The Complexity of Propositional Implication[☆]

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Abstract

The question whether a set of formulae Γ implies a formula φ is fundamental. The present paper studies the complexity of the above implication problem for propositional formulae that are built from a systematically restricted set of Boolean connectives. We give a complete complexity-theoretic classification for all sets of Boolean functions in the meaning of Post's lattice and show that the implication problem is efficiently solvable only if the connectives are definable using the constants $\{0, 1\}$ and only one of $\{\wedge, \vee, \oplus\}$. The problem remains coNP-complete in all other cases. We also consider the restriction of Γ to singletons which makes the problem strictly easier in some cases.

Key words: Computational complexity, propositional implication, Post's lattice

1. Introduction

SAT, the satisfiability problem for propositional formulae, is the most fundamental and historically the first NP-complete problem (proven by S. Cook and L. Levin [6, 7]). A natural question, posed by H. Lewis in 1979, is what the sources of hardness in the Cook-Levin Theorem are. More precisely, Lewis systematically restricted the language of propositional formulae and determined the computational complexity of the satisfiability problem depending on the set of allowed connectives. E. g., if only logical "and" ($\wedge$) and "or" ($\vee$) are allowed, we deal with *monotone formulae* for which the satisfiability problem obviously is easy to solve (in polynomial time). Lewis proved that SAT is NP-complete iff the negation of implication, $x \wedge \neg y$, is among the allowed connectives or can be simulated by the allowed connectives [8]. To simulate a logical connective f by a set of logical connectives (or, in other words, a set of Boolean functions) B formally means that f can be obtained from functions from B by superposition, i. e., general composition of functions. Equivalently, we can express this fact by saying that f is a member of the *clone* generated by B, in symbols $f \in [B]$.

This brings us into the realm of Post's lattice, the lattice of all Boolean clones [11]. In this framework, Lewis' result can be restated as follows. Let SAT(B) denote the satisfiability problem for propositional formulae with connectives restricted to the set B of Boolean functions. Then SAT(B) is NP-complete iff $\mathsf{S}_1 \subseteq [B]$; otherwise the problem is polynomial-time solvable. Note that the 2-ary Boolean function $x \wedge \neg y$ forms a basis for S_1.

Since then, many problems related to propositional formulae and Boolean circuits have been studied for restricted sets of connectives or gates, and their computational complexity has been classified, depending on a parameter B, as just explained for SAT. These include, e. g., the equivalence problem [12], the circuit value problem [13], the quantified Boolean formulae problem QBF [13], but also more recent questions related to non-classical logics like LTL [1], CTL [9], or default logic [**?**]. An important part of the proof of the classification of different reasoning tasks for default logic in the latter paper [**?**] was the identification of the coNP-complete and polynomial-time solvable fragments of the *propositional implication problem*. Though implication is without doubt a very fundamental and natural problem, its computational complexity has not yet been fully classified. This is the purpose of the present note.

We study the problem, given a set Γ of propositional formulae and a formula φ, to decide whether φ is implied by Γ. Depending on the set of allowed connectives in the occurring formulae, we determine the computational complexity of this problem as coNP-complete, $\oplus$L-complete, in $\mathrm{AC}^0[2]$, or in AC^0. The type of reduction we use are *constant-depth reductions* [5] and the weaker AC^0 *many-one reductions*. For both reductions, AC^0 forms the **0**-degree. We also consider the case of the problem restricted to singleton sets Γ, the *singleton-premise implication problem*. Interestingly, the complexity of the previously $\oplus$L-complete cases now drops down to the class $\mathrm{AC}^0[2]$; in all other cases the complexity remains the same as for the unrestricted problem. Finally, as an easy consequence our results give a refinement of Reith's previous classification of the equivalence problem for propositional formulae [12]. While Reith only considered the dichotomy between the coNP-complete and logspace-solvable cases, we show that under constant-depth reductions, three complexity degrees occur: coNP-complete, membership in $\mathrm{AC}^0[2]$, and membership in AC^0.

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2. Preliminaries

In this paper we make use of standard notions of complexity theory. The arising complexity degrees encompass the classes AC^0, $\mathrm{AC}^0[2]$, $\oplus$L, P, and coNP (cf. [10, 15] for background information).

AC^0 forms the class of languages recognizable by logtime-uniform Boolean circuits of constant depth and polynomial size over $\{\wedge, \vee, \neg\}$, where the fan-in of gates of the first two types is not bounded. The class $\mathrm{AC}^0[2]$ is defined similarly as AC^0, but in addition to $\{\wedge, \vee, \neg\}$ we also allow $\oplus$-gates of unbounded fan-in. The class $\oplus$L is defined as the class of languages L for which there exists a nondeterministic logspace Turing machine M such that for all x, $x \in L$ iff $M(x)$ has an odd number of accepting paths.

For the hardness results we use *constant-depth* and AC^0 *many-one reductions*, defined as follows: A language A is *constant-depth reducible* to a language B ($A \leq_{\mathrm{cd}} B$) if there exists a logtime-unifordm AC^0-circuit family $\{C_n\}_{n\geq 0}$ with $\{\wedge, \vee, \neg\}$-gates and oracle gates for B such that for all x, $C_{|x|}(x) = 1$ iff $x \in A$ [15]. A language A is AC^0 *many-one reducible* to a language B ($A \leq_{\mathrm{m}}^{\mathrm{AC}^0} B$) if there exists a function f computable by a logtime-uniform AC^0-circuit family such that $x \in A$ iff $f(x) \in B$.

For both reductions, the class AC^0 forms the **0**-degree. Furthermore, it is easy to see that

$$\mathrm{MOD}_2 := \{w \in \{0,1\}^\star \mid |w|_1 \equiv 1 \pmod 2\},$$

where $|w|_1 = |\{i \mid 1 \leq i \leq n,\ w_i = 1\}|$, is complete for $\mathrm{AC}^0[2]$ under $\leq_{\mathrm{cd}}$-reductions, for $\mathrm{AC}^0[2]$ merely extends AC^0 with oracle gates for MOD_2.

We assume familiarity with propositional logic. The set of all propositional formulae is denoted by $\mathcal{L}$. For $\Gamma \subseteq \mathcal{L}$ and $\varphi \in \mathcal{L}$, we write $\Gamma \models \varphi$ iff all assignments satisfying all formulae in Γ also satisfy φ.

3. Boolean Clones

In order to completely classify the complexity of the implication problem for all possible sets B of Boolean functions, one has to consider an infinite number of parameterized problems. We introduce the notion of a clone to reduce the number of problems to be considered to a finite set.

A propositional formula using only connectives from a finite set B of Boolean functions is called a B-formula. The set of all B-formulae is denoted by $\mathcal{L}(B)$. A *clone* is a set B of Boolean functions that is closed under superposition, i. e., B contains all projections and is closed under arbitrary composition. We denote by $[B]$ the smallest clone containing B and call B a *base* for $[B]$. In [11] Post classified the lattice of all clones and found a finite base for each clone, see Fig. 1. In order to introduce the clones relevant to this paper, we define the following notions for n-ary Boolean functions f:

- f is *c-reproducing* if $f(c, \ldots, c) = c$, $c \in \{0,1\}$.
- f is *monotone* if $a_1 \leq b_1, \ldots, a_n \leq b_n$ implies $f(a_1, \ldots, a_n) \leq f(b_1, \ldots, b_n)$.
- f is *c-separating* if there exists an index $i \in \{1, \ldots, n\}$ such that $f(a_1, \ldots, a_n) = c$ implies $a_i = c$, $c \in \{0,1\}$.
- f is *self-dual* if $f \equiv \mathrm{dual}(f)$, where $\mathrm{dual}(f)(x_1, \ldots, x_n) := \neg f(\neg x_1, \ldots, \neg x_n)$.
- f is *linear* if $f \equiv c_0 \oplus c_1 x_1 \oplus \cdots \oplus c_n x_n \oplus c$ for constants $c_i \in \{0,1\}$, $0 \leq i \leq n$, and variables $x_1, \ldots, x_n$.

The clones relevant to this paper are listed in Table 1. The definition of all Boolean clones can be found, e. g., in [3].

4. The Complexity of the Implication Problem

Let B be a finite set of Boolean functions. The *implication problem* for B-formulae is defined as

Problem:	$\mathrm{IMP}(B)$
Instance:	A finite set Γ of B-formulae and a B-formula φ.
Question:	Does $\Gamma \models \varphi$ hold?

In the general case $[B] = \mathsf{BF}$, verifying an instance $(\Gamma, \varphi) \in \mathrm{IMP}(B)$ amounts to verifying that the formula $\bigwedge\Gamma \rightarrow \varphi$ is tautological. We hence obtain a coNP upper bound. The following theorem classifies the complexity of the implication problem for all possible sets B.

Theorem 4.1. *Let B be a finite set of Boolean functions. Then the implication problem for propositional B-formulae,* $\mathrm{IMP}(B)$, *is*

1. coNP*-complete under* $\leq_{\mathrm{m}}^{\mathrm{AC}^0}$*-reductions if* $\mathsf{S}_{00} \subseteq [B]$ *or* $\mathsf{S}_{10} \subseteq [B]$ *or* $\mathsf{D}_2 \subseteq [B]$,
2. $\oplus$L*-complete under* $\leq_{\mathrm{m}}^{\mathrm{AC}^0}$*-reductions if* $\mathsf{L}_2 \subseteq [B] \subseteq \mathsf{L}$,
3. *in* $\mathrm{AC}^0[2]$ *and* $\mathrm{MOD}_2 \leq_{\mathrm{m}}^{\mathrm{AC}^0} \mathrm{IMP}(B)$ *if* $\mathsf{N}_2 \subseteq [B] \subseteq \mathsf{N}$, *and*
4. *in* AC^0 *for all other cases.*

In contrast to the first two cases, we do not state a completeness result for the third case, where $\mathsf{N}_2 \subseteq [B] \subseteq \mathsf{N}$. Under $\leq_{\mathrm{cd}}$-reductions, however, $\mathrm{IMP}(B)$ is $\mathrm{AC}^0[2]$-complete in this case. For $\leq_{\mathrm{m}}^{\mathrm{AC}^0}$-reductions, the existence of a complete problem A would state that any $\mathrm{AC}^0[2]$-circuit is equivalent to an AC^0-computation followed by a single oracle call to A. To date, there is no such problem known.

We split the proof of Theorem 4.1 into several lemmas.

Lemma 4.2. *Let B be a finite set of Boolean functions such that* $\mathsf{S}_{00} \subseteq [B]$ *or* $\mathsf{S}_{10} \subseteq [B]$. *Then* $\mathrm{IMP}(B)$ *is* coNP*-complete under* $\leq_{\mathrm{m}}^{\mathrm{AC}^0}$*-reductions.*

Proof. Membership in coNP is apparent, because given Γ and φ, we just have to check that for all assignments σ to the variables of Γ and φ, either $\sigma \not\models \Gamma$ or $\sigma \models \varphi$.

The hardness proof is inspired by [12]. Observe that $\mathrm{IMP}(B) \equiv_{\mathrm{cd}} \mathrm{IMP}(B \cup \{1\})$ if $\wedge \in [B]$, and that $\mathrm{IMP}(B) \equiv_{\mathrm{cd}} \mathrm{IMP}(B \cup \{0\})$ if $\vee \in [B]$ (because $\varphi \models \psi \iff \varphi_{[1/t]} \wedge t \models \psi_{[1/t]}$ and $\varphi \models \psi \iff \varphi_{[0/f]} \models \psi_{[0/f]} \vee f$ where t, f are new variables). It hence suffices to show that $\mathrm{IMP}(B)$ is coNP-hard for $\mathsf{M}_0 = [\mathsf{S}_{00} \cup \{0\}] = [\{\wedge, \vee, 0\}]$ and $\mathsf{M}_1 = [\mathsf{S}_{10} \cup \{1\}] = [\{\wedge, \vee, 1\}]$.

Name	Definition	Base
BF	All Boolean functions	$\{\wedge, \neg\}$
M_2	$\{f : f$ is monotone and 0- and 1-reproducing$\}$	$\{\vee, \wedge\}$
S_{00}	$\{f : f$ is 0-separating$\} \cap \mathsf{M}_2$	$\{x \vee (y \wedge z)\}$
S_{10}	$\{f : f$ is 1-separating$\} \cap \mathsf{M}_2$	$\{x \wedge (y \vee z)\}$
D_2	$\{f : f$ is monotone and self-dual$\}$	$\{(x \wedge y) \vee (y \wedge z) \vee (x \wedge z)\}$
L	$\{f : f$ is linear$\}$	$\{\oplus, 1\}$
L_2	$\{f : f$ is linear and 0- and 1-reproducing$\}$	$\{x \oplus y \oplus z\}$
V	$\{f : f \equiv c_0 \vee \bigvee_{i=1}^{n} c_i x_i$ for $c_i \in \{0,1\}, 1 \leq i \leq n\}$	$\{\vee, 0, 1\}$
E	$\{f : f \equiv c_0 \wedge \bigwedge_{i=1}^{n} c_i x_i$ for $c_i \in \{0,1\}, 1 \leq i \leq n\}$	$\{\wedge, 0, 1\}$
N	$\{f : f$ depends on at most one variable$\}$	$\{\neg, 1\}$
N_2	$\{f : f$ is the negation or a projection$\}$	$\{\neg\}$

Table 1: A list of Boolean clones with definitions and bases.

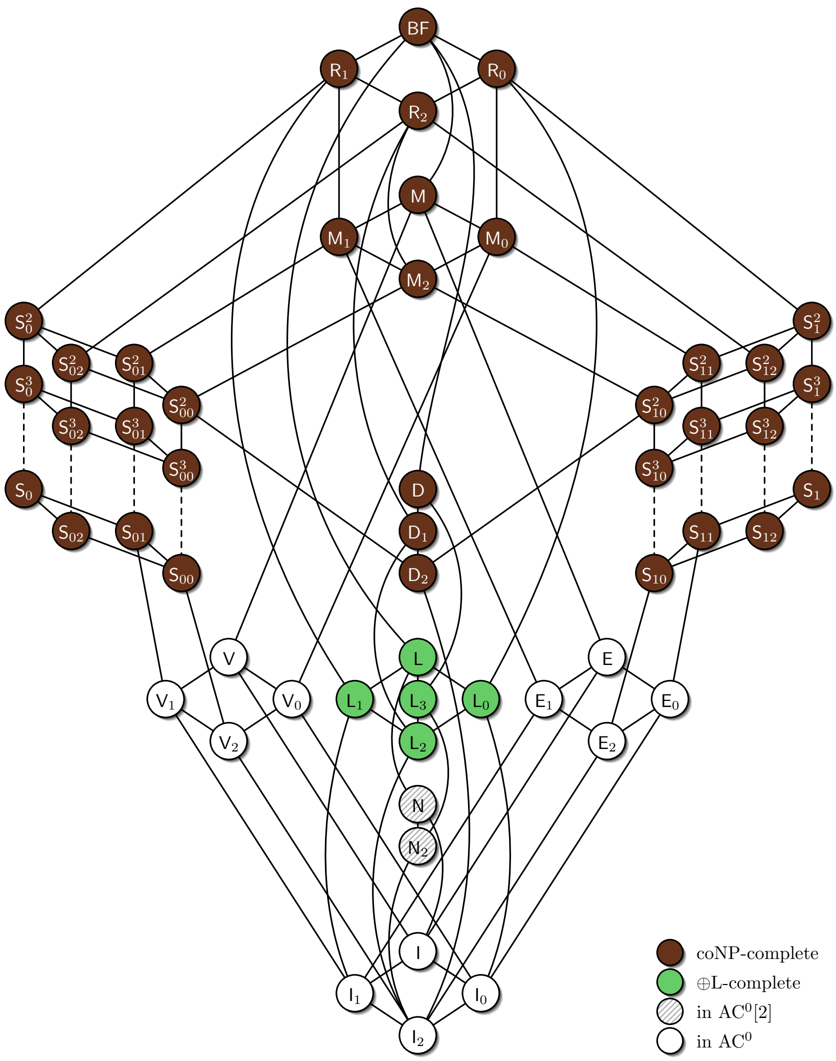

Figure 1: Post's lattice. Colors indicate the complexity of IMP(B), the implication problem for B-formulae.

We will show that IMP(B) is coNP-hard for each base B with $\mathsf{M}_2 = [\{\wedge, \vee\}] \subseteq [B]$. To prove this claim, we will provide a reduction from $\mathrm{TAUT}_{\mathrm{DNF}}$ to IMP(B), where $\mathrm{TAUT}_{\mathrm{DNF}}$ is the coNP-complete problem to decide, whether a given propositional formula in disjunctive normal form is a tautology.

Let φ be a propositional formula in disjunctive normal form over the propositions $X = \{x_1, \dots, x_k\}$. Then $\varphi = \bigvee_{i=1}^{n} \bigwedge_{j=1}^{m} l_{ij}$, where l_{ij} are literals over X. Take new variables $Y = \{y_1, \dots, y_k\}$ and replace in φ each negative literal $l_{ij} = \neg x_l$ by y_l. Define the resulting formula as ψ_2 and let $\psi_1 := \bigwedge_{i=1}^{k}(x_i \vee y_i)$. We claim that $\varphi \in \mathrm{TAUT}_{\mathrm{DNF}} \iff \psi_1 \models \psi_2$.

Let us first assume $\varphi \in \mathrm{TAUT}_{\mathrm{DNF}}$ and let $\sigma\colon X \cup Y \to \{0, 1\}$ be an assignment such that $\sigma \models \psi_1$. As φ is a tautology, $\sigma \models \varphi$. But also $\sigma \models \psi_2$, as we simply replaced the negated variables in φ by positive ones and ψ_2 is monotone. It follows that $\psi_1 \models \psi_2$, since σ was arbitrarily chosen.

For the opposite direction, let $\varphi \notin \mathrm{TAUT}_{\mathrm{DNF}}$. Then there exists an assignment $\sigma\colon X \to \{0, 1\}$ such that $\sigma \not\models \varphi$. We extend σ to an assignment $\sigma'\colon X \cup Y \to \{0, 1\}$ by setting $\sigma'(y_i) = 1 - x_i$ for $i = 1, \dots, k$. Then $\sigma'(x_i) = 0$ iff $\sigma'(y_i) = 1$, and consequently σ' simulates σ on φ'. As a result, $\sigma' \not\models \psi_2$. Yet, either $\sigma'(x_i) = 1$ or $\sigma'(y_i) = 1$ for $i = 1, \dots, k$. Thus $\sigma' \models \psi_1$, yielding $\psi_1 \not\models \psi_2$. □

Lemma 4.3. *Let B be a finite set of Boolean functions such that $\mathsf{D}_2 \subseteq [B]$. Then* IMP($B$) *is* coNP*-complete under $\leq_{\mathrm{m}}^{\mathrm{AC}^0}$-reductions.*

Proof. Again we just have to argue for coNP-hardness of IMP(B). We give a reduction from $\mathrm{TAUT}_{\mathrm{DNF}}$ to IMP(B) for $\mathsf{D}_2 \subseteq [B]$ by modifying the reduction given in the proof of Lemma 4.2.

Given a formula φ in disjunctive normal form, we define the formulae ψ_1 and ψ_2 as above. As $\mathsf{D}_2 \subseteq [B]$, we know that $g(x, y, z) := (x \wedge y) \vee (y \wedge z) \vee (x \wedge z) \in [B]$. Clearly, $g(x, y, 0) \equiv x \wedge y$ and $g(x, y, 1) \equiv x \vee y$. Denote by $\psi_i^B(t, f)$, $i \in \{1, 2\}$, the formula ψ_i with all occurrences of $x \wedge y$ and $x \vee y$ replaced by a B-representation of $g(x, y, f)$ and $g(x, y, t)$, respectively, where t and f are new propositional variables. Then $\psi_i^B(1, 0) \equiv \psi_i$ and $\psi_i^B(0, 1) \equiv \mathrm{dual}(\psi_i)$. The variables x and y occur several times in g, hence $\psi_1^B(t, f)$ and $\psi_2^B(t, f)$ might be exponential in the length of φ (recall that ψ_2 is φ with all negative literals replaced by new variables). That this is not the case follows from the associativity of $\wedge$ and $\vee$: we insert parentheses in such a way that ψ_i^B can be transformed into a tree of logarithmic depth.

We now map a pair (ψ_1, ψ_2) to (ψ_1', ψ_2') where

$$\psi_1' := g(\psi_1^B(t, f), t, f) \text{ and } \psi_2' := g(g(\psi_1^B(t, f), \psi_2^B(t, f), f), t, f).$$

We claim that $\psi_1 \models \psi_2 \iff \psi_1' \models \psi_2'$. To verify this claim, let σ be an arbitrary assignment for the variables in φ. Then σ may be extended to $\{t, f\}$ in the following ways:

$\sigma(t) = 1$ and $\sigma(f) = 0$: This is the intended interpretation. In this case, $g(\psi_1^B(1, 0), 1, 0) \equiv \psi_1 \wedge 1 \equiv \psi_1$ and $g(g(\psi_1^B(1, 0), \psi_2^B(1, 0), 0), 1, 0) \equiv (\psi_1 \wedge \psi_2) \wedge 1 \equiv \psi_1 \wedge \psi_2$. Then $\psi_1 \models \psi_2$ iff $\psi_1 \models \psi_1 \wedge \psi_2$.

$\sigma(t) = 0$ and $\sigma(f) = 1$: In this case, we obtain that $g(\psi_1^B(0, 1), 0, 1) \equiv \mathrm{dual}(\psi_1) \vee 0 \equiv \mathrm{dual}(\psi_1)$ and $g(g(\psi_1^B(0, 1), \psi_2^B(0, 1), 1), 0, 1) \equiv (\mathrm{dual}(\psi_1) \vee \mathrm{dual}(\psi_2)) \vee 0 \equiv \mathrm{dual}(\psi_1) \vee \mathrm{dual}(\psi_2)$. As $\mathrm{dual}(\psi_1) \models \mathrm{dual}(\psi_1) \vee \mathrm{dual}(\psi_2)$ is always valid, we conclude that $\psi_1' \models \psi_2'$ in this case.

$\sigma(t) = \sigma(f) = c$ with $c \in \{0, 1\}$: Then both ψ_1' and ψ_2' are equivalent to c. Thus, as in the previous case, $\psi_1' \models \psi_2'$.

From this analysis, it follows that $\psi_1 \models \psi_2$ iff $\psi_1' \models \psi_2'$. Hence, $\mathrm{TAUT}_{\mathrm{DNF}} \leq_{\mathrm{m}}^{\mathrm{AC}^0}$ IMP(B) via the reduction $\varphi \mapsto (\psi_1', \psi_2')$. □

Lemma 4.4. *Let B be a finite set of Boolean functions such that $\mathsf{L}_2 \subseteq [B] \subseteq \mathsf{L}$. Then* IMP($B$) *is* $\oplus$L*-complete under $\leq_{\mathrm{m}}^{\mathrm{AC}^0}$-reductions.*

Proof. Observe that $\Gamma \models \varphi$ iff $\Gamma \cup \{\varphi \oplus t, t\}$ is inconsistent, where t is a new variable. Let Γ' denote $\Gamma \cup \{\varphi \oplus t, t\}$ rewritten such that for all $\psi \in \Gamma'$, $\psi = c_0 \oplus c_1 x_1 \oplus \cdots \oplus c_n x_n$, where $c_0, \dots, c_n \in \{0, 1\}$. Γ' is logspace constructible, since $c_0 = 1$ iff $\psi(0, \dots, 0) = 1$, and for $1 \leq i \leq n$, $c_i = 1$ iff

$$\psi(0, \dots, 0) \not\equiv \psi(\underbrace{0, \dots, 0}_{i-1}, 1, 0, \dots, 0).$$

Γ' can now be transformed into a system of linear equations S via

$$c_0 \oplus c_1 x_1 \oplus \cdots \oplus c_n x_n \mapsto c_0 + c_1 x_1 + \cdots + c_n x_n = 1 \pmod 2.$$

Clearly, the resulting system of linear equations has a solution iff Γ' is consistent. The equations are furthermore defined over the field $\mathbb{Z}_2$, hence existence of a solution can be decided in $\oplus$L [4].

For the $\oplus$L-hardness, note that solving a system of linear equations over $\mathbb{Z}_2$ is indeed $\oplus$L-complete under $\leq_{\mathrm{m}}^{\mathrm{AC}^0}$-reductions: let $\mathrm{MOD\text{-}GAP}_2$ denote the $\oplus$L-complete problem to decide whether a given directed acyclic graph G with nodes s and t has an odd number of distinct paths leading from s to t. Buntrock et al. [4] give an NC^1-reduction from $\mathrm{MOD\text{-}GAP}_2$ to the problem whether a given matrix over $\mathbb{Z}_2$ is non-singular. The given reduction is actually an AC^0 many-one reduction. We reduce the latter problem to the complement of IMP($\{x \oplus y \oplus z\}$) and then generalize the result to arbitrary finite sets B such that $[B] = \mathsf{L}_2$. The lower bound then follows from $\oplus$L being closed under complement.

First map the system S of linear equations into a set of linear formulae Γ via

$$c_1 x_1 + \cdots + c_n x_n = c \pmod 2 \ \mapsto \ c' \oplus c_1 x_1 \oplus \cdots \oplus c_n x_n,$$

where $c' = 1$ if $c = 0$, and $c' = 0$ otherwise. Next replace the constant 1 with a fresh variable t, pad all formulae having an even number of non-fictive variables with another fresh variable f, and let $\Gamma' := \Gamma \cup \{t\}$. We claim that S has a solution iff $\Gamma' \not\models f$.

Suppose that S has no solutions. If Γ' is inconsistent, then $\Gamma' \models f$. Otherwise, Γ' has a satisfying assignment σ. Clearly, $\sigma(t) = 1$. If $\sigma(f) = 0$, then $\Gamma'[t/1, f/0]$ is equivalent to Γ; hence the transformation of $\Gamma'[t/1, f/0]$ yields a system of linear equations S' that is equivalent to S and that has a solution corresponding to σ — a contradiction to our assumption. Thus $\sigma(f) = 1$ and, consequently, $\Gamma' \models f$.

On the other hand, if S has a solution, then Γ possesses a satisfying assignment σ with $\sigma(t) = 1$ and $\sigma(f) = 0$. Again $\sigma \models \Gamma'$ iff $\sigma \models \Gamma$. Hence, $\Gamma' \not\models f$.

It remains to show that $x \oplus y \oplus z$ can be efficiently expressed in any set B such that $[B] = \mathsf{L}_2$, that is, there exists a function $f_\oplus \in [B]$ such that $f_\oplus$ is equivalent to $x \oplus y \oplus z$ and each variable occurs only once in the body of $f_\oplus$. Let B be such that $[B] = \mathsf{L}_2$ and let $g(x, y, z)$ be a function from $[B]$ depending on three variables. Such a function g exists because $x \oplus y \oplus z \in [B] = \mathsf{L}_2$. As g is a linear function, replacing two occurrences of any variable with a fresh variable t does not change g modulo logical equivalence. Let n denote the number of occurrences of x in g and assume that n is even. Replacing all occurrences of x with an arbitrary symbol yields a formula $g'(y, z) \equiv y \oplus z \notin \mathsf{L}_2$ which gives a contradiction. Analogous arguments hold for the number of occurrences of y and z. Hence, each of the variables x, y, and z occurs an odd number of times, and replacing all but one occurrence of each x, y, and z with t yields a function $g'(x, y, z, t) \equiv x \oplus y \oplus z$ in which each variable occurs exactly once. $\square$

Lemma 4.5. *Let B be a finite set of Boolean functions such that $\mathsf{N}_2 \subseteq [B] \subseteq \mathsf{N}$. Then* IMP($B$) *is contained in* $\mathrm{AC}^0[2]$ *and* $\mathrm{MOD}_2 \leq_m^{\mathrm{AC}^0} \mathrm{IMP}(B)$.

Proof. Let B be a finite set of Boolean functions such that $\mathsf{N}_2 \subseteq [B] \subseteq \mathsf{N}$. Let φ be a B-formula and Γ be a set of B-formulae, both over the set of propositions $\{x_1, \ldots, x_n\}$.

We will argue on membership in $\mathrm{AC}^0[2]$ first. For all $f \in [B]$, f is equivalent to some literal or a constant. Let $L := \{l_i \mid$ there exists $\psi \in \Gamma\colon l_i \equiv \psi\}$, where $l_i = x_i$ or $l_i = \neg x_i$ for $1 \leq i \leq n$. L is computable from Γ using an AC^0-circuit with oracle gates for MOD_2: for each formula in Γ, we determine the atom and count the number of preceding negations modulo 2. In the case that Γ is unsatisfiable, either $L = \emptyset$ or there exist $l_i, l_j \in \Gamma$ with $l_i \equiv \neg l_j$. Both conditions can be checked in AC^0, hence we may w. l. o. g. assume that Γ is satisfiable. It now holds that

$$\Gamma \models \varphi \iff \bigwedge_{l_i \in L} l_i \models \varphi \iff \text{for some } L' \subseteq L\colon \varphi \equiv \bigwedge_{l_i \in L'} l_i.$$

It remains to compute an equivalent formula of the form $\bigwedge_{l_i \in L'} l_i$ from φ and test whether $L' \subseteq L$. It is easy to see that the former task can again be performed in $\mathrm{AC}^0[2]$, and the latter merely requires AC^0. Thus we conclude $\mathrm{IMP}(B) \in \mathrm{AC}^0[2]$.

For $\mathrm{MOD}_2 \leq_m^{\mathrm{AC}^0} \mathrm{IMP}(B)$, we claim that, for $w = w_1 \cdots w_n \in \{0,1\}^n$, $w \in \mathrm{MOD}_2$ iff $t \models \neg^{w_1}\neg^{w_2}\cdots\neg^{w_n}(\neg t)$, where $\neg^1 := \neg$, $\neg^0 :=$ id and t is a variable.

First observe that $t \models \neg^{w_1}\neg^{w_2}\cdots\neg^{w_n}(\neg t)$ iff for all assignments σ of t to $\{0,1\}$, $\sigma \models t$ implies $\sigma \models \neg^{w_1}\neg^{w_2}\cdots\neg^{w_n}(\neg t)$. Now, if $\sigma(t) := 0$, then $t \models \neg^{w_1}\neg^{w_2}\cdots\neg^{w_n}(\neg t)$ is always true, whereas, if $\sigma(t) := 1$, then $t \models \neg^{w_1}\neg^{w_2}\cdots\neg^{w_n}(\neg t)$ iff $1 \models \neg^{w_1}\neg^{w_2}\cdots\neg^{w_n}0$. Hence, the claim applies and $\mathrm{MOD}_2 \leq_m^{\mathrm{AC}^0} \mathrm{IMP}(B)$ follows. $\square$

As an immediate consequence of the above lemma, we obtain the following corollary.

Corollary 4.6. *Let B be a finite set of Boolean functions such that $\mathsf{N}_2 \subseteq [B] \subseteq \mathsf{N}$. Then* IMP($B$) *is* $\mathrm{AC}^0[2]$*-complete under* $\leq_{\mathrm{cd}}$*-reductions.*

Lemma 4.7. *Let B be a finite set of Boolean functions such that $[B] \subseteq \mathsf{V}$ or $[B] \subseteq \mathsf{E}$. Then* IMP(B) *is in* AC^0.

Proof. We prove the claim for $[B] \subseteq \mathsf{V}$ only. The case $[B] \subseteq \mathsf{E}$ follows analogously.

Let B be a finite set of Boolean functions such that $[B] \subseteq \mathsf{V}$. Let further Γ be a finite set of B-formulae and let φ be a B-formula such that Γ and φ only use the variables $x_1, \ldots, x_n$. Let $\varphi \equiv c_0 \vee c_1x_1 \vee \cdots \vee c_nx_n$ with constants $c_i \in \{0,1\}$ for $0 \leq i \leq n$. Equally, every formula from Γ is equivalent to an expression of the form $c'_0 \vee c'_1x_1 \vee \cdots \vee c'_nx_n$ with $c'_i \in \{0,1\}$. Then, $\Gamma \models \varphi$ iff either $c_0 = 1$ or there exists a formula $\psi \equiv c''_0 \vee c''_1x_1 \vee \cdots \vee c''_nx_n$ from Γ such that $c''_i \leq c_i$ for all $0 \leq i \leq n$ and $c''_i \in \{0,1\}$.

The value of c_0 can be determined by evaluating $\varphi(0, \ldots, 0)$. Furthermore, for $1 \leq i \leq n$, $c_i = 0$ iff $c_0 = 0$ and

$$\varphi(\underbrace{0, \ldots, 0}_{i-1}, 1, 0, \ldots, 0) = 0.$$

The values of the coefficients of formulae in Γ can be computed analogously. Thus IMP(B) can be computed in constant depth using oracle gates for B-formula evaluation. As B-formula evaluation is in NLOGTIME [14] and NLOGTIME $\subseteq \mathrm{AC}^0$, the claim follows. $\square$

5. The Complexity of the Singleton-Premise Implication Problem

For a finite set B of Boolean functions, we define the *singleton-premise implication problem* for B-formulae as

Problem:	IMP′(B)
Instance:	Two B-formulae φ and ψ.
Question:	Does $\varphi \models \psi$ hold?

We classify the complexity of this problem as follows:

Theorem 5.1. *Let B be a finite set of Boolean functions. Then* $\mathrm{IMP}'(B) \in \mathrm{AC}^0[2]$ *and* $\mathrm{MOD}_2 \leq_m^{\mathrm{AC}^0} \mathrm{IMP}'(B)$ *if* $\mathsf{L}_2 \subseteq [B] \subseteq \mathsf{L}$. *For all other sets B, the problems* IMP(B) *and* IMP′(B) *are equivalent.*

Before we prove Theorem 5.1, let us try to give an intuitive explanation for the difference in the complexity of IMP′(B) for $\mathsf{L}_2 \subseteq [B] \subseteq \mathsf{L}$ stems from. Deciding IMP(B) is equivalent to solving a set of linear equations corresponding to the set of premises. For IMP′(B), the premise is a single formula. It hences suffice to determine whether there exists an assignment satisfying the premise and setting to true an even (resp. odd) number of variables from the conclusion.

Proof. For $\mathsf{S}_{00} \subseteq [B]$, $\mathsf{S}_{10} \subseteq [B]$, and $\mathsf{D}_2 \subseteq [B]$, observe that the proofs of Lemma 4.2 and Lemma 4.3 actually establish coNP-hardness of IMP′(B). Analogously, for $\mathsf{N}_2 \subseteq [B]$, $\mathrm{MOD}_2 \leq_m^{\mathrm{AC}^0}$ IMP′(B) follows by the same reduction given in the proof of

Lemma 4.5. For $[B] \subseteq \mathsf{V}$ and $[B] \subseteq \mathsf{E}$, we have $\mathrm{IMP}'(B) \leq_{\mathrm{m}}^{\mathrm{AC}^0} \mathrm{IMP}(B) \in \mathrm{AC}^0$. It thus remains to show that $\mathrm{IMP}'(B) \in \mathrm{AC}^0[2]$ for $[B] \subseteq \mathsf{L}$, and that $\mathrm{MOD}_2 \leq_{\mathrm{m}}^{\mathrm{AC}^0} \mathrm{IMP}'(B)$ for $\mathsf{L}_2 \subseteq [B]$.

Let (φ, ψ) be a pair of B-formulae over the variables $\{x_1, \ldots, x_n\}$. As $[B] \subseteq \mathsf{L}$, φ and ψ are equivalent to expressions of the form $\varphi \equiv c_0 \oplus c_1 x_1 \oplus \cdots \oplus c_n x_n$ and $\psi \equiv c'_0 \oplus c'_1 x_1 \oplus \cdots \oplus c'_n x_n$, where $c_i, c'_i \in \{0, 1\}$ for $1 \leq i \leq n$. If $c_0 = \ldots = c_n = 0$, then $\varphi \models \psi$ apparently holds. Therefore, let us assume that not all coefficients c_i are 0. In this situation, we claim that $\varphi \models \psi$ is in fact equivalent to $\varphi \equiv \psi$. To prove this claim observe that $\varphi \models \psi$ iff

$$\chi := (c_0 \oplus c_1 x_1 \oplus \cdots \oplus c_n x_n) \wedge (1 \oplus c'_0 \oplus c'_1 x_1 \oplus \cdots \oplus c'_n x_n)$$

is unsatisfiable. Let us assume now $\varphi \not\equiv \psi$. We will construct a satisfying assignment σ for χ. Let $I := \{i \in \{1, \ldots, n\} \mid c_i = c'_i\}$ and define $\sigma(x_i) := 0$ for $i \in I$. As $\varphi \not\equiv \psi$, the set $\overline{I} := \{1, \ldots, n\} \setminus I$ is nonempty and for all $i \in \overline{I}$, $c_i = 1 \iff c'_i = 0$. Hence, there is a partition $P_1 \uplus P_2 = \overline{I}$ such that

$$\sigma \models \chi \iff \sigma \models (c_0 \oplus \bigoplus_{i \in P_1} c_i x_i) \wedge (1 \oplus c'_0 \oplus \bigoplus_{i \in P_2} c'_i x_i).$$

Here the subformulae $c_0 \oplus \bigoplus_{i \in P_1} c_i x_i$ and $1 \oplus c'_0 \oplus \bigoplus_{i \in P_2} c'_i x_i$ are over disjoint sets of variables. But still, both subformulae are satisfiable using an appropriate completion of σ. Consequently, σ will also satisfy χ and hence the claim holds.

Thus $\varphi \models \psi$ if either $c_0 = \ldots = c_n = 0$ or $\varphi \equiv \psi$. Similarly to the proof of Lemma 4.7, it follows that the latter alternative holds iff $c_i = c'_i$ for all $0 \leq i \leq n$. The coefficients c_i can be determined from $c_0 = \varphi(0, \ldots, 0)$ and

$$c_i = \varphi(\underbrace{0, \ldots, 0}_{i-1}, 1, 0, \ldots, 0) \oplus c_0$$

for $1 \leq i \leq n$. The values of the c'_i's can be computed analogously. As B-formula evaluation is equivalent to MOD_2 [14] in this case, $\mathrm{IMP}'(B) \in \mathrm{AC}^0[2]$.

It remains to prove $\mathrm{MOD}_2 \leq_{\mathrm{m}}^{\mathrm{AC}^0} \mathrm{IMP}'(B)$ for $\mathsf{L}_2 \subseteq [B]$. Consider the mapping $h : \{0, 1\}^\star \to \mathcal{L}(B)$, recursively defined by

$$h(x) = \begin{cases} f & x = \varepsilon \\ h(y) & x = 0y \\ t \oplus f \oplus h(y) & x = 1y \end{cases}$$

where ε denotes the empty word and t, f are propositional variables. We claim that $x \mapsto (t, h(x))$ computes an $\leq_{\mathrm{m}}^{\mathrm{AC}^0}$-reduction from MOD_2 to $\mathrm{IMP}'(B)$. To verify this claim, let $x \in \{0, 1\}^\star$ be an instance of MOD_2. Then

$$\begin{aligned} x \in \mathrm{MOD}_2 &\implies h(x) \equiv t \implies (t, h(x)) \in \mathrm{IMP}'(B), \\ x \notin \mathrm{MOD}_2 &\implies h(x) \equiv f \implies (t, h(x)) \notin \mathrm{IMP}'(B). \end{aligned}$$

Whence, $\mathrm{MOD}_2 \leq_{\mathrm{m}}^{\mathrm{AC}^0} \mathrm{IMP}'(B)$ for $\mathsf{L}_2 \subseteq [B]$. □

Let $\mathrm{EQ}(B)$ denote the equivalence problem for B-formulae. Obviously, $(\varphi, \psi) \in \mathrm{EQ}(B)$ iff $(\varphi, \psi) \in \mathrm{IMP}'(B)$ and $(\psi, \varphi) \in \mathrm{IMP}'(B)$. As AC^0, $\mathrm{AC}^0[2]$, and coNP are all closed under intersection, we obtain as an immediate corollary a finer classification of the complexity of EQ than the one given by Reith [12]. He establishes a dichotomy between coNP-hardness and membership in Ł. We split the second case into two complexity degrees.

Corollary 5.2. *Let B be a finite set of Boolean functions. Then $\mathrm{EQ}(B)$ is* coNP*-complete under $\leq_{\mathrm{cd}}$-reductions if $\mathsf{S}_{00} \subseteq [B]$ or $\mathsf{S}_{10} \subseteq [B]$ or $\mathsf{D}_2 \subseteq [B]$;* $\mathrm{AC}^0[2]$*-complete under $\leq_{\mathrm{cd}}$-reductions if $\mathsf{N}_2 \subseteq [B] \subseteq \mathsf{N}$; and in* AC^0 *for all other cases.*

6. Conclusion

In this paper we provided a complete classification of the complexity of the implication problem, $\mathrm{IMP}(B)$, and the singleton-premise implication problem, $\mathrm{IMP}'(B)$—fundamental problems in the area of propositional logic. Though $\mathrm{IMP}'(B)$ is a restricted version of $\mathrm{IMP}(B)$, the simplification amounts to a difference for $\mathsf{L}_2 \subseteq [B] \subseteq \mathsf{L}$ only: $\mathrm{IMP}'(B)$ is $\mathrm{AC}^0[2]$-complete under constant-depth reductions, whereas $\mathrm{IMP}(B)$ is $\oplus$L-complete under AC^0 many-one reductions and thus strictly harder. For all other clones, both problems have the same complexity.

Due to the close relationship between the implication and the equivalence problem, we were also able to slightly refine the classification of the complexity of the equivalence problem given in [12].

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