The existence of perfect codes in a family of generalized Fibonacci cubes

Michel Mollard

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Abstract

The *Fibonacci cube* of dimension n, denoted as Γ_n, is the subgraph of the n-cube Q_n induced by vertices with no consecutive 1's. In an article of 2016 Ashrafi and his co-authors proved the non-existence of perfect codes in Γ_n for $n \geq 4$. As an open problem the authors suggest to consider the existence of perfect codes in generalization of Fibonacci cubes. The most direct generalization is the family $\Gamma_n(1^s)$ of subgraphs induced by strings without 1^s as a substring where $s \geq 2$ is a given integer. We prove the existence of a perfect code in $\Gamma_n(1^s)$ for $n = 2^p - 1$ and $s \geq 3.2^{p-2}$ for any integer $p \geq 2$.

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1 Introduction and notations

Let G be a connected graph. The *open neighbourhood* of a vertex u is $N(u)$ the set of vertices adjacent to u. The *closed neighbourhood* of u is $N[u] = N(u) \cup \{u\}$. The *distance* between two vertices noted $d_G(x, y)$, or $d(x, y)$ when the graph is unambiguous, is the length of the shortest path between x and y. We have thus $N[u] = \{v \in V(G); d(u, v) \leq 1\}$.

A *dominating set* D of G is a set of vertices such that every vertex of G belongs to the closed neighbourhood of at least one vertex of D. In [2], Biggs initiated the study of perfect codes in graphs a generalization of classical 1-error perfect correcting codes. A *code* C in G is a set of vertices C such that for all pair of distinct vertices c, c' of C we have $N[c] \cap N[c'] = \emptyset$ or equivalently such that $d_G(c, c') \geq 3$.

A *perfect code* of a graph G is both a dominating set and a code. It is thus a set of vertices C such that every vertex of G belongs to the closed neighbourhood of exactly one vertex of C. A perfect code is some time called an efficient dominating set. The existence or non-existence of perfect codes have been considered for many graphs. See the introduction of [1] for some references.

*Institut Fourier, CNRS, Université Grenoble Alpes, France email: michel.mollard@univ-grenoble-alpes.fr

The vertex set of the *n-cube* Q_n is the set $\mathbb{B}_n$ of binary strings of length n, two vertices being adjacent if they differ in precisely one position. Classical 1-error correcting codes and perfect codes are codes and perfect codes in the graph Q_n. The concatenation of strings $\boldsymbol{x}$ and $\boldsymbol{y}$ is noted $\boldsymbol{x}||\boldsymbol{y}$ or just $\boldsymbol{xy}$ when there is no ambiguity. A string $\boldsymbol{f}$ is a *substring* of a string $\boldsymbol{s}$ if there exist strings $\boldsymbol{x}$ and $\boldsymbol{y}$, may be empty, such that $\boldsymbol{s} = \boldsymbol{xfy}$.

A *Fibonacci string* of length n is a binary string $\boldsymbol{b} = b_1 \dots b_n$ with $b_i \cdot b_{i+1} = 0$ for $1 \leq i < n$. In other words a Fibonacci string is a binary string without 11 as substring. The *Fibonacci cube* Γ_n ($n \geq 1$) is the subgraph of Q_n induced by the Fibonacci strings of length n. Fibonacci cubes were introduced as a model for interconnection networks [3] and received a lot of attention afterwards. These graphs also found an application in theoretical chemistry. See the survey [4] for more results and applications about Fibonacci cubes.

The sets $\{00\}$ and $\{010, 101\}$ are perfect codes in respectively Γ_2 and Γ_3. In a recent paper [1] Ashrafi and his co-authors proved the non-existence of perfect codes in Γ_n for $n \geq 4$. As an open problem the authors suggest to consider the existence of perfect codes in generalization of Fibonacci cubes. The most complete generalization proposed in [5] is, for a given string $\boldsymbol{f}$, to consider $\Gamma_n(\boldsymbol{f})$ the subgraph of Q_n induced by strings that do not contain $\boldsymbol{f}$ as substring. Since Fibonacci cubes are $\Gamma_n(11)$ the most immediate generalization [6, 7] is to consider $\Gamma_n(1^s)$ for a given integer s. We will prove the existence of perfect codes in $\Gamma_n(1^s)$ for an infinite family of parameters (n, s).

It will be convenient to consider the binary strings of length n as vectors of $\mathbb{F}^n$ the vector space of dimension n over the field $F = \mathbb{Z}_2$ thus to associate to a string $x_1x_2 \dots x_n$ the vector $\theta(x_1x_2 \dots x_n) = (x_1, x_2, \dots, x_n)$. The *Hamming distance* between two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{F}^n$, $d(\boldsymbol{x}, \boldsymbol{y})$ is the number of coordinates in which they differ. The *parity function* is the function from $\mathbb{F}^n$ to $\mathbb{Z}_2$ defined by $\pi(\boldsymbol{x}) = \pi(x_1, x_2, \dots, x_n) = x_1 + x_2 + \dots + x_n$. By the correspondence θ we can define the sum $\boldsymbol{x} + \boldsymbol{y}$, the Hamming distance $d(\boldsymbol{x}, \boldsymbol{y})$ and the parity $\pi(\boldsymbol{x})$ of strings in $\mathbb{B}_n$. Note that Hamming distance is the usual graph distance in Q_n. The complement of a string $\boldsymbol{x} \in \mathbb{B}_n$ is the string $\overline{\boldsymbol{x}} = \boldsymbol{x} + 1^n$.

We will first recall some basic results about perfect codes in Q_n. Since Q_n is a regular graph of degree n the existence of a perfect code of cardinality $|C|$ implies $|C|(n + 1) = 2^n$ thus a necessary condition of existence is that $n + 1$ is a power of 2 thus that $n = 2^p - 1$ for some integer p.

For any integer p Hamming [8] constructed, a linear subspace of $\mathbb{F}^{2^p-1}$ which is a perfect code. It is easy to prove that all linear perfect codes are Hamming codes.
In 1961 Vasilev [9], and later many authors, see [10, 11] for a survey, constructed perfect codes which are not linear codes. Let us recall Vasilev's construction of perfect codes.

Theorem 1.1 *[9] Let C_r be a perfect code of Q_r. Let f be a function from C_r to $\mathbb{Z}_2$ and π be the* parity function. *Then the set $C_{2r+1} = \{\boldsymbol{x}||\pi(\boldsymbol{x}) + f(\boldsymbol{c})||\boldsymbol{x} + \boldsymbol{c}; \boldsymbol{x} \in \mathbb{B}_r, \boldsymbol{c} \in C_r\}$*

is a perfect code of Q_{2r+1}

We recall also the proof of Theorem 1.1 in such a way our article will be self contained.
Proof. Fist notice that $|C_{2r+1}| = 2^r|C_r| = 2^r\frac{2^r}{r+1} = \frac{2^{2r+1}}{2r+2}$. Thus if is sufficient to prove that the distance between to different elements of C_{2r+1} is at least 3.
Consider $d(\boldsymbol{x}||\pi(\boldsymbol{x}) + f(\boldsymbol{c})||\boldsymbol{x}+\boldsymbol{c}, \boldsymbol{x'}||\pi(\boldsymbol{x'}) + f(\boldsymbol{c'})||\boldsymbol{x'}+\boldsymbol{c'}) = d_1 + d_2 + d_3$ where $d_1 = d(\boldsymbol{x}, \boldsymbol{x'})$, $d_2 = d(\pi(\boldsymbol{x}) + f(\boldsymbol{c}), \pi(\boldsymbol{x'}) + f(\boldsymbol{c'}))$ and $d_3 = d(\boldsymbol{x}+\boldsymbol{c}, \boldsymbol{x'}+\boldsymbol{c'})$.
If $d_1 = 0$ then $\boldsymbol{x} = \boldsymbol{x'}$ thus $d_3 = d(\boldsymbol{c}, \boldsymbol{c'}) \geq 3$.
If $d_1 = 1$ and $\boldsymbol{c} = \boldsymbol{c'}$ then $d_2 = d_3 = 1$
If $d_1 = 1$ and $\boldsymbol{c} \neq \boldsymbol{c'}$ then $d_3 \geq 2$ otherwise $d(\boldsymbol{c}, \boldsymbol{c'}) \leq 2$
If $d_1 = 2$ then $d_3 \neq 0$ otherwise $d(\boldsymbol{c}, \boldsymbol{c'}) = 2$
Thus $d = d_1 + d_2 + d_3 \geq 3$. □

If $f(\boldsymbol{c}) = 0$ for any $\boldsymbol{c} \in C_r$ we obtain the classical inductive construction of Hamming codes with $C_1 = \{0\}$ as basis.

In the next section we will use this construction starting from the Hamming code in Q_r as C_r and a function f chosen in such way that the strings of the constructed code C_{2r+1} has not a too big number of consecutive 1's.

2 Main Result

Lemma 2.1 *Let m be an integer. Let A_0 be the set of strings $A_0 = \{0^{m+1}\boldsymbol{y}; \boldsymbol{y} \in \mathbb{B}_m\}$. For $i \in \{1, \ldots, m\}$ let $A_i = \{\boldsymbol{z}10^{m+1}\boldsymbol{y}; \boldsymbol{z} \in \mathbb{B}_{i-1}, \boldsymbol{y} \in \mathbb{B}_{m-i}\}$. Then the sets A_i are disjoint and any string of B_{2m+1} containing 0^{m+1} as substring belongs to a A_i.*

Proof. Let $\boldsymbol{x}$ be a string of B_{2m+1} containing 0^{m+1} as substring and i be the minimum integer such that $x_{i+1}x_{i+2}\ldots x_{i+m+1} = 0^{m+1}$. Then $i = 0$, and $\boldsymbol{x}$ belongs to A_0, or $m \geq i \geq 1$. In this case $x_i = 1$ thus $\boldsymbol{x} \in A_i$. Assume $\boldsymbol{x} \in A_i \cap A_j$ with $m \geq j > i \geq 0$ then $x_j = 1$ thus $j \geq i + m + 2 > m$ a contradiction.

Theorem 2.2 *Let $n = 2^p - 1$ where $p \geq 2$ and let $s = 3.2^{p-2}$. There exists a perfect code C in Q_n such that no elements of C contains 1^s as substring.*

Proof. Let $m = 2^{p-2} - 1$ thus $2m + 1 = 2^{p-1} - 1$ and $s = 3m + 3$. Let C_{2m+1} be a perfect code in Q_{2m+1}. Let f be the function from $\mathbb{B}_{2m+1}$ to $\mathbb{Z}_2$ defined by

- $f(0^{m+1}\boldsymbol{y}) = 1$ for $\boldsymbol{y} \in \mathbb{B}_m$
- $f(10^{m+1}\boldsymbol{y}) = 0$ for $\boldsymbol{y} \in \mathbb{B}_{m-1}$
- $f(\boldsymbol{z}10^{m+1}\boldsymbol{y}) = \pi(\boldsymbol{z})$ for $\boldsymbol{z} \in \mathbb{B}_{i-1}$ and $\boldsymbol{y} \in \mathbb{B}_{m-i}$ for $i = 2$ to m.
- $f = 0$ otherwise.

Note that from the previous lemma the function is well defined. Let C be the perfect code obtained from Vasilev's construction from C_{2m+1} and f. Assume there exists a string $\boldsymbol{d}$ in C with 1^{3m+3} as substring. Therefore $\boldsymbol{d}$ is obtained from $\boldsymbol{x} = d_1 d_2 \dots d_{2m+1}$ and $\boldsymbol{c} \in C_{2m+1}$. Since $n = 4m+3$ note first that $d_{m+1}d_{m+2}\dots d_{3m+3} = 1^{2m+2}$. Let i be the minimum integer such that $d_i d_{i+1} \dots d_{3m+i+2} = 1^{3m+3}$. We consider 3 cases

- $i = 1$ then $x = d_1 d_2 \dots d_{2m+1} = 1^{2m+1}$ and $d_{2m+2}d_{2m+3}\dots d_{3m+3} = 1^{m+2}$. Since $\boldsymbol{c} + \boldsymbol{x} = 1^{m+1}d_{3m+4}d_{3m+5}\dots d_{4m+3}$ we have $\boldsymbol{c} = 0^{m+1}\boldsymbol{y}$ for some $\boldsymbol{y} \in \mathbb{B}_m$. Thus $f(\boldsymbol{c}) = 1$ and since $\pi(\boldsymbol{x}) = 1$ we obtain $d_{2m+2} = f(\boldsymbol{c}) + \pi(\boldsymbol{x}) = 0$ a contradiction.
- $i = 2$ then $\boldsymbol{x} = 01^{2m}$ and $d_{2m+2}d_{2m+3}\dots d_{3m+4} = 1^{m+3}$. Since $\boldsymbol{c} + \boldsymbol{x} = 1^{m+2}d_{3m+5}d_{3m+6}\dots d_{4m+3}$ we have $\boldsymbol{c} = 10^{m+1}\boldsymbol{y}$ for some $\boldsymbol{y} \in \mathbb{B}_{m-1}$. Thus $f(\boldsymbol{c}) = 0$ and since $\pi(\boldsymbol{x}) = 0$ we obtain $d_{2m+2} = f(\boldsymbol{c}) + \pi(\boldsymbol{x}) = 0$ a contradiction.
- $i \geq 3$ then $\boldsymbol{x} = \boldsymbol{z}01^{2m-i+2}$ for $\boldsymbol{z} \in \mathbb{B}_{i-2}$ and $d_{2m+2}d_{2m+3}\dots d_{3m+2+i} = 1^{m+i+1}$. Since $\boldsymbol{c} + \boldsymbol{x} = 1^{m+i}d_{3m+i+3}d_{3m+i+4}\dots d_{4m+3}$ we have $\boldsymbol{c} = \overline{\boldsymbol{z}}10^{m+1}\boldsymbol{y}$ for some $\boldsymbol{y} \in \mathbb{B}_{m-i+1}$. Thus $f(\boldsymbol{c}) = \pi(\overline{\boldsymbol{z}})$. Since $\pi(\boldsymbol{x}) = \pi(\boldsymbol{z}) + \pi(1^{2m-i+2})$ and $\pi(\overline{\boldsymbol{z}}) + \pi(\boldsymbol{z}) = \pi(1^{i-2})$ we obtain $d_{2m+2} = f(\boldsymbol{c}) + \pi(\boldsymbol{x}) = \pi(1^{2m}) = 0$ a contradiction.

Therefore there exists no string d in C with 1^{3m+3} as substring. □

Corollary 2.3 *Let $n = 2^p - 1$ where $p \geq 2$ and let $s \geq 3.2^{p-2}$. There exists a perfect code in $\Gamma_n(1^s)$.*

Proof. Indeed let C be a perfect code in Q_n such that no element of C contains $1^{3.2^{p-2}}$ as substring. The strings of C are in $V(\Gamma_n(1^s))$. Let x be a vertex of $V(\Gamma_n(1^s))$. If $x \notin C$ then x is adjacent in Q_n to a vertex c in C. Note that x and c are also adjacent in $\Gamma_n(1^s)$ thus C is a dominating set of $\Gamma_n(1^s)$. If c and c' are two strings of C then $d_{\Gamma_n(1^s)}(c, c') \geq d_{Q_n}(c, c') \geq 3$. Therefore C is a perfect code in $\Gamma_n(1^s)$.

3 Concluding remark and open problems

Whenever $n = 2^p - 1$ it will be interesting to determine the minimum s such that there exists a perfect code in $\Gamma_n(1^s)$.

Corollary 2.3 is not always the best result possible. For example for $n = 7$ the code C_7 obtained in Vasilev's construction starting from $C_3 = \{000, 111\}$ with $f(000) = f(111) = 1$ is a perfect code in $\Gamma_n(1^5)$. Indeed

- $11111ab$ or 0011111 cannot be in C_7 since the $P(111) + 1 = P(001) + 1 = 0$
- $011111a$ cannot be in C_7 since the possible codewords begining with 011 are 0111011 and 0111100.

Note that that all strings of this code are obtained from strings in the Hamming code of length 7 by a translation of 0001000. This simple idea can be generalized but is less efficient than our result in the general case.

We propose also the following conjecture:

Conjecture 3.1 *For $n \geq 3$ and $s \geq 1$ if C is a perfect code in $\Gamma_n(1^s)$ then $n = 2^p - 1$ for some integer p and furthermore C is a perfect code in Q_n.*

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References

[1] A.R. Ashrafi, J. Azarija, A. Babai, K. Fathalikhani, S. Klavžar, The (non-) existence of perfect codes in Fibonacci cubes, Information Processing Letters 116 (2016) 387–390.

[2] N. Biggs, Perfect codes in graphs, J. Combin. Theory Ser.B, 5 (1973) 289–296.

[3] W.-J. Hsu, Fibonacci cubes a new interconnection technology, IEEE Trans. Parallel Distrib. Syst. 4 (1993) 3–12.

[4] S. Klavžar, Structure of Fibonacci cubes: a survey, J. Comb. Optim. 25 (2013) 505–522.

[5] A.Ilić, S. Klavžar, Y.Rho, Generalized Fibonacci cubes, Dicrete Math. 312 (2012) 2–11.

[6] W.-J. Hsu, J. Liu, Distributed algorithms for shortest-path, deadlock-free routing and broadcasting in a class of interconnection topologies, in: Parallel Processing Symposium, Beverly Hills, CA, USA (1992) 589-596.

[7] N. Zagaglia Salvi, On the existence of cycles of every even length on generalized Fibonacci cubes, Le Matematiche (Catania) 51 (1996) 241-251.

[8] R. W. Hamming, Error detecting and error correcting codes', Bell Syst. Tech. J. 29 (1950) 147–160.

[9] J. L. Vasilev, On ungrouped, close-packed codes (in Russian), Problemy Kibernet 8 (1962) 337–339.

[10] G. Cohen, I. Honkala, S. Litsyn, A. Lobstein, Covering Codes, Chap. 11, (1997) Elsevier, Amsterdam.

[11] F. I. Soloveva, On perfect binary codes, Discrete Applied Mathematics 156(9)(2008) 1488–1498.