# On error linear complexity of new generalized cyclotomic binary sequences of period $p^{2}$ 

Chenhuang $\mathrm{Wu}^{1,2}$, Chunxiang $\mathrm{Xu}^{1}$, Zhixiong Chen ${ }^{2}$, Pinhui $\mathrm{Ke}^{3}$<br>1. School of Computer Science and Engineering, University of Electronic Science and Technology of China, Chengdu 611731, P. R. China<br>2. Provincial Key Laboratory of Applied Mathematics, Putian University, Putian, Fujian 351100, P. R. China 3. Fujian Provincial Key Laboratory of Network Security and Cryptology, College of Mathematics and Informatics, Fujian Normal University, Fuzhou, Fujian, 350117, P. R. China

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#### Abstract

We consider the $k$-error linear complexity of a new binary sequence of period $p^{2}$, proposed in the recent paper "New generalized cyclotomic binary sequences of period $p^{2}$ ", by Z. Xiao et al., who calculated the linear complexity of the sequences (Designs, Codes and Cryptography, 2017, https://doi.org/10.1007/s10623-017-0408-7). More exactly, we determine the values of $k$-error linear complexity over $\mathbb{F}_{2}$ for almost $k>0$ in terms of the theory of Fermat quotients. Results indicate that such sequences have good stability.


Keywords: cryptography, pseudorandom binary sequences, $k$-error linear complexity, generalized cyclotomic classes, Fermat quotients.

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## 1 Introduction

Cyclotomic and generalized cyclotomic classes are widely adopted in cryptography. They play an important role in the design of pseudorandom sequences. The typical
examples are the Legendre sequences derived from cyclotomic classes modulo an odd prime and the Jacobi sequences derived from generalized cyclotomic classes modulo a product of two odd distinct primes. The generalized cyclotomic classes modulo a general number (such as a prime-power) are also paid attention in the literature, see the related works such as $[2,9,10,16,18,19]$ and references therein.

Recently, a new family of binary sequences were introduced by Xiao, Zeng, Li and Helleseth [17] via defining the generalized cyclotomic classes modulo $p^{2}$ for odd prime $p$. Now we introduce the way of defining generalized cyclotomic classes modulo $p^{2}$.

Let $p-1=e f$ and $g$ be a primitive root ${ }^{11}$ modulo $p^{2}$. The generalized cyclotomic classes for $1 \leq j \leq 2$ is defined by

$$
D_{0}^{\left(p^{j}, f\right)} \triangleq\left\{g^{k f p^{j-1}} \quad\left(\bmod p^{j}\right): 0 \leq k<e\right\}
$$

and

$$
D_{l}^{\left(p^{j}, f\right)} \triangleq g^{l} D_{0}^{\left(p^{j}, f\right)}=\left\{g^{l} \cdot g^{k f p^{j-1}} \quad\left(\bmod p^{j}\right): 0 \leq k<e\right\}, 1 \leq l<f p^{j-1}
$$

Then The authors of [17] chose even $f$ and an integer $b \in \mathbb{Z}: 0 \leq b<f p$ to define a new $p^{2}$-periodic binary sequence $\left(s_{n}\right)$ :

$$
s_{n}= \begin{cases}0, & \text { if } n \quad\left(\bmod p^{2}\right) \in \mathcal{C}_{0},  \tag{1}\\ 1, & \text { if } n \quad\left(\bmod p^{2}\right) \in \mathcal{C}_{1},\end{cases}
$$

where

$$
\mathcal{C}_{0}=\bigcup_{i=f / 2}^{f-1} p D_{i+b}^{(p)}(\bmod f) \cup \bigcup_{i=p f / 2}^{p f-1} D_{i+b}^{\left(p^{2}\right)}(\bmod p f)
$$

and

$$
\mathcal{C}_{1}=\bigcup_{i=0}^{f / 2-1} p D_{i+b}^{(p, f)}(\bmod f) \cup \bigcup_{i=0}^{p f / 2-1} D_{i+b}^{\left(p^{2}, f\right)}(\bmod p f) \cup\{0\} .
$$

The notation $p D_{j}^{(p, f)}$ above means that $p D_{j}^{(p, f)}=\left\{p v: v \in D_{j}^{(p, f)}\right\}$. They determined the linear complexity (see below for the notion) of the proposed sequences $\left(s_{n}\right)$ for $f=2^{r}$ for some integer $r \geq 1$.

Theorem 1. ( 17, Thm. 1]) Let $\left(s_{n}\right)$ be the binary sequence of period $p^{2}$ defined in Eq.(1) with $f=2^{r}$ (integer $r>0$ ) and any $b$ for defining $\mathcal{C}_{0}$ and $\mathcal{C}_{1}$. If $2^{(p-1) / f} \not \equiv 1$ $\left(\bmod p^{2}\right)$, then the linear complexity of $\left(s_{n}\right)$ is

$$
L C^{\mathbb{F}_{2}}\left(\left(s_{n}\right)\right)= \begin{cases}p^{2}-(p-1) / 2, & \text { if } 2 \in D_{0}^{(p, f)}, \\ p^{2}, & \text { if } 2 \notin D_{0}^{(p, f)}\end{cases}
$$

[^0]The linear complexity is an important cryptographic characteristic of sequences and provides information on the predictability and thus unsuitability for cryptography. Here we give a short introduction of the linear complexity of periodic sequences. Let $\mathbb{F}$ be a field. For a $T$-periodic sequence $\left(s_{n}\right)$ over $\mathbb{F}$, we recall that the linear complexity over $\mathbb{F}$, denoted by $L C^{\mathbb{F}}\left(\left(s_{n}\right)\right)$, is the least order $L$ of a linear recurrence relation over $\mathbb{F}$

$$
s_{n+L}=c_{L-1} s_{n+L-1}+\ldots+c_{1} s_{n+1}+c_{0} s_{n} \text { for } n \geq 0
$$

which is satisfied by $\left(s_{n}\right)$ and where $c_{0} \neq 0, c_{1}, \ldots, c_{L-1} \in \mathbb{F}$. Let

$$
S(X)=s_{0}+s_{1} X+s_{2} X^{2}+\ldots+s_{T-1} X^{T-1} \in \mathbb{F}[X]
$$

which is called the generating polynomial of $\left(s_{n}\right)$. Then the linear complexity over $\mathbb{F}$ of $\left(s_{n}\right)$ is computed by

$$
\begin{equation*}
L C^{\mathbb{F}}\left(\left(s_{n}\right)\right)=T-\operatorname{deg}\left(\operatorname{gcd}\left(X^{T}-1, S(X)\right)\right) \tag{2}
\end{equation*}
$$

see, e.g. [8] for details.
For a sequence to be cryptographically strong, its linear complexity should be large, but not significantly reduced by changing a few terms. This directs to the notion of the $k$-error linear complexity. For integers $k \geq 0$, the $k$-error linear complexity over $\mathbb{F}$ of $\left(s_{n}\right)$, denoted by $L C_{k}^{\mathbb{F}}\left(\left(s_{n}\right)\right.$ ), is the smallest linear complexity (over $\mathbb{F}$ ) that can be obtained by changing at most $k$ terms of the sequence per period, see [15], and see [11] for the related even earlier defined sphere complexity. Clearly $L C_{0}^{\mathbb{F}}\left(\left(s_{n}\right)\right)=L C^{\mathbb{F}}\left(\left(s_{n}\right)\right)$ and

$$
T \geq L C_{0}^{\mathbb{F}}\left(\left(s_{n}\right)\right) \geq L C_{1}^{\mathbb{F}}\left(\left(s_{n}\right)\right) \geq \ldots \geq L C_{w}^{\mathbb{F}}\left(\left(s_{n}\right)\right)=0
$$

when $w$ equals the number of nonzero terms of $\left(s_{n}\right)$ per period, i.e., the weight of $\left(s_{n}\right)$.
The main contribution of this work is to determine the $k$-error linear complexity of $\left(s_{n}\right)$ in Eq.(11) for any even number $f$ (including $f=2^{r}$ considered in [17]). The main results are presented in the following two theorems. The proofs appear in Section 4 , Some necessary lemmas are introduced in Section 3. A crucial tool for the proof is the Fermat quotients, which is introduced in Section 2.

Theorem 2. (Main theorem) Let $\left(s_{n}\right)$ be the binary sequence of period $p^{2}$ defined in Eq.(1) with even $f$ and any $b$ for defining $\mathcal{C}_{0}$ and $\mathcal{C}_{1}$. If 2 is a primitive root modulo $p^{2}$, then the $k$-error linear complexity of $\left(s_{n}\right)$ satisfies

$$
L C_{k}^{\mathbb{F}_{2}}\left(\left(s_{n}\right)\right)=\left\{\begin{array}{cl}
p^{2}, & \text { if } k=0 \\
p^{2}-1, & \text { if } 1 \leq k<(p-1) / 2 \\
p^{2}-p, & \text { if }(p-1) / 2 \leq k<\left(p^{2}-p\right) / 2, \\
p-1, & \text { if } k=\left(p^{2}-p\right) / 2, \\
1, & \text { if } k=\left(p^{2}-1\right) / 2 \\
0, & \text { if } k \geq\left(p^{2}+1\right) / 2
\end{array}\right.
$$

if $p \equiv 3(\bmod 8)$, and

$$
L C_{k}^{\mathbb{F}_{2}}\left(\left(s_{n}\right)\right)=\left\{\begin{array}{cl}
p^{2}, & \text { if } k=0, \\
p^{2}-1, & \text { if } 1 \leq k<(p-1) / 2, \\
p^{2}-p+1, & \text { if } k=(p-1) / 2, \\
p^{2}-p, & \text { if }(p+1) / 2 \leq k<\left(p^{2}-p\right) / 2, \\
p, & \text { if } k=\left(p^{2}-p\right) / 2, \\
1, & \text { if } k=\left(p^{2}-1\right) / 2, \\
0, & \text { if } k \geq\left(p^{2}+1\right) / 2,
\end{array}\right.
$$

if $p \equiv 5(\bmod 8)$.

## 2 Fermat quotients

In this section, we interpret that the construction of $\left(s_{n}\right)$ in Eq.(1) is related to Fermat quotients. Certain similar constructions can be found in [3-7, [13].

For integers $u \geq 0$, the Fermat quotient $q_{p}(u)$ is the value in $\{0,1, \ldots, p-1\}$ at $u$ defined by

$$
q_{p}(u) \equiv \frac{u^{p-1}-1}{p} \quad(\bmod p),
$$

where $\operatorname{gcd}(u, p)=1$, if $p \mid u$ we set $q_{p}(u)=0$, see [14].
Thanks to the facts that

$$
\left\{\begin{array}{l}
q_{p}(u+\ell p) \equiv q_{p}(u)-\ell u^{-1} \quad(\bmod p)  \tag{3}\\
q_{p}(u v) \equiv q_{p}(u)+q_{p}(v) \quad(\bmod p)
\end{array}\right.
$$

for $\operatorname{gcd}(u, p)=1$ and $\operatorname{gcd}(v, p)=1$, we define

$$
D_{l}=\left\{u: 0 \leq u<p^{2}, \operatorname{gcd}(u, p)=1, q_{p}(u)=l\right\}, 0 \leq l<p
$$

In fact, together with the second equation in (3) and the primitive root $g$ modulo $p^{2}$ with $q_{p}(g)=12$, we have

$$
D_{l}=\left\{g^{l+i p} \quad\left(\bmod p^{2}\right): 0 \leq i<p-1\right\}, 0 \leq l<p
$$

So according to the definition of $D_{l}^{\left(p^{2}, f\right)}$ in Sect. 1, we see that

$$
\begin{equation*}
D_{l}=\bigcup_{i=0}^{f-1} D_{l+i p}^{\left(p^{2}, f\right)}, 0 \leq l<p \tag{4}
\end{equation*}
$$

The $D_{l}$ 's and Eq.(3) help us to study the $k$-error linear complexity of $\left(s_{n}\right)$ in Eq.(1) in this work.

[^1]
## 3 Auxiliary lemmas

In this section, we present some necessary lemmas needed in the proofs. In the sequel, the notation $|Z|$ denotes the cardinality of the set $Z$.

Lemma 1. Let $D_{l}$ be defined for $0 \leq l<p$ by Fermat quotients as in Sect.2. Then we have for $0 \leq l<p$,

$$
\left\{n \bmod p: n \in D_{l}\right\}=\{1,2, \ldots, p-1\}
$$

Proof. Since $D_{l}=\left\{g^{l+i p}\left(\bmod p^{2}\right): 0 \leq i<p-1\right\}$ for $0 \leq l<p$, we get

$$
\left\{g^{l+i p} \quad(\bmod p): 0 \leq i<p-1\right\}=\left\{g^{l+i} \quad(\bmod p): 0 \leq i<p-1\right\}
$$

which completes the proof.
Lemma 2. Let $D_{l}$ be defined for $0 \leq l<p$ by Fermat quotients as in Sect.2. Let $v \in\{1,2, \ldots, p-1\}$ and $\mathcal{V}_{v}=\{v, v+p, v+2 p, \ldots, v+(p-1) p\}$. Then for each $0 \leq l<p$, we have $\left|\mathcal{V}_{v} \cap D_{l}\right|=1$.

Proof. By the first equation in Eq.(3), if $q_{p}\left(v+i_{1} p\right)=q_{p}\left(v+i_{2} p\right)=l$ for $0 \leq$ $i_{1}, i_{2}<p$, that is $q_{p}(v)-i_{1} v^{-1} \equiv q_{p}(v)-i_{2} v^{-1}(\bmod p)$, then $i_{1}=i_{2}$.

Lemma 3. Let $D_{l}^{\left(p^{j}, f\right)}$ be defined for $0 \leq l<f p^{j-1}$ with even $f$ and $1 \leq j \leq 2$ as in Sect.1. Then we have for any $0 \leq l<f p$

$$
\left\{u \quad(\bmod p): u \in D_{l}^{\left(p^{2}, f\right)}\right\}=D_{l}^{(p, f)}(\bmod f) .
$$

Proof. It is clear.

Lemma 4. Let $D_{l}^{\left(p^{j}, f\right)}$ be defined for $0 \leq l<f p^{j-1}$ with even $f$ and $1 \leq j \leq 2$ as in Sect.1. Let $v \in\{1,2, \ldots, p-1\}$ and $\mathcal{V}_{v}=\{v, v+p, v+2 p, \ldots, v+(p-1) p\}$. If $v \in D_{\ell}^{(p, f)}$ for some $0 \leq \ell<f$, we have for any $0 \leq l<p$

$$
\left|\mathcal{V}_{v} \cap D_{l+i p}^{\left(p^{2}, f\right)}\right|= \begin{cases}1, & \text { if } i \equiv l-\ell \quad(\bmod f) \\ 0, & \text { otherwise },\end{cases}
$$

where $0 \leq i<f$.
Proof. For each $0 \leq l<p$, we have by Lemma $2\left|\mathcal{V}_{v} \cap D_{l}\right|=1$. Then by Eq.(4) there exists some $i_{0}: 0 \leq i_{0}<f$ such that

$$
\left|\mathcal{V}_{v} \cap D_{l+i_{0} p}^{\left(p^{2}, f\right)}\right|=1
$$

Now we determine $i_{0}$. Let $u \in \mathcal{V}_{v} \cap D_{l+i_{0} p}^{\left(p^{2}, f\right)}$. We check that

$$
u \equiv v \quad(\bmod p), u \quad(\bmod p) \in D_{l+i_{0} p}^{(p, f)} \quad(\bmod f)
$$

by Lemma 3, hence we get $\ell \equiv l+i_{0} p \equiv l+i_{0}(\bmod f)$, that is $i_{0} \equiv l-\ell(\bmod f)$. We complete the proof.

For any integer $b$, let

$$
\begin{equation*}
\mathcal{Q}_{b}=\bigcup_{i=0}^{f / 2-1} D_{i+b}^{(p, f)}(\bmod f) \subseteq\{1,2, \ldots, p-1\}, \mathcal{N}_{b}=\{1,2, \ldots, p-1\} \backslash \mathcal{Q}_{b} \tag{5}
\end{equation*}
$$

We see that $\left|\mathcal{Q}_{b}\right|=\left|\mathcal{N}_{b}\right|=(p-1) / 2$.
Lemma 5. Let $D_{l}^{\left(p^{j}, f\right)}$ be defined for $0 \leq l<f p^{j-1}$ with even $f$ and $1 \leq j \leq 2$, and let $\mathcal{C}_{0}$ and $\mathcal{C}_{1}$ be defined with any integer $b$ as in Sect.1. Let $v \in\{1,2, \ldots, p-1\}$ and $\mathcal{V}_{v}=\{v, v+p, v+2 p, \ldots, v+(p-1) p\}$.
(1). If $v \in \mathcal{Q}_{b}$, which is defined in Eq.(5), we have

$$
\left|\mathcal{V}_{v} \cap \mathcal{C}_{0}\right|=(p-1) / 2,\left|\mathcal{V}_{v} \cap \mathcal{C}_{1}\right|=(p+1) / 2
$$

(2). If $v \in \mathcal{N}_{b}$, which is defined in Eq.(5), we have

$$
\left|\mathcal{V}_{v} \cap \mathcal{C}_{0}\right|=(p+1) / 2,\left|\mathcal{V}_{v} \cap \mathcal{C}_{1}\right|=(p-1) / 2
$$

Proof. It follows from Lemma 4
Define polynomials in $\mathbb{F}_{2}[X]$

$$
\begin{equation*}
d_{l}^{\left(p^{j}, f\right)}(X)=\sum_{n \in D_{l}^{\left(p^{j}, f\right)}} X^{n} \tag{6}
\end{equation*}
$$

for $0 \leq l<f p^{j-1}$ and $1 \leq j \leq 2$. Let $\theta \in \overline{\mathbb{F}}_{2}$ be a primitive $p$-th root of unity and

$$
\omega_{b}=\sum_{i=0}^{f / 2-1} d_{i+b}^{(p, f)}(\bmod f)(\theta)=\sum_{n \in \mathcal{Q}_{b}} \theta^{n} \in \overline{\mathbb{F}}_{2}
$$

It is easy to see that

$$
\omega_{b+f / 2}=\sum_{n \in \mathcal{N}_{b}} \theta^{n} \text { and } \omega_{b}+\omega_{b+f / 2}=1
$$

Lemma 6. Suppose that 2 is a primitive root modulo $p$. Let $\theta \in \overline{\mathbb{F}}_{2}$ be a primitive p-th root of unity and $\omega_{b}=\sum_{n \in \mathcal{Q}_{b}} \theta^{n}$. Then $\omega_{b} \notin \mathbb{F}_{2}$ for any integer $b$.

Proof. Write

$$
W_{b}^{(1)}(X)=\sum_{n \in \mathcal{Q}_{b}} X^{n}, W_{b}^{(2)}(X)=\sum_{n \in \mathcal{N}_{b}} X^{n} .
$$

Then we have $1<\operatorname{deg}\left(W_{b}^{(1)}(X)\right) \neq \operatorname{deg}\left(W_{b}^{(2)}(X)\right) \leq p-1$.
Suppose that $\omega_{b}=0$ for some integer $b$, we have for all $i: 0 \leq i<p-1$

$$
W_{b}^{(1)}\left(\theta^{2^{i}}\right)=W_{b}^{(1)}(\theta)^{2^{i}}=0
$$

and

$$
1+W_{b}^{(2)}\left(\theta^{2^{i}}\right)=\left(1+W_{b}^{(2)}(\theta)\right)^{2^{i}}=\left(1+\omega_{b+f / 2}\right)^{2^{i}}=\left(\omega_{b}\right)^{2^{i}}=0
$$

which tell us that both $W_{b}^{(1)}(X)$ and $1+W_{b}^{(2)}(X)$ have $p-1$ many solutions, since 2 is a primitive root modulo $p$. This contradicts to that one of $\operatorname{deg}\left(W_{b}^{(1)}(X)\right)$ and $\operatorname{deg}\left(1+W_{b}^{(2)}(X)\right)$ is smaller than $p-1$.

A similar proof can also lead to a contradiction if $\omega_{b}=1$. We complete the proof.

For any non-zero polynomial $h(X) \in \mathbb{F}_{2}[X]$, the weight of $h(X)$ is referred to as the number of non-zero coefficients of $h(X)$.

Lemma 7. Suppose that 2 is a primitive root modulo $p$. Let $\theta \in \overline{\mathbb{F}}_{2}$ be a primitive $p$-th root of unity and $\omega_{b}=\sum_{n \in \mathcal{Q}_{b}} \theta^{n}$ for any integer $b$. For any non-constant polynomial $h(X) \in \mathbb{F}_{2}[X]$ with $h(\theta)=\omega_{b}$, we have $w t(h(X)) \geq(p-1) / 2$.

Proof. By Lemma 6 we see that $\omega \in \overline{\mathbb{F}}_{2} \backslash \mathbb{F}_{2}$. First, we can choose $h(X) \in \mathbb{F}_{2}[X]$ such that $h(X)=W_{b}^{(1)}(X)=\sum_{n \in \mathcal{Q}_{b}} X^{n}$, then we have $h(\theta)=W_{b}^{(1)}(\theta)=\omega_{b}$. In this case $w t(h(X))=(p-1) / 2$.

Second, suppose that there is an $h_{0}(X) \in \mathbb{F}_{2}[X]$ such that $w t\left(h_{0}(X)\right)<(p-1) / 2$ and $h_{0}(\theta)=\omega_{b}$. Let $\bar{h}_{0}(X) \equiv h_{0}(X)\left(\bmod X^{p}-1\right)$ with $\operatorname{deg}\left(\bar{h}_{0}\right)<p$ and let $H_{0}(X)=$ $\bar{h}_{0}(X)+W_{b}^{(1)}(X)$, the degree of which is $<p$. Clearly $H_{0}(X)$ is non-zero since $\bar{h}_{0}(X) \neq W_{b}^{(1)}(X)$, since the weight of $W_{b}^{(1)}(X)$ is $(p-1) / 2$. Then we derive that $H_{0}(\theta)=0$ and $H_{0}\left(\theta^{2^{i}}\right)=0$ for $1 \leq i<p-1$. Since 2 is a primitive root modulo $p$, we see that

$$
\left(1+X+X^{2}+\ldots+X^{p-1}\right) \mid H_{0}(X)
$$

i.e., due to $\operatorname{deg}\left(H_{0}(X)\right)<p$, we have $H_{0}(X)=1+X+X^{2}+\ldots+X^{p-1}$, from which we get $\bar{h}_{0}(X)=1+W_{b}^{(2)}(X)$ and $w t\left(\bar{h}_{0}(X)\right)=(p+1) / 2$. Therefore, $w t\left(h_{0}\right) \geq w t\left(\bar{h}_{0}\right)=$ $(p+1) / 2$, a contradiction.

Now we turn to prove our main result.

## 4 Proof of the main theorem

(Proof of Theorem (2). From the construction (1), we see that the weight of $\left(s_{n}\right)$ is $\left(p^{2}-1\right) / 2+1$, i.e., there are $\left(p^{2}-1\right) / 2+1$ many 1's in one period. Changing all terms of 0 's of $\left(s_{n}\right)$ will lead to the constant 1 -sequence, whose linear complexity is 1. And changing all terms of 1's will lead to the constant 0 -sequence. So we always assume that $k<\left(p^{2}-1\right) / 2$.

The generating polynomial of $\left(s_{n}\right)$ is of the form

$$
\begin{equation*}
S(X)=1+\sum_{i=0}^{p f / 2-1} d_{i+b}^{\left(p^{2}, f\right)}(\bmod p f)(X)+\sum_{i=0}^{f / 2-1} d_{i+b}^{(p, f)}(\bmod f)\left(X^{p}\right) \in \mathbb{F}_{2}[X] \tag{7}
\end{equation*}
$$

where $d_{l}^{\left(p^{j}, f\right)}(X)$ is defined in Eq.(6). We first recall the linear complexity of $\left(s_{n}\right)$ that $L C_{0}^{\mathbb{F}_{2}}\left(\left(s_{n}\right)\right)=p^{2}$ for $f=2^{r}(r \geq 1)$ from [17, Thm.1] and for even $f$ from [12, Thm.8].

Then we suppose $k>0$. Let

$$
\begin{equation*}
S_{k}(X)=S(X)+e(X) \in \mathbb{F}_{2}[X] \tag{8}
\end{equation*}
$$

be the generating polynomial of the sequence obtained from $\left(s_{n}\right)$ by changing exactly $k$ terms of $\left(s_{n}\right)$ per period, where $e(X)$ is the corresponding error polynomial with $k$ many monomials. We note that $S_{k}(X)$ is a nonzero polynomial due to $k<\left(p^{2}-1\right) / 2$. We will consider the common roots of $S_{k}(X)$ and $X^{p^{2}}-1$, i.e., the roots of the form $\beta^{n}\left(n \in \mathbb{Z}_{p^{2}}\right)$ for $S_{k}(X)$, where $\beta \in \overline{\mathbb{F}}_{2}$ is a primitive $p^{2}$-th root of unity. The number of the common roots will help us to derive the values of $k$-error linear complexity of $\left(s_{n}\right)$ by Eq.(2).

Case I: $k<\left(p^{2}-p\right) / 2$.
On the one hand, we assume that $S_{k}\left(\beta^{n_{0}}\right)=0$ for some $n_{0} \in \mathbb{Z}_{p^{2}}^{*}=\{1 \leq n<$ $\left.p^{2} \mid \operatorname{gcd}(p, n)=1\right\}$. Since 2 is a primitive root modulo $p^{2}$, for each $n \in \mathbb{Z}_{p^{2}}^{*}$, there exists a $0 \leq j_{n}<(p-1) p$ such that $n \equiv n_{0} 2^{j_{n}}\left(\bmod p^{2}\right)$. Then we have

$$
S_{k}\left(\beta^{n}\right)=S_{k}\left(\beta^{n_{0} 2^{j n}}\right)=S_{k}\left(\beta^{n_{0}}\right)^{2^{j n}}=0
$$

that is, all $\left(p^{2}-p\right)$ many elements $\beta^{n}$ with $n \in \mathbb{Z}_{p^{2}}^{*}$ are roots of $S_{k}(X)$. Hence we have

$$
\Phi(X) \mid S_{k}(X) \text { in } \overline{\mathbb{F}}_{2}[X]
$$

where

$$
\Phi(X)=1+X^{p}+X^{2 p}+\ldots+X^{(p-1) p} \in \mathbb{F}_{2}[X]
$$

the roots of which are exactly $\beta^{n}$ for $n \in \mathbb{Z}_{p^{2}}^{*}$. Let

$$
\begin{equation*}
S_{k}(X) \equiv \Phi(X) \pi(X) \quad\left(\bmod X^{p^{2}}-1\right) \tag{9}
\end{equation*}
$$

Since $\operatorname{deg}\left(S_{k}(X)\right)=\operatorname{deg}(\Phi(X))+\operatorname{deg}(\pi(X))<p^{2}$, we see that $\pi(X)$ should be one of the following:

$$
\pi(X)=1 ; \pi(X)=X^{v_{1}}+X^{v_{2}}+\ldots+X^{v_{t}} ; \pi(X)=1+X^{v_{1}}+X^{v_{2}}+\ldots+X^{v_{t}}
$$

where $1 \leq t<p$ and $1 \leq v_{1}<v_{2}<\ldots<v_{t}<p$. Then the exponent of each monomial in $\Phi(X) \pi(X)$ forms the set $\{l p: 0 \leq l \leq p-1\}$ or $\left\{v_{j}+l p: 1 \leq j \leq t, 0 \leq l \leq p-1\right\}$ or $\left\{l p, v_{j}+l p: 1 \leq j \leq t, 0 \leq l \leq p-1\right\}$ for different $\pi(X)$ above.
(i). If $\pi(X)=1$, we see that by (77)-(9)

$$
e(X)=\sum_{j=0}^{p f / 2-1} d_{b+j}^{\left(p^{2}, f\right)}(\bmod p f)(X)+\sum_{n \in \mathcal{N}_{b}} X^{n p}
$$

which implies that $k=\left(p^{2}-1\right) / 2$, where $\mathcal{N}_{b}$ is in Eq.(5).
(ii). If $\pi(X)=X^{v_{1}}+X^{v_{2}}+\ldots+X^{v_{t}}$, we let

$$
\mathcal{I}=\left\{v_{1}, v_{2}, \ldots, v_{t}\right\}, \mathcal{J}=\{1,2, \ldots, p-1\} \backslash \mathcal{I}
$$

and let $z=\left|\mathcal{I} \cap \mathcal{Q}_{b}\right|$. We have $\left|\mathcal{J} \cap \mathcal{Q}_{b}\right|=(p-1) / 2-z,\left|\mathcal{I} \cap \mathcal{N}_{b}\right|=t-z$ and $\left|\mathcal{J} \cap \mathcal{N}_{b}\right|=(p-1) / 2-t+z$. For $\mathcal{V}_{v}=\{v+\ell p: 0 \leq \ell<p\}$, by Lemma 5 we derive that

$$
\begin{aligned}
e(X)= & \sum_{v \in \mathcal{\mathcal { I } \cap \mathcal { Q } _ { b }}} \sum_{n \in \mathcal{V}_{v} \cap \mathcal{C}_{0}} X^{n}+\sum_{v \in \mathcal{J} \cap \mathcal{Q}_{b}} \sum_{n \in \mathcal{V}_{v} \cap \mathcal{C}_{1}} X^{n} \\
& +\sum_{v \in \mathcal{I} \cap \mathcal{N}_{b}} \sum_{n \in \mathcal{V}_{v} \cap \mathcal{C}_{0}} X^{n}+\sum_{v \in \mathcal{J} \cap \mathcal{N}_{b}} \sum_{n \in \mathcal{V}_{v} \cap \mathcal{C}_{1}} X^{n}+1+\sum_{n \in \mathcal{Q}_{b}} X^{n p},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
k= & z \cdot(p-1) / 2+((p-1) / 2-z) \cdot(p+1) / 2+(t-z) \cdot(p+1) / 2 \\
& +((p-1) / 2-t+z) \cdot(p-1) / 2+1+(p-1) / 2 \\
= & \left(p^{2}-1\right) / 2+1+t-2 z .
\end{aligned}
$$

We can check that $-(p-1) / 2 \leq t-2 z \leq(p-1) / 2$. Then $k \geq\left(p^{2}-p\right) / 2+1$.
(iii). For $\pi(X)=1+X^{v_{1}}+X^{v_{2}}+\ldots+X^{v_{t}}$, we can get similarly

$$
\begin{aligned}
e(X)= & \sum_{v \in \mathcal{I} \cap \mathcal{Q}_{b}} \sum_{n \in \mathcal{V}_{v} \cap \mathcal{C}_{0}} X^{n}+\sum_{v \in \mathcal{J} \cap \mathcal{Q}_{b}} \sum_{n \in \mathcal{V}_{v} \cap \mathcal{C}_{1}} X^{n} \\
& +\sum_{v \in \mathcal{I} \cap \mathcal{N}_{b}} \sum_{n \in \mathcal{V}_{v} \cap \mathcal{C}_{0}} X^{n}+\sum_{v \in \mathcal{J} \cap \mathcal{N}_{b}} \sum_{n \in \mathcal{V}_{v} \cap \mathcal{C}_{1}} X^{n}+\sum_{n \in \mathcal{N}_{b}} X^{n p},
\end{aligned}
$$

and $k=\left(p^{2}-1\right) / 2+t-2 z$. As in (ii), we have $k \geq\left(p^{2}-p\right) / 2$.
So putting everything together, if $k<\left(p^{2}-p\right) / 2$, we always have $S_{k}\left(\beta^{n}\right) \neq 0$ for all $n \in \mathbb{Z}_{p^{2}}^{*}$.

On the other hand, we note first that $p \equiv-3(\bmod 8)$ or $p \equiv 3(\bmod 8)$ when 2 is a primitive root modulo $p^{2}$. From Eq.(7) we have by Lemmas 1 and 5

$$
S(X) \equiv 1+\frac{p-1}{2} \sum_{n=1}^{p-1} X^{n}+\sum_{n \in \mathcal{Q}_{b}} X^{n}+\frac{p-1}{2} \quad\left(\bmod X^{p}-1\right)
$$

So let $\theta=\beta^{p}$, we get for $0 \leq i<p$

$$
\begin{align*}
& S_{k}\left(\beta^{i p}\right)=S_{k}\left(\theta^{i}\right)=e\left(\theta^{i}\right)+S\left(\theta^{i}\right) \\
& =e\left(\theta^{i}\right)+ \begin{cases}\left(p^{2}+1\right) / 2, & \text { if } i=0, \\
1+\sum_{n \in \mathcal{Q}_{b}} \theta^{n i}, & \text { if } 1 \leq i<p,\end{cases} \\
& =e\left(\theta^{i}\right)+ \begin{cases}1, & \text { if } i=0, \\
1+\sum_{n \in \mathcal{Q}_{b+\ell}} \theta^{n}, & \text { if } i \in D_{\ell}^{(p, f)}, 0 \leq \ell<f,\end{cases}  \tag{10}\\
& =e\left(\theta^{i}\right)+ \begin{cases}1, & \text { if } i=0, \\
1+\omega_{b+\ell}\left(=\omega_{b+\ell+f / 2}\right), & \text { if } i \in D_{\ell}^{(p, f)}, 0 \leq \ell<f,\end{cases}
\end{align*}
$$

where $\omega_{b}=\sum_{n \in \mathcal{Q}} \theta^{n}$. We want to look for an $e(X)$ with the least $w t(e(X))$ such that $e(1)=1$ and $e\left(\theta^{i}\right)=\omega_{b+\ell+f / 2}$ for $i \in D_{\ell}^{(p, f)}$. These help us to calculate the number of roots for $S_{k}(X)$ of the form $\beta^{i p}(0 \leq i<p)$.

First, we consider the case $1 \leq k<(p-1) / 2$. If $e(X)$ is with $1 \leq w t(e(X))<$ $(p-1) / 2$, we have $e(1)=w t(e(X))$ and $e\left(\theta^{i}\right) \neq \omega_{b+\ell+f / 2}$ for $1 \leq i<p$ by Lemma 7. So we can use any monomial $e(X)$ (i.e., $w t(e(X))=1$ ) to deduce $S_{k}\left(\beta^{0}\right)=0$ but $S_{k}\left(\beta^{i p}\right) \neq 0$ for $1 \leq i<p$. So by Eq.(2) we derive

$$
L C_{(p-3) / 2}^{\mathbb{F}_{2}}\left(\left(s_{n}\right)\right)=L C_{1}^{\mathbb{F}_{2}}\left(\left(s_{n}\right)\right)=p^{2}-1
$$

Second, we consider the case $k=(p-1) / 2$. Let $e(X)$ satisfy $w t(e(X))=(p-1) / 2$ and $e(X) \equiv \sum_{n \in \mathcal{N}_{b}} X^{n}\left(\bmod X^{p}-1\right)$. Then we have $e(1)=(p-1) / 2$ and

$$
e\left(\theta^{i}\right)=\sum_{n \in \mathcal{N}_{b}} \theta^{n i}=\sum_{n \in \mathcal{N}_{b+\ell}} \theta^{n}=\omega_{b+\ell+f / 2} \text { for } 1 \leq i<p,
$$

if $i \in D_{\ell}^{(p, f)}$ for $0 \leq \ell<f$. For such $e(X)$, it indicates that $S_{k}\left(\beta^{0}\right)=(p+1) / 2$ and $S_{k}\left(\beta^{i p}\right)=0$ for $1 \leq i<p$. Hence

$$
L C_{(p-1) / 2}^{\mathbb{F}_{2}}\left(\left(s_{n}\right)\right)= \begin{cases}p^{2}-p+1, & \text { if } p \equiv-3(\bmod 8), \\ p^{2}-p, & \text { if } p \equiv 3 \quad(\bmod 8)\end{cases}
$$

Furtherly, for the case when $p \equiv 5(\bmod 8)$, we choose an $e(X)$ satisfying $w t(e(X))=$ $(p+1) / 2$ and $e(X) \equiv 1+\sum_{n \in \mathcal{Q}_{b}} X^{n}\left(\bmod X^{p}-1\right)$. Then we derive $S_{k}\left(\beta^{i p}\right)=0$ for $0 \leq i<p$ and hence

$$
L C_{(p+1) / 2}^{\mathbb{F}_{2}}\left(\left(s_{n}\right)\right)=p^{2}-p, \text { if } p \equiv-3 \quad(\bmod 8)
$$

Case II: $k=\left(p^{2}-p\right) / 2$.
Now we consider the case $k=\left(p^{2}-p\right) / 2$. From (iii) above, only that $\mathcal{I}=\mathcal{Q}_{b}$ is useful for us, in this case $t=z=(p-1) / 2$ and

$$
e(X)=\sum_{v \in \mathcal{Q}_{b}} \sum_{n \in \mathcal{V}_{v} \cap \mathcal{C}_{0}} X^{n}+\sum_{v \in \mathcal{N}_{b}} \sum_{n \in \mathcal{V}_{v} \cap \mathcal{C}_{1}} X^{n}+\sum_{n \in \mathcal{N}_{b}} X^{n p},
$$

which can guarantee $S_{k}\left(\beta^{n}\right)=0$ for all $n \in \mathbb{Z}_{p^{2}}^{*}$. We also check that by Lemma 5

$$
\begin{aligned}
e\left(\beta^{i p}\right)=e\left(\theta^{i}\right) & =\frac{p-1}{2} \sum_{n \in \mathcal{Q}_{b}} \theta^{n i}+\frac{p-1}{2} \sum_{n \in \mathcal{N}_{b}} \theta^{n i}+\frac{p-1}{2} \\
& =\frac{p-1}{2}\left(\sum_{n \in \mathcal{Q}_{b}} \theta^{n i}+\sum_{n \in \mathcal{N}_{b}} \theta^{n i}\right)+\frac{p-1}{2} \\
& =\frac{p-1}{2}+\frac{p-1}{2}=p-1=0
\end{aligned}
$$

for $1 \leq i<p$ and $e\left(\beta^{0}\right)=e(1)=(p-1) / 2$. Then from Eq.(10), we get $S_{k}\left(\beta^{0}\right)=$ $(p+1) / 2$ and $S_{k}\left(\beta^{i p}\right) \neq 0$ for $1 \leq i<p$. So we have

$$
L C_{\left(p^{2}-p\right) / 2}^{\mathbb{F}_{2}}\left(\left(s_{n}\right)\right)=p-\delta,
$$

where $\delta=1$ if $p \equiv 3(\bmod 8)$ and $\delta=0$ if $p \equiv-3(\bmod 8)$.
Case III: $k>\left(p^{2}-p\right) / 2$.
If $k=\left(p^{2}-1\right) / 2$, after changing the $k$ many 0 's in $\left(s_{n}\right)$, we get the 1 -sequence, whose linear complexity is 1 . And if $k>\left(p^{2}-1\right) / 2$, we can get the 0 -sequence whose linear complexity is 0 . We complete the proof.

We remark that, it seems difficult for us to determine the values of $L C_{k}^{\mathbb{F}_{2}}\left(\left(s_{n}\right)\right)$ for $\left(p^{2}-p\right) / 2<k<\left(p^{2}-1\right) / 2$ here, but it is at most $p$ (or $p-1$ ).

## 5 A lower bound

We have the following lower bound on the $k$-error linear complexity when 2 is not a primitive root modulo $p^{2}$.

Theorem 3. Let $\left(s_{n}\right)$ be the binary sequence of period $p^{2}$ defined in Eq.(1) with even $f$ and any $b$ for defining $\mathcal{C}_{0}$ and $\mathcal{C}_{1}$. If $2^{p-1} \not \equiv 1\left(\bmod p^{2}\right)$, then the $k$-error linear complexity of $\left(s_{n}\right)$ satisfies

$$
L C_{k}^{\mathbb{F}_{2}}\left(\left(s_{n}\right)\right) \geq \lambda p \text { for } 0 \leq k<\left(p^{2}-p\right) / 2
$$

where $1<\lambda<p$ is the order of 2 modulo $p$.

Proof. First we show the order of 2 modulo $p^{2}$ is $\lambda p$. Under the assumption on $2^{p-1} \not \equiv 1\left(\bmod p^{2}\right)$, we see that the order of 2 modulo $p^{2}$ is of the form $m p$ for some $1 \leq m \leq p-1$ and $m \mid(p-1)$. Then $\lambda \mid m$ from $1 \equiv 2^{m p} \equiv 2^{m}(\bmod p)$ since $2^{m p} \equiv 1\left(\bmod p^{2}\right)$, and $m \mid \lambda$ from $2^{\lambda p} \equiv 1\left(\bmod p^{2}\right)$ since otherwise we write for some $1 \leq \epsilon<p$

$$
2^{\lambda p} \equiv 1+\epsilon p \quad\left(\bmod p^{2}\right),
$$

from which we derive

$$
1 \equiv\left(2^{\lambda p}\right)^{m / \lambda} \equiv(1+\epsilon p)^{m / \lambda} \equiv 1+\frac{\epsilon m p}{\lambda} \quad\left(\bmod p^{2}\right)
$$

However $\frac{\epsilon m}{\lambda} \not \equiv 0(\bmod p)$, a contradiction.
Second, the fact $k<\left(p^{2}-p\right) / 2$ implies that there do exist an $n_{0} \in \mathbb{Z}_{p^{2}}^{*}$ such that $S_{k}\left(\beta^{n_{0}}\right) \neq 0$, where $S_{k}(X)$, as before, is the generating polynomial of the sequence obtained from $\left(s_{n}\right)$ by changing exactly $k$ terms per period. Since otherwise, $k \geq$ $\left(p^{2}-p\right) / 2$ according to Case I in the proof of Theorem 2,

Thus there are at least $\lambda p$ many $n \in\left\{n_{0} 2^{j} \bmod p^{2}: 0 \leq j<\lambda p\right\}$ such that $S_{k}\left(\beta^{n}\right) \neq 0$. Then the result follows.

Theorem 3 covers almost all primes. As far as we know, the primes satisfying $2^{p-1} \equiv 1\left(\bmod p^{2}\right)$ are very rare. It was shown that there are only two such primes ${ }^{3}$, 1093 and 3511 , up to $6 \times 10^{17}$ [1].

Finally, we draw a conclusion that we have determined the values of the $k$-error linear complexity of a new generalized cyclotomic binary sequence of period $p^{2}$ discussed recently in the journal Designs, Codes and Cryptography. Results indicate that such sequences have large linear complexity and the linear complexity does not significantly decrease by changing a few terms.

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[^2]
## References

[1] A. Akbary, S. Siavashi. The largest known Wieferich numbers. Integers, 18-\#A3 (2018) 1-6.
[2] H. Cai, X. Zeng, T. Helleseth, X. Tang, Y. Yang. A new construction of zerodifference balanced functions and its applications. IEEE Trans. Inf. Theory 59(8) (2013) 5008-5015.
[3] Z. X. Chen. Trace representation and linear complexity of binary sequences derived from Fermat quotients. Sci. China Inf. Sci., 57 (2014) 11:2109(10)
[4] Z. X. Chen and X. N. Du. On the linear complexity of binary threshold sequences derived from Fermat quotients. Des. Codes Cryptogr., 67 (2013) 317-323.
[5] Z. X. Chen and D. Gómez-Pérez. Linear complexity of binary sequences derived from polynomial quotients. Sequences and Their Applications-SETA 2012, 181189, Lecture Notes in Comput. Sci., 7280, Springer, Berlin, 2012.
[6] Z. X. Chen, Z. H. Niu and C. H. Wu. On the $k$-error linear complexity of binary sequences derived from polynomial quotients. Sci. China Inf. Sci., 58 (2015) 09:2107(15)
[7] Z. X. Chen, A. Ostafe and A. Winterhof. Structure of pseudorandom numbers derived from Fermat quotients. Arithmetic of Finite Fields-WAIFI 2010, 73-85, Lecture Notes in Comput. Sci., 6087, Springer, Berlin, 2010.
[8] T. W. Cusick, C. Ding and A. Renvall. Stream Ciphers and Number Theory. Gulf Professional Publishing, 2004.
[9] C. Ding, T. Helleseth. New generalized cyclotomy and its applications. Finite Fields Appl. 4 (1998) 140-166.
[10] C. Ding, T. Helleseth. Generalized cyclotomy codes of length $p_{1}^{m_{1}} p_{2}^{m_{2}} \cdots p_{t}^{m_{t}}$. IEEE Trans. Inf. Theory 45(2) (1999) 467-474.
[11] C. S. Ding, G. Z. Xiao, W. J. Shan. The stability theory of stream ciphers. Lecture Notes in Computer Science, 561. Springer-Verlag, Berlin, 1991.
[12] V. Edemskiy, C. L. Li, X. Y. Zeng, T. Helleseth. The linear complexity of new binary cyclotomic sequences of period $p^{n}$. Preprint, 2018 (personal communication)
[13] D. Gómez-Pérez and A. Winterhof. Multiplicative character sums of Fermat quotients and pseudorandom sequences, Period. Math. Hungar., 64 (2012) 161-168.
[14] A. Ostafe and I. E. Shparlinski. Pseudorandomness and dynamics of Fermat quotients. SIAM J. Discr. Math., 25 (2011) 50-71.
[15] M. Stamp and C. F. Martin. An algorithm for the $k$-error linear complexity of binary sequences with period $2^{n}$. IEEE Trans. Inform. Theory, 39 (1993) 13981401.
[16] A. L. Whiteman. A family of difference sets. Illionis J. Math., 6(1) (1962) 107121.
[17] Z. B. Xiao, X. Y. Zeng, C. L. Li, T. Helleseth. New generalized cyclotomic binary sequences of period $p^{2}$. Designs, Codes and Cryptography, https://doi.org/10.1007/s10623-017-0408-7.(2017)
[18] S. Xu, X. Cao, G. Xu. Optimal frequency-hopping sequence sets based on cyclotomy. Int. J. Found. Comput. Sci., 27(4) (2016) 443-462.
[19] X. Zeng, H. Cai, X. Tang, Y. Yang. Optimal frequency hopping sequences of odd length. IEEE Trans. Inf. Theory, 59(5) (2013) 3237-3248.


[^0]:    ${ }^{1}$ For our purpose, we will choose $g$ such that the fermat quotient $q_{p}(g)=1$, see the notion in Sect. 2.

[^1]:    ${ }^{2}$ Such $g$ always exists.

[^2]:    ${ }^{3}$ A prime $p$ satisfying $2^{p-1} \equiv 1\left(\bmod p^{2}\right)$ is called a Wieferich prime.

