Information Processing Letters 114 (2014) 450–456

Contents lists available at ScienceDirect

Information Processing Letters

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Triangle strings: Structures for augmentation of vertex-disjoint triangle sets ☆

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ARTICLE INFO

Article history:
Received 25 September 2012
Received in revised form 10 March 2014
Accepted 19 March 2014
Available online 25 March 2014
Communicated by J. Torán

Keywords:
Combinatorial problem
Vertex-disjoint triangle set
Augmentation
Triangle factor
Triangle string

ABSTRACT

Vertex-disjoint triangle sets (triangle sets for short) have been studied extensively. Many theoretical and computational results have been obtained. While the maximum triangle set problem can be viewed as the generalization of the maximum matching problem, there seems to be no parallel result to Berge's augmenting path characterization on maximum matching (C. Berge, 1957 [1]). In this paper, we describe a class of structures called triangle string, which turns out to be equivalent to the class of union of two triangle sets in a graph. Based on the concept of triangle string, a sufficient and necessary condition that a triangle set can be augmented is given. Furthermore, we provide an algorithm to determine whether a graph G with maximum degree 4 is a triangle string, and if G is a triangle string, we compute a maximum triangle set of it. Finally, we give a sufficient and necessary condition for a triangle string to have a triangle factor.

1. Introduction, definitions and terminologies

We consider undirected, simple graphs in this paper. Let G be a graph. A set $\mathcal{T}$ of vertex disjoint triangles in G is called a *vertex-disjoint triangle set* of G. For short, we call $\mathcal{T}$ a *triangle set* of G in this paper. The number of triangles in $\mathcal{T}$, denoted by $|\mathcal{T}|$, is called the *size* of it. A triangle set of G with the maximum size is called a *maximum triangle set* of G. We say that a vertex u is *covered* by a triangle set $\mathcal{T}$, if u is a vertex of a triangle in $\mathcal{T}$. If $\mathcal{T}$ covers all vertices of G, we say that $\mathcal{T}$ is a *triangle factor* of G.

The study on triangle sets and triangle factors has a long history. Important results include sufficient conditions for the existence of triangle factors in graphs, and bounds on the size of the maximum triangle sets in graphs. For example, the following fundamental result is a special case of a theorem in [5].

Theorem 1.1. *(See Corrádi and Hajnal [5], 1963.) If G is a graph with $3k$ vertices and minimum degree of at least $2k$ then G contains a triangle factor.*

While in tripartite graphs the bound can be reduced.

Theorem 1.2. *(See Johansson [8], 2000.) Let G be a tripartite graph with $3k$ vertices, k in each class, such that each vertex is connected to at least $\frac{2}{3}k + \sqrt{k}$ of the vertices in each of the other two classes, then G has a triangle factor.*

Another example is a result of the size of triangle sets in claw-free graphs.

☆ Research supported by Natural Science Foundation of Guangdong Province (9451030007003340), partially supported by NSFC (10801077), the Natural Science Foundation of the Jiangsu Higher Education Institutions of China (13KJB110018), and the Science Foundation of Guangdong Industry Technical College (KJ201219).

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http://dx.doi.org/10.1016/j.ipl.2014.03.009
0020-0190/

Theorem 1.3. *(See Wang [10], 1998.) For any integer $k \geqslant 2$, if G is a claw-free graph of order at least $6(k-1)$ and with minimum degree at least 3, then G contains a triangle set of size k unless G is of order $6(k-1)$ and G belongs to a known class of graphs.*

The problem of computing the maximum triangle sets (called the vertex-disjoint triangles problem, VDT for short) in graphs catches much attention. The VDT problem has many variants such as computing the maximum triangle sets in edge-weighted graphs [7], in degree-bounded graphs ([2–4] and [9]), or in some special classes of graphs [6]. Particularly, in [2], Caprara and Rizzi prove that the VDT problem is APX-hard for graphs with maximum degree 4.

Triangle sets can be viewed as a generalization of matchings in graphs. For matching problems, Berge's famous characterization says that a matching M in a graph G is maximum if and only if G has no M-augmenting path [1]. However, for triangle sets in graphs, there seems no similar augmenting results. In this paper, we describe a class of structures called triangle string, which corresponds to the union of the graphs of two triangle sets. Based on the concept of triangle string, we give a sufficient and necessary condition under which a triangle set $\mathcal{T}$ of a graph G can be augmented. We describe an algorithm which determines whether a given graph G with degree bound 4 is a triangle string, and if G is a triangle string, find out a maximum triangle set of it. Finally we give a sufficient and necessary condition under which a triangle string has a triangle factor.

We use $\langle\mathcal{T}\rangle$ to denote the graph consisting of all the triangles in a triangle set $\mathcal{T}$, and say that it is the *graph* of $\mathcal{T}$. We often consider $\langle\mathcal{T}_1\rangle \cup \langle\mathcal{T}_2\rangle$, the union of the graphs of two triangle sets $\mathcal{T}_1$ and $\mathcal{T}_2$, which is called the *union graph* of $\mathcal{T}_1$ and $\mathcal{T}_2$ in this paper.

Let u be a vertex of degree d in a graph G. Then we say that u is a *d-vertex* in G. Let $T = uvwu$ be a triangle in G, where the degree of u, v and w are d_u, d_v and d_w in G. Then we say that T is a *(d_u, d_v, d_w)-triangle* in G. We also say that the *degree sequence* of T in G is (d_u, d_v, d_w).

2. Triangle strings

In this section we describe the triangle string structure. The following operations are to be used in the construction of triangle strings.

Let G be a graph. The operations (1), (2) and (3) below are called *triangle-additions*.

(1) Let w be a 2-vertex in G, add two new vertices u and v to G, and add edges to form a triangle on u, v and w.
(2) Let v and w be two nonadjacent 2-vertices at odd distance in G, add a new vertex u to G, and add edges to form a triangle on u, v and w.
(3) Let u, v and w be three pairwise nonadjacent 2-vertices at pairwise odd distance in G, add edges to form a triangle on u, v and w.

Denote by K_4^- the graph obtained from K_4 by removing an edge. An *hourglass* is a graph isomorphic to $K_5 - E(C_4)$,

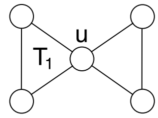

Fig. 1. Case (2): One vertex of T_1 covered by $\mathcal{T}_2$.

i.e. the graph obtained by removing the edges of a 4-cycle from a K_5. The following operations (4) and (5) are called *K_4^--insertions*.

(4) Let w be a 2-vertex in G and u, v the neighbors of w. Replace w with a K_4^- denoted by K, and let u, v be adjacent to one 2-vertex w_1 in K.
(5) Let w be the 4-vertex in an induced hourglass in G. Let the neighbors of w be u_1, u_2, v_1 and v_2, where $u_1v_1 \in E(G)$ and $u_2v_2 \in E(G)$. Replace w with a K_4^- denoted by K, and let u_1 and v_1 be adjacent to a 2-vertex w_1 in K, u_2 and v_2 be adjacent to another 2-vertex w_2 in K.

We call a K_4^- in a graph G connected to the other parts of G through the 2-vertices in it a *swing* K_4^-. Hence, operation (4) and (5) insert swing K_4^-'s into G.

A *vertex-jointed triangle string* S is either a triangle, or obtained from another vertex-jointed triangle string S' by performing one triangle addition.

A *triangle string* S is either a K_4^-, or a vertex-jointed triangle string, or obtained from another triangle string S' by performing a K_4^--insertion.

It is not hard to see the following properties of a triangle string S.

(a) Every vertex in S should be of degree 2, 3 or 4.
(b) S is vertex-jointed if and only if it contains no vertex of degree 3.
(c) Every 2-vertex in S is contained in one triangle.
(d) Every 3-vertex in S is a 3-vertex in a swing K_4^-, and hence it is contained in two triangles.
(e) Every 4-vertex in S is the 4-vertex in an induced hourglass, and hence it is contained in two triangles. Furthermore, operation (5) is valid for every 4-vertex in S.
(f) A triangle in S can only have one of the following six degree sequences in S: $(2,2,2)$, $(2,2,4)$, $(2,4,4)$, $(4,4,4)$, $(2,3,3)$ and $(3,3,4)$.

3. Union graph of two triangle sets and an augmenting theorem

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two triangle sets in a graph G, and $T_1 = uvwu$ be a triangle in $\mathcal{T}_1$. Considering how the vertices of T_1 be covered by the triangles in $\mathcal{T}_2$, we have the following cases on the structure near T_1 and the degree sequence of T_1 in $\langle\mathcal{T}_1\rangle \cup \langle\mathcal{T}_2\rangle$.

(1) All vertices of T_1 are not covered by $\mathcal{T}_2$. Then T_1 forms a component of $\langle\mathcal{T}_1\rangle \cup \langle\mathcal{T}_2\rangle$, and the degree sequence of T_1 in $\langle\mathcal{T}_1\rangle \cup \langle\mathcal{T}_2\rangle$ is $(2,2,2)$.
(2) There is one vertex u of T_1 covered by $\mathcal{T}_2$, as shown in Fig. 1. Then, u is a vertex of degree 4 in $\langle\mathcal{T}_1\rangle \cup$

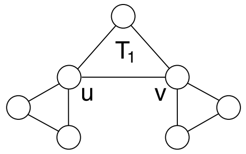

Fig. 2. Case (3.1): Two vertices of T_1 covered by different triangles in $\mathcal{T}_2$.

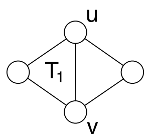

Fig. 3. Case (3.2): Two vertices of T_1 covered by the same triangle in $\mathcal{T}_2$.

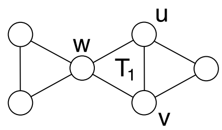

Fig. 4. Case (4.2): Two vertices of T_1 covered by one triangle, and the other one covered by another triangle in $\mathcal{T}_2$.

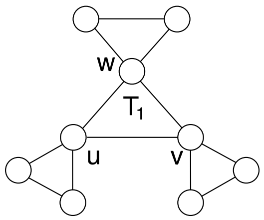

Fig. 5. Case (4.3): All three vertices of T_1 covered by different triangles in $\mathcal{T}_2$.

$\langle\mathcal{T}_2\rangle$, and the degree sequence of T_1 in $\langle\mathcal{T}_1\rangle \cup \langle\mathcal{T}_2\rangle$ is $(2,2,4)$.

(3) There are two vertices u and v of T_1 covered by $\mathcal{T}_2$. There are two possible situations.
  - (3.1) u and v are covered by different triangles in $\mathcal{T}_2$, as shown in Fig. 2. Then the degree sequence of T_1 in $\langle\mathcal{T}_1\rangle \cup \langle\mathcal{T}_2\rangle$ is $(2,4,4)$.
  - (3.2) u and v are covered by the same triangle in $\mathcal{T}_2$, as shown in Fig. 3. Then the degree sequence of T_1 in $\langle\mathcal{T}_1\rangle \cup \langle\mathcal{T}_2\rangle$ is $(2,3,3)$.

(4) All vertices u, v and w of T_1 are covered by $\mathcal{T}_2$. There are three possible situations.
  - (4.1) u, v and w are covered by the same triangle $T_1 \in \mathcal{T}_2$. Then the degree sequence of T_1 in $\langle\mathcal{T}_1\rangle \cup \langle\mathcal{T}_2\rangle$ is $(2,2,2)$, and T_1 forms a component of $\langle\mathcal{T}_1\rangle \cup \langle\mathcal{T}_2\rangle$.
  - (4.2) Two vertices of T_1, say u and v, are covered by one triangle in $\mathcal{T}_2$, while w is covered by another triangle in $\mathcal{T}_2$, as shown in Fig. 4. Then the degree sequence of T_1 in $\langle\mathcal{T}_1\rangle \cup \langle\mathcal{T}_2\rangle$ is $(4,3,3)$.
  - (4.3) u, v and w are covered by different triangles in $\mathcal{T}_2$, as shown in Fig. 5. Then the degree sequence of T_1 in $\langle\mathcal{T}_1\rangle \cup \langle\mathcal{T}_2\rangle$ is $(4,4,4)$.

By the above discussion, we can see that (a), (c), (d), (e) and (f) also hold for the union graph $\langle\mathcal{T}_1\rangle \cup \langle\mathcal{T}_2\rangle$.

We define two operations that will be used in our proofs.

Let T be a triangle of degree sequence $(2,2,4)$ or $(2,4,4)$ or $(4,4,4)$ in a graph G. We removed the edges of T and the 2-vertices in T. Such an operation is called a *triangle-removal* on T. We also say that T is removed from G.

Suppose the G is a graph not isomorphic to K_4^-. Let K be a swing K_4^- in G with 2-vertices u_1 and u_2, and 3-vertices v_1 and v_2. Remove K from G, add a new vertex k to G, and join the neighbors of u_1 and u_2 in G other than v_1 and v_2 to k. Such an operation is called a *K_4^--absorption* on K.

A K_4^--absorption is obviously the reverse operation of a K_4^--insertion. But a triangle-removal on T is not necessarily the reverse operation of a triangle-addition, because the 4-vertices in T, which become 2-vertices after removing T, may not be at pairwise odd distance in the graph obtained. However, we will prove that a triangle-removal in a union graph of two triangle sets is indeed the reverse operation of a triangle-addition.

The following lemma is an easy observation and we omit the proof.

Lemma 3.1. *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two triangle sets in a graph G, then any triangle T in $\langle\mathcal{T}_1\rangle \cup \langle\mathcal{T}_2\rangle$ must be an element of $\mathcal{T}_1$ or $\mathcal{T}_2$.*

Theorem 3.2. *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two triangle sets in a graph G. Let H be a component of $\langle\mathcal{T}_1\rangle \cup \langle\mathcal{T}_2\rangle$. Then, H is a triangle string.*

Proof. We apply mathematical induction on the number of edges in H. It is not hard to verify the conclusion for H having at most 6 edges. Assume that H has at least 7 edges, and the conclusion holds for any components of a union graph of two triangle sets with fewer edges than H.

If there is at least one 3-vertex in H, by (d), there is at least one swing K_4^-, denoted by K, in H. Let the 2-vertices in K be u_1 and u_2, and the 3-vertices in K be v_1 and v_2. Without loss of generality, assume that the triangles $u_1v_1v_2u_1 \in \mathcal{T}_1$ and $u_2v_1v_2u_2 \in \mathcal{T}_2$. We perform a K_4^--absorption by replacing K with a new vertex k and obtain the graph H'. Note that u_1 is either of degree 4 or 2 in H. If $d_H(u_1) = 2$, we delete the triangle $u_1v_1v_2u_1$ from $\mathcal{T}_1$. If u_1 is of degree 4 then it is contained in another triangle $u_1w_1w_2u_1 \in \mathcal{T}_2$, where $\{w_1, w_2\} \cap \{v_1, v_2\} = \emptyset$. We delete the triangle $u_1v_1v_2u_1$ from $\mathcal{T}_1$ and replace the triangle $u_1w_1w_2u_1$ with kw_1w_2k in $\mathcal{T}_2$. Similarly we handle the triangles containing u_2. Let $\mathcal{T}_1'$ and $\mathcal{T}_2'$ be the two triangle sets we obtain. Then H' is a component of $\langle\mathcal{T}_1'\rangle \cup \langle\mathcal{T}_2'\rangle$. By our induction hypothesis, H' is a triangle string. Then H is a triangle string, since it can be obtained from H' by one K_4^--insertion.

Therefore, we can assume that there is no vertex of degree 3 in H.

Let H' be a connected graph obtained from H by performing one triangle-removal which removes the triangle T. By Lemma 3.1, T is either in $\mathcal{T}_1$ or in $\mathcal{T}_2$. Without loss of generality assume that $T \in \mathcal{T}_1$. Then $H' =$

$\langle \mathcal{T}_1 \backslash T \rangle \cup \langle \mathcal{T}_2 \rangle$, and by our induction hypothesis, H' is a triangle string.

If H contains a triangle of degree sequence $(2, 2, 2)$, then H itself must be this triangle. But this is impossible since we have assumed that H contains at least 7 edges.

If H contains a triangle T of degree sequence $(2, 2, 4)$, then we can perform a triangle-removal on T. The graph H' obtained is connected and hence a triangle string. Therefore, H is a triangle string since it is obtained from H' by a triangle-addition adding T to it.

So, we can further assume that H contains $(2, 4, 4)$-triangles and $(4, 4, 4)$-triangles only.

Suppose H contains at least one $(2, 4, 4)$-triangle. We prove that there exists one $(2, 4, 4)$-triangle T, such that removing T from H results in a connected graph H'. Assume that such a triangle does not exist, then removing any $(2, 4, 4)$-triangle from H results in a graph with two components. We choose a $(2, 4, 4)$-triangle T such that removing T from H we obtain a component H_0 of the minimum order. We claim that H_0 has only one 2-vertex, which is a vertex of T with degree 4 in H. Otherwise, we have at least another $(2, 4, 4)$-triangle T_0 in H_0, which also has degree sequence $(2, 4, 4)$ in H. Removing T_0 from H we get a component of order smaller than H_0, contradicting the choice of H_0. Hence H_0 has one 2-vertex, and the other vertices are of degree 4. By Lemma 3.1 T is in $\mathcal{T}_1$ or $\mathcal{T}_2$, so H_0 can be represented as the union graph of two triangle sets, denoted by $\mathcal{T}_1'$ and $\mathcal{T}_2'$. Since there is no 3-vertex in H_0, no triangle shares a common edge. Let the number of triangles in $\mathcal{T}_1'$ and $\mathcal{T}_2'$ be t_1 and t_2, and assume without loss of generality that the unique 2-vertex is covered by $\mathcal{T}_1'$. Since every 4-vertex is covered by one triangle from $\mathcal{T}_1'$ and one from $\mathcal{T}_2'$, counting the edges of the triangles in $\mathcal{T}_1'$ and $\mathcal{T}_2'$, we have $3t_1 = \frac{1}{2}(2(|H_0| - 1) + 2) = |H_0|$ and $3t_2 = \frac{1}{2}(2(|H_0| - 1)) = |H_0| - 1$, which is impossible since t_1, t_2 and $|H_0|$ are integers. Therefore, there must exist a $(2, 4, 4)$-triangle T such that removing it from H results in a connected graph H'. Then H' is a triangle string.

Let the vertices of T with degree 4 in H be u_1 and u_2. Then u_1 and u_2 are 2-vertices in H'. We prove that u_1 and u_2 are at odd distance in H'. Let the shortest path between u_1 and u_2 in H' be $P = x_0x_1 \dots x_l$, where $x_0 = u_1$ and $x_l = u_2$. Since P is the shortest, there could not be any edge x_ix_j in H' for $0 \leqslant i, j \leqslant l$ and $j > i + 1$. If x_1 has degree 2 in H', then x_0 and x_2 must be connected in H', a contradiction. Therefore x_1, and similarly $x_2, \dots, x_{l-1}$, must have degree 4 in H'. Let y_0 be the vertex other than x_0 and x_1 in the triangle covering x_0. Then y_0 cannot be joined to x_i, $2 \leqslant i \leqslant l$, or P cannot be the shortest. Since x_1 is of degree 4, there must be another triangle $x_1x_2y_1$ covering it. Similarly, y_1 cannot be joined to any x_i, $0 \leqslant i \leqslant l$ and $i \neq 1, 2$. By analogous arguments we conclude that there are triangles $x_0x_1y_0x_0, x_1x_2y_1x_1, \dots, x_{l-1}x_ly_{l-1}x_{l-1}$ in H', where all y_i, $0 \leqslant i \leqslant l-1$, are different. All the triangles must also be in H, and by Lemma 3.1, be in $\mathcal{T}_1$ or $\mathcal{T}_2$. Any two triangles sharing a common vertex must be in different triangle sets, hence the triangles $x_{2m}x_{2m+1}y_{2m}x_{2m}$, $0 \leqslant m \leqslant \lfloor (l-1)/2 \rfloor$, must be in the same triangle set. Since $x_0x_1y_0x_0$ and $x_{l-1}x_ly_{l-1}x_{l-1}$ are both adjacent to T in H,

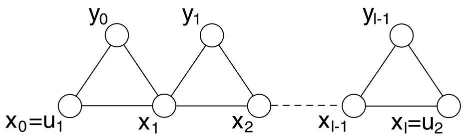

Fig. 6. The triangles on the edges of P.

they must be in the same triangle set. Therefore $l - 1$ must be even, and l, the length of P, is odd. See Fig. 6.

Because the distance between u_1 and u_2 in H' is odd, H is obtained from the triangle string H' by a triangle-addition adding T to it. By definition, H is a triangle string.

Finally, we assume that H contains only $(4, 4, 4)$-triangles. Removing a $(4, 4, 4)$-triangle T from H, the resulting graph H' must be connected, or we will get a component of H' with one vertex of degree 2, and all other vertices of degree 4, which is impossible by the above discussion. Then H' is a triangle string. We can use arguments similar to the above to prove that the distances between any two vertices of T in H' is odd. Hence H is obtained from H' by a triangle-addition, and it is a triangle string. $\square$

Theorem 3.3. *Every triangle string S can be represented as the union graph of two triangle sets. Furthermore, if S contains at least two triangles the representation are unique.*

Proof. It is easy to verify the conclusion for triangle strings with at most three triangles. Let S be a triangle string with at least four triangles, and the conclusion holds for all triangle strings with fewer edges than S. S must be obtained from another triangle string S' by a triangle-addition or a K_4^--insertion, and S' contains at least two triangles.

Suppose that S is a vertex-jointed triangle string. Then S is obtained from S' through any of the triangle-additions (1), (2) or (3), and S' is vertex-jointed. By the induction hypothesis, S' is uniquely represented as the union graph of two triangle sets $\mathcal{T}_1'$ and $\mathcal{T}_2'$. It suffices to prove that any two 2-vertices at odd distance in S' are covered by the same triangle set, say $\mathcal{T}_1'$. Then we can put the added triangle into $\mathcal{T}_2'$ but not into $\mathcal{T}_1'$ to form two triangle sets whose union graph is exactly S. And the uniqueness of these two triangle sets are guaranteed by the fact that we can only add the triangle to one of $\mathcal{T}_1'$ or $\mathcal{T}_2'$.

Let u_1 and u_2 be two 2-vertices at odd distance in S'. We prove that u_1 and u_2 are covered by the same triangle set. Let $P = x_0x_1 \dots x_l$ be a shortest path between u_1 and u_2, where $x_0 = u_1$, $x_l = u_2$ and l is odd. Since S' is vertex-jointed, it contains only 2-vertices and 4-vertices. By similar argument as those in the proof of Theorem 3.2, we get a serial of triangles $x_0x_1y_0x_0$, $x_1x_2y_1x_1, \dots, x_{l-1}x_ly_{l-1}x_{l-1}$, where all y_i, $0 \leqslant i \leqslant l-1$, are different. By Lemma 3.1, the triangles $x_0x_1y_0x_0, x_1x_2y_1x_1, \dots, x_{l-1}x_ly_{l-1}x_{l-1}$ must be in either $\mathcal{T}_1'$ or $\mathcal{T}_2'$, and since they are adjacent one by one they must be in $\mathcal{T}_1'$ and $\mathcal{T}_2'$ alternatively. Since l is odd, $x_0x_1y_1x_0$ and $x_{l-1}x_ly_{l-1}x_{l-1}$ must be in the same triangle set, which implies that $u_1 = x_0$ and $u_2 = x_l$ are covered by the same triangle set.

Now suppose S is not a vertex-jointed triangle string. Then S is obtained from S' by one K_4^--insertion, which replaces a vertex k in S' with a swing $K_4^- K$, with 2-vertices u_1 and u_2, and 3-vertices v_1 and v_2. By the induction hypothesis, S' can be uniquely represented as $S' = \langle \mathcal{T}_1' \rangle \cup \langle \mathcal{T}_2' \rangle$, where $\mathcal{T}_1'$ and $\mathcal{T}_2'$ are two triangle sets. If k is of degree 2, let the triangle containing k in S' be kw_1w_2k. Without loss of generality assume that $kw_1w_2k \in \mathcal{T}_1'$, and w_1 and w_2 are adjacent to u_1 in S. Let $\mathcal{T}_1 = (\mathcal{T}_1' \backslash kw_1w_2k) \cup \{u_1w_1w_2u_1, u_2v_1v_2u_2\}$ and $\mathcal{T}_2 = \mathcal{T}_2' \cup \{u_1v_1v_2u_1\}$, we have $S = \langle \mathcal{T}_1 \rangle \cup \langle \mathcal{T}_2 \rangle$, and the representation is unique since the way we put the triangles into the triangle sets is unique. If the degree of k is 4, then it must be a 4-vertex in an induced hourglass in S'. By arguments similar as in the case that k is of degree 2, we can construct $\mathcal{T}_1$ and $\mathcal{T}_2$, such that $S = \langle \mathcal{T}_1 \rangle \cup \langle \mathcal{T}_2 \rangle$ and the representation is unique. Thus the theorem is proved. □

Let $\mathcal{T}$ be a triangle set of a graph G. We call a triangle string S in G a *$\mathcal{T}$-alternating triangle string* if the following conditions hold.

(1) One of every two adjacent triangles in S belongs to $\mathcal{T}$, and
(2) if a 2-vertex in S is covered by a triangle $T \in \mathcal{T}$ then T is a subgraph of S.

Let S be a $\mathcal{T}$-alternating triangle string, and the set of the triangles in S that belong to $\mathcal{T}$ be $\mathcal{T}_1$, the set of the other triangles in S be $\mathcal{T}_2$. Because one of every two adjacent triangles belongs to $\mathcal{T}$, the triangles in $\mathcal{T}_2$ are independent. So $\mathcal{T}_2$ is a triangle set. Since every edge in S is contained in a triangle, $S = \langle \mathcal{T}_1 \rangle \cup \langle \mathcal{T}_2 \rangle$.

Now we can state our augmentation theorem for triangle sets. Let $V(\mathcal{T})$ denote the set of the vertices of the triangles in $\mathcal{T}$. Let $V_2(S)$ denote the set of the 2-vertices of S.

Theorem 3.4. *Let $\mathcal{T}$ be a triangle set of a graph G. There exists a triangle set $\mathcal{T}'$ of G with $|\mathcal{T}'| > |\mathcal{T}|$, if and only if there exists a $\mathcal{T}$-alternating triangle string S, such that $|V(\mathcal{T}) \cap V_2(S)| < |V_2(S)|/2$.*

Proof. Suppose a triangle string S as described exists. If S is a triangle then it is easily seen that the conclusion holds. Assume that S contains more than two triangles. By Theorem 3.3, S can be uniquely represented as the union graph of two triangle sets, say $S = \langle \mathcal{T}_1 \rangle \cup \langle \mathcal{T}_2 \rangle$, where $\mathcal{T}_1 \subseteq \mathcal{T}$. The 3-vertices and 4-vertices of S are covered by both $\mathcal{T}_1$ and $\mathcal{T}_2$, and every 2-vertex in S is covered by exactly one of $\mathcal{T}_1$ and $\mathcal{T}_2$. Note that by condition (2) in the definition of $\mathcal{T}$-alternating triangle string, $V(\mathcal{T}_1) \cap V_2(S) = V(\mathcal{T}) \cap V_2(S)$. Therefore, $|V(\mathcal{T}_2) \cap V_2(S)| = |V_2(S)| - |V(\mathcal{T}_1) \cap V_2(S)| > |V(\mathcal{T}_1) \cap V_2(S)|$. Let $\mathcal{T}' = (\mathcal{T} \backslash \mathcal{T}_1) \cup \mathcal{T}_2$. $\mathcal{T}'$ is a triangle set of G which covers more vertices than $\mathcal{T}$, hence $|\mathcal{T}'| > |\mathcal{T}|$.

Now suppose that we find a triangle set $\mathcal{T}'$ such that $|\mathcal{T}'| > |\mathcal{T}|$. By Theorem 3.2, every component of $\langle \mathcal{T} \rangle \cup \langle \mathcal{T}' \rangle$ is a triangle string. Since $|\mathcal{T}'| > |\mathcal{T}|$, there must be a component S with more triangles from $\mathcal{T}'$ than from $\mathcal{T}$, which is exactly the triangle string we are looking for. □

4. Triangle sets in triangle strings: an algorithm and a condition for triangle factors

When trying to compute a triangle set in a graph G, we can often ignore the edges that are not contained in any triangle, without affecting the results. Therefore, we assume that all edges of the graph G we consider are contained in some triangles henceforth.

A triangle string has maximum degree no more than 4. As we have stated in Section 1, VDT problem is APX-hard even in graphs with maximum degree 4. We prove in this section that the VDT problem is linear time solvable in triangle strings. Precisely saying, we provide an algorithm to determine whether a given graph G with maximum degree 4 is a triangle string, and if G is a triangle string the algorithm computes its maximum triangle set. A sufficient and necessary condition for a triangle string to have a triangle factor is also described.

We define the *triangle graph* $T(G)$ of G, whose vertex set consists of all triangles in G, and two vertices in $T(G)$ are adjacent if and only if the triangles they represented are adjacent in G.

Theorem 4.1. *G is a triangle string if and only if $T(G)$ is a bipartite graph.*

Proof. Suppose G is a triangle string, by Theorem 3.3, G can be represented as $\langle \mathcal{T}_1 \rangle \cup \langle \mathcal{T}_2 \rangle$, where $\mathcal{T}_1$ and $\mathcal{T}_2$ are triangle sets. By Lemma 3.1, every triangle in G is an element of either $\mathcal{T}_1$ or $\mathcal{T}_2$. Therefore, $T(G)$ is a bipartite graph whose two parts consist of vertices representing the triangles in $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively.

For the reverse, if $T(G)$ is bipartite, let $\mathcal{T}_1$ be the triangle set consisting of all triangles represented by one part of $T(G)$, and $\mathcal{T}_2$ be the triangle set consisting of all triangles represented by the other part of $T(G)$. By the assumption that every edge of G is contained in a triangle, we have that $G = \langle \mathcal{T}_1 \rangle \cup \langle \mathcal{T}_2 \rangle$, and by Theorem 3.2, G is a triangle string. □

By Theorem 4.1, we can construct the triangle graphs $T(G)$ of a graph G and test whether it is bipartite to determine whether G is a triangle string. Furthermore, the part of $T(G)$ with more vertices represents the maximum triangle set of G. To construct $T(G)$, we run a breadth first search on G, and when we first visit a vertex of a triangle in G, we add one vertex in $T(G)$ representing the triangle. When we visit a vertex v, we add edges between every two triangles containing v. The ideas are implemented in Algorithm 1.

Let n be the number of vertices in G. Step 1 takes $\Theta(n)$ time. Since a vertex in G has degree no more than 4, we need to check the existence of at most 6 edges in Step 5. A vertex in a triangle string is contained in at most two triangles. Therefore, if we find a vertex contained in more than two triangles, we exit from the algorithm with a negative answer in Step 6. In the loop from Step 8 to Step 12, we need to check at most two triangles. In Step 10, we need to find a vertex in $T(G)$ representing a triangle T in G. This could be accelerated by maintaining some pointers for every vertex v in G, pointing to the vertices in

Algorithm 1 Determining a triangle string and looking for its maximum triangle set.

Input: A graph G with maximum degree 4;
Output: The judgement that whether G is a triangle string (YES or NO), and the maximum triangle set of G if it is a triangle string;
1: Mark all vertices of G as unvisited; Initial $T(G)$ as an empty graph; and Q as an empty queue;
2: Take a vertex of G and push it into the rear of Q;
3: **while** Q is not empty **do**
4: Pop an element v from the front of Q;
5: Find in G all triangles containing v;
6: **if** v is contained in more than two triangles in G **then return** NO;
7: **end if**
8: **for all** triangle T containing v, **do**
9: **if** all two vertices in T other than v are not visited **then** add a new vertex representing T in $T(G)$;
10: **else** find in $T(G)$ the vertex representing T;
11: **end if**
12: **end for**
13: Add an edge in $T(G)$ between the vertices representing the triangles containing v;
14: Push the unvisited neighbors of v into the rear of Q;
15: Mark v as visited;
16: **end while**
17: Test whether $T(G)$ is bipartite;
18: **if** $T(G)$ is not bipartite **then return** NO;
19: **else return** YES, and the triangle set represented by the part of $T(G)$ with more vertices;
20: **end if**

$T(G)$ representing the (at most two) triangles in G containing v. Then it takes constant time to find the vertex in $T(G)$. Hence, every operation from Step 4 to Step 15 takes $O(1)$ time. Further, the loop from Step 3 to Step 16 is repeated $O(n)$ times. Therefore, the total running time of the loop from Step 3 to Step 16 is $O(n)$.

In Step 17, we need to test the bipartiteness of $T(G)$. One classical method on this problem running in $O(m)$ time, where m is the number of edges in $T(G)$, is as follows.

Run a breadth first search on $T(G)$, and use two colors red and blue to color all vertices of $T(G)$. The first visited vertex is colored red, and when visiting a vertex v, color all its uncolored neighbors different with v. After the breadth first search, all vertices of $T(G)$ are colored. Check all edges of $T(G)$. If there exists at least one edge whose two endvertices colored the same, then $T(G)$ is not bipartite. If the endvertices of every edge receive different colors, $T(G)$ is bipartite.

Since a vertex in G is contained in at most two triangles, the number of triangles in G is less than n. Furthermore, a triangle in G is adjacent to at most three triangles. Therefore, m is linear to n. Hence, Step 17 takes $O(n)$ time and Algorithm 1 is a linear time algorithm.

Finally we give a sufficient and necessary condition that a triangle string has a triangle factor.

Theorem 4.2. *A triangle string S has a triangle factor if and only if it satisfies the following.*

(1) The number of vertices of S is divided by 3.
(2) The distance between every two 2*-vertices is odd.*

Proof. Firstly we prove the theorem for vertex-jointed triangle strings.

Suppose that S is a vertex-jointed triangle string satisfying the given conditions. By Theorem 3.3, $S = \langle \mathcal{T}_1 \rangle \cup \langle \mathcal{T}_2 \rangle$ for two triangle sets $\mathcal{T}_1$ and $\mathcal{T}_2$. All 4-vertices in S is covered by both $\mathcal{T}_1$ and $\mathcal{T}_2$. By the proof of Theorem 3.3, two 2-vertices in a vertex-jointed triangle string at odd distance are covered by the same triangle set. Hence by (2), one of $\mathcal{T}_1$ and $\mathcal{T}_2$ covered all vertices of S, and it is a triangle factor of S.

Now let S be a vertex-jointed triangle string who has a triangle factor. Obviously (1) holds. By Theorem 3.3, $S = \langle \mathcal{T}_1 \rangle \cup \langle \mathcal{T}_2 \rangle$ for two triangle sets $\mathcal{T}_1$ and $\mathcal{T}_2$. Without loss of generality we may assume that $\mathcal{T}_1$ is the triangle factor of S. Using ideas similar to those in the proof of Theorem 3.2, which consider a shortest path between two 2-vertices covered by $\mathcal{T}_1$, we can prove that they are at odd distance.

Now suppose that S is not a vertex-jointed triangle string. Then S contained at least one swing K_4^-. We use mathematical induction on the number of edges in S to prove the theorem. Performing a K_4^--absorption on S, replacing a K_4^- denoted by K with the vertex k, we obtain another triangle string S'.

Suppose that S satisfies the conditions. Since $|S'| = |S| - 3$, and a K_4^--absorption in a triangle string does not affect the parity of the distance between any two 2-vertices, S' satisfies the conditions as well. Therefore, by the induction hypothesis, S' have a triangle factor $\mathcal{T}'$. It is easy to see that a triangle factor $\mathcal{T}$ of S can then be obtained by adding one triangle of K to $\mathcal{T}'$, and replacing the vertex k in a triangle in $\mathcal{T}'$ with one vertex in K.

Suppose that S has a triangle factor $\mathcal{T}$. Then a triangle factor $\mathcal{T}'$ of S' can be obtained by deleting one triangle in K from $\mathcal{T}$, and replacing one vertex, both in K and in a triangle in $\mathcal{T}$, with k. By the induction hypothesis, S' satisfies the conditions. Since $|S'| = |S| - 3$, and a K_4^--absorption in a triangle string does not affect the parity of the distance between any two 2-vertices, S satisfies the conditions. □

Acknowledgements

The authors are indebted to the helpful comments and suggestions of the referees and Professor Hajo Broersma that have helped a lot to improve the quality of the paper.

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