Independent Domination versus Weighted Independent Domination

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Abstract

INDEPENDENT DOMINATION is one of the rare problems for which the complexity of weighted and unweighted versions is known to be different in some classes of graphs. Trying to better understand the gap between the two versions of the problem, in the present paper we prove two complexity results. First, we extend NP-hardness of the weighted version in a certain class to the unweighted case. Second, we strengthen polynomial-time solvability of the unweighted version in the class of $P_2 + P_3$-free graphs to the weighted case. This result is tight in the sense that both versions are NP-hard in the class of $P_3 + P_3$-free graphs, i.e. $P_3 + P_3$ is a minimal graph forbidding of which produces an NP-hard case for both versions of the problem.

1 Introduction

INDEPENDENT DOMINATION is the problem of finding in a graph an inclusionwise maximal independent set of minimum cardinality. This is one of the hardest algorithmic problems and it remains difficult in very restricted graph classes. In particular, it is NP-hard in the class of so called sat-graphs, where the problem is equivalent to SATISFIABILITY [9].

The weighted version of the problem (abbreviated WID) deals with vertex-weighted graphs and asks to find an inclusionwise maximal independent set of minimum total weight. This version is provenly harder, as it remains NP-hard even for chordal graphs [2], where INDEPENDENT DOMINATION can be solved in polynomial time [4].

Recently, we have strengthened the NP-hardness of WID in the class of chordal graphs by showing that the problem is NP-hard in the class of (C_4, Sun_3)-free sat-graphs [5] (see Figure 1 for Sun_3). This is a proper subclass of the intersection of sat-graphs and chordal graphs, and therefore INDEPENDENT DOMINATION admits a polynomial-time solution in this class. Trying to better understand the gap between the two versions of the problem, we slightly extend this class by replacing C_4 with a *domino* in the set of forbidden graphs and show that with this extension the complexity of INDEPENDENT DOMINATION jumps from polynomial-time solvability to NP-hardness.

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On the other hand, INDEPENDENT DOMINATION can be solved in polynomial time in the class of $P_2 + P_3$-free graphs [7], where the complexity of WID is an open question. We answer this question by showing that WID also admits a polynomial-time solution for $P_2 + P_3$-free graphs. This result is tight in the sense that both versions are NP-hard in the class of P_3+P_3-free graphs, because this class contains all sat-graphs. In other words, our result shows that $P_3 + P_3$ is a minimal graph forbidding of which produces an NP-hard case for both versions of the problem.

The organization of the paper is as follows. In the rest of this section, we introduce basic terminology and notation. In Section 2, we prove our NP-hardness results. Section 3 is devoted to the polynomial-time result.

All graphs in this paper are finite, undirected, without loops and multiple edges. Let $G = (V, E)$ be a graph and let v be a vertex in G. By $N_G(v)$ we denote the *neighborhood* of v, i.e. the set of vertices of G adjacent to v. For a set $S \subseteq V$, we denote by $A_G(S)$ the *antineighborhood* of S, i.e. the set of vertices of G that have no neighbours in S. Also by $G[S]$ we denote the subgraph of G induced by S and by $G - S$ the subgraph $G[V \setminus S]$. If S consists of a single element, say $S = \{v\}$, we write $A_G(v)$ and $G - v$, omitting the braces.

As usual, P_n, C_n and K_n denote a chordless path, a chordless cycle and a complete graph on n vertices, respectively. Given two graphs G and H, we denote by $G + H$ the disjoint union of G and H, and by mG the disjoint union of m copies of G.

A set $S \subseteq V$ of pairwise non-adjacent vertices is called an *independent set*. We say that S is a *maximal* independent set if it is not properly contained in any other independent set. Clearly, if S is a maximal independent set, then it is also dominating, i.e. every vertex not in S has a neighbour in S. This is why maximal independent sets also known as independent dominating sets.

The *independent domination number* of G is the size of a minimum independent dominating set in G; we denote it by $id(G)$. If G is a vertex weighted graph with a weight function w, then $id_w(G)$ stands for the minimum weight of an independent dominating set in G.

2 An NP-hardness result

As we mentioned in the introduction, both versions of the problem are NP-hard for general graphs and remain difficult under various restrictions. For instance, both of them are NP-hard for graphs of bounded vertex degree, of large girth [1], line graphs [8], chordal bipartite[1] graphs [3], etc. The weighted version of the problem is also NP-hard in the class of (C_4, Sun_3)-free sat graphs [5], where the unweighted version is polynomial-time solvable, because this is a subclass of chordal graphs. By replacing C_4 in the set of forbidden induced subgraphs with a domino (see Figure 1) we obtain a larger class. Therefore, WID remains NP-hard in this extension. However, the complexity of ID in this class is an open question. In this section, we answer this question by showing that INDEPENDENT DOMINATION is NP-hard in the class of $(domino, Sun_3)$-free sat-graphs

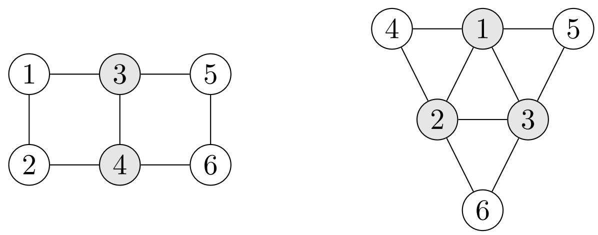
A graph G is called a *sat-graph* if there exists a partition $A \cup B = V(G)$ such that

1. A is a clique (possibly, $A = \emptyset$);
2. $G[B]$ is a 1-regular graph, also known as an induced matching (possibly, $B = \emptyset$);

[1]This is a bipartite analog of chordal graphs and is not the intersection of chordal and bipartite graphs.

3. there are no triangles (a, b, b'), where $a \in A$ and $b, b' \in B$.

We shall refer to the pair (A, B) as a *sat-partition* of G. The NP-hardness of INDEPENDENT DOMINATION (and hence of WID) in the class of sat-graphs was proved in [9], where the author showed that INDEPENDENT DOMINATION restricted to the class of sat-graphs is equivalent to SATISFIABILITY.



(a) *domino* (b) Sun_3

Figure 1: Graphs (a) *domino*, and (b) Sun_3

Observe that no cycle with at least 5 vertices is an induced subgraph of a sat-graph, while each of *domino* and Sun_3 (Figure 1) can be an induced subgraph of a sat-graph. Moreover, it is easy to see that each of *domino* and Sun_3 is partitioned by a sat-partition in a unique way.

Observation 1. *Let G be a sat-graph with a sat-partition (A, B).*

(1) If G contains a domino (see Figure 1 (a)) as an induced subgraph, then $3, 4 \in A$ and $1, 2, 5, 6 \in B$.

(2) If G contains Sun_3 (see Figure 1 (b)) as an induced subgraph, then $1, 2, 3 \in A$ and $4, 5, 6 \in B$.

Since each of *domino* and Sun_3 can be an induced subgraph of a sat-graph, the class of $(domino, Sun_3)$-free sat-graphs form a proper subclass of sat-graphs. To prove the NP-hardness of ID in this class, we reduce the problem from sat-graphs.

Let G be a sat-graph with a sat-partition (A, B), and let $a \in A$ and $b \in B$ be two adjacent vertices. The *transformation* $\gamma(a, b)$ of G consists in

1. adding a new vertex v to A, and connecting it to all other vertices in A;
2. adding new vertices x and y to B and connecting them by an edge;
3. removing the edge (a, b);
4. adding edges (v, b), (v, x) and (a, y).

We say that v is an α-new vertex, x, y are β-new vertices, and the edge (x, y) is a β-new edge. Vertices in A that are not α-new, and vertices and edges in B that are not β-new will be called α-old and β-old, respectively.

Notice that transformation $\gamma(a, b)$ does not change the property of being a sat-graph. In what follows, we show that this transformation increases the independent domination number by exactly one. To this end, we first make the following observation.

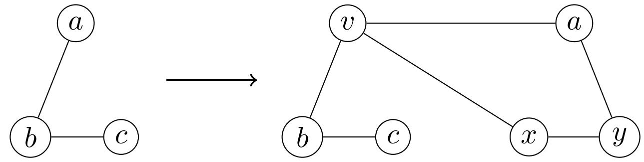


Figure 2: Transformation $\gamma(a, b)$

Observation 2. *Let G be a sat-graph with a sat-partition (A, B) and let s be the number of edges in $G[B]$. Then $s \leq id(G) \leq s + 1$.*

Lemma 1. *Let G be a sat-graph and G' be the sat-graph obtained from G by applying transformation $\gamma(a, b)$. Then $id(G') = id(G) + 1$.*

Proof. Let (A, B) be a sat-partition of G and let s be the number of edges in $G[B]$. By Observation 2 we have $s \leq id(G) \leq s + 1$, and $s + 1 \leq id(G') \leq s + 2$. Therefore, to prove the lemma it is sufficient to show that $id(G) = s$ if and only if $id(G') = s + 1$.

First, assume that $id(G) = s$ and D is a minimum independent dominating set in G. If $b \in D$, then $D' = D \cup \{y\}$ is an independent dominating set in G'. Moreover, D' is minimum, since $id(G') \geq s+1$ and $|D'| = s+1$. Let now $id(G') = s+1$ and D' be a minimum independent dominating set in G'. If $y \in D'$, then $b \in D'$ (to dominate v) and therefore $D = D' \setminus \{y\}$ is a minimum independent dominating set in G. If $x \in D'$, then there exists $c \in D' \setminus \{x, y\}$ that dominates a, and hence $D = D' \setminus \{x\}$ is a minimum independent dominating set in G. □

Let G be a sat-graph with a sat-partition (A, B). By G^* we denote the sat-graph obtained from G by successive applications of transformation $\gamma(a, b)$ to every edge (a, b) with $a \in A$ and $b \in B$. Denote $\mathcal{S}^* = \{G^* \mid G \text{ is a sat-graph with a sat-partition } (A, B)\}$. It follows from Lemma 1 that INDEPENDENT DOMINATION in sat-graphs polynomially reduces to the same problem in the subclass $\mathcal{S}^*$ of sat-graphs. Now we show that graphs in $\mathcal{S}^*$ are $(domino, Sun_3)$-free. First, we note some useful properties of a graph G in $\mathcal{S}^*$:

(1) No α-old vertex is adjacent to a β-old edge.

(2) Every α-new vertex is adjacent to exactly one β-new edge, and to exactly one β-old edge.

(3) In every β-new edge one of its vertices is adjacent to exactly one vertex in A and this vertex is α-new, and the other vertex is adjacent to exactly one vertex in A and this vertex is α-old.

Lemma 2. *Let G be a graph in $\mathcal{S}^*$. Then G is $(domino, Sun_3)$-free.*

Proof. Let H be a sat-graph with a sat-partition (A, B) such that $G = H^*$. Let also (A', B') be a sat-partition of G such that $A \subseteq A'$, $B \subseteq B'$, and all α-new vertices belong to A' and all β-new vertices belong to B'.

Suppose to the contrary that G contains an induced *domino* with vertex set $\{1, 2, 3, 4, 5, 6\}$ as shown in Figure 1 (a). By Observation 1 (1) we have $3, 4 \in A'$ and $1, 2, 5, 6 \in B'$. Assume that 3 is an α-new vertex, then by Property (2) of H^* one of the edges $(1, 2)$ and $(5, 6)$ is β-new and the other one is β-old. Without loss of generality, let $(1, 2)$ be a new edge. Now by Property (1) vertex 4 is also α-new. But then two new vertices $3, 4$ are adjacent to the new edge $(1, 2)$, which contradicts Property (3). Assume now that both vertices 3 and 4 are α-old. Then by Property (1) edges $(1, 2)$ and $(5, 6)$ are β-new, which again contradicts Property (3). This contradiction shows that G is *domino*-free.

Suppose now that G contains the graph Sun_3 induced by vertices $1, 2, 3, 4, 5, 6$ as shown in Figure 1 (b). Then by Observation 1 we have $1, 2, 3 \in A'$ and $4, 5, 6 \in B'$. Let us consider vertex 1. From Properties (1) and (2) we conclude that at least one of its neighbours 4 and 5, say 4, is a β-new vertex. But this is impossible, since by Property (3) every β-new vertex has at most one neighbour in A. This contradiction shows that G is Sun_3-free and completes the proof of the lemma. □

Now the main result of this section follows from Lemmas 1 and 2 and from the fact that INDEPENDENT DOMINATION is NP-hard in sat-graphs.

Theorem 1. INDEPENDENT DOMINATION *is NP-hard in the class of* $(domino, Sun_3)$*-free sat-graphs.*

3 A polynomial-time result

In this section, we study the weighted version of the problem restricted to the class of P_2+P_3-free graphs. This special case of the problem was also studied in [6]. However, a solution presented in [6] turned out to be erroneous. The error was partially corrected in [7], where a solution for the unweighted version of the problem was presented. Now we extend it to WEIGHTED INDEPENDENT DOMINATION. The initial stage of the solution, up to Claim 2, is the same for both versions of the problem, and we start by briefly outlining the main ideas of this stage and stating the necessary results from [7].

The first step in the solution is Algorithm GENERATION, which generates a family $\mathcal{S}$ of vertex subsets of the input graph. To each set $H \in \mathcal{S}$ the algorithm assigns a *special vertex* $\mu(H)$ which does not belong to H. Given a graph G with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$, we denote by G_i the subgraph of G induced by vertices $v_1, v_2, \dots, v_i$. Also, for a subset $U \subseteq V(G)$, we denote by U_0 the set of isolated vertices in $G[U]$. Without loss of generality we assume that the input graph G has no isolated vertices, because all these vertices must belong to any optimal solution. We also assume that the vertices of G are ordered so that v_1 is adjacent to v_2. In the beginning of the algorithm, the family $\mathcal{S}$ includes only the set $\{v_1\}$ and the special vertex assigned to this set is $\mu(\{v_1\}) = v_2$.

Lemma 3. [7] *Let G be a $P_2 + P_3$-free graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and let $\mathcal{S}$ be the family of subsets of $V(G)$ produced by Algorithm* GENERATION*. Then*

(i) for each $H \in \mathcal{S}$, we have $H - H_0 \subseteq A_G(\{v_i, \mu(H)\})$, where i is the iteration at which H was created;

(ii) for each maximal independent set I in G, there is a set $H \in \mathcal{S}$ such that $I \subseteq H$.

Lemma 4. [7] *For a graph G with n vertices, Algorithm* GENERATION *runs in time $O(n^5)$ and the family $\mathcal{S}$ produced by this algorithm contains $O(n^3)$ subsets of $V(G)$.*

By Lemma 3, a solution to WID in a $P_2 + P_3$-free graph G belongs to one of the subsets $H \in \mathcal{S}$ generated by the algorithm. Therefore, the problem can be solved by checking, for each

Algorithm 1 GENERATION

Input: A graph G with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$.
Output: A family $\mathcal{S}$ of subsets of $V(G)$.

1: $\mathcal{S} := \{\{v_1\}\}$ with $\mu(\{v_1\}) = v_2$
2: **for** $i = 2$ to n **do**
3:     [*Extension of some members of* $\mathcal{S}$]
4:     **for each** $H \in \mathcal{S}$ **do**
5:         **if** $\left[N_{G_i}(v_i) \cap H = \emptyset\right]$ or $\left[(N_{G_i}(v_i) \cap H_0 = \emptyset) \text{ and } (\mu(H), v_i) \notin E(G)\right]$ **then**
6:             $H := H \cup \{v_i\}$
7:     [*Addition of new members of* $\mathcal{S}$]
8:     **for each** pair of vertices v_i, u inducing in G_i a P_2 **do**
9:         $H := \{v_i\} \cup A_{G_i}(\{v_i, u\})$
10:         $\mu(H) := u$
11:         $\mathcal{S} := \mathcal{S} \cup \{H\}$
12:     **for each** triple of vertices v_i, u, w inducing in G_i a P_3 with u being the centre **do**
13:         $H := \{v_i, w\} \cup A_{G_i}(\{v_i, u, w\})$
14:         $\mu(H) := u$
15:         $\mathcal{S} := \mathcal{S} \cup \{H\}$
16: **return** $\mathcal{S}$

$H \in \mathcal{S}$, all maximal independent sets in $G[H]$, verifying which of them are also maximal in G and then choosing a set of minimum weight. For sets $H \in \mathcal{S}$ created in Line 13 of the algorithm, this is an easy task, because all these sets are independent.

Now let H be created in Line 9 of the algorithm and denote by A the subset of H obtained by removing isolated vertices from $G[H]$. By Lemma 3, $A \subseteq A_G(\{v_i, \mu(H)\})$, and hence, $G[A]$ is P_3-free, i.e. each connected component of $G[A]$ is a clique with at least two vertices. We denote the components (cliques) of $G[A]$ by $Q_1, \ldots, Q_t$.

The condition in Line 5 of the algorithm implies that every maximal independent set in $G[H]$ dominates all vertices of G except possibly some neighbours of $\mu(H)$. We denote the set of neighbours of $\mu(H)$, which are not dominated by at least one maximal independent set in $G[H]$, by W. With this notion and an argument similar to the one used in [7] (see Proposition 4 of [7]), maximal independent sets in $G[H]$, which also are maximal in G, can be characterized as follows.

Claim 1. *Let $H \in \mathcal{S}$ be a set created in Line 9 of Algorithm* GENERATION*. An independent set I which is maximal in $G[H]$ also is maximal in G if and only if $I \cap A$ dominates W.*

Our next claim characterizes independent sets $I \cap A$ that dominate W.

Claim 2. *For $i = 1, \ldots, t$, let $z_i \in Q_i$ and $Y_i = W \cap A_G(z_i)$. The set $I = \{z_1, \ldots, z_t\}$ dominates W if and only if $Y_1 \cap \ldots \cap Y_t = \emptyset$.*

Proof. Denote $\overline{Y}_i = W - Y_i$. Then I dominates W if and only if $\overline{Y}_1 \cup \ldots \cup \overline{Y}_t = W$. By De Morgan's law, this holds if and only if $Y_1 \cap \ldots \cap Y_t = \emptyset$. □

It remains to show that the problem of finding a maximal independent set in $G[H]$ of minimum weight that dominates W can be solved in polynomial time. To prove this, we first observe that if a vertex $x \in W$ is non-adjacent to two vertices $a, b \in A$, then a and b are non-adjacent to each other, since otherwise $a, b, x, \mu(H), v_i$ induce a $P_2 + P_3$. Therefore, if $Q = \{q_1, \ldots, q_p\}$ is a component (clique) in $G[A]$ and $W_i = W \cap A_G(q_i)$, then $\{W_1, \ldots, W_p\}$ is a partition of W and we denote this partition by $\mathcal{P}(Q)$.

We recall that $Q_1, \ldots, Q_t$ are the components of $G[A]$. The corresponding t partitions $\mathcal{P}(Q_1), \ldots, \mathcal{P}(Q_t)$ of W can be described as a rooted tree T as follows. Let W be the root of T and let v be a node of distance $0 \le i < t$ from the root representing a subset $U \subseteq W$. If $U = \emptyset$, then v is a leaf. Otherwise, v has p children representing the sets $U \cap W_1, \ldots, U \cap W_p$, where $W_1, \ldots, W_p$ are the subsets in the partition $\mathcal{P}(Q_{i+1})$. In other words, if a node v is located at level i of the tree, then it represents the set $Y_1 \cap \ldots \cap Y_i$ for a unique collection of sets $Y_1 \in \mathcal{P}(Q_1), \ldots, Y_i \in \mathcal{P}(Q_i)$ defined by the unique path connecting v to the root of T. Every set $Y_i \in \mathcal{P}(Q_i)$ corresponds to a unique vertex $z_i \in Q_i$ with $Y_i = W \cap A_G(z_i)$ and we denote $A_v = \{z_1, \ldots, z_i\} \cup Q_{i+1} \cup \ldots \cup Q_t$.

From Claims 1 and 2, we derive the following conclusion.

Claim 3. *If I is a maximal independent set in $G[H]$, which is also maximal in G, then $I \cap A \subset A_v$ for a node v of T representing the empty set.*

Each A_v induces a P_3-free graph and hence WID can be solved for such a graph in time $O(n)$. It remains to estimate the number of nodes representing the empty sets (empty nodes, for short). To this end, we first slightly modify the tree T as follows. If a node of T has several children representing the empty set, then we ignore all of them except one, corresponding to a vertex $z_i \in Q_i$ of minimum weight. Clearly, with this modification we do not lose any potential solution. Now, every node of T has at most one child representing the empty set, and hence the number of empty nodes does not exceed the number of non-empty nodes, i.e nodes representing non-empty sets.

Claim 4. *The number of non-empty nodes in T is at most n^2.*

Proof. For each i, the subsets of W represented by the nodes of T at level i form a partition of W. Therefore, for each i, the number of non-empty nodes at level i is at most n. The number of levels (cliques) is at most n, and hence, the total number of non-empty nodes is at most n^2. □

Theorem 2. *The* WEIGHTED INDEPENDENT DOMINATION *problem can be solved for n-vertex $P_3 + P_2$-free graphs in $O(n^6)$ time.*

Proof. To solve the problem for a $P_3 + P_2$-free graph G, first, we run Algorithm GENERATION which produces $O(n^3)$ subsets of G. For each of these subsets, which is not an independent set, we construct a tree T and for each of the $O(n^2)$ empty nodes of T, we solve the problem in $O(n)$ time. Therefore, the total time is $O(n^6)$. □

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