Schütt's theorem for vector-valued sequence spaces

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Abstract

The entropy numbers of certain finite-dimensional operators acting between vector-valued sequence spaces are estimated, thus providing a generalisation of the famous result of Schütt. In addition, two-sided estimates of the entropy numbers of some diagonal operators are obtained.

1 Introduction

The notion of the entropy of a set and the companion idea of the entropy numbers of a bounded linear map between (quasi-) Banach spaces are now of proven importance in analysis, especially in spectral theory and approximation theory: one has only to think of the ground-breaking work of Kolmogorov and Tikhomirov [11], the subsequent study related to Hilbert's thirteenth problem by Vitushkin and Henkin [20], and Birman and Solomyak's celebrated paper [2] on embeddings of Sobolev spaces to have an idea of the possibilities. The theorem of Schütt mentioned in the title relates to the natural embedding id of l_p^m in l_q^m, where $n \in \mathbb{N}$ and $1 \leq p < q \leq \infty$: it asserts that given any $k \in \mathbb{N}$, there are positive constants c_1, c_2, independent of m and n, such that the n^{th} entropy number $e_n(T)$ of T satisfies $c_1 A(m,n) \leq e_n(id) \leq c_2 A(m,n)$, where $A(m,n)$ is an explicit function of m and n (see Theorem 2.1 below). This was proved in [18] by means of volume arguments. Now it is known that the result holds whenever $0 < p < q \leq \infty$: the upper estimate was obtained in [8], Proposition 3.2.2, again by volume arguments, while for the lower estimate we refer to [5], Theorem 2 and [12]. Apart from its intrinsic interest, a good deal of the importance of Schütt's theorem stems from its connection with embeddings of function spaces. In the work of Birman and Solomyak alluded to above, estimates of the entropy numbers of embeddings between Sobolev spaces were obtained by means of piecewise polynomial approximations. To deal with more general spaces, such as those of Besov (perhaps with generalised smoothness) or Lizorkin-Triebel type, it is more convenient to use decompositions of wavelet (see, for example, [4], [15], [19]) or atomic (see [10]) form to reduce questions of embeddings of function spaces to considerations of mappings between sequence spaces. It is in connection with these mappings that the Schütt result plays a part.

In this paper we obtain two-sided estimates for the entropy numbers of certain mappings between vector-valued sequence spaces. More precisely, we consider a mapping

$$T : l_p^m\left(\{X_i\}_{i=1}^m\right) \to l_q^m\left(\{Y_i\}_{i=1}^m\right),$$

where $0 < p < q \leq \infty$, the X_k and Y_k are quasi-Banach spaces and T is defined by $Tx = (T_1 x_1, ..., T_m x_m)$, where $x = (x_1, ..., x_m)$, each T_i being a bounded linear map from X_i to Y_i. Our main focus is on the case when $X = X_1 = ... = X_m$, $Y = Y_1 = ... = Y_m$, $T_i = \lambda_i T_0$ $(i = 1, ..., m)$, where $T_0 : X \to Y$ is a bounded linear operator and the λ_i are real numbers. In particular, when $\lambda_i = 1$ for all $i \in \{1, ..., m\}$ it is shown that knowledge of the entropy numbers $e_1(T_0), ..., e_n(T_0)$ of the operator T_0 leads to two-sided estimates of the entropy numbers $e_n(T)$ $(n \in \mathbb{N})$ of T. In [5] we gave a generalisation of Schütt's theorem to the case of finite-dimensional spaces with symmetric bases: in the present paper we use some ideas from [5] but in a very simple form. Unlike the volume arguments mentioned above, and the interpolation techniques appearing in [9] and [12] (in [12] the same ideas as in [5] were used-see Lemma 2.7 below-but with functions with values in the set $\{-1, 0, 1\}$ instead of the characteristic functions of [5]), our proofs are essentially combinatoric in nature: by specialisation they give an independent proof of Schütt's theorem.

For previous work on mappings between vector-valued sequence spaces we refer to [13], [3] and the references contained in these papers. Interest in the entropy numbers of embeddings of function spaces owes much to [2], in which Sobolev spaces were considered; since the appearance of [2] the literature on the subject has grown enormously. Many papers deal with estimates of the entropy numbers of embeddings of Besov spaces with generalised smoothness; we refer again to [3], [7], [19] and the references given in those works.

2 Preliminaries

2.1 Background

Throughout the paper log is to be understood as $\log_2$, $[x]$ will denote the integer part of the real number x, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $a \asymp b$ means that $c_1 a \leq b \leq c_2 a$ for some positive constants independent of variables occurring in a and b. Given quasi-Banach spaces X and Y, we shall write $B(X, Y)$ for the space of all bounded linear maps from X to Y, abbreviating this to $B(X)$ when $X = Y$; the closed unit ball in X will be denoted by B_X and the quasinorm on X by $\|\cdot\|_X$. We recall that a quasi-Banach space Z is said to be an $r-$Banach space if the quasi-norm $\|\cdot\|_Z$ has the property that for all $z_1, z_2 \in Z$,

$$\|z_1 + z_2\|_Z^r \leq \|z_1\|_Z^r + \|z_2\|_Z^r;$$

the quasi-norm is then said to be an $r-$norm. It is well known (see, for example, [1] and [17]) that if Z is any quasi-Banach space then there exist $r \in (0,]$ and an $r-$norm on Z equivalent to the original quasi-norm. Given a finite set A we shall write $\sharp A$ for the cardinality of the set A.

Let $n \in \mathbb{N}$ and suppose that M is a bounded subset of an $r-$normed quasi-Banach space Y. The n^{th} (dyadic) outer entropy number $e_n(M)$ of M is defined to be the infimum of those $\varepsilon > 0$ such that M can be covered by 2^{n-1} balls in Y of radius ε. The n^{th} outer entropy number of a map $T \in B(X,Y)$ (where X and Y are quasi-Banach spaces) is

$$e_n(T) := e_n\left(T\left(B_X\right)\right);$$

the numbers $e_n(T)$ are monotonic decreasing as n increases, with $e_1(T) = \|T\|$; and T is compact if and only if $\lim_{n\to\infty} e_n(T) = 0$. Moreover, for all $s, n \in \mathbb{N}$, and whenever $T_1 + T_2$ and $R \circ S$ are properly defined bounded linear operators acting between quasi-Banach spaces,

$$e_{s+n-1}\left(R \circ S\right) \leq e_s\left(R\right) e_n\left(S\right)$$

and, if the target space of T_1 and T_2 is an $r-$Banach space,

$$e^r_{s+n-1}\left(T_1 + T_2\right) \leq e^r_s\left(T_1\right) + e^r_n\left(T_2\right).$$

Following Pietsch ([16], 12.1.6), for each $n \in \mathbb{N}$ we denote by $f_n(T)$ the (dyadic) inner entropy number of $T \in B(X,Y)$, defined to be the supremum of all those $\varepsilon > 0$ such that there are $x_1, ..., x_{2^{n-1}+1} \in B_X$ with $\|Tx_i - Tx_j\|_Y \geq 2\varepsilon$ whenever i, j are distinct points of $\left\{1, 2, ..., 2^{n-1}+1\right\}$. If Y is an $r-$Banach space, then the outer and inner entropy numbers are related by

$$f_n(T) \leq 2^{1/r-1} e_n(T) \leq 2^{1/r} f_n(T) \quad (n \in \mathbb{N}).$$

These estimates were proved by Pietsch in the Banach space case ($r = 1$); the proof in the general case merely involves a simple modification of his arguments.

We shall need vector-valued versions of the familiar sequence space l_p and its $m-$dimensional subspace l^m_p. Let $p \in (0, \infty]$, $m \in \mathbb{N}$ and suppose that $X_1, ..., X_m$ are quasi-Banach spaces. Then

$$l^m_p\left(\{X_i\}^m_{i=1}\right) := \{x = (x_1, ..., x_m) : x_i \in X_i \text{ for each } i\};$$

(for simplicity we shall denote this space by $l^m_p\left(X_i\right)$) endowed with the quasi-norm

$$\begin{aligned}\left\|x \mid l^m_p\left(\{X_i\}^m_{i=1}\right)\right\| &:= \left(\sum_{i=1}^m \|x_i\|^p_{X_i}\right)^{1/p} \quad \text{if } 0 < p < \infty,\\ \left\|x \mid l^n_\infty\left(\{X_i\}^m_{i=1}\right)\right\| &:= \sup_{1\leq i\leq m} \|x_i\|_{X_i},\end{aligned}$$

it is a quasi-Banach space. When $X_1 = ... = X_m = X$, we shall simply denote this space by $l^m_p\left(X\right)$.

The theorem of Schütt in which we are interested appears in [18] and asserts the following:

Theorem 2.1 *Let $m, n \in \mathbb{N}$ and $1 \le p < q \le \infty$; denote by id the natural embedding of l_p^m in l_q^m. Then there are positive constants c_1, c_2, independent of m and n, such that*

$$c_1 A(m,n) \le e_n(id) \le c_2 A(m,n),$$

where

$$A(m,n) = \begin{cases} 1, & \text{if} \quad n \le \log m, \\ \left(\frac{\log(m/n+1)}{n}\right)^{1/p-1/q}, & \text{if} \quad \log m \le n \le m, \\ 2^{-n/m} m^{1/q-1/p}, & \text{if} \quad n \ge m. \end{cases}$$

Various authors have contributed to the generalisation of this result to the case $0 < p < q \le \infty$. For the estimate from above, we refer to [8], Proposition 3.2.2; an elementary proof of the lower estimate in the case $\log m \le n \le m$ is given in Theorem 2 of [5], where a generalisation of Schütt's result for the case of quasinormed spaces with a symmetric basis was presented (such a generalisation is still unknown for the case $n \ge m$); a proof of the lower estimate contained in [12]. More detailed estimates of the constants (upper and lower) and a new proof of the whole result for $0 < p < q \le \infty$ are given in [9].

2.2 Preparatory results

Here we present the main ingredients to be used in the proof of the main result.

Lemma 2.2 *Let $m \in \mathbb{N}$. Then there is a set $\Gamma(m) \subset (0,1]^n$ with the following properties:*
(i) $\sharp\Gamma(m) \le 2^{5m/2}$.
(ii) For any sequence $\{\varepsilon_i\}_{i=1}^m$ in $\Gamma(m)$, the numbers $n\varepsilon_i$ are positive integers for all $i \in \{1, 2, ..., n\}$, $\sum_{i=1}^m \varepsilon_i \le 3$ and for all $t > 0$,

$$\sharp\{i \in \{1, 2, ..., m\} : \varepsilon_i \ge t\} \le 2/t.$$

(iii) For any sequence $\{\alpha_i\}_{i=1}^m$ with each $\alpha_i \in [0,1]$ and $\sum_{i=1}^m \alpha_i = 1$ there is a sequence $\{\varepsilon_i\}_{i=1}^m \in \Gamma(m)$ such that $\alpha_i \le \varepsilon_i$ for all $i \in \{1, 2, ..., m\}$.

Proof. Put

$$E = \left\{2^k/m : k \in \mathbb{N}_0, 2^k < m\right\} \cup \{1\}$$

and define $\Gamma(m)$ to be the set of all sequences $\{\varepsilon_i\}_{i=1}^m \in E^m$ such that $\sum_{i=1}^m \varepsilon_i \le 3$ and $\sharp\{i \in \{1, 2, ..., m\} : \varepsilon_i \ge t\} \le 2/t$ for all $t > 0$. This ensures that (ii) holds.
To estimate the number of elements in $\Gamma(m)$ we observe that if $\{\varepsilon_i\}_{i=1}^m \in \Gamma(m)$, $k \in \mathbb{N}_0$, $A(k) := \left\{i \in \{1, 2, ..., m\} : \varepsilon_i = 2^k/m\right\}$ and $B(k) := \left\{i \in \{1, 2, ..., m\} : \varepsilon_i \ge 2^k/m\right\}$, then

$$\sharp A(k) \le \sharp B(k) \le \min\left(m, m/2^{k-1}\right).$$

Fix the sets $A(0), A(1), A(2)$ and $B(3)$; note that $\sharp B(3) \leq m/4$. Then for the choice of the sets $A(k)$ $(k \geq 3)$ there are at most

$$\prod_{k=3}^{\infty} 2^{m/2^{k-1}} = 2^{m/2}$$

possibilities. Since $A(k) \subset B(k)$, $\sharp B(k) \leq m/2^{k-1}$, and it follows that once the sets $A(0), \ldots A(k-1)$ have been chosen, there are at most $2^{m/2^k}$ possibilities for the choice of $A(k, k \geq 3$.

We now claim that given any non-empty finite set S with m elements, there are 2^{2m} distinct representations of S as the union of 4 disjoint subsets. For if $S = \{s_1, ..., s_m\}$ and $S = \cup_{j=1}^{4} S_j$, where the S_j are disjoint, then each s_i has to belong to some S_j, and as there are 4 choices for each s_i, the total number of choices is $4^m = 2^{2m}$. Thus $\sharp\Gamma(n) \leq 2^{2m} \cdot 2^{m/2} = 2^{5m/2}$, so that (i) holds. The final property (iii) is established in a routine fashion and is left to the reader. ■

Lemma 2.3 *Let* $m \in \mathbb{N}\backslash\{1\}$, *suppose that* $0 < p < q \leq \infty$ *and put* $\alpha = 1/p - 1/q$. *For each* $i \in \{1, 2, ..., m\}$ *let* X_i, Y_i *be quasi-Banach spaces and* $T_i \in B(X_i, Y_i)$. *Suppose that for every* $i, s \in \{1, 2, ..., m\}$,

$$e_s(T_i) \leq (m/s)^{\alpha}.$$

Let $T : l_p^m(X_i) \to l_q^m(Y_i)$ *be the linear operator defined by*

$$T(x) = (T_1(x_1), ..., T_m(X_m)), \quad x = (x_1, ..., x_m) \in X_1 \times X_2 \times ...X_m.$$

Then

$$e_{5m}(T) \leq 3^{1/q}.$$

Proof. Let $W = l_p^m(X_i)$. Given any point $x \in B_W$, there is a sequence $\{\alpha_i\}_{i=1}^m$ with each $\alpha_i \in [0, 1]$ and $\sum_{i=1}^{m} \alpha_i = 1$ such that $x \in \prod_{i=1}^{m} \alpha_i^{1/p} B_{X_i}$. By Lemma 2.2, it follows that

$$B_W \subset \bigcup \left(\prod_{i=1}^{m} \varepsilon_i^{1/p} B_{X_i} \right),$$

where the union is taken over all sequences $\{\varepsilon_i\}_{i=1}^m \in \Gamma(m)$, where $\Gamma(m)$ is the set defined in Lemma 2.2. Viewing

$$K := \prod_{i=1}^{m} \varepsilon_k^{1/p} T_i(B_{X_i}).$$

as a subset of $l_q^m(\{Y_i\}_{i=1}^m)$, we estimate $e_{2n+1}(K)$. Put $m_i = m\varepsilon_i$ $(i = 1, 2, ..., m)$. Then

$$e_{m_i}\left(\varepsilon_i^{1/p} T_i(B_{X_i})\right) \leq \varepsilon_i^{1/p} \left(\frac{n}{n\varepsilon_i}\right)^{\alpha} = \varepsilon_i^{1/q}.$$

Since $\sum_{i=1}^{m}(m_i - 1) \leq 3m - m = 2m$, application of the following simple lemma, the proof of which is omitted, shows that

$$e_{2m+1}(K) \leq \left(\sum_{i=1}^{m} \varepsilon_i\right)^{1/q} \leq 3^{1/q}.$$

As $\sharp\Gamma(m) \leq 2^{5m/2}$, we see that $e_N(T) \leq 3^{1/q}$, where $N = \left[\frac{5m}{2}\right] + 2n + 1$: the result follows. ■

Lemma 2.4 *Let $m, n \in \mathbb{N}$ and let $n_1, ..., n_m$ be non-negative integers such that $n - 1 = \sum_{i=1}^{m}(n_i - 1)$; let $q \in (0, \infty]$. For each $i \in \{1, 2, ..., m\}$ suppose that Z_i, Y_i are quasi-Banach spaces and $U_i \in B(Z_i, Y_i)$. Let $U : l_\infty^m(Z_i) \to l_q^m(Y_i)$ be the linear operator defined by*

$$U(z) = (U_1(z_1), ..., U_m(z_m)), \; z = (z_1, ..., z_m) \in Z_1 \times ... \times Z_m.$$

Then

$$e_n(U) \leq \left(\sum_{i=1}^{m} e_{n_i}^q(U_i)\right)^{1/q}.$$

In the next section we shall need the following estimates, proved in [6] (or [5]) and [7].

Lemma 2.5 *(i) If $k, m \in \mathbb{N}$, $k \leq m$, then*

$$\left(\frac{m}{k}\right)^k \leq \binom{m}{k} \leq \left(\frac{em}{k}\right)^k.$$

(ii) There are positive constants c_1, c_2 such that for any $m, n \in \mathbb{N}$ with $2 \leq n \leq m \leq 2^n$ the following estimates hold:

$$2^{c_1 n} \leq \binom{m}{k} \leq 2^{c_2 n},$$

where k is the smallest positive integer such that

$$k \geq A := \frac{n}{2\log(2m/n)}.$$

Proof. As (i) is well known we simply deal with (ii) and suppose that $m \geq 2n$. By (i) we have

$$\begin{aligned}\frac{\log\binom{m}{k}}{n} &\asymp \frac{k\log(m/k)}{n} \asymp \frac{A\log(m/A)}{n} \\ &= \frac{\log(2m/n) - \log\log(2m/n)}{2\log(2m/n)}.\end{aligned}$$

The rest follows easily. ■

Lemma 2.6 *Let $m, n \in \mathbb{N}$, $b \in (0, \infty)$ and $0 < p < q \leq \infty$; put $\alpha = 1/p - 1/q$. For each $i \in \{1, 2, ..., m\}$ let X_i, Y_i be quasi-Banach spaces and $T_i \in B(X_i, Y_i)$. Let $T : l_p(\{X_i\}_{i=1}^m) \to l_q(\{Y_i\}_{i=1}^m)$ be the linear operator defined by*

$$T(x) = (T_1(x_1), T_2(x_2), ..., T_m(x_m)), \; x = (x_1, x_2, ..., x_m) \in X_1 \times X_2 \times ... \times X_m.$$

Suppose that $f_n(T_i) \geq b$ for each $i \in \{1, 2, ..., m\}$. Then

$$f_k(T) \geq 2^{-1/q} b m^{-\alpha}, \text{ where } k = [((n-1)m/6].$$

Lemma 2.7 *Let E be a set, let $v \in \mathbb{N}$ be such that $64e^3 v \leq \sharp E$ and put $\mathcal{L}(E, v) = \{E_1 \subset E : \sharp E_1 = v\}$. Then there is a set $\mathcal{L}(E, v, 1/2) \subset \mathcal{L}(E, v)$ with the following properties:*
(i) for any distinct $E_1, E_2 \in \mathcal{L}(E, v, 1/2)$,

$$\sharp (E_1 \cap E_2) \leq v/2;$$

(ii)

$$(\sharp \mathcal{L}(E, v, 1/2))^4 \geq \sharp \mathcal{L}(E, v) = \binom{\sharp E}{v}.$$

3 The main results

Theorem 3.1 *Let $0 < p < q \leq \infty$, set $\alpha = 1/p - 1/q$ and let $m, n \in \mathbb{N}$, $2 \leq n \leq m \leq 2^n$. Let X, Y be $r-$normed quasi-Banach spaces and suppose that $T_0 \in B(X, Y)$. Let $T(m) : l_p^m(X) \to l_q^m(Y)$ be the linear operator defined by $T(m)(x) = (T_0(x_1), ..., T_0(x_m)), x = (x_1, ..., x_m) \in l_p^m(X)$. Then*

$$c_1 A(n, m, T_0) \leq e_n(T(m)) \leq c_2 A(n, m, T_0). \tag{3.1}$$

Here c_1, c_2 are positive constants which depend on the parameters p, q and r only, and

$$A(n, m, T_0) = \max\left(\|T_0\| \left(\frac{\log(m/n) + 1}{n} \right)^\alpha, \max_{k \in \{1, 2, ..., n\}} ((k/n)^\alpha e_k(T_0)) \right).$$

Proof. First note that given any $a > 1$, there are positive constants $C_1(a), C_2(a)$ such that, for any $m, n, \widetilde{m}, \widetilde{n} \in \mathbb{N}$ with $m \geq n, \widetilde{m} \geq \widetilde{n}, ma \geq \widetilde{m} \geq m/a, na \geq \widetilde{n} \geq n/a$,

$$C_1(a) A(\widetilde{n}, \widetilde{m}, T_0) \geq A(n, m, T_0) \geq C_2 A(\widetilde{n}, \widetilde{m}, T_0). \tag{3.2}$$

Now we show that the required statement is a consequence of the following two assertions.
1. There are positive constants C_3, C_4 and an integer $a > 1$ such that, for any $m, n \in \mathbb{N}$ with $m \geq n$, the following estimates hold:

$$e_{na}(T(m)) \leq C_3 A(n, m, T_0), \; f_{n(a)}(T(m)) \geq C_4 A(n, m, T_0).$$

Here $n(a)$ is the smallest positive integer greater than or equal to n/a.

2. Given any integer $b \geq 1$ (we need this assertion for $b = 1$ only), there is a constant $C_5(b) = C_5$ such that for every $n \in \mathbb{N}$,

$$f_{nb}(T(n)) \geq C_5 A(n, n, T_0).$$

Indeed let us prove that the first estimate in (3.1) is a consequence of the first estimate in assertion 1 and the estimates in (3.2). Let $\widetilde{m}, \widetilde{n} \in \mathbb{N}$ with $\widetilde{m} \geq \widetilde{n}$; without loss of generality we can suppose that $\widetilde{n} \geq 2a, a \geq 2$. Choose $n \in \mathbb{N}$ in such a way that $na \leq \widetilde{n} \leq (n+1)a$. Then

$$\begin{aligned} e_{\widetilde{n}}(T(\widetilde{m}) &\leq e_{na}(T(\widetilde{m}) \leq C_3 A(n, \widetilde{m}, T_0) \leq C_1 C_3 A(na, \widetilde{m}, T_0) \\ &\leq C_1^2 C_3 A(\widetilde{n}, \widetilde{m}, T_0). \end{aligned}$$

To prove the second estimate in (3.1), once more let $\widetilde{m}, \widetilde{n} \in \mathbb{N}$ with $\widetilde{m} \geq \widetilde{n}$. Choose $n \in \mathbb{N}$ in such a way that $n(a) \geq \widetilde{n} \geq n(a) - 1$; without loss of generality we can suppose that $\widetilde{n} \geq 2a, a \geq 2$. There are two possibilities: $n \leq \widetilde{m}$ or $n \geq \widetilde{m}$. In the first case we use the estimates

$$f_{\widetilde{n}}(T(\widetilde{m})) \geq f_{n(a)}(T(\widetilde{m})) \geq C_4 A(n, \widetilde{m}, T_0) \geq C_2 C_4 A(\widetilde{n}, \widetilde{m}, T_0).$$

In the second case we use the estimate $f_n(T(\widetilde{m})) \geq f_{\widetilde{m}}(T(\widetilde{m}))$, and then assertion 2 with $b = 1$ and the estimate (3.1).

Now we prove assertions 1 and 2. We begin with the proof of the upper estimate in statement 1 and let k be the positive integer defined in Lemma 2.5 (ii). For any set $F \subset \{1, 2, ..., m\}$ such that $\sharp F = k$ let $T(m)_F : l_p^m(X) \to l_q^m(Y)$ be the linear operator defined by

$$T(m)_F(x) = (\chi_F(1) T_0(x_1), ..., \chi_F(m) T_0(x_m)), \ x = (x_1, ..., x_m) \in l_p^m(X).$$

Here χ_F is the characteristic function of F. Let $s \in \mathbb{N}, \varepsilon > 0$ and $\eta = e_s(T(m))$; let B_p be the unit ball in $l_p^m(X)$ and denote by $\Gamma(F)$ an $(\eta+\varepsilon)-$net (of cardinality 2^{s-1}) of $T(m)_F(B_p) \subset l_q^m(Y)$ such that $y_i = 0$ for any $i \in \{1, 2, ..., m\} \backslash F$ and $y = (y_1, ..., y_m) \in \Gamma(F)$. Let $\Gamma = \cup \Gamma(F)$, where the union is taken over all sets $F \subset \{1, 2, ..., m\}$ with $\sharp F = k$. Then

$$\sharp \Gamma(F) \leq 2^{s-1} \binom{m}{k}.$$

Much as in [6] (see the proof of Lemma 11) it can be seen that Γ is an ε_0-net of $T(m)(B_p)$ in $l_q^m(Y)$, where

$$\varepsilon_0^r = (\eta + \varepsilon)^r + (\|T_0\| / (k+1)^\alpha)^r.$$

Now let $x = (x_1, ..., x_m) \in B_p$ and let F be any subset of $\{1, 2, ..., m\}$ such that $\sharp F = k$ and $\|x_i\|_X \geq \|x_j\|_X$ whenever $i \in F$ and $j \in \{1, 2, ..., m\} \backslash F$. Then $\|T_0 x_j\|_Y \leq \|T_0\| / (k+1)^{1/p}$ if $j \in \{1, 2, ..., m\} \backslash F$. By Hölder's inequality,

$$\|(T(m) - T(m)_F)(x)\|_{l_q^m(Y)} \leq \|T_0\| / (k+1)^\alpha.$$

In view of Lemma 2.5 these arguments imply that there is a positive integer C_6 such that, for any $s \in \mathbb{N}$,

$$e_{C_6 n+s}(T(m)) \leq 2^{1/q} \max\left(e_s(T(k), \|T_0\| / (k+1)^\alpha\right).$$

Together with Lemma 2.3 this gives the required upper estimate in statement 1. The lower estimate is a consequence of Lemma 2.6.

To prove statement 2, note that because of Lemma 2.6 and the estimates of $A(n, m, T_0)$ given in (3.1), it is enough to show that given any $b \in \mathbb{N}$, there is a positive constant $C_7(b) = C_7$ such that for every $n \in \mathbb{N}$,

$$f_{nb}(T(n)) \geq C_7 \|T_0\| / n^\alpha.$$

Let $n, u \in \mathbb{N}$ with $n > 64e^3$, put $E = \{1, 2, ..., n\}$, suppose that v is the largest integer such that $64e^3 v \leq n$, and let $x \in X$ satisfy $\|x\|_X \leq 1$ and $\|T_0 x\|_Y \geq \|T_0\| / 2$. Define $I(u)$ to be the subset of the unit ball of $l_p^m(X)$ consisting of all points with i^{th} coordinate $(1 \leq i \leq m)$

$$\sum_{j=1}^{u} 2^{-kr} v^{-1/p} \chi_{E(j)}(i) x \text{ for some } E(j) \in \mathcal{L}(E, v, 1/2).$$

Then $\sharp I(u) \geq 2^{C_8 n}$ and $\|T(m)x - T(m)y\|_{l_q^m(Y)} \geq 2^{-ru} v^{-\alpha} C_9$ for all distinct $x, y \in I(u)$. The result follows. ■

To conclude we formulate one more result, the proof of which is similar to that of the last theorem.

Theorem 3.2 *Let $0 < p < q \leq \infty$, set $\alpha = 1/p - 1/q$ and let $m, n \in \mathbb{N}$, $m \leq 2^n$. For each $i \in \{1, 2, ..., m\}$ let X_i, Y_i be $r-$normed quasi-Banach spaces and suppose that $T_i \in B(X_i, Y_i)$. Let $T : l_p^m(X_i) \to l_q^m(Y_s)$ be the linear operator defined by*

$$T(x) = (T_1(x_1), ..., T_m(x_m))\, x = (x_1, ..., x_m) \in l_p^m(X_i).$$

(i) Let $m \geq 2n$ and suppose that $\|T_1\| \geq \|T_2\| \geq ... \geq \|T_m\|$, $\|T_1\| \leq 2 \|T_n\|$; put

$$A(n, m) = \max_{s \in \{n, n+1, ..., m\}, s \leq 2^n} \|T_s\| \left(\frac{\log(2s/n)}{n}\right)^\alpha,$$
$$B(n, m) = \max_{k \in \{1, 2, ..., n\},\ i \in \{1, 2, ..., m\}} \left((k/n)^\alpha e_k(T_i)\right).$$

Then

$$c_1 A(n, m) \leq e_n(T) \leq c_2 \max(A(n, m), B(n, m)),$$

where c_1, c_2 are positive constants which depend on the parameters p and q only.
(ii) Suppose that $m \leq n$ and $T_1 = T_2 = ... = T_m = T_0$. For any $a > 0$ let

$$D(a, n, m) = \max_{k \in \{1, 2, ..., n\},\ k \geq a} \left((k/n)^\alpha e_k(T_0)\right).$$

Then

$$c_5 D(c_3 n/m, n, m) \leq e_n(T) \leq c_6 D(c_4 n/m, n, m),$$

where c_3, c_4 *are absolute constants and the constants* c_5, c_6 *depend on the parameters* p *and* q *only.*

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