

On some Graphs with a Unique Perfect Matching

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Abstract

We show that deciding whether a given graph G of size m has a unique perfect matching as well as finding that matching, if it exists, can be done in time $O(m)$ if G is either a cograph, or a split graph, or an interval graph, or claw-free. Furthermore, we provide a constructive characterization of the claw-free graphs with a unique perfect matching.

Keywords: unique perfect matching; claw-free

1 Introduction

Bartha [1] conjectured that a unique perfect matching of a given graph G of size m , if it exists, can always be found in $O(m)$ time. Gabow, Kaplan, and Tarjan [6] describe a $O(m \log^4 m)$ algorithm for this problem. Furthermore, they show that, if apart from G , some perfect matching M is also part of the input, then one can decide the uniqueness of M in $O(m)$ time. Since maximum matchings can be found in linear time for chordal bipartite graphs [2], cocomparability graphs [11], convex bipartite [13], and cographs [3, 16], also deciding whether these graphs have a unique perfect matching, as well as finding the unique perfect matching, if it exists, is possible in linear time. Also for strongly chordal graphs given a strong elimination order [4], a maximum matching can be found in linear time, and the same conclusion applies. Levit and Mandrescu [9] showed

that unique perfect matchings can be found in linear time for König-Egerváry graphs and unicyclic graphs.

We contribute some structural and algorithmic results concerning graphs with a unique perfect matching. First, we extend a result from [5] to cographs and split graphs, which leads to a very simple linear time algorithm deciding the existence of a unique perfect matching, and finding one, if it exists. For interval graphs, we describe a linear time algorithm that determines a perfect matching, if the input graph has a unique perfect matching. Similarly, for connected claw-free graphs of even order, we describe a linear time algorithm that determines a perfect matching. Together with the result from [6] this implies that for such graphs the existence of a unique perfect matching can be decided in linear time. Finally, we give a constructive characterization of claw-free graphs with a unique perfect matching.

2 Results

For a graph G , we say that a set $U = \{u_1, \dots, u_k\}$ forces a unique perfect matching in G if $n(G) = 2k$, and $d_{G_i}(u_i) = 1$ for every $i \in [k]$, where $G_i = G - \bigcup_{j=1}^{i-1} N_G[u_j]$, and $N_G[u]$ denotes the closed neighborhood of u in G . Clearly, if U forces a unique perfect matching in G , then G has a unique perfect matching u_1v_1, \dots, u_kv_k , where v_i is the only neighbor of u_i in G_i for $i \in [k]$. As shown by Golumbic, Hirst, and Lewenstein (Theorem 3.1 in [5]), a bipartite graph G has a unique perfect matching if and only if some set forces a unique perfect matching in G ; their result actually implies that both partite sets of G force a unique perfect matching. This equivalence easily extends to cographs and split graphs.

Theorem 2.1. *If G is a cograph or a split graph, then G has a unique perfect matching if and only if some set forces a unique perfect matching in G .*

Proof. Since the sufficiency is obvious, we proceed to the proof of the necessity. Therefore, let G be a cograph or a split graph with a unique perfect matching M . In view of an inductive argument, and since the classes of cographs and of split graphs are both hereditary, it suffices to consider the case that G is a connected graph of order at least 4, and to show that G has a vertex of degree 1.

First, suppose that G is a cograph. Since G is connected, it is the join

of two graphs G_1 and G_2 . If G_1 and G_2 both have order at least 2, then M contains either two edges between $V(G_1)$ and $V(G_2)$, or one edge of G_1 as well as one edge of G_2 . In both cases, these two edges are part of an M -alternating cycle of length 4, which is a contradiction. Hence, we may assume that $V(G_2)$ contains exactly one vertex v , which is a universal vertex in G . Let u be such that $uv \in M$. If u' is a neighbor of u in G_1 , and $u'v' \in M$, then $uu'v'vu$ is an M -alternating cycle of length 4, which is a contradiction. Hence, the vertex u has degree 1 in G .

Next, suppose that G is a split graph. Let $V(G) = S \cup C$, where S is an independent set, and C is a clique that is disjoint from S . Since G has a unique perfect matching, it follows easily that $|C| - |S|$ is either 0 or 2. Since G has order at least 4, the set S is not empty. If no vertex in S has degree 1, then it follows, similarly as for bipartite graphs, that G contains an M -alternating cycle, which completes the proof. \square

If G is given by neighborhood lists, then it is straightforward to decide the existence of a set that forces a unique perfect matching in G in linear time, by iteratively identifying a vertex of degree 1, and removing this vertex together with its neighbor from G . Altogether, for a given cograph or split graph, one can decide in linear time whether it has a unique perfect matching, and also find that matching, if it exists.

Our next results concern interval graphs.

Lemma 2.2. *Let G be an interval graph with a unique perfect matching M , and let $\left([\ell_u, r_u]\right)_{u \in V(G)}$ be an interval representation of G such that all $2n(G)$ endpoints of the intervals $[\ell_u, r_u]$ for $u \in V(G)$ are distinct.*

If $u^ \in V(G)$ is such that $r_{u^*} = \min\{r_u : u \in V(G)\}$, and $v^* \in N_G(u^*)$ is such that $r_{v^*} = \min\{r_v : v \in N_G(u^*)\}$, then $u^*v^* \in M$.*

Proof. Suppose, for a contradiction, that $u^*v \in M$ for some neighbor v of u^* that is distinct from v^* . Let u be such that $uv^* \in M$. By the choice of u^* and v^* , and since the intervals $[\ell_u, r_u]$ and $[\ell_{v^*}, r_{v^*}]$ intersect, also the intervals $[\ell_u, r_u]$ and $[\ell_v, r_v]$ intersect, that is, $uv \in E(G)$. Now, $u^*vuv^*u^*$ is an M -alternating cycle of length 4, which is a contradiction. \square

Since, for a given interval graph, an interval representation as in Lemma 2.2 can be found in linear time [7], Lemma 2.2 yields a simple linear time algorithm to determine a perfect matching in a given interval graph G , provided that G has a unique perfect matching.

We proceed to claw-free graphs.

Let G be a graph. Let $P : u_1 \dots u_k$ be a path in G , where we consider u_k to be the *last* vertex of P . We consider two operations replacing P with a longer path P' in G .

- P' arises by applying an *end-extension* to P , if P' is the path $u_1 \dots u_k v$, where v is some neighbor of u_k that does not lie on P .
- P' arises by a *swap-extension* to P , if $k \geq 3$, and P' is the path

$$u_1 \dots u_{k-2} u_k u_{k-1} v,$$

where v is some neighbor of u_{k-1} that does not lie on P . Note that u_{k-2} and u_k need to be adjacent for this operation.

The following lemma is a simple variation of a folklore proof of Sumner's result [14] that connected claw-free graphs of even order have a perfect matching.

Lemma 2.3. *If G is a connected claw-free graph of even order, and $P : u_1 \dots u_k$ is a path in G that does not allow an end-extension or a swap-extension, then the edge $u_{k-1}u_k$ belongs to some perfect matching of G .*

Proof. In view of an inductive argument, it suffices to show that $G' = G - \{u_{k-1}, u_k\}$ is connected. Suppose, for a contradiction, that G' is not connected. Clearly, $k \geq 2$. If $k = 2$, then u_1 has neighbors in two components of G' while u_2 is only adjacent to u_1 , which yields a claw centered at u_1 . Now, let $k \geq 3$. The path $u_1 \dots u_{k-2}$ lies in one component K of G' . Let K' be a component of G' that is distinct from K . Since P allows no end-extension, u_k has no neighbor in K' . Hence, u_{k-1} has a neighbor v in K' . Since u_{k-2} and v are not adjacent, and G is claw-free, u_k is adjacent to u_{k-2} , and P allows a swap-extension, which is a contradiction. \square

Lemma 2.3 is the basis for the simple greedy algorithm PMinCF (cf. Algorithm 1) that determines a perfect matching in connected claw-free graphs of

even order.

Input: A connected claw-free graph G of even order.

Output: A perfect matching M of G .

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1 begin
2    $M \leftarrow \emptyset$ ;  $k \leftarrow 0$ ;
3    $\text{lm\_nb}(u) \leftarrow -1$  for all vertices  $u$  of  $G$ ;
4   while  $n(G) \geq 2$  do
5     if  $k = 0$  then  $k \leftarrow 2$ ;  $u_1 \leftarrow u$ ;  $u_2 \leftarrow v$ , where  $uv$  is some edge of  $G$ ;
6     repeat
7       extend  $\leftarrow 0$ ;
8       if  $\exists v \in N_G(u_k) \setminus \{u_1, \dots, u_k\}$  then
9          $u_{k+1} \leftarrow v$ ;  $k \leftarrow k + 1$ ; extend  $\leftarrow 1$ ;
10      else
11        if  $\text{lm\_nb}(u_k) = -1$  then
12           $\text{lm\_nb}(u_k) \leftarrow \max\{\ell \in [k-1] : u_{k-i} \in N_G(u_k) \text{ for } i \in$ 
13             $[\ell]\}$ ;
14          end
15          if  $\text{lm\_nb}(u_k) \geq 2$  and  $\exists v \in N_G(u_{k-1}) \setminus \{u_1, \dots, u_k\}$  then
16             $\text{lm\_nb}(u_k) \leftarrow \text{lm\_nb}(u_k) - 1$ ;
17            if  $\text{lm\_nb}(u_{k-1}) \neq -1$  then
18               $\text{lm\_nb}(u_{k-1}) \leftarrow \text{lm\_nb}(u_{k-1}) + 1$ 
19            end
20             $x \leftarrow u_k$ ;  $y \leftarrow u_{k-1}$ ;
21             $u_{k-1} \leftarrow x$ ;  $u_k \leftarrow y$ ;  $u_{k+1} \leftarrow v$ ;
22             $k \leftarrow k + 1$ ; extend  $\leftarrow 1$ ;
23          end
24        until extend = 0;
25         $M \leftarrow M \cup \{u_{k-1}u_k\}$ ;  $G \leftarrow G - \{u_{k-1}, u_k\}$ ;  $k \leftarrow k - 2$ ;
26      end
27    return  $M$ ;
28 end

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Algorithm 1: PMinCF

Theorem 2.4. *The algorithm PMinCF works correctly and can be implemented to run in $O(m(G))$ time for a given connected claw-free graph G of even order.*

Proof. Line 2 initializes the matching M as empty and the order k of the path P :

$u_1 \dots u_k$ as 0. The **while**-loop in lines 4 to 26 extends the matching iteratively as long as possible using the last edge $u_{k-1}u_k$ of the path $P : u_1 \dots u_k$. If $k = 0$, which happens in the first execution of the **while**-loop, and possibly also in later executions, then, in line 5, the path P is reinitialized with $k = 2$ using any edge uv of G . The **repeat**-loop in lines 6 to 24 ensures that P allows no end-extension and no swap-extension, which, by Lemma 2.3, implies the correctness of **PMinCF**. The proof of Lemma 2.3 actually implies that G stays connected throughout the execution of **PMinCF**. In line 8 we check for the possibility of an end-extension, which, if possible, is performed in line 9. If no end-extension is possible, we check for the possibility of a swap-extension. The first time that some specific vertex u_k is the last vertex of P , and we check for the possibility of a swap-extension, we set $\mathbf{lm_nb}(u_k)$ to the largest integer ℓ such that u_k is adjacent to $u_{k-1}, \dots, u_{k-\ell}$. Initializing $\mathbf{lm_nb}(u)$ as -1 for every vertex u of G in line 3 indicates that its correct value has not yet been determined. This happens for the first time in lines 11 and 12. Once $\mathbf{lm_nb}(u_k)$ has been determined, it is only updated in line 15 for u_k , and, if necessary, in line 16 for u_{k-1} . Clearly, u_k is adjacent to u_{k-2} if and only if $\mathbf{lm_nb}(u_k) \geq 2$. Therefore, line 14 correctly checks for the possibility of a swap-extension, which, if possible, is performed in lines 19, 20, and 21. Altogether, the correctness follows, and it remains to consider the running time.

We assume that G is given by neighborhood lists, that is, for every vertex u of G , the elements of the neighborhood $N_G(u)$ of u in G are given as an (arbitrarily) ordered list. Checking for the existence of a suitable vertex v within the **if**-statements in lines 8 and 14 can be implemented in such a way that we traverse the neighborhood list of every vertex at most once throughout the entire execution of **PMinCF**. Every time we check for the existence of such a neighbor v of u_k , we only need to consider the neighbors of u_k that have not been considered before, that is, we start with the first not yet considered neighbor of u_k within its neighborhood list, and continue until we either find a suitable neighbor v or reach the end of the list. Since vertices that leave P are also removed from G in line 25, this is correct, and the overall effort spent on checking for such neighbors is proportional to the sum of all vertex degrees, that is, $O(m(G))$. The first computation of $\mathbf{lm_nb}(u_k)$ in line 12 can easily be done in $O(d_G(u_k))$ time. After that, every update of $\mathbf{lm_nb}(u_k)$ only requires constant effort. Since P is extended exactly $n(G) - 2$ times, the overall effort spent on maintaining $\mathbf{lm_nb}(u_k)$ is again proportional to the sum of all vertex degrees. Altogether, it follows that the running time is $O(m(G))$, which completes the

proof. □

Again, it follows using [6] that one can decide in linear time whether a given claw-free graph has a unique perfect matching.

Our final goal is a constructive characterization of the claw-free graphs that have a unique perfect matching. Let \mathcal{G} be the class of graphs G obtained by starting with G equal to K_2 , and iteratively applying the following two operations:

- **Operation 1**

Add to G two new vertices x and y , and the three new edges xy , xu , and yu , where u is a simplicial vertex of G .

- **Operation 2**

Add to G two new vertices x and y , the new edge xy , and new edges between x and all vertices in a set C , where C is a non-empty clique in G such that $N_G(u) \setminus C$ is a clique for every vertex u in C .

Theorem 2.5. *A connected claw-free graph G has a unique perfect matching if and only if $G \in \mathcal{G}$.*

Proof. It is easy to prove inductively that all graphs in \mathcal{G} are connected, claw-free, and have a unique perfect matching. Note that requiring u to be simplicial in Operation 1 ensures that no induced claw is created by this operation. Similarly, the conditions imposed on C in Operation 2 ensure that no induced claw is created.

Now, let G be a connected claw-free graph with a unique perfect matching M . If G has order 2, then, trivially, G is K_2 , which lies in \mathcal{G} . Now, let G have order at least 4. By Kotzig's theorem [8, 12], G has a bridge that belongs to M . In particular, G is not 2-connected. Let B be an endblock of G . If $n(B) \leq 3$, then B is K_2 or K_3 , and the claw-freeness of G easily implies that G arises from a proper induced subgraph of G by applying Operation 1 or 2. Hence, we may assume that $n(B) \geq 4$. If $n(B)$ is even, then, by Kotzig's theorem, B , and hence also G , has two distinct perfect matchings, which is a contradiction. Hence, $n(B)$ is odd, that is $n(B) \geq 5$. If u is the cutvertex of G in B , then, since G is claw-free, $N_B(u)$ is a clique of order at least 2. This implies that $B - u$ is 2-connected. Again, by Kotzig's theorem, $B - u$, and hence also G , has two distinct perfect matchings, which is a contradiction, and completes the proof. □

Note that Theorem 3.4 in [15], and also Theorem 3.2 in [15] restricted to claw-free graphs, follow very easily from Theorem 2.5 by an inductive argument.

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