ON THE COMPLEXITY OF CO-SECURE DOMINATING SET PROBLEM

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ABSTRACT. A set $D \subseteq V$ of a graph G = (V, E) is a dominating set of G if every vertex $v \in V \setminus D$ is adjacent to at least one vertex in D. A set $S \subseteq V$ is a co-secure dominating set (CSDS) of a graph G if S is a dominating set of G and for each vertex $u \in S$ there exists a vertex $v \in V \setminus S$ such that $uv \in E$ and $(S \setminus \{u\}) \cup \{v\}$ is a dominating set of G. The minimum cardinality of a co-secure dominating set of G is the co-secure domination number and it is denoted by $\gamma_{cs}(G)$. Given a graph G = (V, E), the minimum co-secure dominating set problem (MIN CO-SECURE DOM) is to find a co-secure dominating set of minimum cardinality. In this paper, we strengthen the inapproximability result of MIN CO-SECURE DOM for general graphs by showing that this problem can not be approximated within a factor of $(1 - \epsilon) \ln |V|$ for perfect elimination bipartite graphs and star convex bipartite graphs unless P=NP. On the positive side, we show that MIN CO-SECURE DOM can be approximated within a factor of $O(\ln |V|)$ for any graph G with $\delta(G) \geq 2$. For 3-regular and $4\mbox{-regular graphs},$ we show that Min Co-secure Dom is approximable within a factor of $\frac{8}{3}$ and $\frac{10}{3}$, respectively. Furthermore, we prove that MIN CO-SECURE DOM is APX-complete for 3-regular graphs.

Domination, Co-secure domination, Approximation algorithm, Inapproximability, APX-complete

1. INTRODUCTION

Let G = (V, E) be a finite, simple, and undirected graph with vertex set V and edge set E. The graph G considered in this paper is without isolated vertices. A set $D \subseteq V$ is said to be a *dominating set* of G if every vertex v in $V \setminus D$ has an adjacent vertex u in D. The minimum cardinality among all dominating sets of G is the *domination number* of G, and it is denoted by $\gamma(G)$. Given a graph G, in minimum dominating set problem (MIN DOM), it is required to find a dominating set D of minimum cardinality. MIN DOM and its variations are studied extensively because of their real-life applications and theoretical applications. Detailed survey and results are available in [7, 8, 9].

A dominating set $S \subseteq V$ of G = (V, E) is called a secure dominating set of G, if S is a dominating set of G and for every $u \in V \setminus S$ there exists a vertex $v \in S$, adjacent to u such that $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of G. This important variation of domination was introduced by Cockayne et al. [4]. The problem of finding a minimum cardinality secure dominating set of a graph is known as the Minimum Secure Domination Problem. This problem and its many variants have been extensively studied by several researchers [1, 4, 11, 12, 15, 18, 20].

A set $S \subseteq V$ is a co-secure dominating set (CSDS) of a graph G if S is a dominating set and for each vertex $u \in S$ there exists a vertex $v \in V \setminus S$ such that

 $uv \in E$ and $(S \setminus \{u\}) \cup \{v\}$ is a dominating set of G. The minimum cardinality of a co-secure dominating set of G is the co-secure domination number and it is denoted by $\gamma_{cs}(G)$. Given a graph G = (V, E), in minimum co-secure dominating set problem (MIN CO-SECURE DOM), it is required to find a co-secure dominating set S of minimum cardinality. MIN CO-SECURE DOM was introduced by Arumugam et al. [2], where they showed that the decision version of MIN CO-SECURE DOM is NP-complete for bipartite, chordal, and planar graphs. They also determined the co-secure domination number for some families of the standard graph classes such as paths, cycles, wheels, and complete t-partite graphs. Some bounds on the co-secure domination number for certain families of graphs were given by Joseph et al. [10]. Manjusha et al. [14] characterized the Mycielski graphs with the co-secure domination number 2 or 3 and gave a sharp upper bound for $\gamma_{cs}(\mu(G))$, where $\mu(G)$ is the Mycielski of a graph G. Later Zou et al. [22] proved that the co-secure domination number of proper interval graphs can be computed in linear time. In [13], it is proved that MIN CO-SECURE DOM is NP-hard to approximate within a factor of $(1 - \varepsilon) \ln |V|$ for any $\varepsilon > 0$, and it is APX-complete for graphs with maximum degree 4.

In this paper, we extend the algorithmic study of MIN CO-SECURE DOM by using certain properties of minimum double dominating set under some assumptions. The main contributions of the paper are summarised below.

- We prove that MIN CO-SECURE DOM can not be approximated within a factor of $(1-\varepsilon)\ln|V|$ for perfect elimination bipartite graphs and star convex bipartite graphs unless P=NP. This improves the result due to Kusum and Pandey [13].
- We propose an approximation algorithm for MIN CO-SECURE DOM for general graphs G with $\delta(G) \geq 2$, within a factor of $O(\ln |V|)$. In terms of maximum degree Δ , it can be approximated within a factor of $2+2(\ln \Delta+2)$.
- For 3-regular and 4-regular graphs, we show that MIN CO-SECURE DOM is approximable within a factor of ⁸/₃ and ¹⁰/₃, respectively.
 We also prove that MIN CO-SECURE DOM is APX-complete for 3-regular
- graphs.

2. Preliminaries

In this section, we give some pertinent definitions and state some preliminary results. Let G = (V, E) be a finite, simple, and undirected graph with no isolated vertex. The open neighborhood of a vertex v in G is $N(v) = \{u \in V \mid uv \in E\}$ and the closed neighborhood is $N[v] = \{v\} \cup N(v)$. The degree of a vertex v is |N(v)|and is denoted by d(v). If d(v) = 1 then v is called a *pendant vertex* in G. The minimum degree and maximum degree of G are denoted by δ and Δ , respectively. For $D \subseteq V$, G[D] denotes the subgraph induced by D. We use the notation [k] for $\{1, 2, \dots, k\}$. Given $S \subseteq V$ and $v \in S$, a vertex $u \in V \setminus S$ is an S-external private neighbor (S-epn) of v if $N(u) \cap S = \{v\}$. The set of all S-epn of v is denoted by EPN(v, S). Some other notations and terminology which are not introduced here can be found in [21].

A bipartite graph is a graph G = (V, E) whose vertices can be partitioned into two disjoint sets X and Y such that every edge has one endpoint in X and other in Y. We denote a bipartite graph with vertex bi-partition X and Y of V as G = (X, Y, E).

The edge $uv \in E$ is a bi-simplicial edge if $N(u) \cup N(v)$ induces a complete bipartite subgraph in G. Let $\sigma = [e_1, e_2, \cdots, e_k]$ be an ordering of pairwise non-adjacent edges of G. With respect to this ordering σ , we define P_i , $i \in [k]$ as the set of end vertices of the edges $\{e_1, e_2, \ldots, e_i\}$, and let $P_0 = \emptyset$. The ordering σ is said to be a *perfect elimination ordering* for G if $G[(X \cup Y) \setminus P_k]$ has no edge and each edge e_i is bi-simplicial in $G[(X \cup Y) \setminus P_{i-1}]$. A graph G = (V, E) is said to be a *perfect elimination bipartite graph* if and only if it admits a perfect elimination ordering [6]. A bipartite graph G = (X, Y, E) is called a *star convex bipartite graph* if a star graph $H = (X, E_X)$ can be defined such that for every vertex $y \in Y$, N(y) induces a connected subgraph in H.

3. Approximation Algorithms

In this section, we propose an approximation algorithm for MIN CO-SECURE DOM whose approximation ratio is a logarithmic factor of the number of vertices of the input graph. To obtain the approximation ratio of MIN CO-SECURE DOM, we require the approximation ratio of the minimum double dominating set problem (MIN DOUBLE DOM). Given a graph G = (V, E), in MIN DOUBLE DOM, the aim is to find a vertex set $D \subseteq V$ of minimum cardinality such that $|N(v) \cap D| \ge 2$, for all $v \in V \setminus D$. We shall denote $\gamma_2(G)$ as the cardinality of a minimum double dominating set in G. We will use the following proposition and a few lemmas to analyze our approximation algorithms' performance.

Proposition 3.1. ([2]) Let S be a CSDS of G. A vertex $v \in V \setminus S$ replaces $u \in S$ if and only if $v \in N(u)$ and $EPN(u, S) \subseteq N[v]$.

Lemma 3.1. If G is a connected graph with at least 3 vertices then every minimal double dominating set D_2 of G is a proper subset of V. Moreover, if $\delta(G) \ge 2$ then every minimal double dominating set D_2 is a co-secure dominating set of G.

Proof. Suppose there exists a minimal double dominating set D_2 of G such that $|D_2| = |V|$. Since $|V| \ge 3$ and G is connected, there exists a vertex $v \in V$ with $d(v) \ge 2$. Now, $D_2 \setminus \{v\}$ is a double dominating set of G contradicting the minimality of D_2 .

Let D_2 be a minimal double dominating set of G. From the minimality of D_2 , it follows that every vertex $u \in D_2$ has at least one neighbor in $V \setminus D_2$. Suppose there exists a vertex $p \in D_2$ such that $N(p) \subseteq D_2$. Then $D_2 \setminus \{p\}$ is also a double dominating set (as $d(p) \ge 2$.) This contradicts the minimality of D_2 .

Let u be any vertex in D_2 and v be its neighbor not in D_2 . Next, we show that $S = (D_2 \setminus \{u\}) \cup \{v\}$ is a dominating set of G. Suppose not, then there exists a vertex $w \in V \setminus S$ such that no vertex of S dominates w. D_2 is a dominating set of G implies that $N(w) \cap D_2 = \{u\}$. This contradicts the fact that D_2 is a double dominating set of G.

From the above arguments, it follows that D_2 is a co-secure dominating set of G.

In the next lemma, we prove bounds on $\gamma_2(G)$ which we will use in designing approximation algorithms for MIN CO-SECURE DOM.

Lemma 3.2. For every graph G with $\delta(G) \geq 2$, $\gamma_{cs}(G) \leq \gamma_2(G) \leq 2\gamma_{cs}(G)$. Moreover, these bounds are tight. Proof. $\gamma_{cs}(G) \leq \gamma_2(G)$ holds as every minimal double dominating set of G is also a CSDS of G (by Lemma 3.1). Next we will prove that $\gamma_2(G) \leq 2\gamma_{cs}(G)$. Let D be a γ_{cs} set of G. Let $D' = \{x \in D \mid EPN(x, D) \neq \emptyset\}$, and $D'' = D \setminus D'$. Let $A = \bigcup_{x \in D'} EPN(x, D)$. Then, every vertex $v \in (V \setminus \{D \cup A\})$ has at least two neighbors in D''. By Proposition 3.1, for every vertex $x \in S$ there exists at least one vertex $x^* \in V \setminus S$ and $x^* \in EPN(x, S)$ such that $d_G(x^*) \geq |EPN(x, S)|$. Let $A' \subseteq A$ such that A' contains exactly one vertex x^* of each EPN(x, D) for every $x \in D'$. Thus, |A'| = |D'|. Note that, every vertex in $A \setminus A'$ has at least two neighbors in $D' \cup A'$. Let B' be the smallest subset of $(V \setminus D) \setminus A'$ that dominates D''. Since every vertex of D'' has $EPN(x, D'') = \emptyset$, we obtain $|B'| \leq |D''|$. Thus, $D \cup A' \cup B'$ is a double dominating set of G. Hence, $\gamma_2(G) \leq |D| + |A'| + |B'| \leq |D| + |D'| + |D''| = 2|D| = 2\gamma_{cs}(G)$.

These two inequalities are tight for the graphs $K_{2,2}$ and K_n $(n \ge 3)$, respectively.

Theorem 3.1. MIN DOUBLE DOM can be approximated with an approximation ratio of $O(\ln |V|)$, where V is the vertex set of the input graph G. It can also be approximated within a factor of $1 + \ln(\triangle + 2)$, where \triangle is the maximum degree of G.

Proof. Given an instance G = (V, E) of MIN DOUBLE DOM, we construct a multiset multicover problem [19] as follows. We take V as the universe and for each vertex $v \in V$ we construct a multiset $S_v = N[v] \cup \{v\}$. In S_v , v is appearing twice whereas other elements appear exactly once. We set the requirement of each vertex $v \in V$ as 2. Minimum Multiset Multicover problem can be approximated within a factor of $O(\ln |V|)$ (also $1 + \ln(\triangle + 2)$) [19]. Therefore, MIN DOUBLE DOM can be approximated within a factor of $O(\ln |V|)$ (also $1 + \ln(\triangle + 2)$).

Next, we propose an algorithm (described in Algorithm 1) to compute an approximate solution of MIN CO-SECURE DOM. This algorithm computes a minimal double dominating set D_2 of the input graph G (with $\delta(G) \geq 2$) using the approximation algorithm described in Theorem 3.1 and returns it as a CSDS of G. By Lemma 3.1, D_2 is also a CSDS of G. It is easy to observe that Algorithm 1 runs in polynomial time.

Algorithm 1: Approx-CSD
Input: A graph $G = (V, E)$.
Output: A minimum CSDS of G .
begin
Compute a double dominating set D_2 of G (as described in Theorem
3.1);
$S = D_2;$
return S ;
\mathbf{end}

Theorem 3.2. MIN CO-SECURE DOM can be approximated within a factor of $O(\ln |V|)$, for graphs with $\delta(G) \geq 2$. It can also be approximated within a factor of $2 + 2\ln(\triangle + 2)$, where \triangle is the maximum degree of G.

Proof. Let S be the CSDS of G computed by the Algorithm 1. By Theorem 3.1, we have $|S| \leq O(\ln |V|)\gamma_2(G)$. Also, by Lemma 3.2 we have

$$|S| \le O(\ln |V|)\gamma_2(G) \le 2O(\ln |V|)\gamma_{cs}(G) = O(\ln |V|)\gamma_{cs}(G).$$

Similarly, it can be observed that $|S| \leq [2 + 2\ln(\triangle + 2)]\gamma_{cs}(G)$.

4. Lower bound on Approximation ratio

In this section, we obtain a lower bound on the approximation ratio of MIN CO-SECURE DOM for some subclasses of bipartite graphs. To obtain our lower bound, we establish an approximation preserving reduction from MIN DOM to MIN CO-SECURE DOM. We need the following lower bound result on MIN DOM.

Theorem 4.1. ([3, 5]) Unless P=NP, MIN DOM can not be approximated within a factor of $(1 - \varepsilon) \ln |V|$, for any $\varepsilon > 0$. Such a result holds for MIN DOM even when restricted to bipartite graphs.

By using this theorem, we will prove similar lower bound results for MIN CO-SECURE DOM for two subclasses of bipartite graphs, namely perfect elimination bipartite graphs and star convex bipartite graphs.

Theorem 4.2. Unless P=NP, MIN CO-SECURE DOM for a perfect elimination bipartite graph G = (V, E) can not be approximated within $(1 - \varepsilon) \ln |V|$, for any $\varepsilon > 0$.

Proof. Given a graph G = (V, E), an instance of MIN DOM, we construct a graph G' = (V', E'), an instance of MIN CO-SECURE DOM, as follows. Here we assume that $V = \{v_1, v_2, \ldots, v_n\}$. After making a copy of G, we introduce n new vertices a_1, a_2, \ldots, a_n and n edges $v_i a_i$, for $i \in [n]$. Then we introduce 6 vertices s, t, x, y, w, z and the edges st, xy, wz. Finally, we introduce the edge set $\{a_i v_i, v_i s, a_i x, a_i z \mid i \in [n]\}$. It is easy to observe that $V' = V \cup \{a_i \mid i \in [n]\} \cup \{x, y, z, w, s, t\}$ and $E' = E \cup \{a_i v_i, v_i s, a_i x, a_i z \mid i \in [n]\} \cup \{xy, zw, st\}$ and it is a polynomial time construction as |V'| = 2|V| + 6 and |E'| = |E| + 4|V| + 3. G' is a perfect elimination bipartite graph with the perfect elimination ordering $\{st, xy, zw, v_1 a_1, v_2 a_2, \cdots, v_n a_n\}$. For an illustration of this construction, we refer to Figure 1.

Claim 4.1. The graph G has a dominating set of cardinality at most k if and only if G' has a CSDS of cardinality at most k' = k + 3.

Proof. Let D be a minimal dominating set of G. It is easy to check that $S = D \cup \{x, z, s\}$ is a CSDS of G'. Thus, |S| = |D| + 3.

Conversely, let S be a minimal CSDS of G'. $S \cap \{x, y\} = \{x\}$ as y is the only degree 1 vertex adjacent to x. Similarly, $S \cap \{s, t\} = \{s\}$ and $S \cap \{w, z\} = \{z\}$. We will assume that S does not contain any a_i vertex. This is because, each a_i vertex is dominated by at least two vertices x and z, and if $a_i \in S$ then we will replace the vertex a_i with v_i in S. Now, we define $D = S \cap V$. If D is a dominating set of G then we are done. Otherwise, there exists a vertex v_k which is not dominated by any vertex of D. Now, v_k is dominated only by $s \in S$ and $(S \setminus \{s\}) \cup \{v\}$ is not a dominating set, for every $v \in (N_{G'}(s) \setminus S)$. This is a contradiction. Hence, D is a dominating set of G with |S| = |D| + 3.

Let us assume that there exists some (fixed) $\varepsilon > 0$ such that MIN CO-SECURE DOM for perfect elimination bipartite graphs with |V'| vertices can be approximated



FIGURE 1. An illustration of the construction of G^\prime from G in the proof of Theorem 4.2

within a ratio of $\alpha = (1 - \varepsilon) \ln |V'|$ by a polynomial time algorithm A. Let l > 0 be a fixed integer with $l > \frac{1}{\varepsilon}$. By using algorithm A, we construct a polynomial time algorithm for MIN DOM as described in Algorithm 2.

Initially, if there is a minimum dominating set D of G with |D| < l, then it can be computed in polynomial time. Since the algorithm \mathbb{A} runs in polynomial time, the Algorithm 2 also runs in polynomial time. If the returned set D satisfies |D| < lthen D is a minimum dominating set of G and we are done.

Next, we will analyze the case when Algorithm 2 returned the set D with $|D| \ge l$. By Claim 4.1 we have $|S_o| = |D_o| + 3$, where D_o and S_o are minimum dominating set of G and minimum CSDS of G', respectively. Here $|D_o| \ge l$.

Algorithm 2: APPROX-DOM1Input: A graph G = (V, E).Output: A minimum dominating set D of G.beginif there is a minimum dominating set D of G with |D| < l then| return D;else| Construct the graph G' as described above;Compute a CSDS S in G' using \mathbb{A} ; $D = S \cap V$;return D;end

Now, $|D| \leq |S| - 3 < |S| \leq \alpha |S_o| = \alpha (|D_o| + 3) = \alpha (1 + \frac{3}{|D_o|}) |D_o| \leq \alpha (1 + \frac{3}{l}) |D_o|$. This implies that Algorithm 2 approximates MIN DOM within a ratio of $\alpha (1 + \frac{3}{l})$. Since $\frac{1}{l} < \varepsilon$

$$\alpha\left(1+\frac{3}{l}\right) \le (1-\varepsilon)(1+3\varepsilon)\ln|V'| = (1-\varepsilon')\ln|V|,$$

where $\varepsilon' = 3\varepsilon^2 + 2\varepsilon$ as $\ln |V'| = \ln(2|V| + 6) \approx \ln |V|$ for sufficiently large value of |V|.

Therefore, Algorithm 2 approximates MIN DOM within a ratio of $(1 - \varepsilon) \ln |V|$ for some $\varepsilon > 0$. This contradicts the lower bound result in Theorem 4.1.

Next, we prove the inapproximability of MIN CO-SECURE DOM in star convex bipartite graphs by using the Theorem 4.1.

Theorem 4.3. MIN CO-SECURE DOM for a star convex bipartite graph G = (V, E) can not be approximated within $(1 - \varepsilon) \ln |V|$ for any $\varepsilon > 0$, unless P=NP.

Proof. Given a bipartite graph G = (X, Y, E), as an instance of MIN DOM, we obtain a star convex bipartite graph G' = (X', Y', E') such that G has a dominating set of cardinality at most k if and only if G' has a CSDS of cardinality at most k' = k + 2. Now the construction of G' from G is as follows. After making a copy of G, we introduce four vertices x_0, x, y_0, y . Finally, we make every vertex of $X \cup \{x, x_0\}$ adjacent to y and every vertex of $Y \cup \{y, y_0\}$ adjacent to x. Now, $X' = \{X\} \cup \{x, x_0\}$, $Y' = \{Y\} \cup \{y, y_0\}$ and $E' = \{E\} \cup \{x_i y \mid x_i \in X\} \cup \{y_i x \mid y_i \in Y\} \cup \{x_0 y, xy, xy_0\}$. The new graph G' = (V', E') formed from G = (V, E) has |V'| = |V| + 4 and |E'| = |E| + n + 3, which can be constructed in polynomial time. It can be observed that G' is a star convex bipartite graph with the associated star graph which is shown in Figure 2.



FIGURE 2. An illustration of the construction of G' from G in the proof of Theorem 4.3

Claim 4.2. *G* has a dominating set of cardinality at most k if and only if the graph G' has a CSDS of cardinality at most k' = k + 2.

Proof. Suppose D is a minimal dominating set of G and let $S = D \cup \{x, y\}$. Clearly, S is a CSDS of G' with $|S| = |D| + 2 \leq k + 2$. Conversely, let S be a minimal dominating set of G'. Note that, $|S \cap \{x, y_0\}| = 1$, and similarly $|S \cap \{x_0, y\}| = 1$. If $x_0, y_0 \in S$, observe that $EPN(x_0, S) = y$ and $EPN(y_0, S) = x$. So, without loss of generality, assume $\{x, y\} \subseteq S$. Now, let $D = S \setminus \{x, y\}$. Now we show that D is a dominating set of G. If D is dominating set of G, then we are done. Otherwise, suppose D is not a dominating set of G. Then there exists at least one vertex $v_k \in V(G)$ which is not dominated by any vertex of D. Without loss of generality, assume $v_k \in X$, then v_k can only be dominated by $y \in Y'$. Since S is a CSDS of $G', (S \setminus \{y\}) \cup \{v_k\}$ is a dominating set of G', which is a contradiction. Thus, D is a dominating set of G of cardinality $|D| = |S| - 2 \leq k$. Therefore, G has a dominating set D of cardinality at most k if and only if G' has a CSDS of cardinality at most k' = k + 2. This completes the proof of this claim.

Presume that there exists some (fixed) $\varepsilon > 0$ such that MIN CO-SECURE DOM for star convex bipartite graphs having |V'| vertices can be approximated within a ratio of $\alpha = (1 - \varepsilon) \ln |V'|$ by using an algorithm A that runs in polynomial time. Let l > 0 be an integer. By using algorithm A, we construct a polynomial time algorithm Algorithm 3 for MIN DOM.

Firstly, if there is a minimum dominating set D of G with |D| < l, then it can be computed in polynomial time. Moreover, Algorithm 3 runs in polynomial time as \mathbb{A} runs in polynomial time. Note that, if the returned set D satisfies |D| < l then it is a minimum dominating set of G and we are done. Now, let us assume that the returned set D satisfies |D| > l.

Let D_o and S_o be a minimum dominating set of G and a minimum CSDS of G', respectively. Then $|D_o| \ge l$, and $|S_o| = |D_o| + 2$ by the above Claim 4.2. Now,

$$|D| \le |S| - 2 < |S| \le \alpha |S_o| = \alpha (|D_o| + 2) = \alpha \left(1 + \frac{2}{|D_o|}\right) |D_o| \le \alpha \left(1 + \frac{2}{l}\right) |D_o|.$$

Hence, Algorithm 3 approximates MIN DOM for given bipartite graph G = (X, Y, E) within the ratio $\alpha(1 + \frac{2}{l})$. Let l be the positive integer such that $\frac{1}{l} < \varepsilon$. Then

$$\alpha\left(1+\frac{2}{l}\right) \le (1-\varepsilon)(1+2\varepsilon)\ln|X'\cup Y'| = (1-\varepsilon')\ln|X\cup Y|,$$

where $\varepsilon' = 2\varepsilon^2 - \varepsilon$ as $\ln |X' \cup Y'| = \ln(|X \cup Y| + 4) \approx \ln |X \cup Y|$ for sufficiently large value of $|X \cup Y|$.

Therefore, Algorithm 3 approximates MIN DOM within a ratio of $(1-\varepsilon) \ln |X \cup Y|$ for some $\varepsilon > 0$. This contradicts the lower bound result in Theorem 4.1.

5. Complexity on bounded degree graphs

In this section, we show that MIN CO-SECURE DOM is APX-complete for 3-regular graphs. Note that the class APX is the set of all optimization problems which admit a *c*-approximation algorithm, where *c* is a constant. From Theorem 3.2 it follows that MIN CO-SECURE DOM can be approximated within a factor of 5.583 for graphs with maximum degree at most 4. We improve this approximation factor to $\frac{10}{3}$.

We first show that MIN CO-SECURE DOM for 3-regular graphs is approximable within a factor of $\frac{8}{3}$.

Algorithm 4: APPROX-CSD-3RG

Input: A 3-regular graph G = (V, E). Output: A CSDS S of G = (V, E). begin $W' = \emptyset$; while \exists an edge $uv \in E$ do $W' = W' \cup \{u, v\}$; Delete $N[u] \cup N[v]$ from G; end Let T be the remaining vertices; $W = W' \cup T$; $S = V \setminus W$; return S; end

Lemma 5.1. MIN CO-SECURE DOM is approximable within a factor of $\frac{8}{3}$ for 3-regular graphs.

Proof. Let S_o be a minimum CSDS of a 3-regular graph G = (V, E). A vertex $x \in S_o$ can co-securely dominate at most 3 vertices of $V \setminus S_o$. Therefore, $|V \setminus S_o| \leq 3|S_o|$. This implies that

$$(1) |S_o| \ge \frac{n}{4}$$

The set S of vertices returned by Algorithm 4 is a minimal double dominating set in G because each vertex in W' has exactly two neighbors in S. By Lemma 3.1, S is a CSDS of G.

Thus, $W = W' \cup T$. Let $|W' \cup S| = n_1 = n - |T|$. Now $|W'| \ge \frac{n_1}{3}$, since in the while loop, the algorithm has picked two vertices and simultaneously removed at most six vertices from the graph. Now,

$$|W| = |W'| + |T| \ge \frac{n_1}{3} + n - n_1 \ge n - \frac{2n}{3} = \frac{n}{3}.$$

Thus,

(2)
$$|S| = |V| - |W| \le n - \frac{n}{3} = \frac{2n}{3}$$

This yields the upper bound on the size of the CSDS returned. Combining equation (1) and equation (2), we obtain $\frac{|S|}{|S_0|} \leq \frac{8}{3}$, thereby proving the lemma.

Next, we design a constant factor approximation algorithm for MIN CO-SECURE DOM when the input graph is 4-regular.

Algorithm 5: Approx-CSD-4RG

Lemma 5.2. MIN CO-SECURE DOM for 4-regular graphs can be approximated within a factor of $\frac{10}{2}$.

Proof. Given a 4-regular graph G, in polynomial time Algorithm 5 computes a vertex set W such that the degree of each vertex in G[W] is at most 2.

Claim 5.1. S is a CSDS of G.

Proof. By Lemma 3.1, it is enough to show that S is a minimal double dominating set of G.

S is a double dominating set of G as each vertex in W has at least two neighbors in S. Suppose S is not a minimal double dominating set of G. Then there must be a vertex $v \in S$ such that $S \setminus \{v\}$ is a double dominating set of G. This implies that v must have at least two neighbors in S. If $v \in S$ is adjacent to a vertex of degree two in G[W] then $S \setminus \{v\}$ is not a double dominating set of G (because G is 4-regular). This implies that v must be adjacent to at least one end-vertex of an induced path P in G[W']. This contradicts the maximality of P.

Following the proof of Lemma 5.1, it can be proved that $|S_o| \ge \frac{n}{5}$. Let W' be the set of vertices of degree 2 in G[W] and $Q = W \setminus W'$. By setting $n_1 = n - |Q|$ and following the proof of Lemma 5.1, it can be proved that $|W'| \ge \frac{n_1}{3}$. This implies

that
$$|W| \ge \frac{n}{3}$$
 and $|S| \le \frac{2n}{3}$. Therefore, $\frac{|S|}{|S_o|} \le \frac{10}{3}$.

Before we prove that MIN CO-SECURE DOM is APX-complete for 3-regular graphs, we need some terminology and results regarding the partial monopoly set.

Definition 5.1 ([17]). (MIN PARTIAL MONOPOLY PROBLEM) Given a graph G = (V, E), partial monopoly problem is to find a set $M \subseteq V$ of minimum cardinality such that for each $v \in V \setminus M$, $|M \cap N[v]| \ge \frac{1}{2}|N[v]|$.

It is known that for 3-regular graphs MIN PARTIAL MONOPOLY PROBLEM is APX-complete [16]. It is easy to observe the following lemma:

Lemma 5.3. Let G be a 3-regular graph. A partial monopoly set M of G is a double dominating set of G and vice versa.

Lemma 5.4. Let G be a 3-regular graph and $S \subseteq V$ be a minimal CSDS of G. In polynomial time one can construct a double dominating set $S' \subseteq V$ with $|S'| \leq 2|S|$.

Proof. Let S be a minimal CSDS of G. Define S_1 be the set of vertices $v \in S$ such that $EPN(v, S) \neq \emptyset$, and $S_2 = S \setminus S_1$. Now let $A = \bigcup_{v \in S} EPN(v, S)$. Note that every vertex in A has exactly one neighbor in S_1 and every vertex in $(V \setminus S) \setminus A$ has at least two neighbors in S_2 . By Proposition 3.1, for every vertex $x \in S$ there exists at least one vertex $x^* \in V \setminus S$ and $x^* \in EPN(x, S)$ such that $d_G(x^*) \geq |EPN(x, S)|$. Let us define a new set $A' \subseteq A$, such that A' contains that one vertex x^* of each EPN(x, S) for every $x \in S_1$. Thus, $|A'| = |S_1|$. Let $S' = S \cup A'$. Now every vertex in $V \setminus S'$ has at least two neighbors in S'. Hence S' is a double dominating set of G with cardinality $|S| + |A'| = |S| + |S_1| \leq 2|S|$. □

Now, we will prove that MIN CO-SECURE DOM is APX-complete for 3-regular graphs by establishing a reduction from MIN PARTIAL MONOPOLY PROBLEM for 3-regular graphs.

Theorem 5.1. MIN CO-SECURE DOM is APX-complete for 3-regular graphs.

Proof. Because of Lemma 5.1, it is enough to establish a polynomial time approximation ratio preserving reduction from MIN PARTIAL MONOPOLY PROBLEM for 3-regular graphs to MIN CO-SECURE DOM for 3-regular graphs.

Given a 3-regular graph G = (V, E), an instance of MIN PARTIAL MONOPOLY PROBLEM, we take the same graph G as an instance of MIN CO-SECURE DOM. Let M_o be a minimum partial monopoly set of G and S_o be a minimum CSDS of G. Then $|M_o| = \gamma_2(G)$ (by Lemma 5.3). Also, we have $|S_o| \leq |M_o|$, by Lemma 3.2. Given a minimal CSDS S of G, we can construct a partial monopoly set $M \subseteq V$ with $|M| \leq 2|S|$ (from Lemma 5.4 and 5.3) Therefore, $\frac{|M|}{|M_o|} \leq 2\frac{|S|}{|S_o|}$. Hence, MIN CO-SECURE DOM is APX-complete for 3-regular graphs.

6. Conclusion

In this paper, we prove that MIN CO-SECURE DOM is hard to approximate within a factor smaller than $\ln |V|$ for perfect elimination bipartite graphs and star convex bipartite graphs. On the positive side, we have proposed a $O(\ln |V|)$ approximation algorithm for MIN CO-SECURE DOM for any graph. Apart from these, we have shown that for 3-regular graphs and 4-regular graphs MIN CO-SECURE DOM admits a $\frac{8}{3}$ and $\frac{10}{3}$ factor approximation algorithms, respectively. It would be interesting to design a better approximation algorithm for 3-regular graphs. We prove that it is APX-complete for 3-regular graphs. We conjecture that it is APX-hard for 3-regular bipartite graphs.

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