Approximation error of single hidden layer neural networks with fixed weights

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Abstract. This paper provides an explicit formula for the approximation error of single hidden layer neural networks with two fixed weights.

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1. INTRODUCTION

Properties of feedforward neural networks with one hidden layer have been studied quite well. By selecting different activation functions, many authors showed that single hidden layer neural networks possess the universal approximation property. In recent years, the theory of neural networks has been developed further in this direction. For example, from the point of view of practical applications, neural networks with a restricted set of weights have gained special interest.

A single hidden layer neural network with r units in the hidden layer and input $\mathbf{x} = (x_1, ..., x_d)$ computes a function of the form

$$\sum_{i=1}^{r} c_i \sigma(\mathbf{w}^i \cdot \mathbf{x} - \theta_i), \tag{1.1}$$

where the weights \mathbf{w}^i are vectors in \mathbb{R}^d , the thresholds θ_i and the coefficients c_i are real numbers and the activation function σ is a real univariate function. For various activation functions σ , it was shown by many authors that one can approximate arbitrarily well to any continuous function by functions of the form (1.1) (r is not fixed!) over any compact subset of \mathbb{R}^d . That is, the set

$$\mathcal{M}(\sigma) = span \left\{ \sigma(\mathbf{w} \cdot \mathbf{x} - \theta) : \theta \in \mathbb{R}, \mathbf{w} \in \mathbb{R}^d \right\}$$

is dense in the space $C(\mathbb{R}^d)$ in the topology of uniform convergence on compact sets (see, e.g., [5, 6, 9, 15, 23, 38]). The most general and complete result of this type was obtained by Leshno, Lin, Pinkus and Schocken [29]. They proved that a continuous activation function has the *density property* or the *universal approximation property* if and only if it is not a polynomial. This result has shown the power of single hidden layer neural networks within

all possible choices of the continuous activation function σ . For detailed information on this and other density results see [38].

It was formerly believed and particularly emphasized in many works that for the universal approximation property, large networks with sufficiently many hidden neurons are needed. However, the recent papers [12, 13] have shown that there exist neural networks with very few hidden neurons, which can approximate arbitrarily well any continuous function on any compact set. Moreover, it was shown that such networks can be constructed in practice.

A number of authors proved that single hidden layer neural networks with some suitably restricted set of weights also possess the universal approximation property. For example, White and Stinchcombe [41] showed that a single layer network with a polygonal, polynomial spline or analytic activation function and a bounded set of weights has the universal approximation property. Ito [23] investigated this property of networks using a monotone sigmoidal function (any continuous function tending to 0 at minus infinity and 1 at infinity), with weights located only on the unit sphere. Note that sigmoidal functions play an important role in neural network theory and related application areas (see, e.g., [7, 8, 12, 13, 16, 27, 31, 33]). Thus we see that the weights required for the universal approximation property are not necessarily of an arbitrarily large magnitude. But what if they are too restricted. Obviously, in this case, the universal approximation property does not hold, and the problem reduces to the identification of compact subsets in \mathbb{R}^d over which the model preserves its general propensity to approximate arbitrarily well. The first and most interesting case is, of course, neural networks with a finite set of weights. In [19], we considered this problem and gave sufficient and necessary conditions for good approximation by networks with finitely many weights and also with weights varying on finitely many straight lines. For a set W of weights consisting of two vectors or two straight lines, we showed that there is a geometrically explicit solution to this problem (see [19]).

It should be remarked that the above density results do not tell about the degree of approximation. They only provide us with the knowledge if and when single hidden layer neural networks can approximate multivariate functions. The problem of degree of approximation is related to the *problem of complexity*, which is the same as the problem of determining the number of hidden neurons required for approximation within a given accuracy. This problem was investigated in a number of papers (see, e.g., [2, 14, 30, 31, 32, 36]).

In this paper, we consider the uniform approximation of single hidden layer networks with two fixed weights in \mathbb{R}^d . As noted above these networks are not always dense in the space of continuous functions. In fact, the possibility of density depends on a compact set, where all given functions are defined. Characterization of compact sets, for which various density results hold, was given in [19, 22]. Here we are interested in the approximation error, the minimal number within which the considered network can approximate a given multivariate function. We establish an explicit approximation error formula for single hidden layer neural networks with two fixed weights. Our formula is valid for many activation functions. For example, it is valid for all continuous nonconstant activation functions, which have limits at plus and minus infinities.

2. THE MAIN RESULT

Assume σ is a continuous function on \mathbb{R} . Assume, besides, **a** and **b** are two fixed nonzero vectors in \mathbb{R}^d . Consider the set

$$\mathcal{N}(\sigma) = \left\{ \sum_{i=1}^{r} c_i \sigma(\mathbf{w}^i \cdot \mathbf{x} - \theta_i) : r \in \mathbb{N}, \ c_i, \theta_i \in \mathbb{R} \right\},\,$$

where the weights $\mathbf{w}^i = \mathbf{a}$ or $\mathbf{w}^i = \mathbf{b}$. That is, we consider the set of single hidden layer neural networks with weights restricted to only two vectors. In particular, these vectors may coincide, and then we have the set of neural networks with a single fixed weight. Let Q be a compact subset of \mathbb{R}^d and $f \in C(Q)$. Consider the approximation of f by neural networks from $\mathcal{N}(\sigma)$. The approximation error is defined as

$$E\left(f, \mathcal{N}(\sigma)\right) \stackrel{def}{=} \inf_{\Lambda \in \mathcal{N}(\sigma)} \left\| f - \Lambda \right\|.$$

The following objects, called *paths*, were exploited in many papers. We will use these objects in the further analysis.

Definition 2.1. A finite or infinite ordered set $(\mathbf{p}_1, \mathbf{p}_2, ...) \subset Q$ with $\mathbf{p}_i \neq \mathbf{p}_{i+1}$ and either $\mathbf{a} \cdot \mathbf{p}_1 = \mathbf{a} \cdot \mathbf{p}_2$, $\mathbf{b} \cdot \mathbf{p}_2 = \mathbf{b} \cdot \mathbf{p}_3$, $\mathbf{a} \cdot \mathbf{p}_3 = \mathbf{a} \cdot \mathbf{p}_4$, ... or $\mathbf{b} \cdot \mathbf{p}_1 = \mathbf{b} \cdot \mathbf{p}_2$, $\mathbf{a} \cdot \mathbf{p}_2 = \mathbf{a} \cdot \mathbf{p}_3$, $\mathbf{b} \cdot \mathbf{p}_3 = \mathbf{b} \cdot \mathbf{p}_4$, ..., is called a path with respect to the directions \mathbf{a} and \mathbf{b} .

It should be remarked that paths with respect to two directions in \mathbb{R}^2 were first considered by Braess and Pinkus [3]. They proved a theorem, which yields that the idea of paths are essential for deciding if a set of points $\{\mathbf{x}^i\}_{i=1}^m \subset \mathbb{R}^2$ has the interpolation property for so-called ridge functions. Ismailov and Pinkus [21] exploited paths to solve the interpolation problem on straight lines by ridge functions with fixed directions. In the special case, when \mathbf{a} and \mathbf{b} are the coordinate vectors in \mathbb{R}^2 , paths represent bolts of lightning (see, e.g., [1, 4, 35]). Note that bolts, first introduced by Diliberto and Straus [10] under the name of permissible lines, played an essential role in various problems of approximation of multivariate functions by sums of univariate functions (see, e.g., [10, 11, 25, 34, 35]). Note that the name "bolt of lightning" is due to Arnold [1]. There is a useful generalization of closed paths with respect to two directions to those with respect to finitely many functions. This generalization is effective in solutions of some representation problems arising in the theory of linear superpositions (see [17]).

In the following, we consider paths with respect to two directions **a** and **b** in \mathbb{R}^d . A path $(\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_{2n})$ is said to be closed if $(\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_{2n}, \mathbf{p}_1)$ is also a path. The *length* of a path is the number of its points.

We associate each closed path $p = (\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_{2n})$ with the functional

$$G_p(f) = \frac{1}{2n} \sum_{k=1}^{2n} (-1)^{k+1} f(\mathbf{p}_k).$$

In the sequel, we will assume that the considered compact set $Q \subset \mathbb{R}^d$ contains a closed path. This assumption is not too restrictive. Sufficiently many sets in \mathbb{R}^d have this property. For example, any compact set with at least one interior point contains closed paths. Note that if Q does not contain closed paths, then in almost all cases we have $E(f, \mathcal{N}(\sigma)) = 0$ for any $f \in C(Q)$ (see [19]). We say "in almost all cases" because there is a highly nontrivial example of such Q and continuous $f: Q \to \mathbb{R}$, for which $E(f, \mathcal{N}(\sigma)) > 0$ (see [19]).

We also need the concept of extremal paths.

Definition 2.2 (see [18]). A finite or infinite path $(\mathbf{p}_1, \mathbf{p}_2, ...)$ is said to be extremal for a function $u \in C(Q)$ if $u(\mathbf{p}_i) = (-1)^i ||u||$, i = 1, 2, ... or $u(\mathbf{p}_i) = (-1)^{i+1} ||u||$, i = 1, 2, ...

The following definition belongs to Schwartz [40].

Definition 2.3 (see [40]). A function $\rho \in C(\mathbb{R})$ is called mean periodic if the set $span\{\rho(x-\theta): \theta \in \mathbb{R}\}$ is not dense in $C(\mathbb{R})$ in the topology of uniform convergence on compacta.

Properties of mean periodic functions were studied in several papers (see, e.g., [24, 26, 28, 40]). It was proven that the condition in Definition 2.3 is equivalent to each of the following conditions:

a) there exists a non-zero measure μ of compact support such that

$$\int \rho(x-y)\mu(y) = 0,$$

for all $x \in \mathbb{R}$;

b) ρ is the limit in $C(\mathbb{R})$ of a sequence of exponential polynomials $P(x)e^{i\lambda x}$, which are orthogonal to a measure μ with compact support, that is,

$$\int P(y)e^{-i\lambda y}\mu(y) = 0.$$

For equivalence of the above conditions and for detailed information on mean periodic functions see Kahane [24].

In our main result (see Theorem 2.1 below), we assume that the considered function f has a best approximation in the set

$$\mathcal{R}(\mathbf{a}, \mathbf{b}) = \{ g(\mathbf{a} \cdot \mathbf{x}) + h(\mathbf{b} \cdot \mathbf{x}) : g, h \in C(\mathbb{R}) \},$$

that is, there exists $v_0 \in \mathcal{R}(\mathbf{a}, \mathbf{b})$ such that

$$||f - v_0|| = \inf_{v \in \mathcal{R}(\mathbf{a}, \mathbf{b})} ||f - v||.$$

Some results on existence of a best approximation from $\mathcal{R}(\mathbf{a}, \mathbf{b})$ was obtained in our paper [20].

The following lower bound error estimate holds in approximation with elements from $\mathcal{N}(\sigma)$.

Lemma 2.1. Assume σ is an arbitrary continuous activation function. Then

$$\sup_{p \in Q} |G_p(f)| \le E(f, \mathcal{N}(\sigma)), \tag{2.1}$$

for any $f \in C(Q)$. Here the sup is taken over all closed paths.

Proof. Consider an element of $\mathcal{N}(\sigma)$. This is a sum of the functions $f_i(\mathbf{x}) = c_i \sigma(\mathbf{w}^i \cdot \mathbf{x} - \theta_i)$, i = 1, ..., r. Note that for each $c_i, \theta_i \in \mathbb{R}$, $f_i(\mathbf{x})$ is a function of the form $g(\mathbf{w}^i \cdot \mathbf{x})$. Since the weight $\mathbf{w}^i = \mathbf{a}$ or $\mathbf{w}^i = \mathbf{b}$, we have $g(\mathbf{w}^i \cdot \mathbf{x}) = g(\mathbf{a} \cdot \mathbf{x})$ or $g(\mathbf{w}^i \cdot \mathbf{x}) = g(\mathbf{b} \cdot \mathbf{x})$. Thus, any neural network $\sum_{i=1}^r c_i \sigma(\mathbf{w}^i \cdot \mathbf{x} - \theta_i)$ in $\mathcal{N}(\sigma)$ is an element of $\mathcal{R}(\mathbf{a}, \mathbf{b})$.

Assume p is a closed path in Q and Λ is an arbitrary network from $\mathcal{N}(\sigma)$. Since $\Lambda(\mathbf{x}) = g(\mathbf{a} \cdot \mathbf{x}) + h(\mathbf{b} \cdot \mathbf{x})$, it is not difficult to verify that $G_p(\Lambda) = 0$. On the other hand, from the definition of G_p , it follows that $||G_p|| \leq 1$. Thus we obtain that

$$|G_p(f)| = |G_p(f - \Lambda)| \le ||f - \Lambda||.$$

Since the left-hand side and the right-hand side of this inequality do not depend on Λ and p, respectively, it follows that

$$\sup_{p \subset Q} |G_p(f)| \le \inf_{\Lambda \in \mathcal{N}(\sigma)} ||f - \Lambda|| = E(f, \mathcal{N}(\sigma)).$$

The following theorem is valid.

Theorem 2.1. Assume $Q \subset \mathbb{R}^d$ is a compact set and $f \in C(Q)$. Suppose the following conditions hold.

- 1) f has a best approximation in $\mathcal{R}(\mathbf{a}, \mathbf{b})$;
- 2) There exists a positive integer N such that any path $(\mathbf{p}_1,...,\mathbf{p}_n) \subset Q$, n > N, or a subpath of it can be made closed by adding not more than N points of Q.

Then for any activation function σ , which is not mean periodic, the approximation error of the class of single hidden layer networks $\mathcal{N}(\sigma)$ can be computed by the formula

$$E(f, \mathcal{N}(\sigma)) = \sup_{p \in Q} |G_p(f)|,$$

where the sup is taken over all closed paths.

Proof. By assumption, f has a best approximation in $\mathcal{R}(\mathbf{a}, \mathbf{b})$. Denote this function by $v_0(\mathbf{x}) = g_0(\mathbf{a} \cdot \mathbf{x}) + h_0(\mathbf{b} \cdot \mathbf{x})$. Let us concentrate on extremal paths for the function $f_1 = f - v_0$. The main result of [18] says that regarding such paths there may be only two cases

Case 1. There exists a closed path $p_0 = (\mathbf{p}_1, ..., \mathbf{p}_{2n})$ extremal for the function f_1 . In this case, based on Definition 2.2, we can write that

$$|G_{p_0}(f)| = |G_{p_0}(f - v_0)| = ||f - v_0||. (2.2)$$

Since σ is not mean periodic, the $span\{\sigma(x-\theta): \theta \in \mathbb{R}\}$ is dense in $C(\mathbb{R})$ in the topology of uniform convergence on compacta. It follows that for any $\varepsilon > 0$ there exist natural numbers m_1, m_2 and real numbers $c_{ij}, \theta_{ij}, i = 1, 2, j = 1, ..., m_i$, for which

$$\left| g_0(t) - \sum_{j=1}^{m_1} c_{1j} \sigma(t - \theta_{1j}) \right| < \frac{\varepsilon}{2}$$
 (2.3)

and

$$\left| h_0(t) - \sum_{j=1}^{m_2} c_{2j} \sigma(t - \theta_{2j}) \right| < \frac{\varepsilon}{2}$$

$$(2.4)$$

for all $t \in [a, b]$. Here [a, b] is a sufficiently large interval which contains both the sets $\{\mathbf{a} \cdot \mathbf{x} : \mathbf{x} \in Q\}$ and $\{\mathbf{b} \cdot \mathbf{x} : \mathbf{x} \in Q\}$.

Taking $t = \mathbf{a} \cdot \mathbf{x}$ in (2.3) and $t = \mathbf{b} \cdot \mathbf{x}$ in (2.4) we obtain that

$$\left| g_0(\mathbf{a} \cdot \mathbf{x}) + h_0(\mathbf{b} \cdot \mathbf{x}) - \sum_{i=1}^m c_i \sigma \left(\mathbf{w}^i \cdot \mathbf{x} - \theta_i \right) \right| < \varepsilon, \tag{2.5}$$

for all $\mathbf{x} \in Q$ and some $c_i, \theta_i \in \mathbb{R}$ and $\mathbf{w}^i = \mathbf{a}$ or $\mathbf{w}^i = \mathbf{b}$. Clearly,

$$\left\| f - \sum_{i=1}^{m} c_i \sigma \left(\mathbf{w}^i \cdot \mathbf{x} - \theta_i \right) \right\|$$

$$\leq \left\| f - g_0 - h_0 \right\| + \left\| g_0 + h_0 - \sum_{i=1}^{m} c_i \sigma \left(\mathbf{w}^i \cdot \mathbf{x} - \theta_i \right) \right\|. \tag{2.6}$$

It follows from (2.6) that

$$E(f, \mathcal{N}(\sigma)) \le \|f - g_0 - h_0\| + \left\| g_0 + h_0 - \sum_{i=1}^m c_i \sigma \left(\mathbf{w}^i \cdot \mathbf{x} - \theta_i \right) \right\|. \tag{2.7}$$

The last inequality together with (2.2) and (2.5) yield

$$E(f, \mathcal{N}(\sigma)) \leq |G_{p_0}(f)| + \varepsilon.$$

Now since ε is arbitrarily small, we obtain that

$$E(f, \mathcal{N}(\sigma)) \leq |G_{p_0}(f)|$$
.

From this and Lemma 2.1 it follows that

$$E(f, \mathcal{N}(\sigma)) = \sup_{p \in Q} |G_p(f)|,$$

where the sup is taken over all closed paths.

Case 2. There exists an infinite path extremal for f_1 . Assume a path $p = (\mathbf{p}_1, \mathbf{p}_2, ...)$ is infinite and extremal for f_1 . Then by the assumption of the theorem, the finite extremal paths $(\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_n) \subset p$, n = N+1, N+2, ..., or subpaths of them must be made closed by adding not more than N points. Without loss of generality we may assume that these paths themselves can be made closed. That is, for each finite extremal path $p_n = (\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_n)$, n > N, there exists a closed path $l_n = (\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_n, \mathbf{q}_{n+1}, ..., \mathbf{q}_{n+m_n})$, where $m_n \leq N$. The functional G_{l_n} obeys the inequalities

$$|G_{l_n}(f)| = |G_{l_n}(f - v_0)| \le \frac{n \|f - v_0\| + m_n \|f - v_0\|}{n + m_n} = \|f - v_0\|$$
(2.8)

and

$$|G_{l_n}(f)| \ge \frac{n \|f - v_0\| - m_n \|f - v_0\|}{n + m_n} = \frac{n - m_n}{n + m_n} \|f - v_0\|.$$
(2.9)

We obtain from (2.8) and (2.9) that

$$\sup_{l_n} |G_{l_n}(f)| = ||f - v_0||. \tag{2.10}$$

Using the above sum $\sum_{i=1}^{m} c_i \sigma(\mathbf{w}^i \cdot \mathbf{x} - \theta_i)$ and the inequalities (2.5) with (2.7) here, we obtain from (2.10) that

$$E(f, \mathcal{N}(\sigma)) \le \sup_{l_n} |G_{l_n}(f)|. \tag{2.11}$$

The inequality (2.11) together with (2.1) yield that

$$E(f, \mathcal{N}(\sigma)) = \sup_{p \in Q} |G_p(f)|,$$

where the sup is taken over all closed paths. The theorem has been proved. \Box

Corollary 2.1. Let $Q \subset \mathbb{R}^d$ be a compact set, $f \in C(Q)$ and the space $\mathcal{R}(\mathbf{a}, \mathbf{b})$ be proximinal in C(Q) (that is, for any $u \in C(Q)$ there exists a best approximation in $\mathcal{R}(\mathbf{a}, \mathbf{b})$). Let σ be any activation function, which is not mean periodic. Then the approximation error of the class of single hidden layer networks $\mathcal{N}(\sigma)$ can be computed by the formula

$$E(f, \mathcal{N}(\sigma)) = \sup_{p \in Q} |G_p(f)|,$$

where the sup is taken over all closed paths.

Proof. Since $\mathcal{R}(\mathbf{a},\mathbf{b})$ is proximinal in C(Q), the lengths of *irreducible paths* are uniformly bounded by some positive integer N (see [20]). Note that a path $(\mathbf{q}_1,...,\mathbf{q}_m)$ is irreducible if there is not a path connecting \mathbf{q}_1 and \mathbf{q}_m with the length less than m. Take any path $p = (\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_n)$ with the length n > N. Since n > N, the path p is not irreducible. Thus we can join the points \mathbf{p}_1 and \mathbf{p}_n by an irreducible path $q = (\mathbf{q}_1, \mathbf{q}_2, ..., \mathbf{q}_m)$, where $\mathbf{q}_1 = \mathbf{p}_1$ and $\mathbf{q}_m = \mathbf{p}_n$. Note that by the proximinality assumption, $m \le N$. Then the ordered set $(\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_n, \mathbf{q}_{m-1}, ..., \mathbf{q}_2)$ (or some subset $(\mathbf{p}_i, \mathbf{p}_{i+1}, ..., \mathbf{p}_k, \mathbf{q}_j, ..., \mathbf{q}_s)$ of it) is a closed path, where the number of added points is less than N. We see that all the conditions of Theorem 2.1 are satisfied; hence the assertion of Corollary 2.1 is valid.

Many activation functions exploited in neural network theory and applications are not mean periodic. For example, this is true for a number of popular activation functions (such as sigmoid, hyperbolic tangent, Gaussian, etc). The following corollary specifies one class of such functions.

Corollary 2.2. Assume all the conditions of Theorem 2.1 hold. Let $\sigma \in C(\mathbb{R}) \cap L_p(\mathbb{R})$, where $1 \leq p < \infty$, or σ be a continuous, bounded, nonconstant function, which has a limit at infinity (or minus infinity). Then the approximation error of the class of single hidden layer networks $\mathcal{N}(\sigma)$ can be computed by the formula

$$E(f, \mathcal{N}(\sigma)) = \sup_{p \subset Q} |G_p(f)|,$$

where the sup is taken over all closed paths.

The proof can be easily obtained from Theorem 2.1 and the following result of Schwartz [40]: Any continuous and p-th degree Lebesgue integrable univariate function or continuous, bounded, nonconstant function having a limit at infinity (or minus infinity) is not mean periodic (see also [38]).

As an example we show that the sup $|G_p(f)|$ in Theorem 2.1 can be easily computed for some class of functions f. For the sake of simplicity let the space dimension d = 2. Assume we are given linearly independent vectors $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$, and the domain

$$Q = \{ \mathbf{x} \in \mathbb{R}^2 : c_1 \le \mathbf{a} \cdot \mathbf{x} \le d_1, c_2 \le \mathbf{b} \cdot \mathbf{x} \le d_2 \},$$

where $c_1 < d_1$ and $c_2 < d_2$.

Consider the class M(Q) of continuous functions f on Q, which have the continuous partial derivatives $\frac{\partial^2 f}{\partial x_1^2}$, $\frac{\partial^2 f}{\partial x_1 \partial x_2}$, $\frac{\partial^2 f}{\partial x_2^2}$, and for any $\mathbf{x} = (x_1, x_2) \in Q$,

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} (a_1 b_2 + a_2 b_1) - \frac{\partial^2 f}{\partial x_1^2} a_2 b_2 - \frac{\partial^2 f}{\partial x_2^2} a_1 b_1 \ge 0.$$
 (2.12)

Using Theorem 2.1 we want to compute the error in approximating $f \in M(Q)$ by elements of the set

$$\mathcal{N}(\sigma) = span \left\{ \sigma(\mathbf{w} \cdot \mathbf{x} - \theta) : \theta \in \mathbb{R}, \ \mathbf{w} = \mathbf{a} \text{ or } \mathbf{w} = \mathbf{b} \right\}.$$

Here σ is any non-mean periodic activation function (for example, any continuous nonconstant function having limits at plus and minus infinities). Note that all the assumptions of Theorem 2.1 hold, moreover the set $\mathcal{R}(\mathbf{a}, \mathbf{b})$ is proximinal in C(Q) (see [20]).

Consider the following linear transformation

$$y_1 = a_1 x_1 + a_2 x_2, \quad y_2 = b_1 x_1 + b_2 x_2.$$
 (2.13)

Let

$$K = [c_1, d_1] \times [c_2, d_2].$$

Since the vectors (a_1, a_2) and (b_1, b_2) are linearly independent, for any $(y_1, y_2) \in K$ there exists only one solution $(x_1, x_2) \in Q$ of the system (2.13). This solution is given by the formulas

$$x_1 = \frac{y_1 b_2 - y_2 a_2}{a_1 b_2 - a_2 b_1}, \qquad x_2 = \frac{y_2 a_1 - y_1 b_1}{a_1 b_2 - a_2 b_1}.$$
 (2.14)

The linear transformation (2.14) transforms the function $f(x_1, x_2)$ to the function $g(y_1, y_2)$. Besides, this transformation maps paths with respect to the directions (a_1, a_2) and (b_1, b_2) to paths with respect to the coordinate directions (1,0) and (0,1). As we have already known the latter type of paths are called lightning bolts (see Definition 2.1 and the subsequent discussions). Hence,

$$\sup_{p \in Q} |G_p(f)| = \sup_{q \in K} |G_q(g)|, \qquad (2.15)$$

where the sup in the left hand side of (2.15) is taken over closed paths with respect to the directions (a_1, a_2) and (b_1, b_2) , while the sup in the right hand side of (2.15) is taken over closed bolts.

Note that

$$\frac{\partial^2 g}{\partial y_1 \partial y_2} \ge 0,\tag{2.16}$$

for any $(y_1, y_2) \in K$, which easily follows from (2.12).

The sup in the right hand side of (2.15) can be computed by applying theorems of Ofman [37], and Rivlin and Sibner [39]. By Ofman's theorem

$$\sup_{q \subset K} |G_q(g)| = \inf_{g_1 + g_2} ||g(y_1, y_2) - g_1(y_1) - g_2(y_2)||_{C(K)}.$$
(2.17)

By a result of Rivlin and Sibner (see [39]), Eq. (2.16) yields that

$$\inf_{g_1+g_2} \|g(y_1, y_2) - g_1(y_1) - g_2(y_2)\|_{C(K)} = \frac{1}{4} \iint_K \frac{\partial^2 g}{\partial y_1 \partial y_2} dy_1 dy_2. \tag{2.18}$$

It follows from Corollary 2.1 and equations (2.15), (2.17) and (2.18) that

$$E(f, \mathcal{N}(\sigma)) = \frac{1}{4} \iint_{K} \frac{\partial^{2} g}{\partial y_{1} \partial y_{2}} dy_{1} dy_{2}.$$
 (2.19)

The above integral can be computed easily, using values of q at the vertices of K.

Example. Let $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ be the coordinate vectors (1, 0) and (0, 1), respectively. Assume Q is the unit square $[0, 1]^2$ and σ is a sigmoidal function. Assume we are given the function $f(x_1, x_2) = x_1 x_2$. Note that (2.12) holds, hence $f \in M(Q)$. The approximating set of networks $\mathcal{N}(\sigma)$ has members of the form

$$\sum_{i=1}^{n_1} c_i \sigma(x_1 - \theta_i) + \sum_{j=1}^{n_2} d_j \sigma(x_2 - \lambda_j),$$

where $c_i, d_j, \theta_i, \lambda_j$ are arbitrary real numbers and n_1, n_2 are positive integers. Since linear transformation (2.13) does not change the coordinates in our case, we have g = f and K = Q. Thus, by formula (2.19),

$$E(f, \mathcal{N}(\sigma)) = \frac{1}{4} \iint_{O} \frac{\partial^{2} f}{\partial x_{1} \partial x_{2}} dx_{1} dx_{2} = \frac{1}{4}.$$

Remark. The question on computing the approximation error of neural nets with more than two fixed weights is fair, but its solution seems to be beyond the scope of the methods discussed herein. A path with respect to two directions **a** and **b** is constructed as an ordered set of points $(\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_n)$ in \mathbb{R}^d with edges $\mathbf{p}_i \mathbf{p}_{i+1}$ in alternating hyperplanes so that the first, third, fifth and so on hyperplanes (also the second, fourth, sixth and so on hyperplanes) are parallel. If not differentiate between parallel hyperplanes, the path $(\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_n)$ can be considered as a trace of some point traveling in two alternating hyperplanes. In this case, the path functional

$$F(f) = \frac{1}{n} \sum_{i=1}^{n} (-1)^{i+1} f(\mathbf{p}_i),$$

has some important properties, which lead to a geometric criterion for a best approximation from $\mathcal{R}(\mathbf{a}, \mathbf{b})$ (see [18]). Note that our Theorem 2.1 is mainly based on this criterion. The problem becomes complicated when the number of directions is more than two. The simple generalization of paths demands a point traveling in three or more alternating hyperplanes. But in this case the appropriate generalization of the above functional F looses its original useful properties. Some difficulties with a generalization of paths and path functionals were delineated in [17] and [18].

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