ASYMPTOTICS OF SHARP CONSTANTS IN MARKOV–BERNSTEIN–NIKOLSKII TYPE INEQUALITIES WITH EXPONENTIAL WEIGHTS

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ABSTRACT. We prove that the sharp constant in the univariate Bernstein–Nikolskii inequality for entire functions of exponential type is the limit of the sharp constant in the V. A. Markov type inequality with an exponential weight for coefficients of an algebraic polynomials of degree n as $n \to \infty$.

1. INTRODUCTION

In this paper we discuss limit relations between the sharp constants in the univariate V. A. Markov– Bernstein–Nikolskii type inequalities with exponential weights for algebraic polynomials and entire functions of exponential type.

Notation and Preliminaries. Throughout the paper C, C_1, C_2, \ldots denote positive constants independent of essential parameters. Occasionally we indicate dependence on or independence of certain parameters. The same symbol C does not necessarily denote the same constant in different occurrences.

Let $\mathbb{N} := \{1, 2, \ldots\}, \mathbb{Z}_+ := \{0, 1, \ldots\}, \mathbb{R}$ be the set of all real numbers, \mathbb{C} be the set of all complex numbers, \mathcal{P}_n be the set of all algebraic polynomials with complex coefficients of degree at most $n, n \in \mathbb{Z}_+$, and B_{σ} be the set of all complex-valued entire functions of exponential type $\sigma > 0$.

Let $W : \Omega \to [0, \infty]$ be an integrable weight on a measurable subset Ω of \mathbb{R} , and let $L_{r,W}(\Omega)$ be a weighted space of all measurable complex-valued functions $F : \Omega \to \mathbb{C}$ with the finite quasinorm

$$||F||_{L_{r,W}(\Omega)} := \begin{cases} \left(\int_{\Omega} |F(x)W(x)|^r dx \right)^{1/r}, & 0 < r < \infty \\ \operatorname{ess\,sup}_{x \in \Omega} |F(x)|W(x), & r = \infty. \end{cases}$$

In the nonweighted case (W = 1), we set

 $\|\cdot\|_{L_r(\Omega)} := \|\cdot\|_{L_{r,1}(\Omega)}, \quad L_r(\Omega) := L_{r,1}(\Omega), \quad 0 < r \le \infty.$

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The quasinorm $\|\cdot\|_{L_{r,W}(\Omega)}$ allows the following "triangle" inequality:

$$\left\|\sum_{j=1}^{l} F_{j}\right\|_{L_{r,W}(\Omega)}^{\tilde{r}} \leq \sum_{j=1}^{l} \|F_{j}\|_{L_{r,W}(\Omega)}^{\tilde{r}}, \qquad F_{j} \in L_{r,W}(\Omega), \qquad 1 \leq j \leq l,$$
(1.1)

where $l \in \mathbb{N}$ or $l = \infty$ and $\tilde{r} := \min\{1, r\}$ for $r \in (0, \infty]$.

Throughout the paper we assume that $W : I \to (0, \infty)$ is an exponential weight of the form $W(x) = \exp[-Q(x)]$ defined on a bounded or unbounded interval $I = (-c, c), 0 < c \leq \infty$, where Q is a continuous function on I.

A function $F: (0, c) \to (0, \infty)$ is said to be quasi-increasing if there exists a constant C > 0 such that $F(x) \leq CF(y), 0 < x \leq y < c$. The following definition describes the class of weights that we use in this paper (see [16, Definition 1.1] and [7, Definition 1.4.5]).

Definition 1.1. Let $W = e^{-Q}$, where $Q: I \to [0, \infty)$ satisfies the following properties:

- (a) Q is even in I and Q(0) = 0.
- (b) Q' is continuous in I.
- (c) Q''(x) exists and $Q''(x) > 0, x \in (0, c)$.
- (d) $\lim_{x \to c-} Q(x) = \infty$.
- (e) The function

$$T(x) := \frac{xQ'(x)}{Q(x)}, \qquad x \neq 0,$$
 (1.2)

is quasi-increasing in (0, c) with

$$\Lambda := \inf_{x \in (0,c)} T(x) > 1.$$
(1.3)

(f) There exists a constant C > 0 such that for all $x \in (0, c)$,

$$\frac{Q''(x)}{|Q'(x)|} \le C \frac{|Q'(x)|}{Q(x)}$$

Then we write $W \in \mathcal{F}(C^2)$.

Note that properties (a), (b), and (c) of Definition 1.1 imply that Q is positive and increasing on (0, c). More classes of weights are discussed in [16, Sect. 1.2] and [7, Sect. 1.4].

The behaviour of the function T defined by (1.2) divides $\mathcal{F}(C^2)$ into two classes. In the case $I = \mathbb{R}$, a weight $W \in \mathcal{F}(C^2)$, satisfying the condition $\sup_{x \in \mathbb{R}} T(x) < \infty$, is called *a Freud weight*. A typical example of such a weight is

$$W_{\alpha}(x) := \exp(-|x|^{\alpha}), \qquad \alpha > 1, \quad I = \mathbb{R}.$$

A weight $W \in \mathcal{F}(C^2)$, satisfying the condition $\sup_{x \in \mathbb{R}} T(x) = \lim_{x \to \infty} T(x) = \infty$, is called an Erdös weight. In particular, any weight $W \in \mathcal{F}(C^2)$ on a bounded interval (-c, c) is an Erdös weight. A typical example of an Erdös weight for the unbounded interval is

$$W_{\alpha,\ell}(x) := \exp\left(-\exp_{\ell}(|x|^{\alpha}) + \exp_{\ell}(0)\right), \qquad \alpha > 1, \quad \ell \ge 1, \quad I = \mathbb{R},$$

where $\exp_0(x) := x$ and $\exp_k(x) := \exp\left(\exp_{k-1}(x)\right), 1 \le k \le \ell.$

For $I = \mathbb{R}$ and a weight $W = e^{-Q}$, Q has at most polynomial growth on \mathbb{R} if W is a Freud weight, and Q has faster than polynomial growth on \mathbb{R} if W is an Erdös weight. These and other properties of Freud and Erdös weights along with more examples can be found in [16] (see also [7]).

Next, we define two constants that play an important role in orthogonal polynomials for and weighted approximation with exponential weights. Let $W = e^{-Q} \in \mathcal{F}(C^2)$ and let $a_n = a_n(Q) \in$ (0, c) be the *n*th Mhaskar–Rakhmanov–Saff number defined as the positive root of the equation

$$n = \frac{2}{\pi} \int_0^1 \frac{a_n x Q'(a_n x)}{\sqrt{1 - x^2}} \, dx, \qquad n \in \mathbb{N},$$

(see [18, 22, 19, 16]). Further, the number

$$b_n = b_n(Q) := \frac{2}{\pi} \int_0^1 \frac{Q'(a_n x)\sqrt{1 - x^2}}{x} \, dx + \frac{n}{a_n}, \qquad n \in \mathbb{N},$$

was defined in [7, Eq. 1.2.2] in order to replace n in sharp constants of nonweighted approximation theory. Note that for Freud weights,

$$n/a_n \le b_n \le (1+C)n/a_n, \qquad n \in \mathbb{N},\tag{1.4}$$

and for Erdös weights,

$$b_n = (n/a_n)(1 + o(1)) \tag{1.5}$$

as $n \to \infty$. We also note that

$$\lim_{n \to \infty} b_n = \infty \tag{1.6}$$

and

$$b_n \le Cn, \qquad n \in \mathbb{N}.$$
 (1.7)

Relations (1.4), (1.5), and (1.6) were proved in [7, Proposition 3.2.2], while (1.7) follows from (1.4), (1.5), and increasing of a_n , $n \in \mathbb{N}$ (see [16, Lemma 2.13]).

For example (see [7, Sects. 10.1 and 10.5]) for the weight W_{α} ,

$$a_n = \left(\frac{2^{\alpha-2}\Gamma^2(\alpha/2)}{\Gamma(\alpha)}\right)^{1/\alpha} n^{1/\alpha}, \qquad b_n = \frac{\alpha n}{(\alpha-1)a_n} = \frac{\alpha}{(\alpha-1)} \left(\frac{\Gamma(\alpha)}{2^{\alpha-2}\Gamma^2(\alpha/2)}\right)^{1/\alpha} n^{1-1/\alpha},$$

where $\Gamma(z)$ is the gamma function, and for the weight $W_{\alpha,\ell}$,

$$a_n = (\log_l(n))^{1/\alpha} (1 + o(1)), \qquad b_n = \frac{n}{(\log_l(n))^{1/\alpha}} (1 + o(1)), \qquad n \to \infty,$$

where $\log_0(x) := x$, $\log_k(x) = \log(\log_{k-1}(x))$, $k \in \mathbb{N}$. For a bounded interval, $a_n = c(1+o(1))$ and $b_n = (n/c)(1+o(1))$ as $n \to \infty$. Similarly in the nonweighted case (W = 1) and for the interval I = (-1, 1), we assume that $a_n = 1 + o(1)$ and $b_n = n(1+o(1))$ as $n \to \infty$.

V. A. Markov–Bernstein–Nikolskii Type Inequalities. Next, we define sharp constants in univariate V. A. Markov–Bernstein–Nikolskii type inequalities for algebraic polynomials and entire functions of exponential type. Let

$$M_{p,N,n}(W) := b_n^{-N-1/p} \sup_{P \in \mathcal{P}_n \setminus \{0\}} \frac{|P^{(N)}(0)|}{\|P\|_{L_{p,W}(I)}},$$
(1.8)

$$M_{p,N,n}^{*}(W) := b_{n}^{-N-1/p} \sup_{P \in \mathcal{P}_{n} \setminus \{0\}} \frac{\left|P^{(N)}(0)\right|}{\|P\|_{L_{p,W}([-a_{n},a_{n}])}},$$
(1.9)

$$E_{p,N} := \sigma^{-N-1/p} \sup_{f \in (B_{\sigma} \cap L_p(\mathbb{R})) \setminus \{0\}} \frac{\|f^{(N)}\|_{L_{\infty}(\mathbb{R})}}{\|f\|_{L_p(\mathbb{R})}}.$$
 (1.10)

Here, $p \in (0, \infty]$, $N \in \mathbb{Z}_+$, and $n \in \mathbb{N}$. Note that $E_{p,N}$ is a nonweighted sharp constant, and it does not depend on σ (see [13] for the proof). Therefore, we can assume that $\sigma = 1$ in (1.10). The exact values of $E_{p,N}$ are known only in the following cases (see [13, Sect. 1]):

$$E_{\infty,N} = 1, \qquad E_{2,N} = (\pi(2N+1))^{-1/2},$$
 (1.11)

while the close estimates $0.5409/\pi < E_{1,0} < 0.5484/\pi$ were proved by Gorbachev [14].

The first sharp constant in the nonweighted inequality for polynomial coefficients was found by V. A. Markov [17] (see also [20, Eq. (5.1.4.1)]) in the form $(I = (-1, 1) \text{ and } n \in \mathbb{N})$

$$M_{\infty,N,n}(1) = n^{-N} \begin{cases} \left| T_{n-1}^{(N)}(0) \right|, & n-N \text{ is odd,} \\ \left| T_{n}^{(N)}(0) \right|, & n-N \text{ is even} \end{cases} = (1+o(1))E_{\infty,N}$$
(1.12)

as $n \to \infty$, where $T_n \in \mathcal{P}_n$ is the Chebyshev polynomial of the first kind. Labelle [15] found $M_{2,N,n}(1)$ for I = (-1, 1), and it turns out that

$$M_{2,N,n}(1) = (1 + o(1))E_{2,N}$$
(1.13)

as $n \to \infty$. The author [8, Theorem 1.1] extended (1.12) and (1.13) to any $p \in (0, \infty]$ in the form

$$\lim_{n \to \infty} M_{p,N,n}(1) = E_{p,N}, \qquad I = (-1,1).$$
(1.14)

Multivariate versions of (1.14) were obtained in [9, 10, 12].

Certain properties of $M_{p,0,n}(W)$ for ultraspherical weights were discussed by Arestov and Deikalova [2]. Some asymptotics for sharp constants in V. A. Markov–Bernstein–Nikolskii type inequalities with power and ultraspherical weights were obtained by the author [9, Theorems 4.1 and 4.3].

In this paper we obtain a weighted version of relation (1.14) for exponential weights. The following estimates for $W \in \mathcal{F}(C^2)$ and more general weights were obtained by the author [7, Theorem 2.3.2 (b)] (see also Lemma 2.6):

$$M_{p,N,n}^{*}(W) \le M_{p,N,n}(W) \le C\sqrt{N+1}(1-\varepsilon)^{-N}, \quad \varepsilon \in (0,1), \quad 0 \le N \le n, \quad n \in \mathbb{N}.$$
 (1.15)

Main Results and Remarks. Our major result discusses the limit relations between $M_{p,N,n}(W)$, $M_{p,N,n}^*(W)$, and $E_{p,N}$. In particular, we find an asymptotic behaviour of the sharp constants in inequality (1.15).

Theorem 1.2. If $W \in \mathcal{F}(C^2)$, $N \in \mathbb{Z}_+$, and $p \in (0, \infty]$, then

$$\lim_{n \to \infty} M_{p,N,n}(W) = \lim_{n \to \infty} M_{p,N,n}^*(W) = E_{p,N}.$$
(1.16)

Combining Theorem 1.2 with relations (1.11), we arrive at the following corollary:

Corollary 1.3. If $W \in \mathcal{F}(C^2)$ and $N \in \mathbb{Z}_+$, then

$$\lim_{n \to \infty} M_{\infty,N,n}(W) = \lim_{n \to \infty} M_{\infty,N,n}^*(W) = 1,$$
$$\lim_{n \to \infty} M_{2,N,n}(W) = \lim_{n \to \infty} M_{2,N,n}^*(W) = (\pi (2N+1))^{-1/2}.$$

Remark 1.4. In definitions (1.8), (1.9) and (1.10) of the sharp constants we discuss only complexvalued functions P and f. We can define similarly the "real" sharp constants if the suprema in (1.8), (1.9) and (1.10) are taken over all real-valued functions on \mathbb{R} from $\mathcal{P}_n \setminus \{0\}$ and $(B_\sigma \cap L_p(\mathbb{R})) \setminus \{0\}$, respectively. It turns out that the "complex" and "real" sharp constants coincide. For W = 1, I =(-1, 1), this fact was proved in [8, Sect. 1] (cf. [13, Theorem 1.1] and [10, Remark 1.5]), and the case of exponential weights can be proved similarly. In addition, Theorem 1.2 is formulated for the "complex" sharp constants. However, this result remains valid for the "real" ones as well. The proof of the real version of Theorem 1.2 does not change compared with the complex one if we take into account Remark 2.5 from Section 2.

Remark 1.5. Theorem 1.2 shows that the sharp constants in the weighted L_p -inequalities for the Nth coefficient of a polynomial are asymptotically equal to $E_{p,N}b_n^{N+1/p}/N!$, where by (1.4) and (1.5), $b_n \sim n/a_n$ and for Erdös weights $b_n = (n/a_n)(1 + o(1))$, as $n \to \infty$. Note that the sharp dependence on n of the sharp constant in the A. A. Markov–Nikolskii type inequality with an

exponential weight $W \in \mathcal{F}(C^2)$ is supposed to be $((n/a_n)\sqrt{T(a_n)})^{N+1/p}$ (see [16, Corollary 10.2 and Theorem 10.3]) compared with $(n/a_n)^{N+1/p}$ in Theorem 1.2. An asymptotic for this constant is unknown. However, the asymptotic behaviour of the sharp constant in the classical nonweighted inequality of different metrics was found in [9, Corollary 4.6] in the following form:

$$\lim_{n \to \infty} n^{-2/p} \sup_{P \in \mathcal{P}_n \setminus \{0\}} \frac{\|P\|_{L_{\infty}([-1,1])}}{\|P\|_{L_p([-1,1])}} = 2^{1/p} \sup_{f \in (B_1 \cap L_{p,W^*}(\mathbb{R})) \setminus \{0\}} \frac{|f(0)|}{\|f\|_{L_{p,W^*}(\mathbb{R})}}, \qquad p \in [1,\infty), \quad (1.17)$$

where $W^*(x) := |x|^{1/p}$. A different version of (1.17) for $p \in (0, \infty)$ was proved in [8, Theorem 1.4] (see also [8, p. 94]).

The proof of Theorem 1.2 is presented in Section 3. It follows general ideas developed in [11, Corollary 7.1]. Section 2 contains certain properties of functions from B_{σ} and polynomials from \mathcal{P}_n .

2. PROPERTIES OF ENTIRE FUNCTIONS AND POLYNOMIALS

To prove Theorem 1.2, we need several lemmas about certain properties of entire functions of exponential type and algebraic polynomials. We start with known properties of entire functions of exponential type.

Lemma 2.1. (a) The following crude Bernstein and Nikolskii type inequalities hold true:

$$\left\|f^{(s)}\right\|_{L_{\infty}(\mathbb{R})} \le C \left\|f\right\|_{L_{\infty}(\mathbb{R})}, \qquad f \in B_{\sigma} \cap L_{\infty}(\mathbb{R}), \quad s \in Z_{+},$$
(2.1)

$$\|f\|_{L_{\infty}(\mathbb{R})} \le C \|f\|_{L_{p}(\mathbb{R})}, \qquad f \in B_{\sigma} \cap L_{p}(\mathbb{R}), \quad p \in (0, \infty),$$
(2.2)

where C is independent of f. (b) If $f \in B_{\sigma} \cap L_p(\mathbb{R}), p \in (0, \infty)$, then

$$\lim_{|x| \to \infty} f(x) = 0. \tag{2.3}$$

Proof. The proofs of (2.1) and (2.2) can be found in [4, Theorem 11.3.3] and [23, Eq. 4.9(29)], respectively. The proof of a multivariate version of statement (b), given in [21, Theorem 3.2.5] for $p \in [1, \infty)$, is long and difficult. For the reader's convenience, we present a shorter and more elementary proof of (b) for $p \in (0, \infty)$.

If (2.3) is not valid, then there exist $\varepsilon > 0$ and a number sequence $\{x_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} |x_n| = \infty$ and $\inf_{n\in\mathbb{N}} |f(x_n)| \ge \varepsilon$. Without loss of generality we can assume that $0 < x_1 < x_2 < \ldots$. Setting $x_{n_1} := x_1$ and $y_0 := 0$ and recalling that $f \in L_p(\mathbb{R})$ and f is continuous on \mathbb{R} , we can construct

by induction a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ and a sequence $\{y_k\}_{k=1}^{\infty}$ such that

$$\lim_{k \to \infty} x_{n_k} = \infty, \ y_k > x_{n_k} > y_{k-1}, \ |f(x_{n_k})| \ge \varepsilon, \ |f(y_k)| = \varepsilon/2, \ \inf_{x \in [x_{n_k}, y_k]} |f(x)| \ge \varepsilon/2, \ k \in \mathbb{N}.$$

Next, setting $\lambda_k := y_k - x_{n_k}, k \in \mathbb{N}$, we obtain

$$(\varepsilon/2)^p \sum_{k=1}^{\infty} \lambda_k \le \sum_{k=1}^{\infty} \int_{x_{n_k}}^{y_k} |f(x)|^p dx \le \|f\|_{L_p(\mathbb{R})}^p.$$

Therefore, $\lim_{k\to\infty} \lambda_k = 0$, while by (2.1) and (2.2),

$$\varepsilon/2 \le |f(x_{n_k}) - f(y_k)| \le ||f'||_{L_{\infty}(\mathbb{R})} \lambda_k \le C ||f||_{L_p(\mathbb{R})} \lambda_k, \quad k \in \mathbb{N}.$$

This contradiction proves statement (b).

Next, we need the following version of the compactness theorem for entire functions of exponential type.

Lemma 2.2. Let \mathcal{E} be the set of all entire functions $f(z) = \sum_{k=0}^{\infty} c_k z^k$, satisfying the following condition: for any $\delta > 0$ there exists a constant $C(\delta)$, independent of f and k, such that

$$|c_k| \le \frac{C(\delta)(1+\delta)^k}{k!}, \qquad k \in \mathbb{Z}_+.$$
(2.4)

Then for any sequence $\{f_n\}_{n=1}^{\infty} \subseteq \mathcal{E}$ there exist a subsequence $\{f_{n_m}\}_{m=1}^{\infty}$ and a function $f_0 \in B_1$ such that for every $s \in \mathbb{Z}_+$, $\lim_{m\to\infty} f_{n_m}^{(s)} = f_0^{(s)}$ uniformly on each compact subset of \mathbb{C} .

The lemma was proved in [8, Lemma 2.6].

Further, we discuss estimates of the error of weighted polynomial approximation for functions from B_{τ} .

Lemma 2.3. Let $W \in \mathcal{F}(C^2)$. Then there exist numbers $\delta_1 = \delta_1(W) \in \left(0, \frac{2(\Lambda-1)}{3(\Lambda+1)}\right)$, $\delta_2 = \delta_2(W) > 0$, and a constant $C_1 = C_1(W) > 0$ such that for every $\tau \in \left(0, b_n \left(1 - C_1 n^{-\delta_1}\right)\right]$, any $g \in B_{\tau} \cap L_{\infty}(\mathbb{R})$, and each $k \in \mathbb{N}$ there exists a polynomial $P_k \in \mathcal{P}_k$ such that the following estimate holds:

$$\|g - P_k\|_{L_{r,W}(I)} \le C_2 k^{\gamma} \exp\left(-C_3 k^{\delta_2}\right) \|g\|_{L_{\infty}(\mathbb{R})}, \qquad 0 < r \le \infty.$$
(2.5)

Here, Λ is defined by (1.3), and C_2 , C_3 , and γ are positive constants independent of k and g.

This lemma follows from a more general result proved in [7, Theorem 2.2.1]. Lemma 2.3 is a weighted version of estimates obtained by Bernstein [3] (see also [23, Sect. 5.4.4] and [1, Appendix, Sect. 85]). More precise and more general nonweighted inequalities were proved by the author in [5] and [6].

Lemma 2.4. Let $W \in \mathcal{F}(C^2)$. Then for any $\tau \in (0, b_n (1 - C_1 n^{-\delta_1})], g \in B_{\tau} \cap L_{\infty}(\mathbb{R})$ with $\|g\|_{L_{\infty}(\mathbb{R})} \leq C_4$, and $n \in \mathbb{N}$, there exists a polynomial $P_n \in \mathcal{P}_n$ such that for every $s \in \mathbb{Z}_+$, $r \in (0, \infty]$, and $\eta \geq 0$,

$$\lim_{n \to \infty} n^{\eta} \left\| g^{(s)} - P_n^{(s)} \right\|_{L_{r,W}(I)} = 0.$$
(2.6)

Here, C_1 and δ_1 are the constants from Lemma 2.3 and the constants C_4 and η are independent of n.

Proof. First of all, for $P \in \mathcal{P}_k$, $k \in \mathbb{N}$, and $r \in (0, \infty]$ we need the following crude Markov-type inequality:

$$\|P'\|_{L_{r,W}(I)} \le C_5(r,W) \, k \, \|P\|_{L_{r,W}(I)}.$$
(2.7)

This inequality immediately follows from the estimates

$$||P'||_{L_{r,W}(I)} \le C(k/a_k)\sqrt{T(a_k)}||P||_{L_{r,W}(I)}$$

(see [16, Corollary 10.2]) and $T(a_k) \leq Ca_k^2$, $k \in \mathbb{N}$ (see inequality (3.38) in [16, Lemma 3.7]). Here, *T* is defined by (1.2), and *C* are constants independent of *k* and *P*.

Next, let $\{P_k\}_{k=1}^{\infty}$ be the sequence of polynomials from Lemma 2.3. Then using triangle inequality (1.1) and inequalities (2.7) and (2.5), we obtain

$$\begin{split} n^{\eta \tilde{r}} \left\| g^{(s)} - P_{n}^{(s)} \right\|_{L_{r,W}(I)}^{\tilde{r}} &\leq n^{\eta \tilde{r}} \sum_{k=n}^{\infty} \left\| (P_{k} - P_{k+1})^{(s)} \right\|_{L_{r,W}(I)}^{\tilde{r}} \\ &\leq C_{5}^{s \tilde{r}} n^{\eta \tilde{r}} \sum_{k=n}^{\infty} (k+1)^{s \tilde{r}} \left\| P_{k} - P_{k+1} \right\|_{L_{r,W}(I)}^{\tilde{r}} \\ &\leq C_{5}^{s \tilde{r}} n^{\eta \tilde{r}} \sum_{k=n}^{\infty} (k+1)^{s \tilde{r}} \left(\left\| g - P_{k} \right\|_{L_{r,W}(I)}^{\tilde{r}} + \left\| g - P_{k+1} \right\|_{L_{r,W}(I)}^{\tilde{r}} \right) \\ &\leq 2C_{2}^{\tilde{r}} C_{5}^{s \tilde{r}} n^{\eta \tilde{r}} \sum_{k=n}^{\infty} (k+1)^{(s+\gamma)\tilde{r}} \exp\left(-C_{3} \tilde{r} k^{\delta_{2}} \right) \left\| g \right\|_{L_{\infty}(\mathbb{R})}^{\tilde{r}} \\ &\leq C_{6} n^{\eta \tilde{r}} \int_{n}^{\infty} y^{(s+\gamma)\tilde{r}} \exp\left(-C_{3} \tilde{r} y^{\delta_{2}} \right) dy \left\| g \right\|_{L_{\infty}(\mathbb{R})}^{\tilde{r}} \\ &\leq C_{7} C_{4}^{\tilde{r}} n^{(s+\gamma+\eta)\tilde{r}} \exp\left(-C_{3} \tilde{r} n^{\delta_{2}} \right), \end{split}$$

where C_2 , C_3 , and δ_2 are constants from Lemma 2.3 and C_6 and C_7 are constants independent of n. Thus (2.6) is established.

Remark 2.5. Note that if g is a real-valued entire function in Lemmas 2.1 and 2.2, then polynomials $P_n, n \in \mathbb{N}$, can be chosen real-valued as well.

We also need a weighted estimate for coefficients of a polynomial.

Lemma 2.6. Let $W \in \mathcal{F}(C^2)$ and $p \in (0, \infty]$. Then for every polynomial $P(x) = \sum_{k=0}^{n} c_k x^k$ and any $\varepsilon \in (0, 1)$, the following inequality holds true:

$$|c_k| \le C_8(\varepsilon, p, W) \, \frac{b_n^{k+1/p}}{(1-\varepsilon)^k \, k!} \|P\|_{L_{p,W}([-a_n, a_n])}, \qquad 0 \le k \le n.$$
(2.8)

Proof. The inequality

$$|c_k| \le C_9(\varepsilon, p, W) \, \frac{\sqrt{k+1} \, b_n^{k+1/p}}{(1-\varepsilon/2)^k \, k!} \|P\|_{L_{p,W}([-a_n, a_n])}, \qquad \varepsilon \in (0, 1), \quad 0 \le k \le n, \tag{2.9}$$

was proved in [7, Theorem 2.3.2] for more general weights (see also (1.15)). Then (2.8) follows from (2.9) and an elementary inequality

$$\frac{\sqrt{k+1}}{(1-\varepsilon/2)^k} \le \frac{C_{10}(\varepsilon)}{(1-\varepsilon)^k}, \qquad k \ge 0.$$

3. Proof of Theorems 1.2

We first prove the inequality

$$E_{p,N} \le \liminf_{n \to \infty} M_{p,N,n}(W). \tag{3.1}$$

Let f be a function from $B_1 \cap L_p(\mathbb{R})$, $p \in (0, \infty]$. Then $f \in B_1 \cap L_\infty(\mathbb{R})$ by (2.2), and $f^{(N)} \in B_1 \cap L_p(\mathbb{R})$ by (2.1) and (2.2).

Let us first discuss the case $p \in (0, \infty)$. Then by Lemma 2.1 (b), there exists $x_0 \in \mathbb{R}$ such that $\|f^{(N)}\|_{L_{\infty}(\mathbb{R})} = |f^{(N)}(x_0)|$. Without loss of generality we can assume that $x_0 = 0$. Next, setting $\beta_n := b_n (1 - C_1 n^{-\delta_1})$, we see that the function $g_n(x) := f(\beta_n x)$ belongs to $B_{\beta_n} \cap L_{\infty}(\mathbb{R})$ with $\|g_n\|_{L_{\infty}(\mathbb{R})} = \|f\|_{L_{\infty}(\mathbb{R})}$, $n \in \mathbb{N}$. In addition, recall that W(0) = 1 (by Definition 1.1), and $b_n \leq Cn, n \in \mathbb{N}$ (by (1.7)). Therefore, by Lemma 2.4 for $r = \infty$, s = N, $\eta = 0$ and r = p, s = 0, $\eta = 1/p$, there exists a sequence of polynomials $\{P_n\}_{n=1}^{\infty}$ such that

$$\lim_{n \to \infty} \left| g_n^{(N)}(0) - P_n^{(N)}(0) \right| \le \lim_{n \to \infty} \left\| g_n^{(N)} - P_n^{(N)} \right\|_{L_{\infty,W}(I)} = 0,$$
(3.2)

$$\lim_{n \to \infty} b_n^{1/p} \|g_n - P_n\|_{L_{p,W}(I)} = 0.$$
(3.3)

Then using (3.2) and (1.6) and taking account of definition (1.8), we obtain

$$\begin{split} \left\| f^{(N)} \right\|_{L_{\infty}(\mathbb{R})} &= \left| f^{(N)}(0) \right| = \beta_{n}^{-N} \left| g_{n}^{(N)}(0) \right| = \liminf_{n \to \infty} b_{n}^{-N} \left| g_{n}^{(N)}(0) \right| \\ &\leq \lim_{n \to \infty} b_{n}^{-N} \left| g_{n}^{(N)}(0) - P_{n}^{(N)}(0) \right| + \liminf_{n \to \infty} b_{n}^{-N} \left| P_{n}^{(N)}(0) \right| \\ &= \liminf_{n \to \infty} b_{n}^{-N} \left| P_{n}^{(N)}(0) \right| \leq \liminf_{n \to \infty} \left(M_{p,N,n}(W) b_{n}^{1/p} \left\| P_{n} \right\|_{L_{p,W}(I)} \right). \quad (3.4) \end{split}$$

Next, using triangle inequality (1.1) and (3.3), we have

$$\limsup_{n \to \infty} b_n^{\tilde{p}/p} \|P_n\|_{L_{p,W}(I)}^{\tilde{p}} \le \limsup_{n \to \infty} b_n^{\tilde{p}/p} \left(\|g_n - P_n\|_{L_{p,W}(I)}^{\tilde{p}} + \|g_n\|_{L_{p,W}(I)}^{\tilde{p}} \right) \le \|f\|_{L_p(\mathbb{R})}^{\tilde{p}}.$$
 (3.5)

Combining (3.4) with (3.5), we arrive at (3.1) for $p \in (0, \infty)$.

In the case $p = \infty$, for any $\varepsilon > 0$ there exists $x_0 \in \mathbb{R}$ such that $\|f^{(N)}\|_{L_{\infty}(\mathbb{R})} < (1+\varepsilon) |f^{(N)}(x_0)|$. Without loss of generality we can assume that $x_0 = 0$. Then similarly to (3.4) and (3.5) we can obtain the inequality

$$\left\|f^{(N)}\right\|_{L_{\infty}(\mathbb{R})} < (1+\varepsilon) \liminf_{n \to \infty} M_{\infty,N,n}(W) \|f\|_{L_{\infty}(\mathbb{R})}.$$
(3.6)

Finally letting $\varepsilon \to 0+$ in (3.6), we arrive at (3.1) for $p = \infty$. This completes the proof of (3.1).

Further, we will prove the inequality

$$\limsup_{n \to \infty} M_{p,N,n}^*(W) \le E_{p,N} \tag{3.7}$$

by constructing a nontrivial function $f_0 \in B_1 \cap L_p(\mathbb{R})$, such that

$$\limsup_{n \to \infty} M_{p,N,n}^*(W) \le \left\| f_0^{(N)} \right\|_{L_{\infty}(\mathbb{R})} / \left\| f_0 \right\|_{L_p(\mathbb{R})} \le E_{p,N}.$$
(3.8)

Since

$$M_{p,N,n}(W) \le M_{p,N,n}^*(W),$$
(3.9)

inequalities (3.1) and (3.7) imply (1.16). It remains to construct a nontrivial function f_0 , satisfying (3.8). We first note that

$$\inf_{n \in \mathbb{N}} M_{p,N,n}^*(W) \ge C_{11}(p, N, W).$$
(3.10)

This inequality follows immediately from (3.9) and (3.1). Let $P_n \in \mathcal{P}_n$, $n \in \mathbb{N}$, be a polynomial, satisfying the equality

$$M_{p,N,n}^{*}(W) = \frac{\left|P_{n}^{(N)}(0)\right|}{b_{n}^{N+1/p} \left\|P_{n}\right\|_{L_{p,W}([-a_{n},a_{n}])}}.$$
(3.11)

The existence of an extremal polynomial P_n in (3.11) can be proved by the standard compactness argument (cf. [13, Proof of Theorem 1.5]). Next, setting $Q_n(x) := P_n(x/b_n) = \sum_{k=1}^n c_k x^k$, we have from (3.11) that

$$M_{p,N,n}^{*}(W) = \frac{\left|Q_{n}^{(N)}(0)\right|}{\|Q_{n}\|_{L_{p,W(\cdot/b_{n})}([-a_{n}b_{n},a_{n}b_{n}])}}.$$
(3.12)

Without loss of generality we can assume that

$$\left|Q_{n}^{(N)}(0)\right| = 1.$$
 (3.13)

Then it follows from (3.12), (3.13), and (3.10) that

$$\|Q_n\|_{L_{p,W(\cdot/b_n)}([-a_nb_n,a_nb_n])} = 1/M_{p,N,n}^*(W) \le 1/C_{11}(p,N,W).$$
(3.14)

Further, it follows from inequality (2.8) of Lemma 2.6 for $P = P_n$ and estimate (3.14) that for every $\varepsilon \in (0, 1)$ and any $k, 0 \le k \le n, n \in \mathbb{N}$, the following relations hold true:

$$|c_k| = \frac{\left|P_n^{(k)}(0)\right|}{b_n^k k!} \le \frac{C_8(\varepsilon, p, W) b_n^{1/p} \left\|P_n\right\|_{L_{p,W}([-a_n, a_n])}}{(1 - \varepsilon)^k k!} \le \frac{C_8(\varepsilon, p, W)}{C_{11}(p, N, W)(1 - \varepsilon)^k k!}.$$
(3.15)

Inequality (3.15) holds true for all $k \in \mathbb{Z}_+$ if we set $c_k = 0$ for $k > n, n \in \mathbb{N}$. Therefore, the polynomials $Q_n, n \in \mathbb{N}$, satisfy condition (2.4) of Lemma 2.2 with $\delta := \varepsilon/(1-\varepsilon)$ and $C(\delta) = C_8/C_{11}$. Thus Q_n belongs to a set \mathcal{E} of Lemma 2.2, $n \in \mathbb{N}$. Let $\{n_l\}_{l=1}^{\infty}$ be a subsequence of \mathbb{N} such that

$$\limsup_{n \to \infty} M_{p,N,n}^*(W) = \lim_{l \to \infty} M_{p,N,n_l}^*(W).$$
(3.16)

Then the polynomial sequence $\{Q_{n_l}\}_{l=1}^{\infty} \subseteq \mathcal{E}$ satisfies all the conditions of Lemma 2.2. Therefore, there exist a function $f_0 \in B_1$ and a subsequence $\{Q_{n_{l_m}}\}_{m=1}^{\infty}$ such that

$$\lim_{m \to \infty} Q_{n_{l_m}}^{(s)}(x) = f_0^{(s)}(x), \qquad 0 \le s \le N,$$
(3.17)

uniformly on any interval [-A, A], A > 0. Moreover, by (3.13) and (3.17),

$$\left| f_0^{(N)}(0) \right| = 1.$$
 (3.18)

We also need the following relations:

$$\lim_{n \to \infty} a_n b_n = \infty, \qquad \lim_{n \to \infty} \max_{x \in [-A,A]} (1 - W(x/b_n)) = 0.$$
(3.19)

The first relation in (3.19) follows from (1.4) and (1.5), while the second one follows from (1.6) for every fixed A > 0.

Next, using consecutively triangle inequality (1.1) and relations (3.17), (3.19), (3.12), (3.13), and (3.10), we obtain for any interval [-A, A], A > 0,

$$\|f_{0}\|_{L_{p}([-A,A])}^{\tilde{p}} \leq \limsup_{m \to \infty} \left(\|f_{0} - Q_{n_{l_{m}}}\|_{L_{p}([-A,A])}^{\tilde{p}} + \|Q_{n_{l_{m}}}\|_{L_{p}([-A,A])}^{\tilde{p}} \right)$$

$$= \limsup_{m \to \infty} \|Q_{n_{l_{m}}}\|_{L_{p,W}(\cdot/b_{n_{l_{m}}})}^{\tilde{p}} ([-A,A])$$

$$\leq \lim_{m \to \infty} \|Q_{n_{l_{m}}}\|_{L_{p,W}(\cdot/b_{n_{l_{m}}})}^{\tilde{p}} \left(\left[-a_{n_{l_{m}}}b_{n_{l_{m}}}, a_{n_{l_{m}}}b_{n_{l_{m}}} \right] \right)$$

$$= 1/\lim_{m \to \infty} M_{p,N,n_{l_{m}}}^{*}(W) \leq 1/C_{11}.$$

$$(3.20)$$

Therefore, letting $A \to \infty$ in (3.20), we see that f_0 is a nontrivial function from $B_1 \cap L_p(\mathbb{R})$, by (3.20) and (3.18). Thus for any interval [-A, A], A > 0, we obtain consecutively from (3.16), (3.13), (3.12), (3.19), (3.17), and (3.18)

$$\begin{split} \limsup_{n \to \infty} M_{p,N,n}^*(W) &= \lim_{m \to \infty} \|Q_{n_{l_m}}\|_{L_{p,W}(\cdot/b_{n_{l_m}})}^{-1} \left(\left[-a_{n_{l_m}} b_{n_{l_m}}, a_{n_{l_m}} b_{n_{l_m}} \right] \right) \\ &\leq \lim_{m \to \infty} \|Q_{n_{l_m}}\|_{L_{p,W}(\cdot/b_{n_{l_m}})}^{-1} (\left[-A,A \right]) \\ &= \lim_{m \to \infty} \|Q_{n_{l_m}}\|_{L_{p}(\left[-A,A \right])}^{-1} \\ &= \left| f_0^{(N)}(0) \right| / \|f_0\|_{L_{p}(\left[-A,A \right])} \leq \left\| f_0^{(N)} \right\|_{L_{\infty}(\mathbb{R})} / \|f_0\|_{L_{p}(\left[-A,A \right])}. \end{split}$$
(3.21)

Finally, letting $A \to \infty$ in (3.21), we arrive at (3.8).

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