Clique-width and edge contraction

Bruno COURCELLE
LaBRI, CNRS,
351 Cours de la Libération,
33405 Talence, France
courcell@labri.fr

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Abstract : We prove that edge contractions do not preserve the property that a set of graphs has bounded clique-width.

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1 Introduction

Clique-width is, like tree-width, an integer graph invariant that is an appropriate parameter for the contruction of many fixed-parameter tractable algorithms ([4, 6, 7, 11]). It is thus important to know that the graphs of a particular type have bounded tree-width or clique-width. The article [13] is a survey of clique-width bounded classes. Gurski has reviewed in [9] how clique-width behaves under different graph operations. He asks whether, for each k, the class of graphs of clique-width at most k is stable under edge contractions. This is true for $k = 2$, *i.e.*, for cographs, and we prove that this is false for $k = 3$. (For each k, this stability property is true for graphs of tree-width at most k. It is thus natural to ask the question for clique-width.)

Gurski proves that contracting one edge can at most double the clique-width. The conjecture is made in [14] (Conjecture 8) that contracting several edges in a graph of clique-width k yields a graph of clique-width at most $f(k)$ for some fixed function f. We disprove this conjecture and answer Gurski's question by proving the following proposition.

Proposition 1 : The graphs obtained by edge contractions from graphs of clique-width 3 or of linear clique-width 4, have unbounded clique-width.

The graphs of clique-width at most 2 (they are the cographs) are preserved under edge contractions.The validity of the conjecture of [14] would have implied that the *restricted vertex multicut problem* is fixed-parameter tractable if the parameter is the clique-width of a certain graph describing the input in a natural way. This problem consists in finding a set of vertices of given size that meets every path between the two vertices of each pair of a given set and does not contain any vertex of these pairs. Without Conjecture 8, this problem is fixed-parameter tractable under the additional condition that no two vertices from different pairs are adjacent.

For sake of comparison, we also consider contractions of edges, one end of which has degree 2. We say in this case that we *erase a vertex*: we erase x if it has exactly two neighbours ; to do so, we add an edge between them (unless they are adjacent, we only consider graphs without parallel edges) and we delete x and its two incident edges. The graphs obtained from a graph by erasing and deleting vertices are its *induced topological minors*.

Proposition 2: The induced topological minors of the graphs of clique-width k have clique-width at most $2^{k+1} - 1$.

2 Some basic facts

Graphs are finite, undirected, loop-free and without parallel edges. To keep this note as short as possible, we refer the reader to any of [3, 11, 13, 15, 16] for the definitions of *clique-width* and *rank-width*. Other references for clique-width are [1, 5, 8, 14]. A variant of clique-width called *linear clique-width* is studied in [3, 10]. We denote by $cwd(G)$ and $rwd(G)$ the clique-width and, respectively, the rank-width of a graph G. We recall from [16] that we have $rwd(G) \leq cwd(G) \leq 2^{rwd(G)+1} - 1$. Proving that $cwd(G) > k$ for given G and k is rather difficult in most cases. (See for instance the computation of the exact clique-width of a square grid in [8]. The computation of its rank-width in [12] is not easier.) We overcome this difficulty by using *monadic second-order transductions* : they are graph transformations specified by formulas of monadic second-order logic. The (technical) definition is in [2,3]. We will only need the fact that the graphs defined by a monadic second-order transduction τ from graphs of clique-width at most k have clique-width at most $f_\tau(k)$ for some computable function f_τ that can be determined from the formulas forming the definition of τ (Corollary 7.38(2), [3]). However, we also give an alternative proof based on rank-width and vertex-minors that does not use monadic second-order transductions.

A *vertex-minor* of a graph is obtained by deleting vertices (and the incident edges) and performing *local complementations*. (Local complementation exchanges edges and non-edges between the neighbours of a vertex.) These transformations do not increase rank-width [15]. Erasing a vertex x yields a vertex-minor of the considered graph: let y and z be its neighbours; if they are adjacent, erasing x is the same as deleting it because we fuse parallel edges; if they are not, erasing x is the same as performing a local complementation at x (which creates an edge between y and z), and then deleting x. Hence, by transitivity, every induced topological minor is a vertex-minor.

3 Proofs

Definitions and notation

(a) We denote by H/F the graph obtained from a graph H by contracting the edges of a set F. (Parallel edges are fused, no loops are created.) If $\mathcal{H}$ is a set of graphs, we denote by $EC(\mathcal{H})$ the set of graphs H/F such that $H \in \mathcal{H}$ and F is a set of edges of H.

(b) We denote by $\mathcal{R}$ the set of graphs having a *proper edge coloring* with colors in $\{1, ..., 4\}$: every two adjacent edges have different colors. These graphs have unbounded tree-width and clique-width as they include the square grids (the $n \times n$ grid has clique-width $n+1$ if $n \geq 2$ by [8]).

(c) For $n \geq 2$, we define a graph G_n. Its vertices are $x_1, ..., x_n, y_1, ..., y_n$ and its edges are $x_i - y_i, y_i - y_j$ for all $i, j \neq i$. (The notation $x - y$ designates an edge between x and y). We let D consist of 4 vertices and no edge, and we let H_n be obtained from G_n by substituting disjoint copies of D to each vertex y_i. Hence and more precisely, H_n has the $5n$ vertices $x_1, ..., x_n, y_1^1, y_1^2, y_1^3, y_1^4, y_2^1, ..., y_n^4$ and the $8n^2 - 4n$ edges $x_i - y_i^c, y_i^c - y_j^d$ for all $i, j \neq i$ and $c, d = 1, ..., 4$. We denote by $\mathcal{H}$ the set of graphs H_n. It is easy to construct expressions of G_n and H_n showing that they have clique-width at most 3 and linear clique-width at most 4. If $n \geq 3$, they have clique-width 3 because they contain, as an induced subgraph, the path with 4 vertices, so that they do not have clique-width 2, and linear clique-width 4 because they contain, as an induced subgraph, the graph G_3 that is not a cocomparability graph, so that they do not have linear clique-width at most 3 by Proposition 14 of [10].

(d) We define a monadic second-order transduction α with one parameter X. If G is a graph and X is a set of vertices, then the graph $\alpha(G, X)$ is defined if X is stable (no two vertices are adjacent); its vertex set is X and it has an edge between x and y if and only if these vertices are at distance 2 in G. We denote by $\alpha(G)$ the set of all such graphs, and by $\alpha(\mathcal{G})$ the union of the sets $\alpha(G)$ for G in a set $\mathcal{G}$.

Lemma 3 : We have $\alpha(EC(\mathcal{H})) \supseteq \mathcal{R}$.

Proof: Let R be a graph in $\mathcal{R}$ with vertices $x_1, ..., x_n$ and a proper edge coloring with colors 1 to 4. The set $X = \{x_1, ..., x_n\}$ is also a subset of the vertex set of H_n. The four neighbours of x_i in H_n are y_i^1, y_i^2, y_i^3 and y_i^4.

Let F be the set of edges of the form $y_i^c - y_j^c$ such that $x_i - x_j$ is an edge of R colored by c, $c \in \{1, ..., 4\}$ (hence, x_i and x_j are at distance 3 in H_n). The graph $K = H_n/F$ belongs to $EC(\mathcal{H})$ and X is stable in this graph (the vertices $x_1, ..., x_n$ are not affected by the contractions of edges). It is clear that $x_i - x_j$ is an edge of R if and only if there is in K a path $x_i - z - x_j$ where z results from the contraction of the edge $y_i^c - y_j^c$ and c is the color of $x_i - x_j$ in R. It follows that $R = \alpha(K, X)$. $\square$

Proof of Proposition 1: By Lemma 3, the set $\alpha(EC(\mathcal{H}))$ has unbounded clique-width. Hence, so has $EC(\mathcal{H})$ by Corollary 7.38(2) of [3] recalled above. This concludes the proof because the graphs H_n have clique-width 3 and linear clique-width 4 for $n \geq 3$. $\square$

NLC-width and clique-width are linearly related (see [9]). Hence, the graphs obtained by edge contractions from graphs of NLC-width at most 3 have unbounded NLC-width.

Remark: For each n, the graph H_{m^2} of clique-width 3 having $5m^2$ vertices where $m = f_\alpha(n)$ yields by edge contractions a graph of clique-width at least $n+1$. Here, f_α is the computable function of Section 2 that can be assumed monotone. To prove this, we let F be a set of edges such that $\alpha(H_{m^2}/F)$ contains the $m \times m$ grid R_m and $k = cwd(H_{m^2}/F)$. Then, $m+1 = cwd(R_m) \leq f_\alpha(k)$, hence $f_\alpha(n) + 1 \leq f_\alpha(k)$, and so, $k > n$. The function f_α is very fast growing. A much better upper-bound will be obtained from the alternative proof we give next.

Edge contractions can increase rank-width because the same sets of graphs have bounded rank-width and bounded clique-width [16]. The following proof shows this directly.

Alternative proof of Proposition 1 : The construction is similar and we use the same notation. We construct H'_n from G_n by substituting disjoint copies of K_4 to each vertex y_i and by adding a vertex y_0 adjacent to all vertices $y_1^1, y_1^2, y_1^3, y_1^4, y_2^1, ..., y_n^4$. Hence, H'_n has $5n+1$ vertices.We denote by $\mathcal{H}'$ the set of graphs H'_n. They have clique-width 3 and linear clique-width 4 (by the same argument as for H_n).

Let us fix n and let R be the $n \times n$ grid with vertices $x_1, ..., x_{n^2}$. To prove that it is a vertex-minor of H'_{n^2}, we take a proper edge-coloring of R with colors $1, ..., 4$, we contract the edges $y_i^c - y_j^c$ of H'_{n^2} such that $x_i - x_j$ is an edge of R colored by c. This gives a graph R' that has R as vertex-minor. To prove this, we delete the vertices y_i^c such that x_i has no incident edge colored by c, we take a local complementation at y_0, we delete y_0 and finally, we erase the vertices resulting from the contraction of the edges $y_i^c - y_j^c$ after taking local complementations at them.

The rank-width of R is $n-1$ by [12], that of R' is thus at least $n-1$, and so is its clique-width. Hence, by contracting edges in a graph of clique-width

3 (and linear clique-width 4) that has $5n^2+1$ vertices, one can get a graph of clique-width at least $n-1$.□

Remark: An algorithm can determine a graph of clique-width 3 that yields a graph of clique-width more than 3 by the contraction of a single edge. It performs an exhaustive search until some graph is obtained: for each $n = 2, 3$ ... it considers the finitely many sets F of pairwise nonadjacent edges of H_n. By using the polynomial-time algorithm of [1] to check if a graph has clique-width at most 3, it can look for a set F and an edge $f \in F$ such that $H_n/(F - \{f\})$ has clique-width 3 and H_n/F has clique-width more than 3 (actually 4, 5 or 6 by Theorem 4.8 of [9]). By Proposition 1, one must find some n and such F and f. However, we have not implemented this algorithm.

Gurski has proved that erasing a vertex of degree 2 can increase (or decrease) the clique-width by at most 2. In Proposition 2, we consider the effect of erasing several vertices and taking induced subgraphs.

Proof of Proposition 2 : As noted above, an induced topological minor is a vertex-minor. The result follows since, for every graph G, we have $rwd(G) \leq cwd(G) \leq 2^{rwd(G)+1} - 1$.□

This proof leaves open the question of improving the upper bound $2^{k+1} - 1$, possibly to a polynomial in k or even to k.

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