.
Output: A unit interval graph G' which is a supergraph of G.
begin
if $(\Delta_B \le \Delta_A)$ then
Step 1. Generate a permutation π of $\{1, 2, \ldots, n_1\}$ (the vertices of A)
uniformly at random.
Step 2. Return $G' = U(\pi, A, B, G)$.
else
Step 1. Generate a permutation π of $\{1, 2, \ldots, n_2\}$ (the vertices of B)
uniformly at random.

Step 2. Return $G' = U(\pi, B, A, G)$.

end

Lemma 3.1. *Let $a \in A$ and $b \in B$ be such that $e = (a, b) \notin E(G)$. Let G' be the output of* **RANDUNIT(G)**. *Then*

$$\Pr[e \in E(G')] \leq \tfrac{\Delta'}{\Delta'+1}.$$

Proof. **Case I:** $\Delta_B \leq \Delta_A$.

Let π be a permutation of the vertices in A. Let $G' = U(\pi, A, B, G)$. Suppose two vertices $a \in A$ and $b \in B$ are non-adjacent in G. Let $t = \min_{x \in N(b)} \pi(x)$.

Claim: The vertices a and b will be adjacent in G' if and only if $\pi(a) > t$.

If a and b are adjacent in G', then we have $|f(b) - f(a)| = |(n + t) - \pi(a)| \leq n$, i.e., $\pi(a) > t$. Hence a is adjacent to b in G'.

So, $\Pr[e \in E(G')] = \Pr[\pi(a) > t] = 1 - \Pr[\pi(a) < t]$. (Note that $\pi(a) \neq t$, since $a \notin N(b)$.) Let $X = \{a\} \cup N(b)$ and π_X be the projection of π on X. Total number of permutations of X is $(d(b) + 1)!$. Now, it can be easily seen that $\pi(a) < t$ if and only if $\pi_X(a) = 1$. Thus,

$$\begin{aligned}
\Pr[(a, b) \in E(G')] &= 1 - \frac{d(b)!}{(d(b) + 1)!} \\
&= \frac{d(b)}{d(b) + 1} \\
&\leq \frac{\Delta'}{\Delta' + 1}
\end{aligned}$$

Hence the lemma.

Case II: $\Delta_A \leq \Delta_B$.

Let π be the permutation of the vertices in B. Let $G' = U(\pi, B, A, G)$. Proof is similar to case I. □

Lemma 3.2. *Given a bipartite graph $G = (A \cup B, E)$, there exists a super graph G^* of G with $cub(G^*) \leq 2(\Delta' + 1) \ln n_2$, such that if $u \in A$, $v \in B$ and $(u, v) \notin E(G)$, then $(u, v) \notin E(G^*)$.*

Proof. Let $U_1, U_2, \ldots, U_t$ be the unit interval graphs generated by t invocations of **RANDUNIT(G)**. Clearly U_i, for each i, $1 \leq i \leq t$, is a super graph of G by Claim 1. Let $G^* = U_1 \cap U_2 \cap \cdots \cap U_t$. Now let $u \in A$, $v \in B$ and $(u, v) \notin E(G)$. Then,

$\Pr[(u, v) \in G^*] = \Pr\left[\bigwedge_{1 \leq i \leq t} (u, v) \in E(U_i)\right] \leq \left(\frac{\Delta'}{\Delta'+1}\right)^t$ (Applying Lemma 3.1). Now,

$$\begin{aligned}
Pr\left[\bigvee_{u \in A, b \in B, (u,v) \notin E(G)} (u, v) \in E(G^*)\right] &< n_1 n_2 \left(\frac{\Delta'}{\Delta' + 1}\right)^t \\
&\leq n_2^2 \left(1 - \frac{1}{\Delta' + 1}\right)^t \\
&\leq n_2^2 \, e^{-\frac{t}{\Delta'+1}}
\end{aligned}$$

If $t = 2(\Delta' + 1) \ln n_2$ the above probability is < 1. Thus we infer that there exists a super graph G^* of G such that if $u \in A$, $v \in B$ and $(u, v) \notin E(G)$, $(u, v) \notin E(G^*)$ also. From the definition of G^* we have $cub(G^*) \leq 2(\Delta' + 1) \ln n_2$. Hence the Lemma. □

Remark: If we had chosen $t = 3(\Delta' + 1) \ln n_2$ in the above proof, we can substantially reduce the failure probability. More precisely we can get

$$\Pr(G^* \text{ does not satisfy the desired property }) \leq \frac{1}{n_2}$$

Now we will construct two special graphs H_1 and H_2 such that H_i is a super graph of G for $i = 1, 2$.

Definition 3.3. *Let $A = \{v_1, v_2, \ldots, v_{n_1}\}$. For $1 \leq i \leq \lceil \ln n_1 \rceil$ define the function $f_i : A \cup B \to R$ as follows:*

$$\begin{aligned} \textit{For vertices from } A,\ f_i(v_j) &= 0 \textit{ if the } i\textit{th bit of } j \textit{ is } 0 \\ f_i(v_j) &= 2 \textit{ if the } i\textit{th bit of } j \textit{ is } 1 \\ \textit{For vertices in } u \in B,\ f_i(u) &= 1 \end{aligned}$$

Let I_i be the unit interval graph defined on the vertex set $A \cup B$ such that two vertices u and v are adjacent if and only if $|f_i(u) - f_i(v)| \leq 1$.

Now define $H_1 = \bigcap_{i=1}^{\lceil \ln n_1 \rceil} I_i$. Thus we have $cub(H_1) \leq \lceil \ln n_1 \rceil$.

Definition 3.4. *Let $B = \{u_1, u_2, \ldots, u_{n_2}\}$. For $1 \leq i \leq \lceil \ln n_2 \rceil$ define the function $g_i : A \cup B \to R$ as follows:*

$$\begin{aligned} \textit{For vertices from } B,\ g_i(u_j) &= 0 \textit{ if the } i\textit{th bit of } j \textit{ is } 0 \\ g_i(u_j) &= 2 \textit{ if the } i\textit{th bit of } j \textit{ is } 1 \\ \textit{For vertices in } v \in A,\ g_i(v) &= 1 \end{aligned}$$

Let J_i be the unit interval graph defined on the vertex set $A \cup B$ such that two vertices u and v are adjacent if and only if $|g_i(u) - g_i(v)| \leq 1$.

Now define $H_2 = \bigcap_{i=1}^{\lceil \ln n_2 \rceil} J_i$. Thus $cub(H_2) \leq \lceil \ln n_2 \rceil$.

Lemma 3.3. *H_1 is a super graph of G such that if $u, v \in A$, then $(u, v) \notin E(H_1)$.*

Proof. It is easy to check that I_i is a super graph of G for each i. Thus H_1 is clearly a super graph of G. For $u, v \in A$, let $u = v_j$ and $v = v_k$ where $k \neq j$. Then clearly there exists a t, $1 \leq t \leq \lceil \ln n_1 \rceil$ such that j and k differs in the tth bit position. Now it is easy to verify that u and v will not be adjacent in I_t. It follows that for any pair (u, v) where $u, v \in A$ there exists I_t such that $(u, v) \notin E(I_t)$. Then clearly $(u, v) \notin E(H_1)$ also. Hence the Lemma. □

Lemma 3.4. *H_2 is a super graph of G such that if $u, v \in B$, then $(u, v) \notin E(H_2)$.*

Proof. The proof is similar to that of the Lemma 3.3. □

Theorem 3.5. *Given a bipartite graph $G = (A \cup B, E)$, $cub(G) \leq 2(\Delta' + 2)\lceil \ln n_2 \rceil$.*

Proof. By Lemma 3.2, there exists a super graph G^* of G such that if $u \in A$, $v \in B$ and $(u, v) \notin E(G)$, then $(u, v) \notin E(G^*)$. Also let H_1 and H_2 be the super graphs of G, from definitions 3.3 and 3.4 respectively. Now we claim that $G = G^* \cap H_1 \cap H_2$. Cleary $G^* \cap H_1 \cap H_2$ is a super graph of G, because each of them is a super graph of G. Now to see that $G^* \cap H_1 \cap H_2 = G$ we only need to prove that if $(u, v) \notin G$, then (u, v) is not an edge of at least one of these three graphs. Now, if $u \in A$ and $v \in B$, $(u, v) \notin E(G^*)$ by Lemma 3.2. If $u, v \in A$, then $(u, v) \notin E(H_1)$ by Lemma 3.3 and if $u, v \in B$, then $(u, v) \notin E(H_2)$ by Lemma 3.4.

Now, $cub(G) = cub(G^* \cap H_1 \cap H_2) \leq cub(G^*) + cub(H_1) + cub(H_2)$. By Lemma 3.2 $cub(G^*) \leq 2(\Delta' + 1) \ln n_2$. Also by the definition of H_1 and H_2 we have $cub(H_1) \leq \lceil \ln n_1 \rceil$ and $cub(H_2) \leq \lceil \ln n_2 \rceil$. Thus we have,

$$\begin{aligned} cub(G) &\leq 2(\Delta' + 1) \ln n_2 + \lceil \ln n_1 \rceil + \lceil \ln n_2 \rceil \\ &\leq 2(\Delta' + 1) \ln n_2 + 2\lceil \ln n_2 \rceil \qquad \text{as } n_1 \leq n_2 \\ &= 2(\Delta' + 2)\lceil \ln n_2 \rceil \end{aligned}$$

Hence the theorem. ∎

Remark: In view of the Remark after Lemma 3.2, we can infer that if $t \geq 3(\Delta' + 1) \ln n_2$, $G = G^* \cap H_1 \cap H_2$ with high probability. But then the cube representation output by the algorithm will be in $3(\Delta' + 1) \ln n_2 + \lceil \ln n_2 \rceil + \lceil \ln n_1 \rceil \leq 3(\Delta' + 2)\lceil \ln n_2 \rceil$ dimensions. The following Theorem gives the time complexity of our randomized algorithm to construct such a cube representation.

Theorem 3.6. *Let $G = (A \cup B, E)$ be a bipartite graph with $n = n_1 + n_2$ vertices, m edges and let $\Delta' = \min\{\Delta_A, \Delta_B\}$. Then, with high probability, the cube representation of G in $3(\Delta' + 2)\lceil \ln n_2 \rceil$ dimensions can be generated in $O(\Delta'(m + n) \ln n_2)$ time.*

Proof. We assume that a random permutation π on n_1 vertices can be computed in $O(n_1)$ time. Recall that we assign n intervals to n vertices as follows. If $v \in A$, then we assign the interval $[\pi(v), n + \pi(v)]$ to v. If $v \in B$, then let $t = \min_{x \in N(v)} \pi(x)$. Now, the interval $[t + n, t + 2n]$ is given to the vertex v. Since number of edges in the graph $m = \frac{1}{2} \sum_{u \in A \cup B} d(u)$, one invocation of **RANDUNIT(G)** needs $O(m+n)$ time. Since we need to invoke the algorithm **RANDUNIT(G)** $O(\Delta' \ln n_2)$ times, the overall algorithm that generates the cube representation in $3(\Delta' + 2)\lceil \ln n_2 \rceil$ dimensions runs in $O(\Delta'(m + n) \ln n_2)$ time ∎

References

[1] S. Bellantoni, Irith Ben-Arroyo Hartman, T. M. Przytycka, and S. Whitesides. Grid intersection graphs and boxicity *Discrete Mathematics*, 114(1-3):41–49, 1993.

[2] L. S. Chandran, C. Mannino, and G. Oriolo. On the cubicity of certain graphs. *Inform. Process. Lett.*, 94(3):113–118, 2005.

[3] L. Sunil Chandran and Naveen Sivadasan. The cubicity of hypercube graphs. *Discrete Mathematics*, 308(23):5795–5800, 2008.

[4] L. Sunil Chandran, Mathew C. Francis, and Naveen Sivadasan. Representing graphs as the intersection of axis-parallel cubes. Manuscript, available at http://arxiv.org/abs/cs/0607092.

[5] M. B. Cozzens. *Higher and multidimensional analogues of interval graphs*. PhD thesis, Rutgers University, New Brunswick, NJ, 1981.

[6] J. Kratochvíl. A special planar satisfiability problem and a consequence of its NP-completeness. *Discrete Appl. Math.*, 52(3):233–252, 1994.

[7] H. Maehara. Sphericity exceeds cubicity for almost all complete bipartite graphs. *J. Combin. Theory Ser. B*, 40(2):231–235, 1986.

[8] T. S. Michael and T. Quint. Sphericity, cubicity, and edge clique covers of graphs. *Discrete Appl. Math.*, 154(8):1309–1313, 2006.

[9] F. S. Roberts. On the boxicity and cubicity of a graph. In *Recent Progress in Combinatorics (Proc. Third Waterloo Conf. on Combinatorics, 1968)*, pages 301–310. Academic Press, New York, 1969.

[10] W. T. Trotter. A chanracterization of Roberts' inequality fro boxicity. In *Discrete Mathematics* 28:303–313, 1979.

[11] M. Yannakakis. The complexity of the partial order dimension problem. *SIAM J. Algebraic Discrete Methods*, 3(3):351–358, 1982.