# Homogeneous Faults, Colored Edge Graphs, and Cover Free Families 

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#### Abstract

In this paper, we use the concept of colored edge graphs to model homogeneous faults in networks. We then use this model to study the minimum connectivity (and design) requirements of networks for being robust against homogeneous faults within certain thresholds. In particular, necessary and sufficient conditions for most interesting cases are obtained. For example, we will study the following cases: (1) the number of colors (or the number of non-homogeneous network device types) is one more than the homogeneous fault threshold; (2) there is only one homogeneous fault (i.e., only one color could fail); and (3) the number of non-homogeneous network device types is less than five.


## 1 Background and colored edge graph

In network communications, the communication could fail if some nodes or some edges are broken. Though the failure of a modem could be considered the failure of a node, we can model this scenario also as the failure of the communication link (the edge) attached to this modem. Thus it is sufficient to consider edge failures in communication networks. It is also important to note that several nodes (or edges) in a network could fail at the same time. For example, all brand X routers in a network could fail at the same time due to a platform dependent computer worm (virus) attack. In order to design survivable communication networks, it is essential to consider this kind of homogeneous faults for networks. Existing works on network quality of services have not addressed this issue in detail and there is no existing model to study network reliability in this aspect. In this paper, we use the colored edge graphs which could be used to model homogeneous faults in networks. The model is then used to optimize the design of survivable networks and to study the minimum connectivity (and design) requirements of networks for being robust against homogeneous faults within certain thresholds.

Definition 1.1 A colored edge graph is a tuple $G(V, E, C, f)$, with $V$ the node set, $E$ the edge set, $C$ the color set, and $f$ a map from $E$ onto $C$. The structure

$$
\mathcal{Z}_{C, t}=\{Z: Z \subseteq E \text { and }|f(Z)| \leq t\}
$$

is called a t-color adversary structure. Let $A, B \in V$ be distinct nodes of $G$. $A, B$ are called $(t+1)$-color connected for $t \geq 1$ if for any color set $C_{t} \subseteq C$ of size $t$, there is a path $p$ from $A$ to $B$ in $G$ such that the edges on $p$ do not contain any color in $C_{t}$. A colored edge graph $G$ is $(t+1)$-color connected if and only if for any two nodes $A$ and $B$ in $G$, they are $(t+1)$-color connected.

The interpretation of the above definition is as follows. In a network, if two edges have the same color, then they could fail at the same time. This may happen when the two edges are designed with same technologies (e.g., with same operating systems, with same application software, with same hardware, or with same hardware and software). If a colored edge network is $(t+1)$-color connected, then the network communication is robust againt the failure of edges of any $t$ colors (that is, the adversary may tear down any $t$ types of devices).

In practice, one communication link may be attached to different brands of network devices (e.g., routers, modems) on both sides. For this case, the edge can have two different colors. If any of these colors is broken, the edge is broken. Thus from a reliability viewpoint, if one designs networks with two colors on the same edge, the same reliability/security can be obtained by having only one color on each edge. In the following discussion, we will only consider the case with one color on each edge. Meanwhile, multiple edges between two nodes are not allowed either.

We are interested in the following practical questions. For a given number $n$ of nodes in $V$ (i.e., the number of network nodes), a given number $m$ of the colors (e.g., the number of network device types), and a given number $t$, how can we design a $(t+1)$-color connected colored edge graphs $G(V, E)$ with minimum number $\lambda$ of edges? In another words, how can we use minimum resources (e.g., communication links) to design a network that will keep working even if $t$ types of devices in the network fail?

For practical network designs, one needs first to have an estimate on the number of homogeneous faults. For example, the number $t$ of brands of routers that could fail at the same time. Then it is sufficient to design a $(t+1)$-color connected network with $m=t+1$ colors (e.g., with $t+1$ different brands of routers). Necessary and sufficient conditions for this kind of network design will be obtained in this paper.

Another important issue that should be taken into consideration in practical network designs is that the number $m$ of colors (e.g., the number of brands for routers) is quite small. For example, $m$ is normally less than five. Necessary and sufficient conditions for network designs with $m \leq 5$ and with optimized resources will be obtained in this paper. Note that for cases with small $m$, we may have $m>t+1$.

The outline of the paper is as follows. Section 3 describes the necessary and sufficient conditions for the case of $m=t+1$ without optimizing the number of edges in the networks. Section 4 gives a necessary condition for colored edge networks in terms of optimized number of edges. Section 5 shows that the necessary conditions in Section 4 are also sufficient for the most important three cases: (1) $m=t+1$; (2) $t=1$; and (3) $m \leq 5$. Section 6 shows that it is NP-hard to determine whether a given colored edge graph is $(t+1)$-connected.

## 2 Related works

Though colored-edge graph is a new concept which we used to model network survivability issues, there are related research topics in this field. For example, edge-disjoint (colorful) spanning trees have been extensively studied in the literature (see, e.g., [2]). These results are mainly related to our discussion in the next section for the case of $m=t+1$. A colored edge graph $G$ is proper if whenever two edges share an end point they carry different colors. A spanning tree for a colored edge graph is called colorful if no two of its edges have the same color. Two spanning trees of a graph are edge disjoint if they do not share common edges. For a non-negative integer $s$, let $K_{s}$ denote the complete graph on $s$ vertices. A classical result of Euler states that the edges of $K_{2 n}$ can be partitioned into $n$ isomorphic spanning trees (paths, for example) and each of these spanning trees can easily be made colorful, but the resulting edge colored graph usually fails to be proper.

Though it is important to design colored edge graphs with required security parameters, for several scenarios it is also important to calculate the robustness of a given colored edge graphs. Roskind and Tarjan [9] designed a greedy algorithm to find $(t+1)$-edge disjoint spanning trees in a given graph. This is related to the questions $(t+1)$-color connectivity for the case of $m=t+1$. We are not aware of any approximate algorithms for deciding $(t+1)$-color connectivity of a given colored edge graph. Indeed, we will show that this problem is NP-hard.

## 3 Necessary and sufficient conditions for $m=t+1$

In this section, we show necessary and sufficient conditions for some special cases.
Lemma 3.1 A colored edge graphs $G(V, E, C, f)$ is $(t+1)$-color connected if and only if, for all $i_{1}, i_{2}, \ldots, i_{m-t} \leq m$, $\left(V, E_{i_{1}} \cup E_{i_{2}} \cup \cdots \cup E_{i_{m-t}}\right)$ is a connected graph, where $E_{1}, E_{2}, \ldots, E_{m}$ is a partition of $E$ under the $m$ different colors.

As we have mentioned in the previous section, the classical result by Euler states that $K_{2 n}$ can be partitioned into $n$ spanning trees. Thus, by Lemma3.1 we have the following theorem.

Theorem 3.2 (Euler) For $n=2 m$, there is a coloration $G(V, E, C, f)$ of $K_{n}$ such that $G$ is $(m-1)$-color connected.
In the following, we extend Theorem 3.2 to the general case of $n \geq 2 m$.
Lemma 3.3 For $n \geq 2 m$ and $m \geq 2$, there exists a graph $G(V, E)$ with $|V|=n,|E|=m(n-1)$, and $E=E_{1} \cup E_{2} \cup$ $\cdots \cup E_{m}$ such that the following conditions are satisfied:

1. $G\left(V, E_{i}\right)$ is a connected graph for all $0<i \leq m$;
2. $E_{i} \cap E_{j}=\emptyset$ for all $i, j \leq m$.

Proof. We prove the Lemma by induction on $n$ and $m$. For $n=2$ and $m=1$, the Lemma holds obviously. Assume that the Lemma holds for $n_{0}=2 m_{0}$.

In the following, we show that the Lemma holds for $n=n_{0}+1, m=m_{0}$ and for $n=n_{0}+2, m=m_{0}+1$. Let $G\left(V_{0}, E_{0}\right)$ be the graph with $\left|V_{0}\right|=n_{0},\left|E_{0}\right|=m_{0}\left(n_{0}-1\right)$, and $E_{0}=E_{1}^{0} \cup E_{2}^{0} \cup \cdots \cup E_{m_{0}}^{0}$ such that the conditions in the Lemma are satisfied:

For the case of $n=n_{0}+1$ and $m=m_{0}$, let $V=V_{0} \cup\{u\}$ where $u$ is a new node that is not in $V_{0}$, and let $E_{1}=E_{1}^{0} \cup\left\{\left(u, u_{1}\right)\right\}, E_{2}=E_{2}^{0} \cup\left\{\left(u, u_{2}\right)\right\}, \ldots, E_{m_{0}}=E_{m_{0}}^{0} \cup\left\{\left(u, u_{m_{0}}\right)\right\}$ where $u_{1}, u_{2}, \ldots, u_{m_{0}}$ are distinct nodes from
$V_{0}$. It is straightforward to show that $|V|=n,|E|=m(n-1), G\left(V, E_{i}\right)$ is a connected graph, and $E_{i} \cap E_{j}=\emptyset$ for all $i, j \leq m$. Thus the Lemma holds for this case.

For the case of $n=n_{0}+2$ and $m=m_{0}+1$, let $V=V_{0} \cup\{u, v\}$ where $u, v$ are new nodes that are not in $V_{0}$, and define $E_{1}, \ldots, E_{m}$ as follows.

1. Set $E_{m}=\emptyset$ and $U=\emptyset$, where $U$ is a temporary variable.
2. Define $E_{1}$ :
(a) Select an edge $\left(v_{1}, v_{2}\right) \in E_{1}^{0}$.
(b) Let $E_{1}=\left(E_{1}^{0} \backslash\left\{\left(v_{1}, v_{2}\right)\right\}\right) \bigcup\left\{\left(v_{1}, u\right),(u, v),\left(v, v_{2}\right)\right\}$.
(c) Let $E_{m}=E_{m} \cup\left\{\left(v, v_{1}\right),\left(v_{1}, v_{2}\right),\left(v_{2}, u\right)\right\}$ and $U=U \cup\left\{v_{1}, v_{2}\right\}$.
3. Define $E_{i}$ for $2 \leq i \leq m_{0}$ :
(a) Select $v_{2 i-1}, v_{2 i} \notin U$.
(b) Let $E_{i}=E_{i}^{0} \cup\left\{\left(u, v_{2 i-1}\right),\left(v, v_{2 i}\right)\right\}$.
(c) Let $E_{m}=E_{m} \cup\left\{\left(v, v_{2 i-1}\right),\left(u, v_{2 i}\right)\right\}$ and $U=U \cup\left\{v_{2 i-1}, v_{2 i}\right\}$.

It is straightforward to show that $|V|=n,\left|E_{i}\right|=(n-1)$ (thus $|E|=m(n-1)$ ), $G\left(V, E_{i}\right)$ is a connected graph, and $E_{i} \cap E_{j}=\emptyset$ for all $i, j \leq m$. This completes the proof of the Lemma.
Q.E.D.

Theorem 3.4 Given $n, m$, $t$ with $m=t+1$, there exists a $(t+1)$-color connected colored edge graphs $G(V, E, C, f)$ with $|V|=n$ and $|C|=m$ if and only if $n \geq 2 m$.

Proof. By Lemma3.1, a $(t+1)$-color connected colored edge graphs $G(V, E, C, f)$ with $|V|=n$ and $|C|=m=t+1$ contains at least $m(n-1)$ edges. Meanwhile, $G(V, E, C, f)$ contains at most $n(n-1) / 2$ edges. Thus for $n<2 m$, we have $n(n-1) / 2<m(n-1)$. In nother words, for $n<2 m$, there is no $(t+1)$-color connected colored edge graphs $G(V, E, C, f)$ with $|V|=n$ and $|C|=m=t+1$. Now the theorem follows from Lemmas 3.1 and 3.3
Q.E.D

## 4 Necessary conditions for general cases

First we note that for a colored edge graph $G$ to be $(t+1)$-color connected, each node must have a degree of at least $t+1$. Thus the total degree of an $n$-node graph should be at least $n(t+1)$. This implies the following lemma.

Lemma 4.1 For $m \geq t+1>1$, and a $(t+1)$-color connected colored edge graph $G(V, E, C, f)$ with $|V|=n,|E|=\lambda$, and $|C|=m$, we have $2 \lambda \geq(t+1) n$.

In the following, we use cover free family concepts to study the necessary conditions for colored edge graphs connectivity.

Definition 4.2 Let $X$ be a finite set with $|X|=\lambda$ and $\mathcal{F}$ be a set of mutually disjoint subsets of $X$ with $|\mathcal{F}|=m$. Then $(X, \mathcal{F})$ is called a $(\lambda, m)$-partition of $X$ if $X=\bigcup_{P \in \mathcal{F}} P$. Let $n, t$ be positive integers. An $(\lambda, m)$-partition $(X, \mathcal{F})$ is called $a(t ; n-1)$-cover free family (or $(t ; n-1) \operatorname{CFF}(\lambda, m)$ ) if, for any $t$ elements $B_{1}, \ldots, B_{t} \in \mathcal{F}$, we have that

$$
\left|X \backslash\left(\bigcup_{i=1}^{t} B_{i}\right)\right| \geq n-1 \quad\left(\text { or }\left|\bigcap_{i=1}^{t}\left(X \backslash B_{i}\right)\right| \geq n-1\right)
$$

It should be noted that our above definition of cover-free family is different from the generalized cover-free family definition for set systems in the literature (see, e.g., [6, 10]). In [10], a set system $(X, \mathcal{F})$ is called a $(w, t ; n-1)$-cover free family if for any $w$ blocks $A_{1}, \ldots, A_{w} \in \mathcal{F}$ and any $t$ blocks $B_{1}, \ldots, B_{t} \in \mathcal{F}$, one has $\left|\left(\cap_{j=1}^{w} A_{j}\right) \backslash\left(\cup_{i=1}^{t} B_{i}\right)\right| \geq n-1$. Specifically, there are two major differences between our $(\lambda, m)$-partition system and the set systems in the literature $\mathbb{1}^{1}$.

1. For a set system $(X, \mathcal{F}), \mathcal{F}$ may contain repeated elements.
2. For a set system $(X, \mathcal{F})$, the elements in $\mathcal{F}$ are not necessarily mutually disjoint.
[^0]It is straightforward to show that a colored edge graph $G$ is $(t+1)$-color connected if and only if for any color set $C_{t} \subseteq C$ of size $t$, after the removal of edges in $G$ with colors in $C_{t}, G$ remains connected. Assume that $G$ contains $n$ nodes. Then a necessary condition for connectivity is that $G$ contains at least $n-1$ edges. From this discussion, we get the following lemma.

Lemma 4.3 For a colored edge graph $G(V, E, C, f)$, with $|V|=n,|E|=\lambda,|C|=m$, a necessary condition for $G(V, E, C, f)$ to be $(t+1)$-color connected is that the $(\lambda, m)$-partition $(X, \mathcal{F})$ is $a(t ; n-1)$-CFF $(\lambda, m)$ with $X=E$ and $\mathcal{F}=\left\{E_{c}: c \in C\right\}$ where $E_{c}=\{e: f(e)=c, e \in E\}$.

In the following, we analyze lower bounds for the number $\lambda$ of edges for the existence of a $(t ; n-1)-\mathrm{CFF}(\lambda, m)$. For a set partition $(X, \mathcal{F})$ and a positive integer $t$, let

$$
\mu(X, \mathcal{F} ; t)=\min \left\{\left|X \backslash\left(\bigcup_{i=1}^{t} B_{i}\right)\right|: B_{1}, \ldots, B_{t} \in \mathcal{F}\right\}
$$

It is straightforward to see that a $(\lambda, m)$-partition $(X, \mathcal{F})$ is a $(t ; n-1)-\operatorname{CFF}(\lambda, m)$ if and only if $\mu(X, \mathcal{F} ; t) \geq n-1$.
Given positive integers $\lambda, m, t$, let

$$
\mu(\lambda, m ; t)=\max \{\mu(X, \mathcal{F} ; t):(X, \mathcal{F}) \text { is a }(\lambda, m) \text {-partition }\}
$$

From the above discussion and Lemma 4.1, we have the following theorem.
Theorem 4.4 Let $\lambda, m, t$ be given positive integers. $\mu(\lambda, m ; t) \geq n-1$ and $2 \lambda \geq(t+1) n$ are necessary conditions for the existence of a $(t+1)$-color connected colored edge graph $G(V, E, C, f)$, with $|V|=n,|E|=\lambda,|C|=m$.

Theorem 4.5 Let $\lambda, m, t$ be given positive integers. Then we have

$$
\mu(\lambda, m ; t)= \begin{cases}(m-t) \cdot\left\lfloor\frac{\lambda}{m}\right\rfloor & \text { if } t \geq \lambda-\left\lfloor\frac{\lambda}{m}\right\rfloor \cdot m \\ (m-t) \cdot\left\lfloor\frac{\lambda}{m}\right\rfloor+\left(\lambda-\left\lfloor\frac{\lambda}{m}\right\rfloor \cdot m-t\right) & \text { otherwise }\end{cases}
$$

Proof. For a given $(\lambda, m)$-partition $(X, \mathcal{F})$, let $B_{1}, \ldots, B_{m}$ be an enumeration of elements in $\mathcal{F}$ such that $\left|B_{i}\right| \leq\left|B_{i+1}\right|$ for all $i<m$. It is straightforward to show that $\mu(X, \mathcal{F} ; t)=\sum_{i=1}^{m-t}\left|B_{i}\right|$. Thus $\mu(\lambda, m ; t)$ takes the maximum value if $\sum_{i=1}^{m-t}\left|B_{i}\right|$ is maximized. It is straightforward to show that this value is maximized when the $(\lambda, m)$-partition $(X, \mathcal{F})$ satisfies the following conditions:

1. $\left|B_{i}\right|=\left\lfloor\frac{\lambda}{m}\right\rfloor$ for $i \leq m-\left(\lambda-\left\lfloor\frac{\lambda}{m}\right\rfloor \cdot m\right)$, and
2. $\left|B_{i}\right|=\left\lfloor\frac{\lambda}{m}\right\rfloor+1$ for $m \geq i>m-\left(\lambda-\left\lfloor\frac{\lambda}{m}\right\rfloor \cdot m\right)$.

The theorem follows from the above discussion.
Q.E.D.

Example 1 For $n=7, \lambda=10, m=5$, and $t=2$, we have $\mu(10,5 ; 2)=6=n-1$. However, $2 \lambda=20<(t+1) n=21$. This shows that the condition $2 \lambda \geq(t+1) n$ in Theorem4.4 is not redundant.

Example 2 There are no $(t+1)$-color connected colored edge graphs $G(V, E, C, f)$ for the following special cases:

1. $m=2, t=1, n=3$.
2. $m=4, t=2, n=4$.
3. $m=3, t=2, n \leq 5$.

Proof. Before we consider the specific cases, we observe that, when $m$ and $t$ are fixed, the function $\mu$ is nondecreasing when $\lambda$ increases.

1. In this case, the maximum value that $\lambda$ could take is 3 . Thus $\mu(3,2 ; 1)=1<n-1=2$. That is, there is no (1;2)-CFF(3, 2), which implies the claim. Note that this result also follows from Theorem 3.4
2. In this case, the maximum value that $\lambda$ could take is 6 . Thus $\mu(6,4 ; 2)=2<n-1=3$.
3. We only show this for the case $m=3, t=2, n=5$. In this case, the maximum value that $\lambda$ could take is 10 . Thus $\mu(10,3 ; 2)=1<n-1=4$. Note that this result also follows from Theorem 3.4
Q.E.D

The following theorem is a variant of Theorem4.4

Theorem 4.6 For $m-1>t>0$, a necessary condition for the existence of a $(t+1)$-color connected colored edge graph $G(V, E, C, f)$ with $|V|=n,|E|=\lambda$, and $|C|=m$ is that $2 \lambda \geq(t+1) n$ and the following conditions are satisfied:

- If $n=(m-t) k$ for some integer $k>0$, then $\lambda \geq m k-1$.
- If $n=(m-t) k+1$ for some integer $k>0$, then $\lambda \geq m k$.
- If $n=(m-t) k+2$ for some integer $k>0$, then $\lambda \geq m k+t+1$.
- .......
- If $n=(m-t) k+m-t-1$ for some integer $k>0$, then $\lambda \geq m k+m-2$.

Proof. For $m>t+1$, by Theorem 4.5, we have

$$
\mu(\lambda, m ; t)= \begin{cases}(m-t) k^{\prime} & \text { if } \lambda=m k^{\prime}+i \text { for } 0 \leq i \leq t \\ (m-t) k^{\prime}+1 & \text { if } \lambda=m k^{\prime}+t+1 \\ \cdots \cdots & \\ (m-t) k^{\prime}+m-t-1 & \text { if } \lambda=m k^{\prime}+m-1\end{cases}
$$

Thus the necessary condition $\mu(\lambda, m ; t) \geq n-1$ in Theorem4.4 can be interpreted as the following conditions:

$$
k^{\prime} \geq \begin{cases}\frac{n-1}{m-t} & \text { if } \lambda=m k^{\prime}+i \text { for } 0 \leq i \leq t \\ \frac{n-2}{m-t} & \text { if } \lambda=m k^{\prime}+t+1 \\ \cdots \cdots & \\ \frac{n-m+t}{m-t} & \text { if } \lambda=m k^{\prime}+m-1\end{cases}
$$

In other words, for a $(t+1)$-color connected colored edge graphs $G(V, E, C, f)$, the following $m-t$ conditions (the disjunction not conjunction) are satisfied:

- $|V|=n,|E| \geq m\left\lceil\frac{n-1}{m-t}\right\rceil$, and $|C|=m$.
- $|V|=n,|E| \geq m\left\lceil\frac{n-2}{m-t}\right\rceil+t+1$, and $|C|=m$.
- .......
- $|V|=n,|E| \geq m\left\lceil\frac{n-m+t}{m-t}\right\rceil+m-1$, and $|C|=m$.

By distinguishing the cases for $n=(m-t) k, n=(m-t) k+1, \cdots$, and $n=(m-t) k+m-t-1$, and by reorganizing above lines, these necessary conditions can be interpreted as the following $m-t$ conditions:

- $n=(m-t) k$ and $\lambda \geq m k-1$ for some $k>0$. Note that this follows from the last line of the above conditions (one can surely take other lines, but then the value of $\lambda$ would be larger). This comment applies to following cases also.
- $n=(m-t) k+1$ and $\lambda \geq m k$ for some $k>0$.
- $n=(m-t) k+2$ and $\lambda \geq m k+t+1$ for some $k>0$.
- .......
- $n=(m-t) k+m-t-1$ and $\lambda \geq m k+m-2$ for some $k>0$.
Q.E.D.


## 5 Necessary and sufficient conditions for practical cases (with small $m$ and $t$ )

Generally we are interested in the question whether the necessary condition in Theorems 4.4 and 4.6 are also sufficient. In the following, we show that this is true for several important practical cases.

Theorem 5.1 The necessary condition in Theorem 4.4 is sufficient for the case of $m=t+1$.

Proof. Since $\lambda-\left\lfloor\frac{\lambda}{m}\right\rfloor \cdot m$ is the remainder of $\lambda$ divided by $m$, we trivially have $t=m-1 \geq \lambda-\left\lfloor\frac{\lambda}{m}\right\rfloor \cdot m$. Now assume that $m>\frac{n}{2}$. By Theorem4.5, we have $\mu(\lambda, m ; t)=\left\lfloor\frac{\lambda}{m}\right\rfloor \leq\left\lfloor\frac{n(n-1)}{2 m}\right\rfloor<n-1$. The rest follows from Theorem[3.4. Q.E.D.

Before we show that the necessary conditions in Theorems 4.4 and 4.6 are sufficient for the case of $t=1$, we first present two lemmas whose proofs are straightforward.

Lemma 5.2 For $n=m=\lambda \geq 3$ and $t=1$, the following m-node circle graph is $(1+1)$-color connected:

$$
\left\{\left(v_{1}, v_{2}\right),\left(v_{2}, v_{3}\right), \ldots,\left(v_{m}, v_{1}\right)\right\}
$$

with $f\left(v_{i}, v_{i+1}\right)=c_{i}$ for $i<m$ and $f\left(v_{m}, v_{1}\right)=c_{m}$.
Lemma 5.3 For $t=1, m \geq 3$, and $m<n \leq 2 m-2$, the graph in Figure $\square$ that is defined in the following is $(1+1)$-color connected

$$
\left\{\left(v_{1}, v_{2}\right),\left(v_{2}, v_{3}\right), \ldots,\left(v_{m}, v_{1}\right)\right\} \cup\left\{\left(v_{m}, v_{m+1}\right),\left(v_{m+1}, v_{m+2}\right), \ldots,\left(v_{n}, v_{1}\right)\right\}
$$

with

$$
\begin{aligned}
f\left(v_{i}, v_{i+1}\right) & & =c_{i} & \\
f\left(v_{m}, v_{1}\right) & & =c_{m} & \\
f\left(v_{m+i-1}, v_{m+i}\right) & & =c_{i} & \\
f\left(\left(v_{n}, v_{1}\right)\right) & & =c_{n-m+1} &
\end{aligned}
$$



Figure 1: Graph for Lemma 5.3

Theorem 5.4 The necessary conditions in Theorems 4.4 and 4.6 are sufficient for the case of $t=1$.
Proof. For the case of $m=2$ and $t=1$, it follows from Theorem 5.1 Now assume that $m>2$ and $t=1$. In this special case, the necessary conditions in Theorem4.6 is as follows:

- $n=(m-1) k$ and $\lambda \geq m k-1$ for some $k>0$.
- $n=(m-1) k+1$ and $\lambda \geq m k$ for some $k>0$.
- $n=(m-1) k+2$ and $\lambda \geq m k+2$ for some $k>0$.
- .......
- $n=(m-1) k+m-2$ and $\lambda \geq m k+m-2$ for some $k>0$.

In the following we first show that the condition " $n=(m-1) k+1$ and $\lambda \geq k m$ " is sufficient. Let the graph in Figure 2 be defined as follows:


Figure 2: Graph for the case $n=(m-1) k+1$ and $\lambda \geq k m$

$$
\begin{aligned}
& V=\left\{v_{0}, v_{1} \cdots, v_{(m-1) k}\right\} \\
& E_{1}=\left\{\left(v_{0}, v_{(m-1) i+1}\right): 0 \leq i \leq k-1\right\} \\
& E_{j}=\left\{\left(v_{(m-1) i+j-1}, v_{(m-1) i+j)}: 0 \leq i \leq k-1\right\} \text { for } 2 \leq j \leq m-1\right. \\
& E_{m}=\left\{\left(v_{(m-1) i}, v_{0}\right): 1 \leq i \leq k\right\} \\
& E=E_{1} \cup E_{2} \cup \cdots \cup E_{m}
\end{aligned}
$$

For each $e \in E_{j}$ with $i \leq m$, let $f(e)=c_{j}$. Then it is straightforward to check that the colored edge graphs $G(V, E, C, f)$ is $(1+1)$-color connected, $|V|=(m-1) k+1$, and $|E|=m k$.

Now we show that the condition " $n=(m-1) k+j$ and $\lambda \geq k m+j$ for $2 \leq j \leq m-1$ " is sufficient. Let $G(V, E, C, f)$ be the colored edge graph that we have just constructed with $|V|=(m-1) k+1$, and $|E|=m k$.

Let $V^{\prime}=V \cup\left\{v_{(m-1) k+1}, \ldots, v_{(m-1) k+j-1}\right\}$. Define a new colored edge graph $G\left(V^{\prime}, E^{\prime}, C, f^{\prime}\right)$ (see Figure 3) by attaching the following edges to the $m$-node circle $\left\{\left(v_{0}, v_{1}\right),\left(v_{1}, v_{2}\right), \ldots,\left(v_{m-1}, v_{0}\right)\right\}$ :

$$
\left\{\left(v_{m-1}, v_{(m-1) k+1}\right),\left(v_{(m-1) k+1}, v_{(m-1) k+2}\right), \ldots,\left(v_{(m-1) k+j-1}, v_{0}\right)\right\}
$$

The colors for the new edges are defined by letting $f^{\prime}\left(v_{(m-1) k+i}, v_{(m-1) k+i+1}\right)=c_{i+1}$ for $0 \leq i \leq j-2$ and $f^{\prime}\left(v_{(m-1) k+j-1}, v_{0}\right)=$ $c_{j}$. It is straightforward to check that $G\left(V^{\prime}, E^{\prime}, C, f^{\prime}\right)$ is $(1+1)$-color connected, $|V|=(m-1) k+j$, and $|E|=m k+j$.
Q.E.D.


Figure 3: Graph for case $n=(m-1) k+j$ and $\lambda \geq k m+j$ for $2 \leq j \leq m-1$

Corollary 5.5 For $t=1$ and $m, n, \lambda>1$, there exists an $(1+1)$-color connected colored edge graph $G(V, E, C, f)$ with $|V|=n$ and $|E|=\lambda$ if and only if

$$
\lambda \geq \min \left\{m\left\lceil\frac{n-1}{m-1}\right\rceil, m\left\lceil\frac{n-2}{m-1}\right\rceil+2, \ldots, m\left\lceil\frac{n-m+1}{m-1}\right\rceil+m-1\right\} .
$$

Proof. It follows from the proof of Theorem 5.4
Q.E.D

Theorem 5.6 The conditions in Theorems 4.4 and 4.6 are sufficient for the case of $m=4, t=2$.
Proof. It is sufficient to show that both of the conditions " $n=(m-t) k+1$ and $\lambda \geq k m$ " and " $n=(m-t) k+2$ and $\lambda \geq m k+t+1$ " are sufficient (note that $m=4$ and $t=2$ ). In the following we first show that the condition " $n=(m-t) k+1$ and $\lambda \geq k m$ " is sufficient by induction on $k$.

For the case of $k=2$, we have $n=5, \lambda=8, m=4$, and $t=2$. Let the graph $G_{1}$ in Figure 4 be defined as

$$
G_{1}=\left\{\left(v_{1}, v_{2}\right)_{1},\left(v_{2}, v_{3}\right)_{2},\left(v_{3}, v_{4}\right)_{1},\left(v_{4}, v_{5}\right)_{3},\left(v_{5}, v_{1}\right)_{2},\left(v_{1}, v_{3}\right)_{3},\left(v_{1}, v_{4}\right)_{4},\left(v_{2}, v_{5}\right)_{4}\right\}
$$

where $\left(v, v^{\prime}\right)_{i}$ means that the edge $\left(v, v^{\prime}\right)$ takes color $c_{i}$. It is straightforward to check that $G_{1}$ is $(2+1)$-color connected.
For the case of $k=3$, we have $n=7, \lambda=12, m=4$, and $t=2$. Let the graph $G_{2}$ in Figure 5 be defined as

$$
\begin{aligned}
& \left\{\left(v_{1}, v_{2}\right)_{1},\left(v_{2}, v_{3}\right)_{2},\left(v_{4}, v_{5}\right)_{3},\left(v_{5}, v_{1}\right)_{2},\left(v_{1}, v_{3}\right)_{3},\left(v_{1}, v_{4}\right)_{4},\right. \\
& \left.\quad\left(v_{2}, v_{5}\right)_{4},\left(v_{3}, v_{6}\right)_{1},\left(v_{6}, v_{7}\right)_{3},\left(v_{7}, v_{4}\right)_{1},\left(v_{4}, v_{6}\right)_{4},\left(v_{3}, v_{7}\right)_{2}\right\}
\end{aligned}
$$

where $\left(v, v^{\prime}\right)_{i}$ means that the edge $\left(v, v^{\prime}\right)$ takes color $c_{i}$. It is straightforward to check that $G_{2}$ is $(2+1)$-color connected.
Now for $k=2 r(r \geq 2)$, we have $n=(m-t) k+1=4 r+1$ and $\lambda=k m=8 r$. If we glue the $v_{1}$ node of $r$ copies of $G_{1}$, we get a $(t+1)$-color connected colored graph $G$ with $n=4 r+1$ and $\lambda=8 r$. Thus the condition for the case of $k=2 r$ holds.

For $k=2 r+1(r \geq 2)$, we have $n=(m-t) k+1=4 r+3$ and $\lambda=k m=8 r+4$. If we glue glue the $v_{1}$ node of $r-1$ copies of $G_{1}$ and one copy of $G_{2}$, we get a $(t+1)$-color connected colored graph $G$ with $n=4(r-1)+1+6=4 r+3$ and $\lambda=8(r-1)+12=8 r+4$. Thus the condition for the case of $k=2 r+1$ holds. This completes the induction.

For the condition " $n=(m-t) k+2$ and $\lambda \geq m k+t+1$ ", one can add one node to the graph for the case " $n=(m-t) k+1$ and $\lambda \geq k m$ " with 3 edges (with distinct colors) to any three nodes. The resulting graph meets the requirements. Q.E.D.

Theorem5.6 could be extended to the case of $m=5$ and $t=3$.


Figure 4: Graph $G_{1}$ for the case $n=5, m=4, t=2$


Figure 5: Graph $G_{2}$ for the case $n=7, m=4, t=2$

Theorem 5.7 The conditions in Theorems 4.4 and 4.6 are sufficient for the case of $m=5$ and $t=3$.
Proof. It is sufficient to show that both of the conditions " $n=(m-t) k+1$ and $\lambda \geq k m$ " and " $n=(m-t) k+2$ and $\lambda \geq m k+t+1$ " are sufficient (note that $m-t=2$ ). In the following we first show that the condition " $n=2 k+1$ and $\lambda \geq k m$ " is sufficient by induction on $k$ and $m$.

For $m=5$ and $k=2$, we have $n=5, \lambda=10$. The graph in Figure 6 shows that the condition is sufficient also. For the


Figure 6: Graph $G_{5,1}$ for the case $n=5, m=5, t=3$
case of $k=3$, we have $n=7, \lambda=15$. The graph in Figure 7 shows that the condition is sufficient also.
For $k=2 r(r \geq 2)$, the condition becomes $n=(m-t) k+1=4 r+1$ and $\lambda=k m=10 r$. If we glue the $v_{1}$ node of $r$ copies of $G_{5,1}$, we get a $(t+1)$-color connected colored graph $G$ with $n=4 r+1$ and $\lambda=10 r$. Thus the condition for the case of $k=2 r$ holds.

For $k=2 r+1(r \geq 2)$, the condition becomes $n=(m-t) k+1=4 r+3$ and $\lambda=k m=10 r+5$. If we glue glue the $v_{1}$ node of $r-1$ copies of $G_{5,1}$ and one copy of $G_{5,2}$, we get a $(t+1)$-color connected colored graph $G$ with $n=4(r-1)+1+6=4 r+3$ and $\lambda=10(r-1)+15=10 r+5$. Thus the condition for the case of $k=2 r+1$ holds. This completes the induction.

For the condition " $n=(m-t) k+2$ and $\lambda \geq m k+t+1$ ", we have $n=2 k+2$ and $\lambda \geq 5 k+4$. We can add one node to the graph for the case " $n=(m-t) k+1$ and $\lambda \geq k m$ " with 4 edges (with distinct colors) to any four nodes. The resulting graph meets the requirements.
Q.E.D.

Open Questions: We showed in this section that the conditions in Theorems 4.4 and 4.6 are sufficient for practical cases. It would be interesting to show that these conditions are also sufficient for general cases. We leave this as an open question.


Figure 7: Graph $G_{5,2}$ for the case $n=7, m=5, t=3$

## 6 Hardness results

We have given necessary and sufficient conditions for $(t+1)$-color connected colored edge graphs. Sometimes, it is also important to determine whether a given graph is $(t+1)$-color connected. Unfortunately, the following Theorem shows that the problem ceConnect is coNP-complete. The ceConnect problem is defined as follows.
INSTANCE: A colored edge graph $G=G(V, E, C, f)$, two nodes $A, B \in V$, and a positive integer $t \leq|C|$.
QUESTION: Are $A$ and $B t$-color connected?
Before we prove the hardness result, we first introduce the concept of color separator. For a colored edge graph $G=$ $G(V, E, C, f)$, a color separator for two nodes $A$ and $B$ of the graph $G$ is a color set $C^{\prime} \subseteq C$ such that the removal of all edges with colors in $C^{\prime}$ from the graph $G$ will disconnect $A$ and $B$. It is straightforward to observe that $A$ and $B$ are $(t+1)$-color connected if and only there is no $t$-size color separator for $A$ and $B$.

Theorem 6.1 The problem ceConnect is coNP-complete.
Proof. It is straightforward to show that the problem is in coNP. Thus it is sufficient to show that it is NP-hard. The reduction is from the Vertex Cover problem. The VC problem is as follows (definition taken from [7]):
INSTANCE: A graph $G=(V, E)$ and a positive integer $t \leq|V|$.
QUESTION: Is there a vertex cover of size $t$ or less for $G$, that is, a subset $V^{\prime} \subseteq V$ such that $\left|V^{\prime}\right| \leq t$ and, for each edge $(u, v) \in E$, at least one of $u$ and $v$ belongs to $V^{\prime}$ ?
For a given instance $G=(V, E)$ of VC, we construct a colored edge graph $G_{c}=\left(V_{c}, E_{c}, f, C\right)$ as follows. First assume that the vertex set $V$ is ordered as in $V=\left\{v_{1}, \ldots, v_{n}\right\}$. Let

$$
\begin{aligned}
V_{c} & =\{A, B\} \bigcup\left\{e_{\left(v_{i}, v_{j}\right)}:\left(v_{i}, v_{j}\right) \in E \text { and } i<j\right\} \\
E_{c} & =\left\{\left(A, e_{\left(v_{i}, v_{j}\right)}\right),\left(e_{\left(v_{i}, v_{j}\right)}, B\right):\left(v_{i}, v_{j}\right) \in E\right\} \\
C & =\left\{c_{v}: v \in V\right\} \\
f & =\left\{f\left(A, e_{\left(v_{i}, v_{j}\right)}\right)=c_{v_{i}}, f\left(e_{\left(v_{i}, v_{j}\right)}, B\right)=c_{v_{j}}:\left(v_{i}, v_{j}\right) \in E, i<j\right\}
\end{aligned}
$$

In the following, we show that there is a vertex cover of size $t$ in $G$ if and only if there is a $t$-color edge separator for $G_{c}$.
Without loss of generality, assume that $V^{\prime}=\left\{v_{1}^{\prime}, \ldots, v_{k}^{\prime}\right\}$ is a vertex cover for $G$. Then it is straightforward to show that $C^{\prime}=\left\{c_{v_{i}^{\prime}}: v_{i}^{\prime} \in V^{\prime}\right\}$ is a color separator for $G_{c}$ since each incoming path for $B$ in $G_{c}$ contains two colors corresponding to one edge $\left(v_{i}, v_{j}\right)$ in $G$.

For the other direction, assume that $C^{\prime}=\left\{c_{v_{i}^{\prime}}: i=1, \ldots, t\right\}$ is a $t$-color separator for $G_{c}$. Let $V^{\prime}=\left\{v_{i}^{\prime}: c_{v_{i}^{\prime}} \in C^{\prime}\right\}$. By the fact that $C^{\prime}$ is a color separator for $G_{c}$, for each edge $\left(v_{i}, v_{j}\right) \in E$ in $G$, the path $\left(A, e_{\left(v_{i}, v_{j}\right)}, B\right)$ in $G_{c}$ contains at least one color from $C^{\prime}$. Since this path contains only two colors $c_{v_{i}}$ and $c_{v_{j}}$, we know that $v_{i}$ or $v_{j}$ or both belong to $V^{\prime}$. In another word, $V^{\prime}$ is a $t$-size vertex cover for $G$. This completes the proof of the Theorem.
Q.E.D.

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