

Parameterized Algorithms for Load Coloring Problem

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Abstract

One way to state the Load Coloring Problem (LCP) is as follows. Let $G = (V, E)$ be graph and let $f : V \rightarrow \{\text{red}, \text{blue}\}$ be a 2-coloring. An edge $e \in E$ is called red (blue) if both end-vertices of e are red (blue). For a 2-coloring f , let r'_f and b'_f be the number of red and blue edges and let $\mu_f(G) = \min\{r'_f, b'_f\}$. Let $\mu(G)$ be the maximum of $\mu_f(G)$ over all 2-colorings.

We introduce the parameterized problem k -LCP of deciding whether $\mu(G) \geq k$, where k is the parameter. We prove that this problem admits a kernel with at most $7k$. Ahuja et al. (2007) proved that one can find an optimal 2-coloring on trees in polynomial time. We generalize this by showing that an optimal 2-coloring on graphs with tree decomposition of width t can be found in time $O^*(2^t)$. We also show that either G is a Yes-instance of k -LCP or the treewidth of G is at most $2k$. Thus, k -LCP can be solved in time $O^*(4^k)$.

1 Introduction

For a graph $G = (V, E)$ with n vertices, m edges and maximum vertex degree Δ , the load distribution of a 2-coloring $f : V \rightarrow \{\text{red}, \text{blue}\}$ is a pair (r_f, b_f) , where r_f is the number of edges with at least one end-vertex colored red and b_f is the number of edges with at least one end-vertex colored blue. We wish to find a coloring f such that the function $\lambda_f(G) := \max\{r_f, b_f\}$ is minimized. We will denote this minimum by $\lambda(G)$ and call this problem LOAD COLORING PROBLEM (LCP). The LCP arises in Wavelength Division Multiplexing, the technology used for constructing optical communication networks [1, 9]. Ahuja et al. [1] proved that the problem is NP-hard and gave a polynomial time algorithm for optimal colorings of trees. For graphs G with genus $g > 0$, Ahuja et al. [1] showed that a 2-coloring f such that $\lambda_f(G) \leq \lambda(G)(1 + o(1))$ can be computed in $O(n + g \log n)$ -time, if the maximum degree satisfies $\Delta = o(\frac{m^2}{ng})$ and an embedding is given.

For a 2-coloring $f : V \rightarrow \{\text{red}, \text{blue}\}$, let r'_f and b'_f be the number of edges whose end-vertices are both red and blue, respectively (we call such edges *red* and *blue*, respectively). Let $\mu_f(G) := \min\{r'_f, b'_f\}$ and let $\mu(G)$ be the maximum of $\mu_f(G)$ over all 2-colorings of V . It is not hard to see (and it is proved in Remark 1.1 of [1]) that $\lambda(G) = m - \mu(G)$ and so the LCP is equivalent to maximizing $\mu_f(G)$ over all 2-colorings of V .

In this paper we introduce and study the following parameterization of LCP.

k -LOAD COLORING PROBLEM (k -LCP)	
<i>Input:</i>	A graph $G = (V, E)$ and an integer k .
<i>Parameter:</i>	k
<i>Question:</i>	Is $\mu(G) \geq k$? (Equivalently, is $\lambda(G) \leq m - k$?)

We provide basics on parameterized complexity and tree decompositions of graphs in the next section. In Section 3, we show that k -LCP admits a kernel with at most $7k$ vertices. Interestingly, to achieve this linear bound, only two simple reduction rules are used. In Section 4, we generalise the result of Ahuja et al. [1] on trees by showing that an optimal 2-coloring for graphs with tree decomposition of width t can be obtained in time $2^t n^{O(1)}$. We also show that either G is a Yes-instance of k -LCP or the treewidth of G is at most $2k$. As a result, k -LCP can be solved in time $4^k n^{O(1)}$. We conclude the paper in Section 5 by stating some open problems.

2 Basics on Fixed-Parameter Tractability, Kernelization and Tree Decompositions

A *parameterized problem* is a subset $L \subseteq \Sigma^* \times \mathbb{N}$ over a finite alphabet Σ . L is *fixed-parameter tractable* if the membership of an instance (x, k) in $\Sigma^* \times \mathbb{N}$ can be decided in time $f(k)|x|^{O(1)}$, where f is a function of the parameter k only. It is customary in parameterized algorithms to often write only the exponential part of $f(k)$: $O^*(t(k)) := O(t(k)(kn)^{O(1)})$.

Given a parameterized problem L , a *kernelization* of L is a polynomial-time algorithm that maps an instance (x, k) to an instance (x', k') (the *kernel*) such that (i) $(x, k) \in L$ if and only if $(x', k') \in L$, (ii) $k' \leq g(k)$, and (iii) $|x'| \leq g(k)$ for some function g . The function $g(k)$ is called the *size* of the kernel.

It is well-known that a parameterized problem L is fixed-parameter tractable if and only if it is decidable and admits a kernelization. Due to applications, low degree polynomial size kernels are of main interest. Unfortunately, many fixed-parameter tractable problems do not have kernels of polynomial size unless the polynomial hierarchy collapses to the third level, see, e.g., [2, 3, 4]. For further background and terminology on parameterized complexity we refer the reader to the monographs [5, 6, 8].

Definition 1. A tree decomposition of a graph $G = (V, E)$ is a pair $(\mathcal{X}, \mathcal{T})$, where $\mathcal{T} = (I, F)$ is a tree and $\mathcal{X} = \{X_i : i \in I\}$ is a collection of subsets of V called bags, such that:

1. $\bigcup_{i \in I} X_i = V$;
2. For every edge $xy \in E$, there exists $i \in I$ such that $\{x, y\} \subseteq X_i$;
3. For every $x \in V$, the set $\{i : x \in X_i\}$ induces a connected subgraph of \mathcal{T} .

The width of $(\mathcal{T}, \mathcal{X})$ is $\max_{i \in I} |X_i| - 1$. The treewidth of a graph G is the minimum width of all tree decompositions of G .

To distinguish between vertices of G and \mathcal{T} , we call vertices of \mathcal{T} *nodes*. We will often speak of a bag X_i interchangeably with the node i to which it corresponds in \mathcal{T} . Thus, for example, we might say two bags are *neighbors* if they correspond to nodes in \mathcal{T} which are neighbors. We define the *descendants* of a bag X_i as follows: every child of X_i is a descendant of X_i , and every child of a descendant of X_i is a descendant of X_i .

Definition 2. A nice tree decomposition of a graph $G = (V, E)$ is a tree decomposition $(\mathcal{X}, \mathcal{T})$ such that \mathcal{T} is a rooted tree, and each node i falls under one of the following classes:

- **i is a Leaf node:** Then i has no children;
- **i is an Introduce node:** Then i has a single child j , and there exists a vertex $v \notin X_j$ such that $X_i = X_j \cup \{v\}$;
- **i is a Forget node:** Then i has a single child j , and there exists a vertex $v \in X_j$ such that $X_i = X_j \setminus \{v\}$;
- **i is a Join node:** Then i has two children h and j , and $X_i = X_h = X_j$.

It is known that any tree decomposition of a graph can be transformed into a tree decomposition of the same width.

Lemma 1. [7] Given a tree decomposition with $O(n)$ nodes of a graph G with n vertices, we can construct, in time $O(n)$, a nice tree decomposition of G of the same width and with at most $4n$ nodes.

3 Linear Kernel

For a vertex v of a graph $G = (V, E)$ and set $X \subseteq V$, let $\deg_X(v)$ denote the number of neighbors of v in X . If $X = V$, we will write $\deg(v)$ instead of $\deg_V(v)$.

Lemma 2. Let $G = (V, E)$ be a graph with no isolated vertices, with maximum degree $\Delta \geq 2$ and let $|V| \geq 5k$. If $|V| \geq 4k + \Delta$, then (G, k) is a Yes-instance of k -LCP.

Proof. Suppose that $|V| \geq 4k + \Delta$, but (G, k) is a No-instance of k -LCP.

Let M be a maximum matching in G and let Y be the set of vertices which are not end-vertices of edges in M . If M has at least $2k$ edges, then we may color half of them blue and half of them red, so we conclude that $|M| < 2k$.

For an edge $e = uv$ in M , let $\deg_Y(e) = \deg_Y(u) + \deg_Y(v)$, that is the number of edges between a vertex in Y and a vertex of e .

Claim For any e in M , $\deg_Y(e) \leq \max\{\Delta - 1, 2\}$.

Proof of Claim: Suppose that $\deg_Y(e) \geq \Delta$ and let $e = uv$. As u and v are adjacent, $d_Y(u)$ and $d_Y(v)$ are each less than Δ . But as $\deg_Y(u) + \deg_Y(v) = \deg_Y(e) \geq \Delta$, it follows that $\deg_Y(u) \geq 1$ and $\deg_Y(v) \geq 1$. Then either u and v have only one neighbor in Y , which is adjacent to both of them (in which case $\deg_Y(e) = 2$), or there exist vertices $x \neq y \in Y$ such that x is adjacent to u and y is adjacent to v .

Then M is not a maximum matching, as $xvvy$ is an augmenting path, which proves the claim.

Now let M' be a subset of edges of M such that

$$\sum_{e' \in M'} \deg_Y(e') \geq k - |M'|, \quad (1)$$

and

$$\left[\sum_{e' \in M'} \deg_Y(e') \right] - \deg_Y(e) < k - |M'|, \text{ for any } e \in M'. \quad (2)$$

To see that M' exists observe first that $M' = M$ satisfies (1). Indeed, suppose it is not true. Then $|V| < |V(M)| + k - |M| = k + |M| < 5k$, a contradiction with our assumption that $|V| \geq 5k$. Now let M' be the minimal subset of M that satisfies (1), and observe that by minimality M' also satisfies (2).

Observe that $|M'| \leq k$. Then by the Claim, we have that $\sum_{e' \in M'} \deg_Y(e') \leq k - |M'| + \Delta$. Color M' and all neighbors of M' in Y red, and note that there are at most $k - |M'| + \Delta + 2|M'| = k + |M'| + \Delta \leq 2k + \Delta$ such vertices. The number of red edges is at least $k - |M'| + |M'| = k$.

Color the remaining vertices of G blue. By assumption there are at least $4k + \Delta - 2k - \Delta \geq 2k$ such vertices. As G contains no isolated vertices and M is a maximum matching, the blue vertices in Y have neighbors in the vertices of $M \setminus M'$. Thus, every blue vertex has a blue neighbor. It follows that there are at least $2k/2 = k$ blue edges. Thus, (G, k) is a Yes-instance of k -LCP. \square

We will use the following reduction rules for a graph G .

Reduction Rule 1. *Delete isolated vertices.*

Reduction Rule 2. *If there exists a vertex x and set of vertices S such that $|S| > k$ and every $s \in S$ has x as its only neighbor, delete a vertex from S .*

Theorem 1. *The problem k -LCP has a kernel with at most $7k$ vertices.*

Proof. Assume that G is a graph reduced by Rules 1 and 2. Assume also that G is a No-instance. We will prove that G has at most $7k$ vertices.

Claim A. *There is no pair x, y of distinct vertices such that $\deg(x) > 2k$ and $\deg(y) > k$.*

Proof of Claim: Suppose such a pair x, y exists. Color y and k of its neighbors, not including x , red. This leaves x and at least k of its neighbors uncolored. Color x and k of its neighbors blue.

Construct $G' = (V', E')$ as follows. Let x be a vertex in G of maximum degree. Let S be the vertices of G whose only neighbor is x . Then let $G' = G - (S \cup \{x\})$. The next claim follows from the definition of G' .

Claim B. *The graph G' has no isolated vertices.*

The next claim follows from the definition of G' and Claim A.

Claim C. *If the maximum degree in G' is at least $2k$, then G is a Yes-instance of k -LCP.*

The next claim follows from the definition of G' and Rule 2.

Claim D. *We have $|V| \leq |V'| + k + 1$.*

Observe that if G' was a Yes-instance of k -LCP then so would be G . Thus, G' is a No-instance of k -LCP. If the maximum degree in G' is 1, then we may assume that $|V'| < 4k$ as otherwise by Claim B G' is a matching with at least $2k$ edges and so (G', k) is a Yes-instance. So, the maximum degree of G' is at least 2. By Claim C and Lemma 2, we may assume that $|V'| \leq 4k + 2k - 1 = 6k - 1$. Then by Claim D, $|V| \leq 6k - 1 + k + 1 = 7k$. \square

Using the $7k$ kernel of this section we can get a simple algorithm that tries all 2-colourings of vertices of the kernel. The running time is $O^*(2^{7k}) = O^*(128^k)$. In the next section, we obtain an algorithm of running time $O^*(4^k)$.

4 Load Coloring Parameterized by Treewidth

Theorem 2. *Given a tree decomposition of G of width t , we can solve LCP in time $O(2^{t+1}(k+1)^4 n^3)$.*

Proof. Let $G = (V, E)$ be graph and let $(\mathcal{X}, \mathcal{T})$ be a tree decomposition of G of width t , where $\mathcal{T} = (I, F)$ and $\mathcal{X} = \{X_i : i \in I\}$. By Lemma 1, we may assume that $(\mathcal{X}, \mathcal{T})$ is a nice tree decomposition.

Let $\psi(X_i)$ denote the set of vertices in V which appear in either X_i or a descendant of X_i . For each $i \in I$, each $S \subseteq X_i$ and each $r, b \in \{0, 1, \dots, k\}$, define the boolean-valued function $F(X_i, S, r, b)$ to be true if there exists a 2-coloring $f : \psi(X_i) \rightarrow \{\text{red}, \text{blue}\}$ such that $f^{-1}(\text{red}) \cap X_i = S$ and there are at least r red edges and at least b blue edges in $G[\psi(X_i)]$. We will say such an f satisfies $F(X_i, S, r, b)$.

Let X_0 denote the bag which is the root of \mathcal{T} , and observe that G is a YES-instance if and only if $F(X_0, S, k, k)$ is true for some $S \subseteq X_0$. We now show how to calculate $F(X_i, S, r, b)$ for each X_i, S, r and b . Assume we have already calculated $F(X_j, S', r', b')$ for all descendants j of i and all values of S', r', b' . Our calculation of $F(X_i, S, r, b)$ depends on whether i is a Leaf, Introduce, Forget or Join node.

i is a Leaf node: As $\psi(X_i) = X_i$ there is exactly one 2-coloring $f : \psi(X_i) \rightarrow \{\text{red}, \text{blue}\}$ such that $f^{-1}(\text{red}) \cap X_i = S$. It is sufficient to set $F(X_i, S, r, b)$ to be true if and only if this coloring gives at least r red edges and at least b blue edges.

i is an Introduce node: Let j be the child of i and let v be the vertex such that $X_i \setminus X_j = \{v\}$. If $v \in S$, let r^* be the number of neighbors of v in S . Then for any 2-coloring on $\psi(X_i)$, the number of red edges in $G[\psi(X_i)]$ is exactly the number of red edges in $G[\psi(X_j)]$ plus r^* , and the number of blue edges is the same in $G[\psi(X_i)]$ and $G[\psi(X_j)]$. Therefore we may set $F(X_i, S, r, b)$ to be true if and only if $F(X_j, S \setminus \{v\}, \max(r - r^*, 0), b)$ is true. Similarly if $v \notin S$, we may set $F(X_j, S, r, b)$ to be true if and only if $F(X_j, S \setminus \{v\}, r, \max(b - b^*, 0))$ is true, where b^* is the number of neighbors of v in $X_i \setminus S$.

i is a Forget node: Let j be the child of i and let v be the vertex such that $X_j \setminus X_i = \{v\}$. Observe that $\psi(X_i) = \psi(X_j)$, and so any 2-coloring of $\psi(X_i)$ has exactly the same number of red edges or blue edges in $G[\psi(X_i)]$ and $G[\psi(X_j)]$. Therefore we may set $F(X_i, S, r, b)$ to be true if and only if $F(X_j, S, r, b)$ or $F(X_j, S \cup \{v\}, r, b)$ is true.

i is a Join node: Let h and j be the children of i . Let r^* be the number of red edges in X_i .

Then observe that if there is a 2-coloring on $\psi(X_i)$ consistent with S such that there are r_h red edges in $G[\psi(X_h)]$ and r_j red edges in $G[\psi(X_j)]$, then $r_h, r_j \geq r^*$, and the number of red edges in $G[\psi(X_i)]$ is $r_h + r_j - r^*$.

Let $r' = \min(r^*, k)$. Then if there are at least $r \leq k$ red edges in $G[\psi(X_i)]$, there are at least r_h red edges in $G[\psi(X_h)]$ for some r_h such that $r' \leq r_h \leq k$, and there are at least $r - r_h + r'$ red edges in $G[\psi(X_j)]$. Similarly let b^* be the number of blue edges in X_i , and let $b' = \min(b^*, k)$. Then if there are at least $b \leq k$ blue edges in $G[\psi(X_i)]$, there are at least b_h blue edges in $G[\psi(X_j)]$ for some b_h such that $b' \leq b_h \leq k$, and there are at least $b - b_h + b'$ blue edges in $G[\psi(X_j)]$.

Therefore, we may set $F(X_i, S, r, b)$ to be true if and only if there exist r_h, b_h such that $r' \leq r_h \leq k$, $b' \leq b_h \leq k$, and both $F(X_h, S, r_h, b_h)$ and $F(X_j, S, \max(r - r_h + r', 0), \max(b - b_h + b', 0))$ are true.

It remains to analyse the running time of the algorithm.

We first analyse the running time of calculating a single value $F(X_i, S, r, b)$ assuming we have already calculated $F(X_j, S', r', b')$ for all descendants j of i and all values of S', r', b' . In the case of a Leaf node, we can calculate $F(X_i, S, r, b)$ in $O(n + m)$ by checking a single 2-coloring. In the case of an Introduce node, we need to check a single value for the child of i , and in the case of a Forget node we need to check two values for the child of i . Thus, these cases can be calculated in $O(n + m)$ time. Finally, for a Join node, we need to check a value from both children of i for every possible way of choosing r_h, b_h such that $r' \leq r_h \leq k$ and $b' \leq b_h \leq k$. There are at most $(k + 1)^2$ such choices and so we can calculate $F(X_i, S, r, b)$ in $O((k + 1)^2(n + m))$ time.

It remains to check how many values need to be calculated. As there are at most $O(n)$ bags X_i , at most 2^{t+1} choices of $S \subseteq X_i$, and at most $k + 1$ choices for each of r and b , the number of values $F(X_i, S, r, b)$ we need to calculate is $O(n2^{t+1}(k + 1)^2)$. As each value can be calculated in polynomial time, overall we have running time $O(2^{t+1}(k + 1)^4 n(n + m)) = O(2^{t+1}(k + 1)^4 n^3)$. \square

We will combine Theorem 2 with the following lemma to obtain Theorem 3.

Lemma 3. *For a graph G , in polynomial time, we can either determine that G is a Yes-instance of k -LCP, or construct a tree decomposition of G of width at most $2k$.*

Proof. If every component of G has at most $k - 1$ edges then we may easily construct a tree decomposition of G of width at most $k - 1$ (as each component has at most k vertices).

Now assume that G has a component with at least k edges. By starting with a single vertex in a component with at least k edges, and adding adjacent vertices one at a time, construct a minimal set of vertices X such that $G[X]$ is connected and $|E(X)| \geq k$, where $E(X)$ is the set of edges with both end-vertices in X . Let v be

the last vertex added to X . Then $G[X \setminus \{v\}]$ is connected and $|E(X \setminus \{v\})| < k$. It follows that $|X \setminus \{v\}| \leq k$ and so $|X| \leq k + 1$.

Now if $|E(V \setminus X)| \geq k$, then we may obtain a solution for k -LCP by coloring all of X red and all of $V \setminus X$ blue. Otherwise, we may construct a tree decomposition of $G[V \setminus X]$ of width at most $k - 1$. Now add X to every bag in this tree decomposition. Observe that the result satisfies the conditions of a tree decomposition and has width at most $k - 1 + |X| \leq 2k$. \square

Theorem 3. *There is an algorithm of running time $O^*(4^k)$ to solve k -LCP.*

5 Open Problems

Our kernel and fixed-parameter algorithm seem to be close to optimal: we do not believe that k -LCP admits $o(k)$ -vertex kernel or $2^{o(k)}$ running time algorithm unless the Exponential Time Hypothesis fails. It would be interesting to prove or disprove it. It would also be interesting to obtain a smaller kernel or faster algorithm for k -LCP.

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