Improved approximation algorithms for k-connected m-dominating set problems

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Abstract. A graph is k-connected if it has k internally-disjoint paths between every pair of nodes. A subset S of nodes in a graph G is a kconnected set if the subgraph G[S] induced by S is k-connected; S is an m-dominating set if every $v \in V \setminus S$ has at least m neighbors in S. If S is both k-connected and m-dominating then S is a k-connected m-dominating set, or (k, m)-cds for short. In the k-CONNECTED m-DOMINATING SET ((k, m)-CDS) problem the goal is to find a minimum weight (k, m)-cds in a node-weighted graph. We consider the case $m \ge k$ and obtain the following approximation ratios. For unit disc-graphs we obtain ratio $O(k \ln k)$, improving the ratio $O(k^2 \ln k)$ of [5,15]. For general graphs we obtain the first non-trivial approximation ratio $O(k^2 \ln n)$.

1 Introduction

A graph is k-connected if it has k internally disjoint paths between every pair of its nodes. A subset S of nodes in a graph G is a k-connected set if the subgraph G[S] induced by S is k-connected; S is an m-dominating set if every $v \in V \setminus S$ has at least m neighbors in S. If S is both k-connected and m-dominating set then S is a k-connected m-dominating set, or (k, m)-cds for short. A graph is a unit-disk graph if its nodes can be located in in the Euclidean plane such that there is an edge between nodes u and v iff the Euclidean distance between u and v is at most 1. We consider the following problem for $m \ge k$ both in general graphs and in unit-disc graphs.

k-CONNECTED *m*-DOMINATING SET ((k, m)-CDS) Input: A graph G = (V, E) with node weights $\{w_v : v \in V\}$ and integers k, m. Output: A minimum weight (k, m)-cds $S \subseteq V$.

The case k = 0 is the *m*-DOMINATING SET problem. Let α_m denote the best known ratio for *m*-DOMINATING SET; currently $\alpha_m = O(1)$ in unit-disc graphs [5] and $\alpha_m = \ln(\Delta + m) + 1 < \ln \Delta + 1.7$ in general graphs [4], where Δ is the maximum degree of the input graph. The (k, m)-CDS problem with $m \ge k$ was studied extensively. In recent papers Zhang, Zhou, Mo, and Du [15] and Fukunaga [5] obtained ratio $O(k^2 \ln k)$ for the problem in unit-disc graphs. For unit-disc graphs and k = 2 Zhang et al. [15] also obtained an improved ratio $\alpha_m + 5$. In a related paper Zhang et al. [16] obtained ratio $O(k \ln \Delta)$ in general graphs with unit weights, mentionning that no non-trivial approximation algorithm for arbitrary weights is known.

Let us say that a graph with a designated set T of terminals and a root node r is k-(T, r)-connected if it contains k internally-disjoint rt-paths for every $t \in T$. Our ratios for (k, m)-CDS are expressed in terms of α_m and the best ratio for the following known problem:

Rooted Subset k -Connectivity
Input: A graph $G = (V, E)$ with edge-costs/node-weights, a set $T \subseteq V$ of
terminals, a root node $r \in V \setminus T$, and an integer k.
<i>Output:</i> A minimum cost/weight k - (T, r) -connected subgraph of G .

Let β_k and β'_k denote the best known ratios for the ROOTED SUBSET *k*-CONNECTIVITY problem with edge-costs and node-weights, respectively. Currently, $\beta_m = O(1)$ in unit-disc graphs [5], while in general graphs $\beta_2 = 2$ [3], $\beta_3 = 6\frac{2}{3}$ [13], and $\beta_k = O(k \ln k)$ for $k \ge 4$ [11]. We also have $\beta'_k = O(k^2 \ln n)$ by [11] and the correction of Vakilian [14] to the algorithm and the analysis of [11]; see also [6].

Our main results are summarized in the following theorem.

Theorem 1. Suppose that the m-DOMINATING SET problem admits ratio α_m and that the ROOTED SUBSET k-CONNECTIVITY problem admits ratios β_k for edge-costs and β'_k for node-weights. Then (k, m)-CDS with $m \ge k$ admits ratios $\alpha_m + \beta'_k + 2(k-1) = O(k^2 \ln n)$ for general graphs and $\alpha_m + 5\beta_k + 2(k-1) = O(k \ln k)$ for unit-disc graphs. Furthermore, (3, m)-CDS on unit-disc graphs admits ratio $\alpha_m + 5\beta_3 = \alpha_m + 33\frac{1}{3}$.

Our algorithm uses the main ideas as well as partial results from the papers of Zhang et al. [15] and Fukunaga [5]. Let us say that a graph G is k-T-connected if G contains k internally-disjoint paths between every pair of nodes in T. Both papers [15,5] consider unit-disc graphs and reduce the (k, m)-CDS problem with $m \geq k$ to the SUBSET k-CONNECTIVITY problem: given a graph with edge costs and a subset T of terminals, find a minimum cost k-T-connected subgraph. The problem admits a trivial ratio $|T|^2$ for both edge-costs and node-weights, while for |T| > k the best known ratios are $\frac{|T|}{|T|-k}O(k\ln k) = O(k^2\ln k)$ for edge-costs and $\frac{|T|}{|T|-k}O(k^2\ln n) = O(k^3\ln n)$ for node-weights [12]; see also [8]. In fact, these ratios are derived by applying O(k) times the algorithm for the ROOTED SUBSET k-CONNECTIVITY problem. The main reason for our improvement over the ratios of [15,5] is a reduction to the easier ROOTED SUBSET k-CONNECTIVITY problem. For small values of k we present a refined reduction, but for unit disc graphs and k = 2 the performance of our algorithm and that of [15] coincide, since for k = 2 and edge-costs both SUBSET k-CONNECTIVITY and ROOTED SUBSET k-CONNECTIVITY admit ratio 2 [3].

2 Proof of Theorem 1

For an arbitrary graph H = (U, F) and $u, v \in U$ let $\kappa_H(u, v)$ denote the maximum number of internally disjoint uv-paths in H. We say that H is k-inconnected to r if H is k- $(U \setminus \{r\}, r)$ -connected, namely, if $\kappa_H(v, r) \geq k$ every $v \in U \setminus \{r\}$. For $A \subseteq U$ let $\Gamma_H(A)$ denote the set of neigbors of A in H. The proof of the following known statement can be found in [7], see also [1,2]; part (i) of the lemma relies on the Maders Undirected Critical Cycle Theorem [9].

Lemma 1. Let H_r be k-in-connected to r and let $R = \Gamma_{H_r}(r)$.

- (i) The graph $H = H_r \setminus \{r\}$ can be made k-connected by adding a set J of new edges on R; furthermore, if J is inclusionwise-minimal then J is a forest.
- (ii) Suppose that |R| = k. If k = 2, 3 then H_r is k-connected.

Note that an inclusionwise-minimal edge set J as in Lemma 1(i) can be computed in polynomial time, by starting with J being a clique on R and repeatedly removing from J an edge e if $H \cup (J \setminus e)$ remains k-connected.

A reason why the case $m \ge k$ is easier is given in the following lemma.

Lemma 2. If a graph H = (V, E) has a k-dominating set T such that H is k-T-connected then H is k-connected.

Proof. By a known characterization of k-connected graphs, it is sufficient to show that $|V \setminus (A \cup B)| \ge k$ holds for any subpartition A, B of V such that E has no edge between A and B. If both $A \cap T, B \cap T$ are non-empty, this is so since H is k-T-connected. Otherwise, if say $A \cap T = \emptyset$, then since T is a k-dominating set we have $|\Gamma_H(A)| \ge k$, and the result follows. \Box

Finally, we will need the following known fact, c.f. [11].

Lemma 3. Given a pair s,t of nodes in a node-weighted graph G, the problem of finding a minimum weight node set P_{st} such that $G[P_{st}]$ has k internallydisjoint st-paths admits a 2-approximation algorithm.

For arbitrary k, we will show that the following algorithm achieves the desired approximation ratio.

Algorithm 1: $(G = (V, E), w, m \ge k)$
1 compute an α_m -approximate <i>m</i> -dominating set <i>T</i>

2 construct a graph G_r by adding to G a new node r connected to a set $R \subseteq T$ of k nodes by a set $F_r = \{rv : v \in R\}$ of new edges

- **3** compute a β'_k -approximate node set $S \subseteq V \setminus T$ such that the subgraph H_r of G_r induced by $T \cup S \cup \{r\}$ is k-(T, r)-connected
- 4 let $H = H \setminus \{r\} = G[T \cup S]$ and let J be a forest of new edges on R as in Lemma 1(i) such that the graph $H \cup J$ is k-connected
- **5** for every $uv \in J$ find a 2-approximate node set P_{uv} such that

 $G[T \cup S \cup P_{uv}]$ has k internally-disjoint uv-paths; let $P = \bigcup_{v \in T} P_{uv}$

6 return $T \cup S \cup P$

We now prove that the solution computed is feasible.

Lemma 4. The computed solution is feasible, namely, at the end of the algorithm $T \cup S \cup P$ is a (k, m)-cds.

Proof. Since T is an m-dominating set, so is any superset of T. Thus the node set $T \cup S \cup P$ returned by the algorithm is an m-dominating set.

It remains to prove that $T \cup S \cup P$ is a k-connected set. We first prove that the graph H_r computed at step 3 is k-in-connected to r. By Menger's Theorem, $\kappa_{H_r}(v,r) \ge k$ iff for all $A \subseteq T \cup S$ with $v \in A$

$$|\Gamma_{H_r \setminus R}(A)| + |A \cap R| \ge k . \tag{1}$$

Let $\emptyset \neq A \subseteq T \cup S$. If $A \cap T \neq \emptyset$ then (1) holds since H_r is k-(T, r)-connected. If $A \cap S \neq \emptyset$ then $|\Gamma_{H_r \setminus R}(A)| \geq m \geq k$, since T is an m-dominating set and thus every node in $A \cap S$ has at least m neighbors in T. In both cases, (1) holds, hence H_r is k-in-connected to r.

The graph $H \cup J$ is k-connected, which implies that the graph $G[T \cup S \cup P]$ is $(T \cup S)$ -k-connected and thus T-k-connected. Furthermore, T is a k-dominating set, since $m \ge k$. Applying Lemma 2 on the graph $G[T \cup S \cup P]$ we get that this graphs is k-connected, as required.

Lemma 5. Algorithm 1 has ratio $\alpha_m + \beta'_k + 2(k-1)$.

Proof. Let S^* be an optimal solution to (k, m)-CDS. Clearly, $w(T) \leq \alpha_m w(S^*) \leq \beta'_k w(S^*)$. We claim that $w(S) \leq \beta'_k w(S^* \setminus T)$. For this note that $S^* \setminus T$ is a feasible solution to the problem considered at step 3 of the algorithm, while S is a β'_k -approximate solution. For the same reason, for each $uv \in J$ the set $S^* \setminus (T \cup S)$ is a feasible solution to the problem considered at step 5, while the set P_{uv} computed is a 2-approximate solution; thus $w(P_{uv}) \leq 2w(S^* \setminus (T \cup S))$. Finally, note that $|J| \leq k - 1$, and thus $w(P) \leq 2(k - 1)w(S^*)$. The lemma follows. □

This concludes the proof of the case of general k and general graphs. Let us now consider unit disc graphs. Then we use the following result of [15].

Theorem 2 (Zhang, Zhou, Mo, and Du [15]). Any k-connected unit-disc graph has a k-connected spanning subgraph of maximum degree at most 5 if k = 2, and at most 5k if $k \ge 3$.

Note that any k-connected graph has minimum degree k. Thus Theorem 2 implies that when searching for a k-connected subgraph in a unit disc graph, one can convert node-weights to edge-costs while invoking in the ratio only a factor of 5/2 in the case k = 2 and 5 in the case $k \ge 3$. Specifically, given node weights $\{w_v : v \in V\}$ define edge-costs $c_{uv} = w_u + w_v$. Then for any subgraph (S, F) of G with maximum degree Δ and minimum degree δ we have:

$$\delta w(S) \le c(F) \le \Delta w(S)$$

since $w_v \ge 0$ for all $v \in V$ and since

$$c(F) = \sum_{uv \in E} (w_u + w_v) = \sum_{v \in V} d_F(v)w_v .$$

We may use this conversion in some steps of our algorithm, and specifically in step 3, which concludes the proof of the case of general k and unit-disc graphs.

In the case k = 3 we use a result of Mader [10] that any edge-minimal k-connected graph has at least $\frac{(k-1)n+2}{2k-1}$ nodes of degree k. At step 3 of the algorithm we "guess" such a node r and the 3 edges incident to r in some edge-minimal optimal solution, remove from G all other edges incident to r, and run step 3 while omitting steps 4 and 5. By Lemma 1(ii) the graph $G[S \cup T]$ will be already 3-connected.

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