Counting the Number of Crossings in Geometric Graphs

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September 4, 2020

Abstract

A geometric graph is a graph whose vertices are points in general position in the plane and its edges are straight line segments joining these points. In this paper we give an $O(n^2 \log n)$ time algorithm to compute the number of pairs of edges that cross in a geometric graph on nvertices. For layered graphs and convex geometric graphs the algorithm takes $O(n^2)$ time.

1 Introduction

A geometric graph is a graph whose vertices are points in general position in the plane; and its edges are straight line segments joining these points. A pair of edges of a geometric graph cross if they intersect in their interior; the number of crossings of a geometric graph is the number of pairs of its edges that cross. Let G := (V, E) be a geometric graph on n vertices; and let H be a graph. If G and H are isomorphic as graphs then we say that G is a rectilinear drawing of H. The rectilinear crossing number of H is the minimum number of crossings that appear in all of its rectilinear drawings. We abuse notation and use $\overline{\operatorname{cr}}(H)$ and $\overline{\operatorname{cr}}(G)$ to denote the rectilinear crossing number of Crossings of G, respectively.

Computing the rectilinear crossing number of the complete graph K_n on n vertices is an important and well known problem in Combinatorial Geometry. The current best bounds on $\overline{\operatorname{cr}}(K_n)$ are

$$0.379972\binom{n}{4} < \overline{\operatorname{cr}}(K_n) < 0.380473\binom{n}{4} + \Theta(n^3).$$

The lower bound is due to Ábrego, Fernández-Merchant, Leaños and Salazar [1]. The upper bound is due to Fabila-Monroy and López [6]. In an upcoming paper, Aichholzer, Duque, Fabila-Monroy,

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This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 734922.

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García-Quintero and Hidalgo-Toscano [2] have further improved the upper bound to

$$\overline{\operatorname{cr}}(K_n) < 0.3804491 \binom{n}{4} + \Theta(n^3).$$

For more information on crossing numbers (rectilinear or other variants) we recommend the survey by Schaefer [8].

A notable property of the improvements of [6, 2] on the upper bound of $\overline{cr}(K_n)$ is that they rely on finding rectilinear drawings of K_n with few crossings, for some specific values of n; this is done via heuristics that take a rectilinear drawing of K_n and move its vertices in various ways with the aim of decreasing the number of crossings. In this approach it is instrumental that the computation of the number of crossings is done as fast as possible. In this paper we present an algorithm to compute $\overline{cr}(G)$ in $O(n^2 \log n)$ time. We hope that our algorithm will pave the way for finding new upper bounds on the rectilinear crossing number of various classes of graphs.

The current best algorithm for counting the number of intersections among a set of m line segments in the plane runs in $O(m^{4/3} \log^{1/3} m)$ time; it was given by Chazelle [4]. This provides an $O(|E|^{4/3} \log^{1/3} |E|)$ time algorithm for computing $\overline{cr}(G)$; this running time can be as high as $\Theta(n^{8/3} \log^{1/3} n)$ when G has a quadratic number of edges. The running time can be improved for some classes of geometric graphs: when G is a complete geometric graph, Rote, Woeginger, Zhu, and Wang [7] give an $O(n^2)$ time algorithm for computing $\overline{cr}(G)$; and Waddle and Malhotra [9] give an $O(|E|\log |E|)$ algorithm for computing $\overline{cr}(G)$ when G is a bilayered graph. For layered graphs and convex geometric graphs a slight modification of our algorithm runs in $O(n^2)$ time.

2 The algorithm

In what follows let G := (V, E) be a geometric graph on n vertices. We make some general position assumptions: no two vertices of G have the same x-coordinate nor the same y-coordinate; and no two edges of G are parallel. In [6], the authors give an $O(n^2)$ time algorithm for computing $\overline{\operatorname{cr}}(G)$ when G is a complete geometric graph; the authors define "patterns" on the set of vertices of G; these patterns can be computed in $O(n^2)$ time and $\overline{\operatorname{cr}}(G)$ depends on the number of these patterns. We follow a similar approach.

Let p and q be two points in the plane. Let pq be the ray with apex p and that passes through q; let pq be the ray with apex p, parallel to pq, and with opposite direction to pq. Let (u, v, e) be a triple where u and v are a pair of adjacent vertices of G, and e is an edge of G not incident to u or v. We say that (u, v, e) is a pattern of

- **Type** A) if \overrightarrow{uv} intersects e; and
- **Type** B) if \overleftarrow{uv} intersects e.

See Figure 1.

Consider a pair of non-incident edges of G. This pair of edges provides patterns of type A and type B according to the following three cases. If the pair of edges is crossing then they provide exactly four patterns of type A and zero patterns of type B; if the pair of edges is non-crossing and the line containing one of them intersects the interior of the other edge, then they provide exactly one pattern of type A and one pattern of type B; and if the pair of edges is non-crossing and the intersection point of the two supporting lines of these edges lies outside the union of these edges,



Figure 1: (u, v, e_1) , (v, u, e_1) , and (u, v, e_2) are of type A; (v, u, e_2) is of type B; and neither the endpoints of e_1 and the edge e_2 , nor the endpoints of e_2 and the edge e_1 produce patterns of type A or type B.

then they do not provide patterns of type A nor patterns of type B. See Figure 1. Let A(G) and B(G) be the number of patterns of type A and type B defined by G, respectively; we have the following.

Proposition 1.

$$\overline{\operatorname{cr}}(G) = \frac{A(G) - B(G)}{4}.$$

Before proceeding we give some definitions. Let u and v be vertices of G. Let B(u, v) and A(u, v) be the number of edges of G that intersect \overline{uv} and \overline{uv} , respectively. Note that

$$A(G) = \sum_{u \in V} \sum_{v \in N(u)} A(u, v) \text{ and } B(G) = \sum_{u \in V} \sum_{v \in N(u)} B(u, v).$$

Let $N_{\text{LEFT}}(v, uv)$ be set of neighbors of v to the left of the directed line from u to v; and let $N_{\text{RIGHT}}(v, uv)$ be set of neighbors of v to the right of the directed line from u to v. See Figure 2. Let \overleftarrow{u} and \overrightarrow{u} be the open horizontal rays with apex u that go left and right, respectively. Finally, let E(u) be the set of edges of G that are incident to u. Thus, $|\overleftarrow{u} \cap E|$ and $|\overrightarrow{u} \cap E|$ are equal to the number of edges of G that cross \overleftarrow{u} and \overrightarrow{u} , respectively; and $|\overrightarrow{u} \cap E(v)|$ and $|\overrightarrow{u} \cap E(v)|$ are equal to the number of edges of G incident to v that cross \overleftarrow{u} and \overrightarrow{u} , respectively.



Figure 2: An illustration of $N_{\text{LEFT}}(v, uv) = 5$ and $N_{\text{RIGHT}}(v, uv) = 4$

Consider the following well known duality transform. Let p := (m, b) be a point; the dual of p is the line $p^* : y = mx - b$. Let $\ell : y = mx + b$ be a line; the dual of ℓ is the point $\ell^* := (m, -b)$. This

duality transform preserves various geometric relationships between points and lines. In particular, it preserves point-line incidences. It also preserves above-below relationships between lines and points: if point p is above (resp. below) line ℓ , then point ℓ^* is above (resp. below) line p^* . We compute the line arrangement \mathcal{L} of the dual lines of the points in V. This can be done in $O(n^2)$ time; see for example Chapter 8 of [5]. Afterwards, we compute A(G) and B(G) in the following four steps.

Lemma 2 (Step 1). In $O(n^2)$ time we can compute, for all $u \in V$, the counterclockwise order of the vertices in $V \setminus \{u\}$ by angle around u.

Proof. Let u be a vertex of G and let v_1, \ldots, v_{n-1} be the vertices $V \setminus \{u\}$. For $1 \le i \le n-1$ be ℓ_i be the line through u and v_i . Since all the lines ℓ_i pass through u, all the points ℓ_i^* are contained in the line u^* . Moreover, the order by slope of the lines $\ell_1, \ldots, \ell_{n-1}$ coincides with the order of the points $\ell_1^*, \ldots, \ell_{n-1}^*$ from left to right along the line u^* . This order can be obtained from \mathcal{L} in O(n) time. From the slope order of the lines ℓ_i it is straightforward to compute in O(n) time the counterclockwise order of the vertices in $V \setminus \{u\}$ by angle around u.

Lemma 3 (Step 2). In $O(n^2)$ time we can compute, for all pairs of vertices $u, v \in G$, $|N_{\text{LEFT}}(v, uv)|$ and $|N_{\text{RIGHT}}(v, uv)|$.

Proof. Let *u* be a vertex of *G*. Let v_1, \ldots, v_{n-1} be the vertices of $V \setminus \{u\}$ ordered so that the points $v_1^* \cap u^*, v_2^* \cap u^*, \ldots, v_{n-1}^* \cap u^*$ are encountered in this order when traversing the line u^* from left to right. If the *x*-coordinate of *u* is less than the *x*-coordinate of v_i , then the set $N_{\text{LEFT}}(v_i, uv_i)$ (resp. $N_{\text{RIGHT}}(v_i, uv_i)$) corresponds to the neighbors v_j of v_i such that the lines v_j^* are below (resp. above) the point $v_i^* \cap u^*$. If the *x*-coordinate of *u* is greater than the *x*-coordinate of v_i , then the set $N_{\text{LEFT}}(v_i, uv_i)$ (resp. $N_{\text{RIGHT}}(v_i, uv_i)$) corresponds to the neighbors v_j of v_i such that the lines v_j^* are below (resp. above) the point $v_i^* \cap u^*$. We compute $N_{\text{LEFT}}(v_1, uv_1)$ and $N_{\text{RIGHT}}(v_1, uv_1)$ in linear time. Afterwards, iteratively for $j = 2, \ldots, n - 1$ from $N_{\text{LEFT}}(v_{j-1}, uv_{j-1})$ and $N_{\text{RIGHT}}(v_{j-1}, uv_{j-1})$ we compute $N_{\text{LEFT}}(v_j, uv_j)$ and $N_{\text{RIGHT}}(v_j, uv_j)$ in constant time, respectively. We store the cardinality of these sets. Thus, in O(n) time we compute $|N_{\text{LEFT}}(v_j, uv_j)|$ and $|N_{\text{RIGHT}}(v_j, uv_j)|$ for all v_j . By iterating over all vertices $u \in G$ the result follows.



Figure 3: The information stored on T at line 1 of HORIZONTALRAYCROSSINGS for i = 2

Lemma 4 (Step 3). In $O(n^2 \log n)$ time, we can compute the sets of values

$$\{|\overleftarrow{v} \cap E| : v \in V\} \text{ and } \{|\overrightarrow{v} \cap E| : v \in V\}.$$

Proof. We show how to compute $\{|\overrightarrow{v} \cap E| : v \in V\}$; the computation of $\{|\overleftarrow{v} \cap E| : v \in V\}$ is analogous. We begin by sorting the vertices of G by decreasing y-coordinate. For every vertex $u \in V$ we compute the set of values

$$\{ | \vec{v} \cap E(u) | : v \in V \text{ and } v \text{ is below } u \},\$$

as follows.

Let $v_1, \ldots v_m$ be the vertices of G that lie below u in decreasing order of their y-coordinate. In O(n) time we construct a binary tree, T, of minimum height and whose leaves are v_1, \ldots, v_m . The left to right order of v_1, \ldots, v_m in T coincides with their counterclockwise order around u starting from the first vertex encountered when rotating \overleftarrow{u} counterclockwise around u. We mark all the v_i that are adjacent to u. At each interior node w of T we store in two fields MARKEDLEFT(w) and MARKEDRIGHT(w), the number of marked leaves in the left and right subtrees of w, respectively. By proceeding by increasing height, these values can be computed in O(n) time for every vertex of T.

We execute the procedure HORIZONTALRAYCROSSINGS(u) whose pseudocode is described below. Suppose that we are at the *i*-th execution of the *for* of line 1. We have that: a vertex v_j is marked if and only if $j \ge i$ and v_j is adjacent to u; and for every node $w \in T$, MARKEDLEFT(w) and MARKEDRIGHT(w), store the number of currently marked leaves in the left and right subtrees of w, respectively. Lines 4, 11 and 15 ensure that these invariants are kept at every execution of line 1. In this iteration we compute (and store) the value of $|\vec{v}_i \cap E(u)|$. Note that an edge uv_j intersects \vec{v}_i if and only if v_j comes after v_i in the counterclockwise order around u and j > i. The first condition is equivalent to v_j being to the right of v_i in T; this happens if and only if there exists a common ancestor w in T of v_i and v_j , for which v_i is in the left subtree of w and v_j is in the right subtree of w; see Figure 3. The condition of j being greater than i is equivalent to v_j being marked. We count the number of nodes v_j that satisfy both conditions by traversing the path from v_i to the root. We begin by unmarking v_i ; at each node w in this path we update the fields MARKEDLEFT(w) and MARKEDRIGHT(w); if v_i is in the left subtree of w we also update our counting. Having reached the root we store the value of $|\vec{v}_i \cap E(u)|$ in line 20. Since T has height $O(\log n)$ the execution of HORIZONTALRAYCROSSINGS(u) takes $O(n \log n)$ time. Since

$$\overrightarrow{v} \cap E| = \sum_{u \text{ is above } v} |\overrightarrow{v} \cap E(u)|,$$

the result follows.

Lemma 5 (Step 3). In $O(n^2)$ time we can compute A(u, v) and B(u, v) for all pairs of vertices $u, v \in V$.

Proof. For every $u \in V$ we proceed as follows. Let v_1, \ldots, v_{n-1} be the vertices of $V \setminus \{u\}$, in the order as they are encountered by rotating a horizontal line counterclockwise around u. This order can be obtained from \mathcal{L} in O(n) time. Suppose that the angle from \overrightarrow{u} to $\overrightarrow{uv_1}$ is less than π . Then the only vertex of $V \setminus \{u\}$ in the closed wedge bounded by the rays \overrightarrow{u} and $\overrightarrow{uv_1}$ is v_1 ; and there are no vertices of $V \setminus \{u\}$ in the closed wedge bounded by the rays \overleftarrow{u} and $\overleftarrow{uv_1}$. This implies that

$$A(u, v_1) = |\overrightarrow{u} \cap E| - |N_{\text{RIGHT}}(v_1, uv_1)| \text{ and } B(u, v_1) = |\overleftarrow{u} \cap E|.$$

Procedure HorizontalRayCrossings(u)

1 for $i \leftarrow 1$ to m do $cr \leftarrow 0$; $\mathbf{2}$ if v_i is a neighbor of u then 3 Unmark v_i ; $\mathbf{4}$ end $\mathbf{5}$ 6 $w \leftarrow v_i;$ while $w \neq T$.ROOT do $\mathbf{7}$ if w is the left child of w_{PARENT} then 8 $cr \leftarrow cr + \text{MARKEDRIGHT}(w_{\text{PARENT}});$ 9 if v_i is a neighbor of v then 10 MARKEDLEFT $(w_{\text{PARENT}}) \leftarrow \text{MARKEDLEFT}(w_{\text{PARENT}}) - 1;$ 11 end $\mathbf{12}$ else $\mathbf{13}$ if v_i is a neighbor of u then $\mathbf{14}$ MARKEDRIGHT $(w_{\text{PARENT}}) \leftarrow \text{MARKEDRIGHT}(w_{\text{PARENT}}) - 1;$ $\mathbf{15}$ end 16 end $\mathbf{17}$ $\mathbf{18}$ $w \leftarrow w_{\text{PARENT}};$ end 19 $|\overrightarrow{v_i} \cap E(u)| \leftarrow cr$ $\mathbf{20}$ 21 end

Suppose that the angle from \overrightarrow{u} to $\overrightarrow{uv_1}$ is greater than π . Then there are no vertices $V \setminus \{u\}$ in the closed wedge bounded by the rays \overrightarrow{u} and $\overleftarrow{uv_1}$; and the only vertex of $V \setminus \{u\}$ in the closed wedge bounded by the rays \overleftarrow{u} and $\overrightarrow{uv_1}$ is v_1 . This implies that

$$A(u, v_1) = |\overleftarrow{u} \cap E| - |N_{\text{RIGHT}}(v_1, uv_1)|$$
 and $B(u, v_1) = |\overrightarrow{u} \cap E|$

In general, for i = 2, ..., n - 1, we have the following. Suppose that the angle from $\overrightarrow{uv_{i-1}}$ to $\overrightarrow{uv_i}$ is less than π . Then the only vertices of $V \setminus \{u\}$ in the closed wedge bounded by the rays $\overrightarrow{uv_{i-1}}$ and $\overrightarrow{uv_i}$ are v_{i-1} and v_i ; and there are no vertices of $V \setminus \{u\}$ in the closed wedge bounded by the rays $\overrightarrow{uv_{i-1}}$ and $\overrightarrow{uv_i}$. This implies that

$$A(u, v_i) = A(u, v_{i-1}) + |N_{\text{LEFT}}(v_{i-1}, uv_{i-1})| - |N_{\text{RIGHT}}(v_i, uv_i)|$$

and

$$B(u, v_i) = B(u, v_{i-1}).$$

Suppose that the angle from $\overrightarrow{uv_{i-1}}$ to $\overrightarrow{uv_i}$ is greater than π . Then the only vertex of $V \setminus \{u\}$ in the closed wedge bounded by the rays $\overrightarrow{uv_{i-1}}$ and $\overleftarrow{uv_i}$ is v_{i-1} ; and the only vertex of $V \setminus \{u\}$ in the closed wedge bounded by the rays $\overleftarrow{uv_{i-1}}$ and $\overrightarrow{uv_i}$ is v_{i-1} . This implies that

$$A(u, v_i) = B(u, v_{i-1}) - |N_{\text{RIGHT}}(v_i, uv_i)|$$

and

$$B(u, v_i) = A(u, v_{i-1}) + |N_{\text{LEFT}}(v_{i-1}, uv_{i-1})|$$

See Figure 4. Thus, having computed $A(u, v_{i-1})$ and $B(u, v_{i-1})$ we can compute $A(u, v_i)$ and $B(u, v_i)$ in constant time. The result follows.



Figure 4: The two cases in computing $A(u, v_i)$ and $B(u, v_i)$ in Step 4

Summarizing, we have the following result.

Theorem 6. $\overline{\operatorname{cr}}(G)$ can be computed $O(n^2 \log n)$ time.

3 Faster algorithms for special classes of graphs

Of the four steps of our algorithm, only Step 3 takes superquadratic time. Moreover, the numbers $|\vec{u} \cap E|$ and $|\overleftarrow{u} \cap E|$ are only used as a starting point for computing A(u, v) and B(u, v) in Lemma 5. If we can compute in O(n) time, for each vertex u of G, the number of edges of G that cross two open rays with apex u and in parallel but opposite directions then we can compute $\overline{\mathrm{cr}}(G)$ in $O(n^2)$ time. We pose the following conjecture.

Conjecture 1. For every geometric graph G on n vertices, $\overline{\operatorname{cr}}(G)$ can be computed in $O(n^2)$ time.

To finalize the paper we mention two instances for which is the case.

• Convex Geometric Graphs

A convex geometric graph is a convex geometric graph whose vertices are in convex position. Let G = (V, E) be a convex geometric graph on n vertices. For every u in V, let \overrightarrow{u} and \overleftarrow{u} be two parallel open rays with apex u, in opposite directions and that do not intersect the convex hull of V. These rays can be found in constant time by first computing the convex hull of V (which takes $O(n \log n)$ time). We have that $|\overrightarrow{u} \cap E|$ and $|\overleftarrow{u} \cap E|$ are both equal to zero in this case and $\overrightarrow{cr}(G)$ can be computed in $O(n^2)$ time.

• Layered Graphs

A layered graph is a geometric graph whose vertex set is partitioned into sets L_1, \ldots, L_r called *layers* such that the following holds.

- The vertices in layer L_i have the same y-coordinate y_i^1 ;
- $y_1 < y_2 < \cdots < y_r;$
- vertices in layer L_i are only adjacent to vertices in layers L_{i-1} and L_{i+1} .

Suppose that G is a layered graph with layers L_1, \ldots, L_r . Let G_i be the subgraph of G induced by L_i and L_{i+1} . Note that G_i can be regarded as a convex geometric graph. Since $\overline{\operatorname{cr}}(G) = \sum_{i=1}^{r-1} \overline{\operatorname{cr}}(G_i)$, $\overline{\operatorname{cr}}(G)$ can be computed in $O(n^2)$ time. Problem 33 of [3] asks whether the number of crossings of layered graphs can be computed in time $o(|E| \log |V|)$. For layered graphs with $\omega(n^2/\log n)$ edges we provide an affirmative answer to the question posed in [3].

Acknowledgments. This work was initiated at the 2nd Reunion of Optimization, Mathematics, and Algorithms (ROMA 2018), held in Mexico city, 2018. We thank the anonymous reviewer whose comments substantially improved the presentation of this paper.

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 $^{^1\}mathrm{Although}$ this violates our general position assumption, a simple perturbation argument suffices to apply our algorithm.

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