# ON THE COMPLEXITY OF CO-SECURE DOMINATING SET PROBLEM 

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#### Abstract

A set $D \subseteq V$ of a graph $G=(V, E)$ is a dominating set of $G$ if every vertex $v \in V \backslash D$ is adjacent to at least one vertex in $D$. A set $S \subseteq V$ is a co-secure dominating set (CSDS) of a graph $G$ if $S$ is a dominating set of $G$ and for each vertex $u \in S$ there exists a vertex $v \in V \backslash S$ such that $u v \in E$ and $(S \backslash\{u\}) \cup\{v\}$ is a dominating set of $G$. The minimum cardinality of a co-secure dominating set of $G$ is the co-secure domination number and it is denoted by $\gamma_{c s}(G)$. Given a graph $G=(V, E)$, the minimum co-secure dominating set problem (Min Co-secure Dom) is to find a co-secure dominating set of minimum cardinality. In this paper, we strengthen the inapproximability result of Min Co-secure Dom for general graphs by showing that this problem can not be approximated within a factor of $(1-\epsilon) \ln |V|$ for perfect elimination bipartite graphs and star convex bipartite graphs unless $\mathrm{P}=\mathrm{NP}$. On the positive side, we show that Min Co-secure Dom can be approximated within a factor of $O(\ln |V|)$ for any graph $G$ with $\delta(G) \geq 2$. For 3-regular and 4-regular graphs, we show that Min Co-secure Dom is approximable within a factor of $\frac{8}{3}$ and $\frac{10}{3}$, respectively. Furthermore, we prove that Min Co-secure Dom is APX-complete for 3-regular graphs. Domination, Co-secure domination, Approximation algorithm, Inapproximability, APX-complete


## 1. Introduction

Let $G=(V, E)$ be a finite, simple, and undirected graph with vertex set $V$ and edge set $E$. The graph $G$ considered in this paper is without isolated vertices. A set $D \subseteq V$ is said to be a dominating set of $G$ if every vertex $v$ in $V \backslash D$ has an adjacent vertex $u$ in $D$. The minimum cardinality among all dominating sets of $G$ is the domination number of $G$, and it is denoted by $\gamma(G)$. Given a graph $G$, in minimum dominating set problem (Min Dom), it is required to find a dominating set $D$ of minimum cardinality. Min Dom and its variations are studied extensively because of their real-life applications and theoretical applications. Detailed survey and results are available in $[7,8,9]$.

A dominating set $S \subseteq V$ of $G=(V, E)$ is called a secure dominating set of $G$, if $S$ is a dominating set of $G$ and for every $u \in V \backslash S$ there exists a vertex $v \in S$, adjacent to $u$ such that $(S \backslash\{v\}) \cup\{u\}$ is a dominating set of $G$. This important variation of domination was introduced by Cockayne et al. [4]. The problem of finding a minimum cardinality secure dominating set of a graph is known as the Minimum Secure Domination Problem. This problem and its many variants have been extensively studied by several researchers [1, 4, 11, 12, 15, 18, 20].

A set $S \subseteq V$ is a co-secure dominating set (CSDS) of a graph $G$ if $S$ is a dominating set and for each vertex $u \in S$ there exists a vertex $v \in V \backslash S$ such that
$u v \in E$ and $(S \backslash\{u\}) \cup\{v\}$ is a dominating set of $G$. The minimum cardinality of a co-secure dominating set of $G$ is the co-secure domination number and it is denoted by $\gamma_{c s}(G)$. Given a graph $G=(V, E)$, in minimum co-secure dominating set problem (Min Co-secure Dom), it is required to find a co-secure dominating set $S$ of minimum cardinality. Min Co-Secure Dom was introduced by Arumugam et al. [2], where they showed that the decision version of Min Co-SECure Dom is NP-complete for bipartite, chordal, and planar graphs. They also determined the co-secure domination number for some families of the standard graph classes such as paths, cycles, wheels, and complete $t$-partite graphs. Some bounds on the co-secure domination number for certain families of graphs were given by Joseph et al. [10]. Manjusha et al. [14] characterized the Mycielski graphs with the co-secure domination number 2 or 3 and gave a sharp upper bound for $\gamma_{c s}(\mu(G))$, where $\mu(G)$ is the Mycielski of a graph $G$. Later Zou et al.[22] proved that the co-secure domination number of proper interval graphs can be computed in linear time. In [13], it is proved that Min Co-SEcure Dom is NP-hard to approximate within a factor of $(1-\varepsilon) \ln |V|$ for any $\varepsilon>0$, and it is APX-complete for graphs with maximum degree 4.

In this paper, we extend the algorithmic study of Min Co-SEcure Dom by using certain properties of minimum double dominating set under some assumptions. The main contributions of the paper are summarised below.

- We prove that Min Co-Secure Dom can not be approximated within a factor of $(1-\varepsilon) \ln |V|$ for perfect elimination bipartite graphs and star convex bipartite graphs unless $\mathrm{P}=\mathrm{NP}$. This improves the result due to Kusum and Pandey [13].
- We propose an approximation algorithm for Min Co-SECURE Dom for general graphs $G$ with $\delta(G) \geq 2$, within a factor of $O(\ln |V|)$. In terms of maximum degree $\Delta$, it can be approximated within a factor of $2+2(\ln \Delta+2)$.
- For 3-regular and 4-regular graphs, we show that Min Co-secure Dom is approximable within a factor of $\frac{8}{3}$ and $\frac{10}{3}$, respectively.
- We also prove that Min Co-secure Dom is APX-complete for 3-regular graphs.


## 2. Preliminaries

In this section, we give some pertinent definitions and state some preliminary results. Let $G=(V, E)$ be a finite, simple, and undirected graph with no isolated vertex. The open neighborhood of a vertex $v$ in $G$ is $N(v)=\{u \in V \mid u v \in E\}$ and the closed neighborhood is $N[v]=\{v\} \cup N(v)$. The degree of a vertex $v$ is $|N(v)|$ and is denoted by $d(v)$. If $d(v)=1$ then $v$ is called a pendant vertex in $G$. The minimum degree and maximum degree of $G$ are denoted by $\delta$ and $\Delta$, respectively. For $D \subseteq V, G[D]$ denotes the subgraph induced by $D$. We use the notation $[k]$ for $\{1,2, \cdots, k\}$. Given $S \subseteq V$ and $v \in S$, a vertex $u \in V \backslash S$ is an $S$-external private neighbor ( $S$-epn) of $v$ if $N(u) \cap S=\{v\}$. The set of all $S$-epn of $v$ is denoted by $E P N(v, S)$. Some other notations and terminology which are not introduced here can be found in [21].

A bipartite graph is a graph $G=(V, E)$ whose vertices can be partitioned into two disjoint sets $X$ and $Y$ such that every edge has one endpoint in $X$ and other in $Y$. We denote a bipartite graph with vertex bi-partition $X$ and $Y$ of $V$ as $G=(X, Y, E)$.

The edge $u v \in E$ is a bi-simplicial edge if $N(u) \cup N(v)$ induces a complete bipartite subgraph in $G$. Let $\sigma=\left[e_{1}, e_{2}, \cdots, e_{k}\right]$ be an ordering of pairwise non-adjacent edges of $G$. With respect to this ordering $\sigma$, we define $P_{i}, i \in[k]$ as the set of end vertices of the edges $\left\{e_{1}, e_{2}, \ldots, e_{i}\right\}$, and let $P_{0}=\emptyset$. The ordering $\sigma$ is said to be a perfect elimination ordering for $G$ if $G\left[(X \cup Y) \backslash P_{k}\right]$ has no edge and each edge $e_{i}$ is bi-simplicial in $G\left[(X \cup Y) \backslash P_{i-1}\right]$. A graph $G=(V, E)$ is said to be a perfect elimination bipartite graph if and only if it admits a perfect elimination ordering [6]. A bipartite graph $G=(X, Y, E)$ is called a star convex bipartite graph if a star graph $H=\left(X, E_{X}\right)$ can be defined such that for every vertex $y \in Y, N(y)$ induces a connected subgraph in $H$.

## 3. Approximation Algorithms

In this section, we propose an approximation algorithm for Min Co-SECURE Dom whose approximation ratio is a logarithmic factor of the number of vertices of the input graph. To obtain the approximation ratio of Min Co-secure Dom, we require the approximation ratio of the minimum double dominating set problem (Min Double Dom). Given a graph $G=(V, E)$, in Min Double Dom, the aim is to find a vertex set $D \subseteq V$ of minimum cardinality such that $|N(v) \cap D| \geq 2$, for all $v \in V \backslash D$. We shall denote $\gamma_{2}(G)$ as the cardinality of a minimum double dominating set in $G$. We will use the following proposition and a few lemmas to analyze our approximation algorithms' performance.

Proposition 3.1. ([2]) Let $S$ be a CSDS of $G$. A vertex $v \in V \backslash S$ replaces $u \in S$ if and only if $v \in N(u)$ and $E P N(u, S) \subseteq N[v]$.

Lemma 3.1. If $G$ is a connected graph with at least 3 vertices then every minimal double dominating set $D_{2}$ of $G$ is a proper subset of $V$. Moreover, if $\delta(G) \geq 2$ then every minimal double dominating set $D_{2}$ is a co-secure dominating set of $G$.

Proof. Suppose there exists a minimal double dominating set $D_{2}$ of $G$ such that $\left|D_{2}\right|=|V|$. Since $|V| \geq 3$ and $G$ is connected, there exists a vertex $v \in V$ with $d(v) \geq 2$. Now, $D_{2} \backslash\{v\}$ is a double dominating set of $G$ contradicting the minimality of $D_{2}$.

Let $D_{2}$ be a minimal double dominating set of $G$. From the minimality of $D_{2}$, it follows that every vertex $u \in D_{2}$ has at least one neighbor in $V \backslash D_{2}$. Suppose there exists a vertex $p \in D_{2}$ such that $N(p) \subseteq D_{2}$. Then $D_{2} \backslash\{p\}$ is also a double dominating set (as $d(p) \geq 2$.) This contradicts the minimality of $D_{2}$.

Let $u$ be any vertex in $D_{2}$ and $v$ be its neighbor not in $D_{2}$. Next, we show that $S=\left(D_{2} \backslash\{u\}\right) \cup\{v\}$ is a dominating set of $G$. Suppose not, then there exists a vertex $w \in V \backslash S$ such that no vertex of $S$ dominates $w . D_{2}$ is a dominating set of $G$ implies that $N(w) \cap D_{2}=\{u\}$. This contradicts the fact that $D_{2}$ is a double dominating set of $G$.

From the above arguments, it follows that $D_{2}$ is a co-secure dominating set of $G$.

In the next lemma, we prove bounds on $\gamma_{2}(G)$ which we will use in designing approximation algorithms for Min Co-secure Dom.

Lemma 3.2. For every graph $G$ with $\delta(G) \geq 2, \gamma_{c s}(G) \leq \gamma_{2}(G) \leq 2 \gamma_{c s}(G)$. Moreover, these bounds are tight.

Proof. $\gamma_{c s}(G) \leq \gamma_{2}(G)$ holds as every minimal double dominating set of $G$ is also a CSDS of $G$ (by Lemma 3.1). Next we will prove that $\gamma_{2}(G) \leq 2 \gamma_{c s}(G)$. Let $D$ be a $\gamma_{c s}$ set of $G$. Let $D^{\prime}=\{x \in D \mid E P N(x, D) \neq \emptyset\}$, and $D^{\prime \prime}=D \backslash D^{\prime}$. Let $A=\bigcup_{x \in D^{\prime}} E P N(x, D)$. Then, every vertex $v \in(V \backslash\{D \cup A\})$ has at least two neighbors in $D^{\prime \prime}$. By Proposition 3.1, for every vertex $x \in S$ there exists at least one vertex $x^{*} \in V \backslash S$ and $x^{*} \in E P N(x, S)$ such that $d_{G}\left(x^{*}\right) \geq|E P N(x, S)|$. Let $A^{\prime} \subseteq A$ such that $A^{\prime}$ contains exactly one vertex $x^{*}$ of each $E P N(x, D)$ for every $x \in D^{\prime}$. Thus, $\left|A^{\prime}\right|=\left|D^{\prime}\right|$. Note that, every vertex in $A \backslash A^{\prime}$ has at least two neighbors in $D^{\prime} \cup A^{\prime}$. Let $B^{\prime}$ be the smallest subset of $(V \backslash D) \backslash A^{\prime}$ that dominates $D^{\prime \prime}$. Since every vertex of $D^{\prime \prime}$ has $E P N\left(x, D^{\prime \prime}\right)=\emptyset$, we obtain $\left|B^{\prime}\right| \leq\left|D^{\prime \prime}\right|$. Thus, $D \cup A^{\prime} \cup B^{\prime}$ is a double dominating set of $G$. Hence, $\gamma_{2}(G) \leq|D|+\left|A^{\prime}\right|+\left|B^{\prime}\right| \leq$ $|D|+\left|D^{\prime}\right|+\left|D^{\prime \prime}\right|=2|D|=2 \gamma_{c s}(G)$.

These two inequalities are tight for the graphs $K_{2,2}$ and $K_{n}(n \geq 3)$, respectively.

Theorem 3.1. Min Double Dom can be approximated with an approximation ratio of $O(\ln |V|)$, where $V$ is the vertex set of the input graph $G$. It can also be approximated within a factor of $1+\ln (\triangle+2)$, where $\triangle$ is the maximum degree of $G$.

Proof. Given an instance $G=(V, E)$ of Min Double Dom, we construct a multiset multicover problem [19] as follows. We take $V$ as the universe and for each vertex $v \in V$ we construct a multiset $S_{v}=N[v] \cup\{v\}$. In $S_{v}, v$ is appearing twice whereas other elements appear exactly once. We set the requirement of each vertex $v \in V$ as 2. Minimum Multiset Multicover problem can be approximated within a factor of $O(\ln |V|)$ (also $1+\ln (\triangle+2)$ ) [19]. Therefore, Min Double Dom can be approximated within a factor of $O(\ln |V|)($ also $1+\ln (\triangle+2))$.

Next, we propose an algorithm (described in Algorithm 1) to compute an approximate solution of Min Co-Secure Dom. This algorithm computes a minimal double dominating set $D_{2}$ of the input graph $G$ (with $\delta(G) \geq 2$ ) using the approximation algorithm described in Theorem 3.1 and returns it as a CSDS of $G$. By Lemma 3.1, $D_{2}$ is also a CSDS of $G$. It is easy to observe that Algorithm 1 runs in polynomial time.

```
Algorithm 1: Approx-CSD
    Input: A graph \(G=(V, E)\).
    Output: A minimum CSDS of \(G\).
    begin
        Compute a double dominating set \(D_{2}\) of \(G\) (as described in Theorem
        3.1);
        \(S=D_{2}\);
        return \(S\);
    end
```

Theorem 3.2. Min Co-SEcure Dom can be approximated within a factor of $O(\ln |V|)$, for graphs with $\delta(G) \geq 2$. It can also be approximated within a factor of $2+2 \ln (\triangle+2)$, where $\triangle$ is the maximum degree of $G$.

Proof. Let $S$ be the CSDS of $G$ computed by the Algorithm 1. By Theorem 3.1, we have $|S| \leq O(\ln |V|) \gamma_{2}(G)$. Also, by Lemma 3.2 we have

$$
|S| \leq O(\ln |V|) \gamma_{2}(G) \leq 2 O(\ln |V|) \gamma_{c s}(G)=O(\ln |V|) \gamma_{c s}(G)
$$

Similarly, it can be observed that $|S| \leq[2+2 \ln (\triangle+2)] \gamma_{c s}(G)$.

## 4. LOWER BOUND ON APPROXimation Ratio

In this section, we obtain a lower bound on the approximation ratio of MIN Co-SECURE Dom for some subclasses of bipartite graphs. To obtain our lower bound, we establish an approximation preserving reduction from Min Dom to Min Co-secure Dom. We need the following lower bound result on Min Dom.

Theorem 4.1. ([3, 5]) Unless $P=N P$, Min Dom can not be approximated within a factor of $(1-\varepsilon) \ln |V|$, for any $\varepsilon>0$. Such a result holds for MIN Dom even when restricted to bipartite graphs.

By using this theorem, we will prove similar lower bound results for Min CoSECURE DOM for two subclasses of bipartite graphs, namely perfect elimination bipartite graphs and star convex bipartite graphs.

Theorem 4.2. Unless $P=N P$, Min Co-SECURE Dom for a perfect elimination bipartite graph $G=(V, E)$ can not be approximated within $(1-\varepsilon) \ln |V|$, for any $\varepsilon>0$.

Proof. Given a graph $G=(V, E)$, an instance of Min Dom, we construct a graph $G^{\prime}=\left(V^{\prime}, E^{\prime}\right)$, an instance of Min Co-secure Dom, as follows. Here we assume that $V=\left\{v_{1}, v_{2}, \ldots, v_{n}\right\}$. After making a copy of $G$, we introduce $n$ new vertices $a_{1}, a_{2}, \ldots, a_{n}$ and $n$ edges $v_{i} a_{i}$, for $i \in[n]$. Then we introduce 6 vertices $s, t, x, y, w, z$ and the edges $s t, x y, w z$. Finally, we introduce the edge set $\left\{a_{i} v_{i}, v_{i} s, a_{i} x, a_{i} z \mid i \in\right.$ $[n]\}$. It is easy to observe that $V^{\prime}=V \cup\left\{a_{i} \mid i \in[n]\right\} \cup\{x, y, z, w, s, t\}$ and $E^{\prime}=E \cup$ $\left\{a_{i} v_{i}, v_{i} s, a_{i} x, a_{i} z \mid i \in[n]\right\} \cup\{x y, z w, s t\}$ and it is a polynomial time construction as $\left|V^{\prime}\right|=2|V|+6$ and $\left|E^{\prime}\right|=|E|+4|V|+3 . G^{\prime}$ is a perfect elimination bipartite graph with the perfect elimination ordering $\left\{s t, x y, z w, v_{1} a_{1}, v_{2} a_{2}, \cdots, v_{n} a_{n}\right\}$. For an illustration of this construction, we refer to Figure 1.
Claim 4.1. The graph $G$ has a dominating set of cardinality at most $k$ if and only if $G^{\prime}$ has a CSDS of cardinality at most $k^{\prime}=k+3$.
Proof. Let $D$ be a minimal dominating set of $G$. It is easy to check that $S=$ $D \cup\{x, z, s\}$ is a CSDS of $G^{\prime}$. Thus, $|S|=|D|+3$.

Conversely, let $S$ be a minimal CSDS of $G^{\prime}$. $S \cap\{x, y\}=\{x\}$ as $y$ is the only degree 1 vertex adjacent to $x$. Similarly, $S \cap\{s, t\}=\{s\}$ and $S \cap\{w, z\}=\{z\}$. We will assume that $S$ does not contain any $a_{i}$ vertex. This is because, each $a_{i}$ vertex is dominated by at least two vertices $x$ and $z$, and if $a_{i} \in S$ then we will replace the vertex $a_{i}$ with $v_{i}$ in $S$. Now, we define $D=S \cap V$. If $D$ is a dominating set of $G$ then we are done. Otherwise, there exists a vertex $v_{k}$ which is not dominated by any vertex of $D$. Now, $v_{k}$ is dominated only by $s \in S$ and $(S \backslash\{s\}) \cup\{v\}$ is not a dominating set, for every $v \in\left(N_{G^{\prime}}(s) \backslash S\right)$. This is a contradiction. Hence, $D$ is a dominating set of $G$ with $|S|=|D|+3$.

Let us assume that there exists some (fixed) $\varepsilon>0$ such that Min Co-SECURE Dom for perfect elimination bipartite graphs with $\left|V^{\prime}\right|$ vertices can be approximated


Figure 1. An illustration of the construction of $G^{\prime}$ from $G$ in the proof of Theorem 4.2
within a ratio of $\alpha=(1-\varepsilon) \ln \left|V^{\prime}\right|$ by a polynomial time algorithm $\mathbb{A}$. Let $l>0$ be a fixed integer with $l>\frac{1}{\varepsilon}$. By using algorithm $\mathbb{A}$, we construct a polynomial time algorithm for Min Dom as described in Algorithm 2.

Initially, if there is a minimum dominating set $D$ of $G$ with $|D|<l$, then it can be computed in polynomial time. Since the algorithm $\mathbb{A}$ runs in polynomial time, the Algorithm 2 also runs in polynomial time. If the returned set $D$ satisfies $|D|<l$ then $D$ is a minimum dominating set of $G$ and we are done.

Next, we will analyze the case when Algorithm 2 returned the set $D$ with $|D| \geq l$. By Claim 4.1 we have $\left|S_{o}\right|=\left|D_{o}\right|+3$, where $D_{o}$ and $S_{o}$ are minimum dominating set of $G$ and minimum CSDS of $G^{\prime}$, respectively. Here $\left|D_{o}\right| \geq l$.

```
Algorithm 2: Approx-DOM1
    Input: A graph \(G=(V, E)\).
    Output: A minimum dominating set \(D\) of \(G\).
    begin
        if there is a minimum dominating set \(D\) of \(G\) with \(|D|<l\) then
            return \(D\);
        else
            Construct the graph \(G^{\prime}\) as described above;
            Compute a CSDS \(S\) in \(G^{\prime}\) using \(\mathbb{A}\);
            \(D=S \cap V\);
            return \(D\);
        end
    end
```

Now, $|D| \leq|S|-3<|S| \leq \alpha\left|S_{o}\right|=\alpha\left(\left|D_{o}\right|+3\right)=\alpha\left(1+\frac{3}{\left|D_{o}\right|}\right)\left|D_{o}\right| \leq \alpha\left(1+\frac{3}{l}\right)\left|D_{o}\right|$. This implies that Algorithm 2 approximates Min Dom within a ratio of $\alpha\left(1+\frac{3}{l}\right)$. Since $\frac{1}{l}<\varepsilon$

$$
\alpha\left(1+\frac{3}{l}\right) \leq(1-\varepsilon)(1+3 \varepsilon) \ln \left|V^{\prime}\right|=\left(1-\varepsilon^{\prime}\right) \ln |V|
$$

where $\varepsilon^{\prime}=3 \varepsilon^{2}+2 \varepsilon$ as $\ln \left|V^{\prime}\right|=\ln (2|V|+6) \approx \ln |V|$ for sufficiently large value of $|V|$.

Therefore, Algorithm 2 approximates Min Dom within a ratio of $(1-\varepsilon) \ln |V|$ for some $\varepsilon>0$. This contradicts the lower bound result in Theorem 4.1.

Next, we prove the inapproximability of Min Co-SECURE Dom in star convex bipartite graphs by using the Theorem 4.1.

Theorem 4.3. Min Co-secure Dom for a star convex bipartite graph $G=(V, E)$ can not be approximated within $(1-\varepsilon) \ln |V|$ for any $\varepsilon>0$, unless $P=N P$.

Proof. Given a bipartite graph $G=(X, Y, E)$, as an instance of Min Dom, we obtain a star convex bipartite graph $G^{\prime}=\left(X^{\prime}, Y^{\prime}, E^{\prime}\right)$ such that $G$ has a dominating set of cardinality at most $k$ if and only if $G^{\prime}$ has a CSDS of cardinality at most $k^{\prime}=k+2$. Now the construction of $G^{\prime}$ from $G$ is as follows. After making a copy of $G$, we introduce four vertices $x_{0}, x, y_{0}, y$. Finally, we make every vertex of $X \cup\left\{x, x_{0}\right\}$ adjacent to $y$ and every vertex of $Y \cup\left\{y, y_{0}\right\}$ adjacent to $x$. Now, $X^{\prime}=\{X\} \cup\left\{x, x_{0}\right\}$, $Y^{\prime}=\{Y\} \cup\left\{y, y_{0}\right\}$ and $E^{\prime}=\{E\} \cup\left\{x_{i} y \mid x_{i} \in X\right\} \cup\left\{y_{i} x \mid y_{i} \in Y\right\} \cup\left\{x_{0} y, x y, x y_{0}\right\}$. The new graph $G^{\prime}=\left(V^{\prime}, E^{\prime}\right)$ formed from $G=(V, E)$ has $\left|V^{\prime}\right|=|V|+4$ and $\left|E^{\prime}\right|=|E|+n+3$, which can be constructed in polynomial time. It can be observed that $G^{\prime}$ is a star convex bipartite graph with the associated star graph which is shown in Figure 2.

$G^{\prime}=\left(V^{\prime}, E^{\prime}\right)$


Figure 2. An illustration of the construction of $G^{\prime}$ from $G$ in the proof of Theorem 4.3

Claim 4.2. G has a dominating set of cardinality at most $k$ if and only if the graph $G^{\prime}$ has a CSDS of cardinality at most $k^{\prime}=k+2$.

Proof. Suppose $D$ is a minimal dominating set of $G$ and let $S=D \cup\{x, y\}$. Clearly, $S$ is a CSDS of $G^{\prime}$ with $|S|=|D|+2 \leq k+2$. Conversely, let $S$ be a minimal dominating set of $G^{\prime}$. Note that, $\left|S \cap\left\{x, y_{0}\right\}\right|=1$, and similarly $\left|S \cap\left\{x_{0}, y\right\}\right|=1$. If $x_{0}, y_{0} \in S$, observe that $E P N\left(x_{0}, S\right)=y$ and $E P N\left(y_{0}, S\right)=x$. So, without loss of generality, assume $\{x, y\} \subseteq S$. Now, let $D=S \backslash\{x, y\}$. Now we show that $D$ is a dominating set of $G$. If $D$ is dominating set of $G$, then we are done. Otherwise, suppose $D$ is not a dominating set of $G$. Then there exists at least one vertex $v_{k} \in V(G)$ which is not dominated by any vertex of $D$. Without loss of generality, assume $v_{k} \in X$, then $v_{k}$ can only be dominated by $y \in Y^{\prime}$. Since $S$ is a CSDS of $G^{\prime},(S \backslash\{y\}) \cup\left\{v_{k}\right\}$ is a dominating set of $G^{\prime}$, which is a contradiction. Thus, $D$ is a dominating set of $G$ of cardinality $|D|=|S|-2 \leq k$. Therefore, $G$ has a dominating
set $D$ of cardinality at most $k$ if and only if $G^{\prime}$ has a CSDS of cardinality at most $k^{\prime}=k+2$. This completes the proof of this claim.

Presume that there exists some (fixed) $\varepsilon>0$ such that Min Co-secure Dom for star convex bipartite graphs having $\left|V^{\prime}\right|$ vertices can be approximated within a ratio of $\alpha=(1-\varepsilon) \ln \left|V^{\prime}\right|$ by using an algorithm $\mathbb{A}$ that runs in polynomial time. Let $l>0$ be an integer. By using algorithm $\mathbb{A}$, we construct a polynomial time algorithm Algorithm 3 for Min Dom.

```
Algorithm 3: Approx-DOM2
    Input: A bipartite graph \(G=(X, Y, E)\).
    Output: A minimum dominating set \(D\) of \(G\).
    begin
        if there is a minimum dominating set \(D\) of \(G\) with \(|D|<l\) then
            return \(D\);
        else
            Construct the graph \(G^{\prime}\) as described above;
            Compute a CSDS \(S\) in \(G^{\prime}\) using the algorithm \(\mathbb{A}\);
            \(D=S \cap(X \cup Y)\);
            return \(D\);
        end
    end
```

Firstly, if there is a minimum dominating set $D$ of $G$ with $|D|<l$, then it can be computed in polynomial time. Moreover, Algorithm 3 runs in polynomial time as $\mathbb{A}$ runs in polynomial time. Note that, if the returned set $D$ satisfies $|D|<l$ then it is a minimum dominating set of $G$ and we are done. Now, let us assume that the returned set $D$ satisfies $|D| \geq l$.

Let $D_{o}$ and $S_{o}$ be a minimum dominating set of $G$ and a minimum CSDS of $G^{\prime}$, respectively. Then $\left|D_{o}\right| \geq l$, and $\left|S_{o}\right|=\left|D_{o}\right|+2$ by the above Claim 4.2. Now,

$$
|D| \leq|S|-2<|S| \leq \alpha\left|S_{o}\right|=\alpha\left(\left|D_{o}\right|+2\right)=\alpha\left(1+\frac{2}{\left|D_{o}\right|}\right)\left|D_{o}\right| \leq \alpha\left(1+\frac{2}{l}\right)\left|D_{o}\right| .
$$

Hence, Algorithm 3 approximates Min Dom for given bipartite graph $G=(X, Y, E)$ within the ratio $\alpha\left(1+\frac{2}{l}\right)$. Let $l$ be the positive integer such that $\frac{1}{l}<\varepsilon$. Then

$$
\alpha\left(1+\frac{2}{l}\right) \leq(1-\varepsilon)(1+2 \varepsilon) \ln \left|X^{\prime} \cup Y^{\prime}\right|=\left(1-\varepsilon^{\prime}\right) \ln |X \cup Y|
$$

where $\varepsilon^{\prime}=2 \varepsilon^{2}-\varepsilon$ as $\ln \left|X^{\prime} \cup Y^{\prime}\right|=\ln (|X \cup Y|+4) \approx \ln |X \cup Y|$ for sufficiently large value of $|X \cup Y|$.

Therefore, Algorithm 3 approximates Min Dom within a ratio of $(1-\varepsilon) \ln |X \cup Y|$ for some $\varepsilon>0$. This contradicts the lower bound result in Theorem 4.1.

## 5. Complexity on bounded degree graphs

In this section, we show that Min Co-secure Dom is APX-complete for 3regular graphs. Note that the class APX is the set of all optimization problems which admit a $c$-approximation algorithm, where $c$ is a constant. From Theorem 3.2 it follows that Min Co-secure Dom can be approximated within a factor of 5.583 for graphs with maximum degree at most 4 . We improve this approximation factor to $\frac{10}{3}$.

We first show that Min Co-secure Dom for 3-regular graphs is approximable within a factor of $\frac{8}{3}$.

```
Algorithm 4: Approx-CSD-3RG
    Input: A 3-regular graph \(G=(V, E)\).
    Output: A CSDS \(S\) of \(G=(V, E)\).
    begin
        \(W^{\prime}=\emptyset ;\)
        while \(\exists\) an edge \(u v \in E\) do
            \(W^{\prime}=W^{\prime} \cup\{u, v\}\);
            Delete \(N[u] \cup N[v]\) from \(G\);
        end
        Let \(T\) be the remaining vertices;
        \(W=W^{\prime} \cup T\);
        \(S=V \backslash W\);
        return \(S\);
    end
```

Lemma 5.1. Min Co-Secure Dom is approximable within a factor of $\frac{8}{3}$ for 3 regular graphs.

Proof. Let $S_{o}$ be a minimum CSDS of a 3-regular graph $G=(V, E)$. A vertex $x \in S_{o}$ can co-securely dominate at most 3 vertices of $V \backslash S_{o}$. Therefore, $\left|V \backslash S_{o}\right| \leq$ $3\left|S_{o}\right|$. This implies that

$$
\begin{equation*}
\left|S_{o}\right| \geq \frac{n}{4} \tag{1}
\end{equation*}
$$

The set $S$ of vertices returned by Algorithm 4 is a minimal double dominating set in $G$ because each vertex in $W^{\prime}$ has exactly two neighbors in $S$. By Lemma 3.1, $S$ is a CSDS of $G$.

Thus, $W=W^{\prime} \cup T$. Let $\left|W^{\prime} \cup S\right|=n_{1}=n-|T|$. Now $\left|W^{\prime}\right| \geq \frac{n_{1}}{3}$, since in the while loop, the algorithm has picked two vertices and simultaneously removed at most six vertices from the graph. Now,

$$
|W|=\left|W^{\prime}\right|+|T| \geq \frac{n_{1}}{3}+n-n_{1} \geq n-\frac{2 n}{3}=\frac{n}{3}
$$

Thus,

$$
\begin{equation*}
|S|=|V|-|W| \leq n-\frac{n}{3}=\frac{2 n}{3} \tag{2}
\end{equation*}
$$

This yields the upper bound on the size of the CSDS returned. Combining equation (1) and equation (2), we obtain $\frac{|S|}{\left|S_{0}\right|} \leq \frac{8}{3}$, thereby proving the lemma.

Next, we design a constant factor approximation algorithm for Min Co-SECURE Dom when the input graph is 4-regular.

```
Algorithm 5: Approx-CSD-4RG
    Input: A 4-regular graph \(G=(V, E)\).
    Output: A CSDS \(S\) of \(G=(V, E)\).
    begin
        \(W^{\prime}=\emptyset ;\)
        while \(\exists\) a maximal induced path \(P\left(u_{1}, u_{k}\right)=\left(u_{1}, u_{2}, \ldots, u_{k}\right)\) or an
            induced cycle \(C=\left(u_{1}, u_{2}, \ldots, u_{k}, u_{1}\right)\) do
            \(W^{\prime}=W^{\prime} \cup\left\{u_{1}, u_{2}, \ldots, u_{k}\right\} ;\)
            Delete the vertex set \(\left\{u_{1}, u_{2}, \ldots, u_{k}\right\}\) and their neighbors from \(G\);
        end
        Let \(T\) be the remaining vertices;
        \(W=W^{\prime} \cup T\);
        \(S=V \backslash W ;\)
        return \(S\);
    end
```

Lemma 5.2. Min Co-Secure Dom for 4-regular graphs can be approximated within a factor of $\frac{10}{3}$.

Proof. Given a 4-regular graph $G$, in polynomial time Algorithm 5 computes a vertex set $W$ such that the degree of each vertex in $G[W]$ is at most 2 .

Claim 5.1. $S$ is a $C S D S$ of $G$.
Proof. By Lemma 3.1, it is enough to show that $S$ is a minimal double dominating set of $G$.
$S$ is a double dominating set of $G$ as each vertex in $W$ has at least two neighbors in $S$. Suppose $S$ is not a minimal double dominating set of $G$. Then there must be a vertex $v \in S$ such that $S \backslash\{v\}$ is a double dominating set of $G$. This implies that $v$ must have at least two neighbors in $S$. If $v \in S$ is adjacent to a vertex of degree two in $G[W]$ then $S \backslash\{v\}$ is not a double dominating set of $G$ (because $G$ is 4-regular). This implies that $v$ must be adjacent to at least one end-vertex of an induced path $P$ in $G\left[W^{\prime}\right]$. This contradicts the maximality of $P$.

Following the proof of Lemma 5.1, it can be proved that $\left|S_{o}\right| \geq \frac{n}{5}$. Let $W^{\prime}$ be the set of vertices of degree 2 in $G[W]$ and $Q=W \backslash W^{\prime}$. By setting $n_{1}=n-|Q|$ and following the proof of Lemma 5.1, it can be proved that $\left|W^{\prime}\right| \geq \frac{n_{1}}{3}$. This implies that $|W| \geq \frac{n}{3}$ and $|S| \leq \frac{2 n}{3}$. Therefore, $\frac{|S|}{\left|S_{o}\right|} \leq \frac{10}{3}$.

Before we prove that Min Co-SECURE Dom is APX-complete for 3-regular graphs, we need some terminology and results regarding the partial monopoly set.

Definition 5.1 ([17]). (Min Partial Monopoly Problem) Given a graph $G=$ $(V, E)$, partial monopoly problem is to find a set $M \subseteq V$ of minimum cardinality such that for each $v \in V \backslash M,|M \cap N[v]| \geq \frac{1}{2}|N[v]|$.

It is known that for 3-regular graphs Min Partial Monopoly Problem is APX-complete [16]. It is easy to observe the following lemma:

Lemma 5.3. Let $G$ be a 3-regular graph. A partial monopoly set $M$ of $G$ is a double dominating set of $G$ and vice versa.

Lemma 5.4. Let $G$ be a 3-regular graph and $S \subseteq V$ be a minimal CSDS of $G$. In polynomial time one can construct a double dominating set $S^{\prime} \subseteq V$ with $\left|S^{\prime}\right| \leq 2|S|$.
Proof. Let $S$ be a minimal CSDS of $G$. Define $S_{1}$ be the set of vertices $v \in S$ such that $E P N(v, S) \neq \emptyset$, and $S_{2}=S \backslash S_{1}$. Now let $A=\bigcup_{v \in S} E P N(v, S)$. Note that every vertex in $A$ has exactly one neighbor in $S_{1}$ and every vertex in $(V \backslash S) \backslash A$ has at least two neighbors in $S_{2}$. By Proposition 3.1, for every vertex $x \in S$ there exists at least one vertex $x^{*} \in V \backslash S$ and $x^{*} \in E P N(x, S)$ such that $d_{G}\left(x^{*}\right) \geq|E P N(x, S)|$. Let us define a new set $A^{\prime} \subseteq A$, such that $A^{\prime}$ contains that one vertex $x^{*}$ of each $E P N(x, S)$ for every $x \in S_{1}$. Thus, $\left|A^{\prime}\right|=\left|S_{1}\right|$. Let $S^{\prime}=S \cup A^{\prime}$. Now every vertex in $V \backslash S^{\prime}$ has at least two neighbors in $S^{\prime}$. Hence $S^{\prime}$ is a double dominating set of $G$ with cardinality $|S|+\left|A^{\prime}\right|=|S|+\left|S_{1}\right| \leq 2|S|$.

Now, we will prove that Min Co-SEcure Dom is APX-complete for 3-regular graphs by establishing a reduction from Min Partial Monopoly Problem for 3-regular graphs.

Theorem 5.1. Min Co-secure Dom is APX-complete for 3 -regular graphs.
Proof. Because of Lemma 5.1, it is enough to establish a polynomial time approximation ratio preserving reduction from Min Partial Monopoly Problem for 3-regular graphs to Min Co-SECURE Dom for 3-regular graphs.

Given a 3-regular graph $G=(V, E)$, an instance of Min Partial Monopoly Problem, we take the same graph $G$ as an instance of Min Co-secure Dom. Let $M_{o}$ be a minimum partial monopoly set of $G$ and $S_{o}$ be a minimum CSDS of $G$. Then $\left|M_{o}\right|=\gamma_{2}(G)$ (by Lemma 5.3). Also, we have $\left|S_{o}\right| \leq\left|M_{o}\right|$, by Lemma 3.2. Given a minimal CSDS $S$ of $G$, we can construct a partial monopoly set $M \subseteq V$ with $|M| \leq 2|S|$ (from Lemma 5.4 and 5.3) Therefore, $\frac{|M|}{\left|M_{o}\right|} \leq 2 \frac{|S|}{\left|S_{o}\right|}$. Hence, Min Co-secure Dom is APX-complete for 3-regular graphs.

## 6. Conclusion

In this paper, we prove that Min Co-Secure Dom is hard to approximate within a factor smaller than $\ln |V|$ for perfect elimination bipartite graphs and star convex bipartite graphs. On the positive side, we have proposed a $O(\ln |V|)$ approximation algorithm for Min Co-SECURE Dom for any graph. Apart from these, we have shown that for 3-regular graphs and 4-regular graphs Min CoSECURE DOM admits a $\frac{8}{3}$ and $\frac{10}{3}$ factor approximation algorithms, respectively. It would be interesting to design a better approximation algorithm for 3-regular graphs. We prove that it is APX-complete for 3-regular graphs. We conjecture that it is APX-hard for 3-regular bipartite graphs.

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