# Improved approximation algorithms for $\boldsymbol{k}$-connected $\boldsymbol{m}$-dominating set problems 

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#### Abstract

A graph is $k$-connected if it has $k$ internally-disjoint paths between every pair of nodes. A subset $S$ of nodes in a graph $G$ is a $k$ connected set if the subgraph $G[S]$ induced by $S$ is $k$-connected; $S$ is an $m$-dominating set if every $v \in V \backslash S$ has at least $m$ neighbors in $S$. If $S$ is both $k$-connected and $m$-dominating then $S$ is a $k$-connected $m$-dominating set, or $(k, m)$-cds for short. In the $k$-Connected $m$ Dominating Set ( $(k, m)$-CDS) problem the goal is to find a minimum weight $(k, m)$-cds in a node-weighted graph. We consider the case $m \geq k$ and obtain the following approximation ratios. For unit disc-graphs we obtain ratio $O(k \ln k)$, improving the ratio $O\left(k^{2} \ln k\right)$ of $5[15$. For general graphs we obtain the first non-trivial approximation ratio $O\left(k^{2} \ln n\right)$.


## 1 Introduction

A graph is $k$-connected if it has $k$ internally disjoint paths between every pair of its nodes. A subset $S$ of nodes in a graph $G$ is a $k$-connected set if the subgraph $G[S]$ induced by $S$ is $k$-connected; $S$ is an $m$-dominating set if every $v \in V \backslash S$ has at least $m$ neighbors in $S$. If $S$ is both $k$-connected and $m$-dominating set then $S$ is a $k$-connected $m$-dominating set, or $(k, m)$-cds for short. A graph is a unit-disk graph if its nodes can be located in in the Euclidean plane such that there is an edge between nodes $u$ and $v$ iff the Euclidean distance between $u$ and $v$ is at most 1 . We consider the following problem for $m \geq k$ both in general graphs and in unit-disc graphs.

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k-Connected m-Dominating Set ((k,m)-CDS)
Input: A graph G=(V,E) with node weights {\mp@subsup{w}{v}{}:v\inV} and integers k,m.
Output: A minimum weight ( }k,m)\mathrm{ -cds S}\subseteqV\mathrm{ .
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The case $k=0$ is the $m$-Dominating Set problem. Let $\alpha_{m}$ denote the best known ratio for $m$-Dominating SET; currently $\alpha_{m}=O(1)$ in unit-disc graphs [5] and $\alpha_{m}=\ln (\Delta+m)+1<\ln \Delta+1.7$ in general graphs [4], where $\Delta$ is the maximum degree of the input graph. The $(k, m)$-CDS problem with $m \geq k$ was studied extensively. In recent papers Zhang, Zhou, Mo, and Du 15 and Fukunaga [5] obtained ratio $O\left(k^{2} \ln k\right)$ for the problem in unit-disc graphs. For unit-disc graphs and $k=2$ Zhang et al. [15] also obtained an improved ratio $\alpha_{m}+5$. In a related paper Zhang et al. [16] obtained ratio $O(k \ln \Delta)$ in
general graphs with unit weights, mentionning that no non-trivial approximation algorithm for arbitrary weights is known.

Let us say that a graph with a designated set $T$ of terminals and a root node $r$ is $k$ - $(T, r)$-connected if it contains $k$ internally-disjoint $r t$-paths for every $t \in T$. Our ratios for $(k, m)$-CDS are expressed in terms of $\alpha_{m}$ and the best ratio for the following known problem:

## Rooted Subset $k$-Connectivity <br> Input: A graph $G=(V, E)$ with edge-costs/node-weights, a set $T \subseteq V$ of terminals, a root node $r \in V \backslash T$, and an integer $k$. <br> Output: A minimum cost/weight $k-(T, r)$-connected subgraph of $G$.

Let $\beta_{k}$ and $\beta_{k}^{\prime}$ denote the best known ratios for the Rooted Subset $k$ Connectivity problem with edge-costs and node-weights, respectively. Currently, $\beta_{m}=O(1)$ in unit-disc graphs [5], while in general graphs $\beta_{2}=2$ [3], $\beta_{3}=6 \frac{2}{3}$ [13], and $\beta_{k}=O(k \ln k)$ for $k \geq 4$ [11]. We also have $\beta_{k}^{\prime}=O\left(k^{2} \ln n\right)$ by [11] and the correction of Vakilian [14] to the algorithm and the analysis of [11]; see also [6].

Our main results are summarized in the following theorem.

Theorem 1. Suppose that the m-Dominating Set problem admits ratio $\alpha_{m}$ and that the Rooted Subset $k$-Connectivity problem admits ratios $\beta_{k}$ for edge-costs and $\beta_{k}^{\prime}$ for node-weights. Then ( $k, m$ )-CDS with $m \geq k$ admits ratios $\alpha_{m}+\beta_{k}^{\prime}+2(k-1)=O\left(k^{2} \ln n\right)$ for general graphs and $\alpha_{m}+5 \beta_{k}+2(k-1)=$ $O(k \ln k)$ for unit-disc graphs. Furthermore, $(3, m)$-CDS on unit-disc graphs admits ratio $\alpha_{m}+5 \beta_{3}=\alpha_{m}+33 \frac{1}{3}$.

Our algorithm uses the main ideas as well as partial results from the papers of Zhang et al. [15] and Fukunaga [5]. Let us say that a graph $G$ is $k-T$-connected if $G$ contains $k$ internally-disjoint paths between every pair of nodes in $T$. Both papers [15]5] consider unit-disc graphs and reduce the $(k, m)$-CDS problem with $m \geq k$ to the SUBSET $k$-CONNECTIVITY problem: given a graph with edge costs and a subset $T$ of terminals, find a minimum cost $k$ - $T$-connected subgraph. The problem admits a trivial ratio $|T|^{2}$ for both edge-costs and node-weights, while for $|T|>k$ the best known ratios are $\frac{|T|}{|T|-k} O(k \ln k)=O\left(k^{2} \ln k\right)$ for edge-costs and $\frac{|T|}{|T|-k} O\left(k^{2} \ln n\right)=O\left(k^{3} \ln n\right)$ for node-weights [12]; see also [8]. In fact, these ratios are derived by applying $O(k)$ times the algorithm for the Rooted Subset $k$-Connectivity problem. The main reason for our improvement over the ratios of [15]5] is a reduction to the easier Rooted Subset $k$-Connectivity problem. For small values of $k$ we present a refined reduction, but for unit disc graphs and $k=2$ the performance of our algorithm and that of [15] coincide, since for $k=2$ and edge-costs both Subset $k$-Connectivity and Rooted Subset $k$-Connectivity admit ratio 2 [3].

## 2 Proof of Theorem 1

For an arbitrary graph $H=(U, F)$ and $u, v \in U$ let $\kappa_{H}(u, v)$ denote the maximum number of internally disjoint $u v$-paths in $H$. We say that $H$ is $k$-inconnected to $r$ if $H$ is $k$ - $(U \backslash\{r\}, r)$-connected, namely, if $\kappa_{H}(v, r) \geq k$ every $v \in U \backslash\{r\}$. For $A \subseteq U$ let $\Gamma_{H}(A)$ denote the set of neigbors of $A$ in $H$. The proof of the following known statement can be found in [7], see also [12]; part (i) of the lemma relies on the Maders Undirected Critical Cycle Theorem [9].
Lemma 1. Let $H_{r}$ be $k$-in-connected to $r$ and let $R=\Gamma_{H_{r}}(r)$.
(i) The graph $H=H_{r} \backslash\{r\}$ can be made $k$-connected by adding a set $J$ of new edges on $R$; furthermore, if $J$ is inclusionwise-minimal then $J$ is a forest.
(ii) Suppose that $|R|=k$. If $k=2,3$ then $H_{r}$ is $k$-connected.

Note that an inclusionwise-minimal edge set $J$ as in Lemma 1 (i) can be computed in polynomial time, by starting with $J$ being a clique on $R$ and repeatedly removing from $J$ an edge $e$ if $H \cup(J \backslash e)$ remains $k$-connected.

A reason why the case $m \geq k$ is easier is given in the following lemma.
Lemma 2. If a graph $H=(V, E)$ has a $k$-dominating set $T$ such that $H$ is $k$-T-connected then $H$ is $k$-connected.
Proof. By a known characterization of $k$-connected graphs, it is sufficient to show that $|V \backslash(A \cup B)| \geq k$ holds for any subpartition $A, B$ of $V$ such that $E$ has no edge between $A$ and $B$. If both $A \cap T, B \cap T$ are non-empty, this is so since $H$ is $k$ - $T$-connected. Otherwise, if say $A \cap T=\emptyset$, then since $T$ is a $k$-dominating set we have $\left|\Gamma_{H}(A)\right| \geq k$, and the result follows.

Finally, we will need the following known fact, c.f. 11.
Lemma 3. Given a pair $s, t$ of nodes in a node-weighted graph $G$, the problem of finding a minimum weight node set $P_{s t}$ such that $G\left[P_{s t}\right]$ has $k$ internallydisjoint st-paths admits a 2-approximation algorithm.

For arbitrary $k$, we will show that the following algorithm achieves the desired approximation ratio.

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Algorithm 1: \((G=(V, E), w, m \geq k)\)
    1 compute an \(\alpha_{m}\)-approximate \(m\)-dominating set \(T\)
    2 construct a graph \(G_{r}\) by adding to \(G\) a new node \(r\) connected to a set
        \(R \subseteq T\) of \(k\) nodes by a set \(F_{r}=\{r v: v \in R\}\) of new edges
    3 compute a \(\beta_{k}^{\prime}\)-approximate node set \(S \subseteq V \backslash T\) such that the subgraph \(H_{r}\)
        of \(G_{r}\) induced by \(T \cup S \cup\{r\}\) is \(k\) - \((T, r)\)-connected
    4 let \(H=H \backslash\{r\}=G[T \cup S]\) and let \(J\) be a forest of new edges on \(R\) as in
        Lemma 1 (i) such that the graph \(H \cup J\) is \(k\)-connected
    5 for every \(u v \in J\) find a 2-approximate node set \(P_{u v}\) such that
        \(G\left[T \cup S \cup P_{u v}\right]\) has \(k\) internally-disjoint \(u v\)-paths; let \(P=\bigcup_{u v \in J} P_{u v}\)
    6 return \(T \cup S \cup P\)
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We now prove that the solution computed is feasible.
Lemma 4. The computed solution is feasible, namely, at the end of the algorithm $T \cup S \cup P$ is a $(k, m)$-cds.

Proof. Since $T$ is an $m$-dominating set, so is any superset of $T$. Thus the node set $T \cup S \cup P$ returned by the algorithm is an $m$-dominating set.

It remains to prove that $T \cup S \cup P$ is a $k$-connected set. We first prove that the graph $H_{r}$ computed at step 3 is $k$-in-connected to $r$. By Menger's Theorem, $\kappa_{H_{r}}(v, r) \geq k$ iff for all $A \subseteq T \cup S$ with $v \in A$

$$
\begin{equation*}
\left|\Gamma_{H_{r \backslash R}}(A)\right|+|A \cap R| \geq k . \tag{1}
\end{equation*}
$$

Let $\emptyset \neq A \subseteq T \cup S$. If $A \cap T \neq \emptyset$ then (1) holds since $H_{r}$ is $k$-( $(, r)$-connected. If $A \cap S \neq \emptyset$ then $\left|\Gamma_{H_{r} \backslash R}(A)\right| \geq m \geq k$, since $T$ is an $m$-dominating set and thus every node in $A \cap S$ has at least $m$ neighbors in $T$. In both cases, (1) holds, hence $H_{r}$ is $k$-in-connected to $r$.

The graph $H \cup J$ is $k$-connected, which implies that the graph $G[T \cup S \cup P]$ is ( $T \cup S$ )-k-connected and thus $T$ - $k$-connected. Furthermore, $T$ is a $k$-dominating set, since $m \geq k$. Applying Lemma 2 on the graph $G[T \cup S \cup P]$ we get that this graphs is $k$-connected, as required.

Lemma 5. Algorithm $\mathbb{\square}$ has ratio $\alpha_{m}+\beta_{k}^{\prime}+2(k-1)$.
Proof. Let $S^{*}$ be an optimal solution to $(k, m)$-CDS. Clearly, $w(T) \leq \alpha_{m} w\left(S^{*}\right) \leq$ $\beta_{k}^{\prime} w\left(S^{*}\right)$. We claim that $w(S) \leq \beta_{k}^{\prime} w\left(S^{*} \backslash T\right)$. For this note that $S^{*} \backslash T$ is a feasible solution to the problem considered at step 3 of the algorithm, while $S$ is a $\beta_{k}^{\prime}$-approximate solution. For the same reason, for each $u v \in J$ the set $S^{*} \backslash(T \cup S)$ is a feasible solution to the problem considered at step 5 , while the set $P_{u v}$ computed is a 2-approximate solution; thus $w\left(P_{u v}\right) \leq 2 w\left(S^{*} \backslash(T \cup S)\right)$. Finally, note that $|J| \leq k-1$, and thus $w(P) \leq 2(k-1) w\left(S^{*}\right)$. The lemma follows.

This concludes the proof of the case of general $k$ and general graphs. Let us now consider unit disc graphs. Then we use the following result of [15].

Theorem 2 (Zhang, Zhou, Mo, and Du [15]). Any $k$-connected unit-disc graph has a $k$-connected spanning subgraph of maximum degree at most 5 if $k=2$, and at most $5 k$ if $k \geq 3$.

Note that any $k$-connected graph has minimum degree $k$. Thus Theorem [2 implies that when searching for a $k$-connected subgraph in a unit disc graph, one can convert node-weights to edge-costs while invoking in the ratio only a factor of $5 / 2$ in the case $k=2$ and 5 in the case $k \geq 3$. Specifically, given node weights $\left\{w_{v}: v \in V\right\}$ define edge-costs $c_{u v}=w_{u}+w_{v}$. Then for any subgraph $(S, F)$ of $G$ with maximum degree $\Delta$ and minimum degree $\delta$ we have:

$$
\delta w(S) \leq c(F) \leq \Delta w(S)
$$

since $w_{v} \geq 0$ for all $v \in V$ and since

$$
c(F)=\sum_{u v \in E}\left(w_{u}+w_{v}\right)=\sum_{v \in V} d_{F}(v) w_{v} .
$$

We may use this conversion in some steps of our algorithm, and specifically in step 3 , which concludes the proof of the case of general $k$ and unit-disc graphs.

In the case $k=3$ we use a result of Mader [10] that any edge-minimal $k$-connected graph has at least $\frac{(k-1) n+2}{2 k-1}$ nodes of degree $k$. At step 3 of the algorithm we "guess" such a node $r$ and the 3 edges incident to $r$ in some edgeminimal optimal solution, remove from $G$ all other edges incident to $r$, and run step 3 while omitting steps 4 and 5 . By Lemma 1 (ii) the graph $G[S \cup T]$ will be already 3 -connected.

## References

1. V. Auletta, Y. Dinitz, Z. Nutov, and D. Parente. A 2-approximation algorithm for finding an optimum 3-vertex-connected spanning subgraph. J. of Algorithms, 32(1):21-30, 1999.
2. Y. Dinitz and Z. Nutov. A 3-approximation algorithm for finding optimum 4, 5-vertex-connected spanning subgraphs. J. of Algorithms, 32(1):31-40, 1999.
3. L. Fleischer, K. Jain, and D. Williamson. Iterative rounding 2-approximation algorithms for minimum-cost vertex connectivity problems. J. Computer and System Sciences, 72(5):838-867, 2006.
4. K.-T. Förster. Approximating fault-tolerant domination in general graphs. In ANALCO, pages 25-32, 2013.
5. T. Fukunaga. Constant-approximation algorithms for highly connected multidominating sets in unit disk graphs. arXiv:1511.09156[cs.DS].
6. T. Fukunaga. Spider covers for prize-collecting network activation problem. In SODA, pages 9-24, 2015.
7. G. Kortsarz and Z. Nutov. Approximating node connectivity problems via set covers. Algorithmica, 37:75-92, 2003.
8. B. Laekhanukit. An improved approximation algorithm for minimum-cost subset $k$-connectivity. In $I C A L P$, pages 13-24, 2011.
9. W. Mader. Ecken vom grad $n$ in minimalen $n$-fach zusammenhängenden graphen. Archive der Mathematik, 23:219224, 1972.
10. W. Mader. On vertices of degree $n$ in minimally $n$-connected graphs and digraphs. Combinatorics, Paul Erdös is eighty, 2:423449, 1993.
11. Z. Nutov. Approximating minimum cost connectivity problems via uncrossable bifamilies. ACM Transactions on Algorithms, 9(1):1, 2012.
12. Z. Nutov. Approximating subset $k$-connectivity problems. J. Discrete Algorithms, 17:5159, 2012.
13. Z. Nutov. Improved approximation algorithms for min-cost connectivity augmentation problems. In CSR, pages 324-339, 2016.
14. A. Vakilian. Node-weighted prize-collecting survivable network design problems. Master's thesis, University of Illinois at Urbana-Champaign, 2013.
15. Z. Zhang, J. Zhou, Y. Mo, and D.-Z. Du. Approximation algorithm for minimum weight fault-tolerant virtual backbone in unit disk graphs. IEEE/ACM Transactions on networking, 2016. To appear.
16. Z. Zhang, J. Zhou, Y. Mo, and D.-Z. Du. Performance-guaranteed approximation algorithm for fault-tolerant connected dominating set in wireless networks. In INFOCOM, pages 1-8, 2016.
